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The Star of David rule

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Abstract

In this note, a new concept called *SDR-matrix* is proposed, which is an infinite lower triangular matrix obeying the generalized rule of David star. Some basic properties of *SDR*-matrices are discussed and two conjectures on *SDR*-matrices are presented, one of which states that if a matrix is a *SDR*-matrix, then so is its matrix inverse (if exists).

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Keywords: Narayana triangle; Pascal triangle; Lah triangle; *SDR*-matrix

1. Introduction

The *Star of David rule* (see [1,14,15] and references therein), originally stated by Gould in 1972, is given by

$$\binom{n}{k}\binom{n+1}{k-1}\binom{n+2}{k+1} = \binom{n}{k-1}\binom{n+1}{k+1}\binom{n+2}{k}$$

for any k and n, which implies that

$$\binom{n}{k+1}\binom{n+1}{k}\binom{n+2}{k+2} = \binom{n}{k}\binom{n+1}{k+2}\binom{n+2}{k+1}.$$

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In 2003, the author observed in his Master dissertation [13] that if multiplying the above two identities and dividing by $n(n+1)(n+2)$, one can arrive at

$$N_{n,k+1}N_{n+1,k}N_{n+2,k+2} = N_{n,k}N_{n+1,k+2}N_{n+2,k+1},$$

where $N_{n,k} = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$ is the Narayana Number [10, A001263].

In the summer of 2006, the author asked Mansour [5] for a combinatorial proof of the above Narayana identity to be found. Later, by Chen's bijective algorithm for trees [2], Li and Mansour [4] provided a combinatorial proof of a general identity

$$\begin{aligned} &N_{n,k+m-1}N_{n+1,k+m-2}N_{n+2,k+m-3}\cdots N_{n+m-2,k+1}N_{n+m-1,k}N_{n+m,k+m} \\ &\quad = N_{n,k}N_{n+1,k+m}N_{n+2,k+m-1}\cdots N_{n+m-2,k+3}N_{n+m-1,k+2}N_{n+m,k+1}. \end{aligned}$$

This motivates the author to reconsider the Star of David rule and to propose a new concept called *SDR-matrix* which obeys the generalized rule of David star.

Definition 1.1. Let $\mathscr{A} = (A_{n,k})_{n\geqslant k\geqslant 0}$ be an infinite lower triangular matrix, for any given integer $m \geqslant 3$, if there hold

$$\prod_{i=0}^{r} A_{n+i,k+r-i} \prod_{i=0}^{p-r-1} A_{n+p-i,k+r+i+1} = \prod_{i=0}^{r} A_{n+p-i,k+p-r+i} \prod_{i=0}^{p-r-1} A_{n+i,k+p-r-i-1}$$

for all $2 \leqslant p \leqslant m-1$ and $0 \leqslant r \leqslant p-1$, then $\mathscr{A}$ is called an *SDR-matrix of order m*.

In order to give a more intuitive view on the definition, we present a pictorial description of the generalized rule for the case $m = 5$. See Fig. 1.

Let SDR_m denote the set of SDR-matrices of order m and SDR_∞ be the set of SDR-matrices $\mathscr{A}$ of order ∞, that is $\mathscr{A} \in SDR_m$ for any $m \geqslant 3$. By our notation, it is obvious that the Pascal

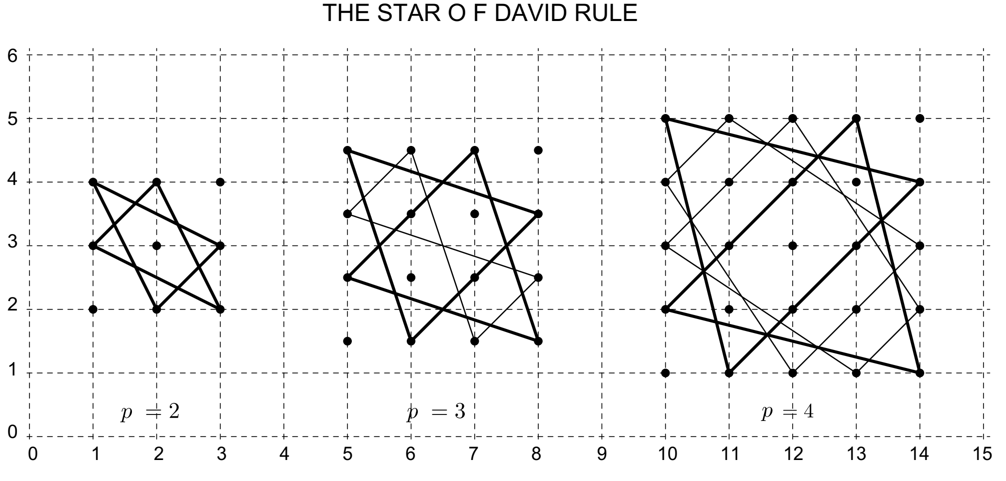

Fig. 1. The case $m = 5$.

triangle $\mathscr{P} = \left(\binom{n}{k}\right)_{n \geqslant k \geqslant 0}$ and the Narayana triangle $\mathscr{N} = (N_{n+1,k+1})_{n \geqslant k \geqslant 0}$ are SDR-matrices of order 3. In fact, both of them will be proved to be SDR-matrices of order ∞

$$\mathscr{P} = \begin{pmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 1 & 2 & 1 & & & \\ 1 & 3 & 3 & 1 & & \\ 1 & 4 & 6 & 4 & 1 & \\ 1 & 5 & 10 & 10 & 5 & 1 \\ & & \cdots & & & \end{pmatrix}, \qquad \mathscr{N} = \begin{pmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 1 & 3 & 1 & & & \\ 1 & 6 & 6 & 1 & & \\ 1 & 10 & 20 & 10 & 1 & \\ 1 & 15 & 50 & 50 & 15 & 1 \\ & & \cdots & & & \end{pmatrix}.$$

In this paper, we will discuss some basic properties of the sets SDR_m and propose two conjectures on SDR_m for $3 \leqslant m \leqslant \infty$ in the next section. We also give some comments on relations between SDR-matrices and Riordan arrays in Section 3.

2. The basic properties of SDR-matrices

For any infinite lower triangular matrices $\mathscr{A} = (A_{n,k})_{n \geqslant k \geqslant 0}$ and $\mathscr{B} = (B_{n,k})_{n \geqslant k \geqslant 0}$, define $\mathscr{A} \circ \mathscr{B} = (A_{n,k}B_{n,k})_{n \geqslant k \geqslant 0}$ to be the Hadamard product of $\mathscr{A}$ and $\mathscr{B}$, denote by $\mathscr{A}^{\circ j}$ the jth Hadamard power of $\mathscr{A}$; If $A_{n,k} \neq 0$ for $n \geqslant k \geqslant 0$, then define $\mathscr{A}^{\circ(-1)} = (A_{n,k}^{-1})_{n \geqslant k \geqslant 0}$ to be the Hadamard inverse of $\mathscr{A}$.

From Definition 1.1, one can easily derive the following three lemmas.

Lemma 2.1. *For any $\mathscr{A} \in SDR_m$, $\mathscr{B} \in SDR_{m+i}$ with $i \geqslant 0$, there hold $\mathscr{A} \circ \mathscr{B} \in SDR_m$, and $\mathscr{A}^{\circ(-1)} \in SDR_m$ if it exists.*

Lemma 2.2. *For any $\mathscr{A} = (A_{n,k})_{n \geqslant k \geqslant 0} \in SDR_m$, then $(A_{n+i,k+j})_{n \geqslant k \geqslant 0} \in SDR_m$ for fixed $i, j \geqslant 0$.*

Lemma 2.3. *Given any sequence $(a_n)_{n \geqslant 0}$, let $A_{n,k} = a_n$, $B_{n,k} = a_k$ and $C_{n,k} = a_{n-k}$ for $n \geqslant k \geqslant 0$, then $(A_{n,k})_{n \geqslant k \geqslant 0}, (B_{n,k})_{n \geqslant k \geqslant 0}, (C_{n,k})_{n \geqslant k \geqslant 0} \in SDR_\infty$.*

Example 2.4. Let $a_n = n!$ for $n \geqslant 0$, then we have

$$\mathscr{P} = (n!)_{n \geqslant k \geqslant 0} \circ (k!)^{\circ(-1)}_{n \geqslant k \geqslant 0} \circ ((n-k)!)^{\circ(-1)}_{n \geqslant k \geqslant 0},$$
$$\mathscr{N} = \left(\frac{1}{k+1}\right)_{n \geqslant k \geqslant 0} \circ \mathscr{P} \circ \left(\binom{n+1}{k}\right)_{n \geqslant k \geqslant 0},$$
$$\mathscr{L} = ((n+1)!)_{n \geqslant k \geqslant 0} \circ \mathscr{P} \circ ((k+1)!)^{\circ(-1)}_{n \geqslant k \geqslant 0},$$

which, by Lemmas 2.1–2.3, produce that the Pascal triangle $\mathscr{P}$, the Narayana triangle $\mathscr{N}$ and the Lah triangle $\mathscr{L}$ belong to SDR_∞, where $(\mathscr{L})_{n,k} = \binom{n}{k}\frac{(n+1)!}{(k+1)!}$ is the Lah number [3].

Theorem 2.5. *For any sequences $(a_n)_{n \geqslant 0}$, $(b_n)_{n \geqslant 0}$ and $(c_n)_{n \geqslant 0}$ such that $b_0 = 1$, $a_n \neq 0$ and $c_n \neq 0$ for $n \geqslant 0$, let $\mathscr{A} = (a_k b_{n-k} c_n)_{n \geqslant k \geqslant 0}$, then $\mathscr{A}^{-1} \in SDR_\infty$.*

Proof. By Lemmas 2.1 and 2.3, we have $\mathscr{A} \in SDR_\infty$. It is not difficult to derive the matrix inverse $\mathscr{A}^{-1}$ of $\mathscr{A}$ with the generic entries

$$(\mathscr{A}^{-1})_{n,k} = a_n^{-1} B_{n-k} c_k^{-1},$$

where B_n with $B_0 = 1$ are given by

$$B_n = \sum_{j=1}^{n} (-1)^j \sum_{i_1+i_2+\cdots+i_j=n, i_1,\ldots,i_j \geqslant 1} b_{i_1} b_{i_2} \cdots b_{i_j} \quad (n \geqslant 1). \tag{2.1}$$

Hence, by Lemmas 2.1 and 2.3, one can deduce that

$$\mathscr{A}^{-1} = (a_n^{-1})_{n\geqslant k\geqslant 0} \circ (B_{n-k})_{n\geqslant k\geqslant 0} \circ (c_k^{-1})_{n\geqslant k\geqslant 0} \in SDR_\infty,$$

as desired. □

Specially, when $c_n := 1$ or $a_n := \frac{a_n}{n!}$, $b_n := \frac{b_n}{n!}$, $c_n := n!$, both $\mathscr{B} = (a_k b_{n-k})_{n\geqslant k\geqslant 0}$ and $\mathscr{C} = \left(\binom{n}{k} a_k b_{n-k}\right)_{n\geqslant k\geqslant 0}$ are in SDR_∞, then so $\mathscr{B}^{-1}$ and $\mathscr{C}^{-1}$. More precisely, let $a_n^{-1} = b_n^{-1} = c_n = n!(n+1)!$ for $n \geqslant 0$, note that the Narayana triangle $\mathscr{N} \in SDR_\infty$ and

$$N_{n+1,k+1} = \frac{1}{n+1}\binom{n+1}{k+1}\binom{n+1}{k} = \frac{n!(n+1)!}{k!(k+1)!(n-k)!(n-k+1)!}.$$

Then one has $\mathscr{N}^{-1} \in SDR_\infty$ by Theorem 2.5.

Theorem 2.5 suggests the following conjecture.

Conjecture 2.6. *For any $\mathscr{A} \in SDR_m$, if the inverse $\mathscr{A}^{-1}$ of $\mathscr{A}$ exists, then $\mathscr{A}^{-1} \in SDR_m$.*

Theorem 2.7. *For any sequences $(a_n)_{n\geqslant 0}$, $(b_n)_{n\geqslant 0}$ with $b_0 = 1$ and $a_n \neq 0$ for $n \geqslant 0$, let $\mathscr{A} = (a_n b_{n-k} a_k^{-1})_{n\geqslant k\geqslant 0}$, then the matrix power $\mathscr{A}^j \in SDR_\infty$ for any integer j.*

Proof. By Lemmas 2.1 and 2.3, we have $\mathscr{A} \in SDR_\infty$. Note that it is trivially true for $j = 1$ and $j = 0$ (where $\mathscr{A}^0$ is the identity matrix by convention). It is easy to obtain the (n,k)-entries of $\mathscr{A}^j$ for $j \geqslant 2$,

$$\begin{aligned}(\mathscr{A}^j)_{n,k} &= \sum_{k\leqslant k_{j-1}\leqslant\cdots\leqslant k_1\leqslant n} \mathscr{A}_{n,k_1}\mathscr{A}_{k_1,k_2}\cdots\mathscr{A}_{k_{j-2},k_{j-1}}\mathscr{A}_{k_{j-1},k} \\ &= a_n C_{n-k} a_k^{-1},\end{aligned}$$

where C_n with $C_0 = 1$ is given by $C_n = \sum_{i_1+i_2+\cdots+i_j=n, i_1,\ldots,i_j\geqslant 0} b_{i_1} b_{i_2} \cdots b_{i_j}$ for $n \geqslant 1$.

By Lemmas 2.1 and 2.3, one can deduce that

$$\mathscr{A}^j = (a_n)_{n\geqslant k\geqslant 0} \circ (C_{n-k})_{n\geqslant k\geqslant 0} \circ (a_k^{-1})_{n\geqslant k\geqslant 0} \in SDR_\infty.$$

By Theorem 2.5 and its proof, we have $\mathscr{A}^{-1} \in SDR_\infty$ and $(\mathscr{A}^{-1})_{n,k} = a_n B_{n-k} a_k^{-1}$, where B_n is given by (2.1). Note that $\mathscr{A}^{-1}$ has the form as required in Theorem 2.7, so by the former part of this proof, we have $\mathscr{A}^{-j} \in SDR_\infty$ for $j \geqslant 1$. Hence we are done. □

Let $a_n = b_n = n!$, $a_n = b_n = n!(n+1)!$ or $a_n = n!(n+1)!$ and $b_n^{-1} = n!$ for $n \geqslant 0$ in Theorem 2.7, one has

Corollary 2.8. *For $\mathscr{P}$, $\mathscr{N}$ and $\mathscr{L}$, then $\mathscr{P}^j$, $\mathscr{N}^j$, $\mathscr{L}^j \in SDR_\infty$ for any integer j.*

Remark 2.9. In general, for $\mathscr{A}, \mathscr{B} \in SDR_m$, their matrix product $\mathscr{A}\mathscr{B}$ is possibly not in SDR_m. For example, $\mathscr{P}, \mathscr{N} \in SDR_3$, but

$$\mathscr{P}\mathscr{N} = \begin{pmatrix} 1 & & & & & \\ 2 & 1 & & & & \\ 4 & 5 & 1 & & & \\ 8 & 18 & 9 & 1 & & \\ 16 & 56 & 50 & 14 & 1 & \\ 32 & 160 & 220 & 110 & 20 & 1 \\ & & \cdots & & & \end{pmatrix} \notin SDR_3.$$

Theorem 2.10. *For any $\mathscr{A} = (A_{n,k})_{n \geqslant k \geqslant 0}$ with $A_{n,k} \neq 0$ for $n \geqslant k \geqslant 0$, then $\mathscr{A} \in SDR_{m+1}$ if and only if $\mathscr{A} \in SDR_m$.*

Proof. Note that $SDR_{m+1} \subset SDR_m$, so the necessity is clear. It only needs to prove the sufficient condition. For the symmetry, it suffices to verify

$$\prod_{i=0}^{r} A_{n+i,k+r-i} \prod_{i=0}^{m-r} A_{n+m-i+1,k+r+i+1} = \prod_{i=0}^{r} A_{n+m-i+1,k+m-r+i+1} \prod_{i=0}^{m-r} A_{n+i,k+m-r-i}$$

for $0 \leqslant r \leqslant [m/2] - 1$. We just take the case $r = 0$ for example, others can be done similarly. It is trivial when $A_{n,k+m} = A_{n+1,k+m+1} = 0$. So we assume that $A_{n,k+m} \neq 0$, $A_{n+1,k+m+1} \neq 0$, then all $A_{n+i,k+j}$ to be considered, except for $A_{n,k+m+1}$, must not be zero. By Definition 1.1, we have

$$\begin{aligned} &A_{n+m-i,k+i} A_{n+m-i-1,k+i+1} A_{n+m-i+1,k+i+2} \\ &\quad = A_{n+m-i+1,k+i+1} A_{n+m-i,k+i+2} A_{n+m-i-1,k+i} \quad (0 \leqslant i \leqslant m-1), \end{aligned} \tag{2.2}$$

$$A_{n+m+1,k+m+1} \prod_{i=0}^{m-1} A_{n+i,k+m-i} = A_{n+1,k+1} \prod_{i=0}^{m-1} A_{n+m-i+1,k+i+2}, \tag{2.3}$$

$$A_{n+1,k+1} \prod_{i=0}^{m-1} A_{n+m-i,k+i+1} = A_{n+m,k+m} \prod_{i=0}^{m-1} A_{n+i+1,k+m-i-1}, \tag{2.4}$$

$$A_{n+m,k+m} \prod_{i=0}^{m-1} A_{n+i,k+m-i-1} = A_{n,k} \prod_{i=0}^{m-1} A_{n+m-i,k+i+1}. \tag{2.5}$$

Multiplying (2.2)–(2.5) together, after cancellation, one can get

$$A_{n,k} \prod_{i=0}^{m} A_{n+m-i+1,k+i+1} = A_{n+m+1,k+m+1} \prod_{i=0}^{m} A_{n+i,k+m-i},$$

which confirms the case $r = 0$. $\square$

Remark 2.11. The condition $A_{n,k} \neq 0$ for $n \geqslant k \geqslant 0$ in Theorem 2.10 is necessary. The following example verifies this claim

$$\left(\binom{\frac{n+k}{2}}{\frac{n-k}{2}}\right)_{n \geqslant k \geqslant 0} = \begin{pmatrix} 1 & & & & & \\ 0 & 1 & & & & \\ 1 & 0 & 1 & & & \\ 0 & 2 & 0 & 1 & & \\ 1 & 0 & 3 & 0 & 1 & \\ 0 & 3 & 0 & 4 & 0 & 1 \\ & & \cdots & & & \end{pmatrix} \in SDR_3, \text{ but not in } SDR_4.$$

Recall that the Narayana number $\mathscr{N}_{n+1,k+1}$ can be represented as

$$\mathscr{N}_{n+1,k+1} = \frac{1}{n+1}\binom{n+1}{k+1}\binom{n+1}{k} = \det\begin{pmatrix} \binom{n}{k} & \binom{n}{k+1} \\ \binom{n+1}{k} & \binom{n+1}{k+1} \end{pmatrix},$$

so we can come up with the following definition.

Definition 2.12. Let $\mathscr{A} = (A_{n,k})_{n \geqslant k \geqslant 0}$ be an infinite lower triangular matrix, for any integer $j \geqslant 1$, define $\mathscr{A}_{[j]} = (A^{[j]}_{n,k})_{n \geqslant k \geqslant 0}$, where

$$A^{[j]}_{n,k} = \det\begin{pmatrix} A_{n,k} & \cdots & A_{n,k+j-1} \\ \vdots & \cdots & \vdots \\ A_{n+j-1,k} & \cdots & A_{n+j-1,k+j-1} \end{pmatrix}.$$

Theorem 2.13. *For any sequences $(a_n)_{n \geqslant 0}$, $(b_n)_{n \geqslant 0}$ and $(c_n)_{n \geqslant 0}$ such that $b_0 = 1$, $a_n \neq 0$ and $c_n \neq 0$ for $n \geqslant 0$, let $\mathscr{A} = (a_k b_{n-k} c_n)_{n \geqslant k \geqslant 0}$, then $\mathscr{A}_{[j]} \in SDR_\infty$ for any integer $j \geqslant 1$.*

Proof. By Lemmas 2.1 and 2.3, we have $\mathscr{A} \in SDR_\infty$. It is easy to derive the determinant

$$\det\begin{pmatrix} a_k b_{n-k} c_n & \cdots & a_{k+j-1} b_{n-k-j+1} c_n \\ \vdots & \cdots & \vdots \\ a_k b_{n-k+j-1} c_{n+j-1} & \cdots & a_{k+j-1} b_{n-k} c_{n+j-1} \end{pmatrix} = B_{n-k} \prod_{i=0}^{j-1} a_{k+i} c_{n+i},$$

where B_n with $B_0 = 1$ are given by

$$B_n = \det\begin{pmatrix} b_n & \cdots & b_{n-j+1} \\ \vdots & \cdots & \vdots \\ b_{n+j-1} & \cdots & b_n \end{pmatrix}.$$

Hence, by Lemmas 2.1 and 2.3, one can deduce that

$$\mathscr{A}_{[j]} = \left(\prod_{i=0}^{j-1} a_{k+i}\right)_{n \geqslant k \geqslant 0} \circ (B_{n-k})_{n \geqslant k \geqslant 0} \circ \left(\prod_{i=0}^{j-1} c_{n+i}\right)_{n \geqslant k \geqslant 0} \in SDR_\infty,$$

as desired. □

Let $a_n^{-1} = b_n^{-1} = c_n = n!$, $a_n^{-1} = b_n^{-1} = c_n = n!(n+1)!$ or $a_n^{-1} = c_n = n!(n+1)!$ and $b_n^{-1} = n!$ for $n \geqslant 0$ in Theorem 2.13, one has

Corollary 2.14. *For $\mathscr{P}$, $\mathscr{N}$ and $\mathscr{L}$, then $\mathscr{P}_{[j]}, \mathscr{N}_{[j]}, \mathscr{L}_{[j]} \in SDR_\infty$ for any integer $j \geqslant 1$.*

Theorem 2.13 suggests the following conjecture.

Conjecture 2.15. *If $\mathscr{A} \in SDR_\infty$, then $\mathscr{A}_{[j]} \in SDR_\infty$ for any integer $j \geqslant 1$.*

Remark 2.16. The conjecture on SDR_m is generally not true for $3 \leqslant m < \infty$. For example, let $\mathscr{A} = (A_{n,k})_{n \geqslant k \geqslant 0}$ with $A_{n,k} = \binom{\frac{n+k}{2}}{\frac{n-k}{2}}$, then we have $\mathscr{A} \in SDR_3$, but

$$\mathscr{A}_{[2]} = \begin{pmatrix} 1 \\ -1 & 1 \\ \mathbf{2} & -2 & 1 \\ -2 & 6 & -\mathbf{3} & 1 \\ 3 & -\mathbf{9} & 12 & -4 & 1 \\ & & \cdots \end{pmatrix} \notin SDR_3,$$

$$\mathscr{A}_{[3]} = \begin{pmatrix} 1 \\ 0 & 1 \\ 2 & 0 & 1 \\ 0 & 15 & 0 & 1 \\ 9 & 0 & 36 & 0 & 1 \\ & & \cdots \end{pmatrix} \in SDR_3.$$

3. Further comments

We will present some further comments on the connections between SDR-matrices and Riordan arrays. The concept of Riordan array introduced by Shapiro et al [9], plays a particularly important role in studying combinatorial identities or sums and also is a powerful tool in study of many counting problems [6–8]. For examples, Sprugnoli [7,11,12] investigated Riordan arrays related to binomial coefficients, colored walks, Stirling numbers and Abel–Gould identities.

To define a Riordan array we need two analytic functions, $d(t) = d_0 + d_1 t + d_2 t^2 + \cdots$ and $h(t) = h_1 t + h_2 t^2 + \cdots$ A *Riordan array* is an infinite lower triangular array $\{d_{n,k}\}_{n,k \in \mathbb{N}}$, defined by a pair of formal power series $(d(t), h(t))$, with the generic element $d_{n,k}$ satisfying

$$d_{n,k} = [t^n] d(t)(h(t))^k \quad (n,\ k \geqslant 0).$$

Assume that $d_0 \neq 0 \neq h_1$, then $(d(t), h(t))$ is an element of the *Riordan group* [9], under the group multiplication rule:

$$(d(t), h(t))(g(t), f(t)) = (d(t)g(h(t)), f(h(t))).$$

This indicates that the identity is $I = (1, t)$, the usual matrix identity, and that

$$(d(t), h(t))^{-1} = \left(\frac{1}{d(\bar{h}(t))}, \bar{h}(t)\right),$$

where $\bar{h}(t)$ is the compositional inverse of $h(t)$, i.e., $\bar{h}(h(t)) = h(\bar{h}(t)) = t$.

By our notation, we have

$$\mathscr{P} = \left(\frac{1}{1-t}, \frac{t}{1-t}\right) \in SDR_\infty,$$

$$\mathscr{P}^j = \left(\frac{1}{1-jt}, \frac{t}{1-jt}\right) \in SDR_\infty,$$

$$\left(\binom{\frac{n+k}{2}}{\frac{n-k}{2}}\right)_{n\geqslant k\geqslant 0} = \left(\frac{1}{1-t^2}, \frac{t}{1-t^2}\right) \in SDR_3,$$

$$\left(\frac{1}{1-t^2}, \frac{t}{1-t^2}\right)^{-1} = \left(\frac{1-\sqrt{1-4t^2}}{2t^2}, \frac{1-\sqrt{1-4t^2}}{2t}\right) \in SDR_3,$$

$$(d_{n-k})_{n\geqslant k\geqslant 0} = (d(t), t) \in SDR_\infty.$$

Hence, it is natural to ask the following question.

Question 3.1. Given a formal power series $d(t)$, what conditions $h(t)$ should satisfy, such that $(d(t), h(t))$ forms an SDR-matrix.

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