ELSEVIER
Available online at www.sciencedirect.com
SCIENCE DIRECT®
Topology and its Applications 132 (2003) 17–27
TOPOLOGY AND ITS APPLICATIONS
www.elsevier.com/locate/topol

# Spaces $u$-equivalent to the $n$-cube

Rafał Górak

*Institute of Mathematics of the Polish Academy of Sciences, ul. Śniadeckich 8, P.O. Box 21, 00-956 Warszawa, Poland*

Received 6 May 2002; received in revised form 25 September 2002

**Abstract**

We give an internal characterization of spaces $X$ such that the space $C_p(X)$ of continuous real-valued functions on $X$, endowed with the pointwise convergence topology, is uniformly homeomorphic to the space $C_p(\boldsymbol{I}^n)$ of functions on the $n$-dimensional cube $\boldsymbol{I}^n = [0,1]^n$.


*MSC:* 54C35

*Keywords:* Function spaces; Pointwise topology; Uniform homeomorphisms; Cantor–Benedixson derivative

## 1. Introduction

For a completely regular space $X$, $C_p(X)$ denotes the space of all continuous real-valued functions on $X$, equipped with the pointwise convergence topology.

Spaces $X$ and $Y$ are called $u$-equivalent ($l$-equivalent) if spaces $C_p(X)$ and $C_p(Y)$ are uniformly (linearly) homeomorphic. We write $X \sim^u Y$ if the spaces $X$ and $Y$ are $u$-equivalent and $X \sim^l Y$ when $X$ and $Y$ are $l$-equivalent. Let us recall that the map $\varphi : E \to L$, where $E$ and $L$ are linear topological spaces, is uniformly continuous if for every neighborhood $U$ of zero in $L$ there is a neighborhood $V$ of zero in $E$ such that, for every $f, g \in E$ with $f - g \in V$ we have $\varphi(f) - \varphi(g) \in U$.

The aim of this paper is to prove the following characterization:

**Main Theorem 1.** *For every positive integer $n$, a space $X$ is $u$-equivalent to $\boldsymbol{I}^n$ if and only if the following conditions are satisfied*:

(a) *$X$ is $n$-dimensional, compact and metrizable,*

*E-mail address:* r.gorak@impan.gov.pl (R. Górak).

0166-8641/02/$ – see front matter 
doi:10.1016/S0166-8641(02)00359-0

(b) *every nonempty closed subset $A$ of $X$ contains a nonempty relatively open subset $U$ which can be embedded into the $n$-cube $\boldsymbol{I}^n$.*

The "only if" part of this theorem is already known and it is essentially due to Gul'ko (see [8]). More precisely, he proved in [8] that $u$-equivalence preserves the dimension. However, from the results in [8] also follows part (b) of the Main Theorem. Therefore, to prove the Main Theorem it is enough to construct a uniform homeomorphism between $C_p(X)$ and $C_p(\boldsymbol{I}^n)$ having in hand conditions (a) and (b). The construction is based on the technique of Gul'ko from the paper [7] where he proved that the relations of $u$- and $l$-equivalence are different even on the class of countable compacta. Namely he showed that all the countable compacta are mutually $u$-equivalent which is not true for the relation of $l$-equivalence (see [4]).

The problem of characterizing spaces which are $u$-equivalent to the $n$-cube was motivated by the question of Arhangel'skiĭ [3] (Problem 30) who asked about the similar (internal) characterization for $l$-equivalence. As the notion of $u$-equivalence is a generalization of the relation of $l$-equivalence, it seemed natural to consider such a problem. However, a satisfactory answer to the original problem of Arhangel'skiĭ has not been given yet. Some partial results can be found in [2,9–12] and [14]. In Section 4, we prove another partial result concerning Arhangel'skii's question. We also formulate the hypothesis how such a characterization can look like.

## 2. Preliminaries

Let us recall that the compactness is preserved by the $u$-equivalence (see [13]). Besides, if $X$ is $u$-equivalent to $\boldsymbol{I}^n$ then it must have a countable weight (see [1]). Thus without loss of generality we can assume that the space $X$ from Main Theorem is compact and metrizable. In the sequel $X$ will always denote such a space.

Denote by **Ord** and **Lim** the classes of all ordinals and all limit ordinals, respectively. We start with the following definition which generalizes the idea of Cantor–Benedixson derivative:

**Definition 2.1.** For every space $X$ we put: $I_n(X) = \bigcup_{s\in S} U_s$ where $\{U_s\colon s \in S\}$ is the family all open subsets of $X$ which can be embedded into $\boldsymbol{I}^n$.

**Definition 2.2.** For a given $n \in \mathbb{N}$ we define the $\alpha$th embedding derivative $X^{[\alpha,n]}$ in the following way:
$X^{[0,n]} = X$;
$X^{[\alpha+1,n]} = X^{[\alpha,n]} \setminus I_n(X^{[\alpha,n]})$;
$X^{[\alpha,n]} = \bigcap_{\beta<\alpha} X^{[\beta,n]}$ for $\alpha \in \mathbf{Lim}$.

As in the case of Cantor–Benedixson derivative, $(X^{[\alpha,n]})_{\alpha\in\mathbf{Ord}}$ stabilizes on some countable ordinal if the space $X$ has a countable base. So let us define this space at which our new derivative stabilizes.

**Definition 2.3.** $\mathrm{ed}_n(X) = \bigcap_{\alpha<\omega_1} X^{[\alpha,n]}$.

Let us point out that embedding derivative has the following obvious property which will be useful in the proof of the main theorem:

**Fact 2.4.** *For every subset $A$ of the space $X$ and for every $n \in \mathbb{N}$ we have $A^{[\alpha,n]} \subset X^{[\alpha,n]} \cap A$.*

Let us introduce the following relation:

**Definition 2.5** (*Gul'ko* [7]). Let $E$ and $F$ be linear topological spaces and $\|\cdot\|_1$, $\|\cdot\|_2$ be norms, on $E$ and $F$, respectively, not necessarily related to the topologies. We write $(E, \|\cdot\|_1) \doteq (F, \|\cdot\|_2)$ if, for every $\varepsilon > 0$, there exists a uniform homeomorphism $u_\varepsilon : E \to F$ satisfying the following condition:

$$(a_\varepsilon) \quad (1+\varepsilon)^{-1}\|f\|_1 \leqslant \|u_\varepsilon(f)\|_2 \leqslant \|f\|_1 \quad \text{for every } f \in E.$$

If it is clear which norms are considered on $E$ and $F$ we write $E \doteq F$.

This relation appeared for the first time in [7] and plays the key role in the proof of the main theorem.

Let us fix that for every two linear topological spaces $E$ and $F$ equipped with norms $\|\cdot\|_0$ and $\|\cdot\|_1$, respectively, on the space $E \times F$ we consider the norm $\|(e,f)\| = \max(\|e\|_0, \|f\|_1)$.

Let us recall the definition of $c_0$-product:

**Definition 2.6.** For every $i \in \mathbb{N}$, let $E_i$ be a linear topological space and $\|\cdot\|_i$ be a norm on $E_i$, not necessarily related to the topology. Let us define: $\prod^*_{i\in\mathbb{N}} E_i = \{(f_i)_{i\in\mathbb{N}} \in \prod_{i\in\mathbb{N}} E_i\colon \lim_{i\to\infty} \|f_i\|_i = 0\}$.

The topology on $\prod^*_{i\in\mathbb{N}} E_i$ is the standard product topology. Usually on $\prod^*_{i\in\mathbb{N}} E_i$ we consider the norm $\|(f_i)_{i\in\mathbb{N}}\| = \max_{i\in\mathbb{N}} \|f_i\|_i$.

**Definition 2.7.** For every linear topological spaces $X$ and $Y$ equipped with some norms (see Definition 2.5) we write $X \leqslant Y$ if there exists a linear topological space $F$ (see Definition 2.5) such that $Y \doteq X \times F$.

Let us point out some obvious, but important, properties of the relations $\doteq$ and $\leqslant$.

**Fact 2.8.** *If $X \doteq X_1$, $Y \doteq Y_1$ (respectively $X \geqslant X_1$, $Y \geqslant Y_1$) then $X \times Y \doteq X_1 \times Y_1$ (respectively $X \times Y \geqslant X_1 \times Y_1$).*

**Fact 2.9.** *If, for every $i \in \mathbb{N}$, $X_i \doteq Y_i$ (respectively $X_i \geqslant Y_i$) then $\prod^*_{i\in\mathbb{N}} X_i \doteq \prod^*_{i\in\mathbb{N}} Y_i$ (respectively $\prod^*_{i\in\mathbb{N}} X_i \geqslant \prod^*_{i\in\mathbb{N}} Y_i$).*

**Theorem 2.10** (Dugundji [5]). *Let $Y$ be a metrizable space and $A$ a closed subset of $Y$. Then there is a continuous linear function $\Phi : C_p(A) \to C_p(Y)$ such that for each $f \in C_p(A)$, $\Phi(f)|A = f$ and $\Phi(f)(Y) \subset \mathrm{conv}(f(A))$.*

**Lemma 2.11** (Gul'ko [7, Lemma 1]). *Let $\mathbb{R}^2$ be the real plane equipped with the norm $\|(x_1,x_2)\| = \max(|x_1|,|x_2|)$ and let $\varepsilon > 0$. Then there exist functions $\varphi_\varepsilon : \mathbb{R}^2 \to \mathbb{R}$ and $\psi_\varepsilon : \mathbb{R}^2 \to \mathbb{R}$ such that the following conditions are satisfied*:

(a) *The mapping $(x_1,x_2) \mapsto (x_1,\varphi_\varepsilon(x_1,x_2))$ is a uniform homeomorphism of the plane with the inverse of the form $(x_1,x_2) \mapsto (x_1,\psi_\varepsilon(x_1,x_2))$*
(b) *$\varphi_\varepsilon(x_1,x_2) = 0$ if $x_1 = x_2$*
(c) *$\psi_\varepsilon(y_1,0) = y_1$*
(d) *$(1+\varepsilon)^{-1}\|(x_1,x_2)\| \leqslant \|(x_1,\varphi_\varepsilon(x_1,x_2))\| \leqslant \|(x_1,x_2)\|$ for $(x_1,x_2) \in \mathbb{R}^2$.*

It is easy to see that condition (c) follows from conditions (a) and (b). Let us remained that in this paper $X$ denotes always compact metrizable space.

**Definition 2.12.** For a closed subset $A$ of a space $X$, let $C_p(X,A) = \{f \in C_p(X)\colon f|A \equiv 0\}$ with the topology of the pointwise convergence. We also equip $C_p(X,A)$ with the standard sup norm. We denote $C_p(X,\{x\})$ by $C_p(X,x)$ for $x \in X$.

Before formulating the next result let us set the following notation. Every continuous function $\varphi : \mathbb{R}^n \to \mathbb{R}$ induces the map $\bar{\varphi} : C_p(X)^n \to C_p(X)$ defined by the formula $\bar{\varphi}(f_1,\ldots,f_n)(x) = \varphi(f_1(x),\ldots,f_n(x))$. It is easy to check that if $\varphi$ is uniformly continuous then $\bar{\varphi}$ is also.

**Proposition 2.13.** *Let $A$, $B$ be closed subsets of a space $X$ such that $B \subset A$. Then $C_p(X,B) \simeq C_p(A,B) \times C_p(X,A)$.*

**Proof.** Fix $\varepsilon > 0$. Define $u_\varepsilon : C_p(X,B) \to C_p(A,B) \times C_p(X,A)$ and $w_\varepsilon : C_p(A,B) \times C_p(X,A) \to C_p(X,B)$ by the formulas $u_\varepsilon(f) = (\rho(f), \bar{\varphi}_\varepsilon(\Phi(\rho(f)), f))$ and $w_\varepsilon(f,g) = \bar{\psi}_\varepsilon(\Phi(f), g)$ where $\rho : C_p(X,B) \to C_p(A,B)$ is defined by $\rho(f) = f|A$, $\varphi_\varepsilon$ is as in Lemma 2.11 and $\Phi : C_p(A,B) \to C_p(X,B)$ is as in Theorem 2.10 (it is possible because $\Phi(C_p(A,B)) \subset C_p(X,B)$). By condition (b) from Lemma 2.11 we have $\bar{\varphi}_\varepsilon(\Phi(\rho(f)), f) \in C_p(X,A)$, hence $u_\varepsilon$ is well defined. The condition (c) from Lemma 2.11 implies that $w_\varepsilon$ is well defined. Let us verify that $w_\varepsilon \circ u_\varepsilon \equiv \mathrm{id}_{C_p(X,B)}$ and $u_\varepsilon \circ w_\varepsilon \equiv \mathrm{id}_{C_p(A,B)\times C_p(X,A)}$. By condition (a) from Lemma 2.11 we have $\psi_\varepsilon(x_1, \varphi_\varepsilon(x_1, x_2)) = x_2$. Therefore, for $f \in C_p(X,B)$, we have

$$w_\varepsilon \circ u_\varepsilon(f) = \bar{\psi}_\varepsilon\big(\Phi\big(\rho(f)\big), \bar{\varphi}_\varepsilon\big(\Phi\big(\rho(f)\big), f\big)\big) = f.$$

Take $(f,g) \in C_p(A,B) \times C_p(X,A)$. Since $\rho(g) \equiv 0$, condition (c) from Lemma 2.11 implies that $\rho(\bar{\psi}_\varepsilon(\Phi(f), g)) = \rho(\Phi(f)) = f$. By condition (a) from Lemma 2.11 we obtain that

$$\begin{aligned} u_\varepsilon \circ w_\varepsilon(f,g) &= \big(\rho\big(\bar{\psi}_\varepsilon\big(\Phi(f), g\big)\big), \bar{\varphi}_\varepsilon\big(\Phi\big(\rho\big(\bar{\psi}_\varepsilon\big(\Phi(f), g\big)\big)\big), \bar{\psi}_\varepsilon\big(\Phi(f), g\big)\big) \\ &= \big(f, \bar{\varphi}_\varepsilon\big(\Phi(f), \bar{\psi}_\varepsilon\big(\Phi(f), g\big)\big)\big) = (f,g). \end{aligned}$$

The fact that $w_\varepsilon$, $u_\varepsilon$ are uniformly continuous follows from Lemma 2.11(a). Finally we will show that $u_\varepsilon$ satisfies condition $(a_\varepsilon)$ from Definition 2.5. Let us check that

$\|u_\varepsilon(f)\| \leqslant \|f\|$. Observe that $\|\Phi(\rho(f))\| \leqslant \|f\|$ (see 2.10). By Lemma 2.11 (c) we have $\|u_\varepsilon(f)\| = \max(\|\rho(f)\|, \|\overline{\varphi}_\varepsilon(\Phi(\rho(f)), f)\|) \leqslant \|f\|$. On the other side by Lemma 2.11 (d) and Theorem 2.10 we get

$$\begin{aligned}\big\|u_\varepsilon(f)\big\| &= \max\big(\big\|\rho(f)\big\|, \big\|\overline{\varphi}_\varepsilon\big(\Phi\big(\rho(f)\big), f\big)\big\|\big)\\ &= \max\big(\big\|\Phi\big(\rho(f)\big)\big\|, \big\|\overline{\varphi}_\varepsilon\big(\Phi\big(\rho(f)\big), f\big)\big\|\big)\\ &\geqslant \sup_{x\in X}\max\big(\big|\Phi\big(\rho(f)\big)(x)\big|, \big|\varphi_\varepsilon\big(\Phi\big(\rho(f)\big)(x), f(x)\big)\big|\big)\\ &\geqslant (1+\varepsilon)^{-1}\sup_{x\in X}\max\big(\big|\Phi\big(\rho(f)\big)(x)\big|, \big|f(x)\big|\big) \geqslant (1+\varepsilon)^{-1}\|f\|. \qquad \square\end{aligned}$$

Taking $B = \emptyset$ in Proposition 2.13 we get the following:

**Corollary 2.14.** *For every closed subset $A$ of a space $X$ we have $C_p(X) \simeq C_p(X, A) \times C_p(A)$.*

**Corollary 2.15.** *$C_p(X, x_0) \simeq C_p(X)$ for every $x_0 \in X$, where $X$ is nondiscrete.*

**Proof.** Let $L \subset X$ be a topological copy of the space $\boldsymbol{S} = \{\frac{1}{n};\ n \in \mathbb{N}\} \cup \{0\} \subset \mathbb{R}$. Then

$$\begin{aligned}C_p(X, x_0) &\simeq C_p\big(X, L \cup \{x_0\}\big) \times C_p\big(L \cup \{x_0\}, x_0\big)\\ &\simeq C_p\big(X, L \cup \{x_0\}\big)C_p\big(L \cup \{x_0\}, x_0\big) \times \mathbb{R}\\ &\simeq C_p(X, x_0) \times \mathbb{R} \simeq C_p(X). \qquad \square\end{aligned}$$

**Fact 2.16.** *For every positive integer $n$ we have $C_p(\boldsymbol{I}^n) \simeq \prod^*_{i\in\mathbb{N}} C_p(\boldsymbol{I}^n)$.*

**Proof.** The proof of the above fact is similar to the well-known linear case. Let us define $\boldsymbol{I}^{n-1}_i = \boldsymbol{I}^{n-1} \times \{i\} \subset \boldsymbol{I}^n$, where $i \in \{0, 1\}$.

Using Proposition 2.13 we have:

$$\begin{aligned}\prod_{i\in\mathbb{N}}{}^{*} C_p\big(\boldsymbol{I}^n\big) &\simeq \prod_{i\in\mathbb{N}}{}^{*}\big(C_p\big(\boldsymbol{I}^n, \boldsymbol{I}^{n-1}_0 \cup \boldsymbol{I}^{n-1}_1\big) \times C_p\big(\boldsymbol{I}^{n-1}_0 \cup \boldsymbol{I}^{n-1}_1\big)\big)\\ &\simeq \bigg(\prod_{i\in\mathbb{N}}{}^{*} C_p\big(\boldsymbol{I}^n, \boldsymbol{I}^{n-1}_0 \cup \boldsymbol{I}^{n-1}_1\big)\bigg) \times \bigg(\prod_{i\in\mathbb{N}}{}^{*} C_p\big(\boldsymbol{I}^{n-1}_0 \cup \boldsymbol{I}^{n-1}_1\big)\bigg). \qquad (1)\end{aligned}$$

Consider the cone over $\boldsymbol{I}^{n-1}$ (here we identify $\boldsymbol{I}^{n-1}$ with $\{(x_1, \ldots, x_n) \in \mathbb{R}^n\colon 0 \leqslant x_i \leqslant 1$ for $i \leqslant n-1$ and $x_n = 0\}$) i.e., the following set: $\mathrm{c}(\boldsymbol{I}^{n-1}) = \{tx + (1-t)p \in \mathbb{R}^n\colon x \in \boldsymbol{I}^{n-1} \subset \mathbb{R}^n, t \in \mathbb{R}\}$; $p = (0, \ldots, 0, 1) \in \mathbb{R}^n$. We define, for $k \in \mathbb{N} \setminus \{0\}$, $I_k = \{\frac{1}{k}x + (1 - \frac{1}{k})p\colon x \in \boldsymbol{I}^{n-1} \subset \mathbb{R}^n\}$ and $I^<_k = \{tx + (1-t)p\colon x \in \boldsymbol{I}^{n-1} \subset \mathbb{R}^n, \frac{1}{k+1} \leqslant t \leqslant \frac{1}{k}\}$.

Let $h_k : \boldsymbol{I}^n \to I^<_k$ be a homeomorphism such that $h_k(\boldsymbol{I}^{n-1}_i) = I_{k+i}$, where $i \in \{0, 1\}$. Let us define the linear homeomorphism: $\Phi : C_p(\mathrm{c}(\boldsymbol{I}^{n-1}), \bigcup_{k\geqslant 1} I_k \cup \{p\}) \to (\prod^*_{i\in\mathbb{N}} C_p(\boldsymbol{I}^n, \boldsymbol{I}^{n-1}_0 \cup \boldsymbol{I}^{n-1}_1))$ by the formula:

$$\Phi(f) = (f \circ h_k)_{k\in\mathbb{N}\setminus\{0\}}$$

It is easy to check that $\|\Phi(f)\|_0 = \|f\|_1$ where each norm comes from the sup norm in a way mentioned before. Applying this to (1) we get:

$$\prod_{i\in\mathbb{N}}^{*} C_p\big(\boldsymbol{I}^n\big) \simeq C_p\Big(\mathrm{c}\big(\boldsymbol{I}^{n-1}\big), \bigcup_{k\geqslant 1} I_k \cup \{p\}\Big) \times \prod_{i\in\mathbb{N}}^{*} C_p\big(\boldsymbol{I}_0^{n-1} \cup \boldsymbol{I}_1^{n-1}\big)$$
$$\simeq C_p\Big(\mathrm{c}\big(\boldsymbol{I}^{n-1}\big), \bigcup_{k\geqslant 1} I_k \cup \{p\}\Big) \times \prod_{k\in\mathbb{N}}^{*} C_p(I_k). \tag{2}$$

In the above we used the obvious fact that $\prod_{i\in\mathbb{N}}^{*} C_p(\boldsymbol{I}_0^{n-1} \cup \boldsymbol{I}_1^{n-1})$ can be identified with $\prod_{i\in\mathbb{N}}^{*} C_p(\boldsymbol{I}^{n-1})$ and that $I_k$ is homeomorphic to $\boldsymbol{I}^{n-1}$. Moreover $\prod_{k\in\mathbb{N}}^{*} C_p(I_k)$ can be seen as $C_p(\bigcup_{k\geqslant 1} I_k \cup \{p\}, \{p\})$. Thus, according to (2), we have:

$$\prod_{i\in\mathbb{N}}^{*} C_p(\boldsymbol{I}^n) \simeq C_p\Big(\mathrm{c}\big(\boldsymbol{I}^{n-1}\big), \bigcup_{k\geqslant 1} I_k \cup \{p\}\Big) \times C_p\Big(\bigcup_{k\geqslant 1} I_k \cup \{p\}, \{p\}\Big)$$
$$\simeq C_p\big(\mathrm{c}\big(\boldsymbol{I}^{n-1}\big), \{p\}\big).$$

By Corollary 2.15 we know that $C_p(\mathrm{c}(\boldsymbol{I}^{n-1}), \{p\}) \simeq C_p(\mathrm{c}(\boldsymbol{I}^{n-1}))$. However $\mathrm{c}(\boldsymbol{I}^{n-1})$ and $\boldsymbol{I}^n$ are homeomorphic which finishes the proof. □

**Corollary 2.17.** $C_p(\boldsymbol{I}^n) \simeq C_p(\boldsymbol{I}^n)^2$.

**Lemma 2.18** (Decomposition scheme). *Let us consider spaces $E$ and $F$ as in Definition* 2.5. *If there exist spaces $Z$ and $V$* (*as in Definition* 2.5) *such that*:

(i) $E \leqslant F$ *and* $F \leqslant E$,
(ii) $E \simeq \prod_{i\in\mathbb{N}}^{*} E$,

*then* $E \simeq F$.

For the proof of Decomposition scheme it is enough to repeat the reasoning for the isomorphisms (see [15]) and replace the isomorphism symbol by $\simeq$.

**Corollary 2.19.** *If* $C_p(\boldsymbol{I}^n) \leqslant C_p(X) \leqslant C_p(\boldsymbol{I}^n)$ *then* $C_p(X) \simeq C_p(\boldsymbol{I}^n)$.

**Lemma 2.20.** *Let $X$ be a space and $(U_i)_{i\leqslant k}$ an open cover of $X$ such that for each $i \leqslant k$ there is an embedding of $U_i$ into $\boldsymbol{I}^n$. Then $C_p(\boldsymbol{I}^n) \geqslant C_p(X)$.*

**Proof.** Let us take an open covering $(V_i^{'})_{i\leqslant m}$ of $X$ with the following properties (observe that by our assumptions $\dim X \leqslant n$):

(a) for each $i \leqslant m$ there exists $j \leqslant k$ such that $\mathrm{cl}\, V_i^{'} \subset U_j$;
(b) for each $i \leqslant m$ $\dim \mathrm{bd}\, V_i^{'} \leqslant n-1$.

Define $(V_i)_{i\leqslant m}$ as follows:
$V_0 = V_0'$,
$V_i = V_i' \setminus \bigcup_{j=0}^{i-1} \operatorname{cl} V_j'$.

It is obvious that:

(i) the sets $(V_i)_{i\leqslant m}$ are pairwise disjoint;
(ii) $(\operatorname{cl} V_i)_{i\leqslant m}$ is a covering of the space $X$;
(iii) for each $i \leqslant m$, $\dim \operatorname{bd} V_i \leqslant n-1$;
(iv) for every $i \leqslant m$, there is an embedding of $\operatorname{cl} V_i$ into $\boldsymbol{I}^n$.

Take $A_{n-1} = \bigcup_{i=0}^m \operatorname{bd} V_i$. Of course $\dim A_{n-1} \leqslant n-1$. By (i) and (ii) it is obvious that $C_p(X, A_{n-1}) \doteq \prod_{i=0}^m C_p(\operatorname{cl} V_i, \operatorname{bd} V_i)$. Using Corollary 2.14 we get $C_p(X) \doteq C_p(X, A_{n-1}) \times C_p(A_{n-1}) \leqslant \prod_{i=0}^m C_p(\operatorname{cl} V_i, \operatorname{bd} V_i) \times C_p(A_{n-1}) \times \prod_{i=0}^m C_p(\operatorname{bd} V_i) \doteq \prod_{i=0}^m C_p(\operatorname{cl} V_i) \times C_p(A_{n-1})$. According to (iv) and Corollary 2.14 for each $i \leqslant m$ $C_p(\operatorname{cl} V_i) \leqslant C_p(\boldsymbol{I}^n)$ and by Corollary 2.17 we have $C_p(X) \leqslant C_p(\boldsymbol{I}^n) \times C_p(A_{n-1})$. $A_{n-1}$ as a closed subset of $X$ satisfies the assumptions of our lemma (we take the cover $(U_i \cap A_{n-1})_{i\leqslant k}$). Therefore we can prove in the similar way that $C_p(A_{n-1}) \leqslant C_p(\boldsymbol{I}^n) \times C_p(A_{n-2})$ where $\dim A_{n-2} \leqslant n-2$ and we get that $C_p(X) \leqslant C_p(\boldsymbol{I}^n) \times C_p(A_{n-2})$. By repeating this reasoning we get $C_p(X) \leqslant C_p(\boldsymbol{I}^n) \times C_p(A_0)$ where $\dim A_0 \leqslant 0$ but because there is an embedding of $A_0$ into $\boldsymbol{I}^n$ then $C_p(A_0) \leqslant C_p(\boldsymbol{I}^n)$ and finally $C_p(X) \leqslant C_p(\boldsymbol{I}^n)$. □

Let us mention that the above lemma holds also when in the definition of $\leqslant$ we replace the relation $\doteq$ by the $\approx^l$ (the relation of being linearly homeomorphic).

**Theorem 2.21** (Gul'ko [8]). *Let $M$, $N$ be metrizable spaces with countable basis. If $M$ is u-equivalent to $N$, then the space $M$ (respectively, $N$) is a countable union of closed subsets which homeomorphically embed into the space $N$ (respectively, $M$). Therefore,* $\dim M = \dim N$.

**Corollary 2.22.** *If two compact, metrizable spaces $X$ and $Y$ are u-equivalent, then for each non-empty closed set $A$ in $X$, there exists a nonempty open set $V$ in $A$ which can be embedded in $Y$.*

**Proof.** By Theorem 2.21 we know that there exists closed covering $(F_n)_{n\in\mathbb{N}}$ of $X$ such that each element of this covering embeds into the space $Y$. Let $A$ be a closed nonempty subset of $X$. By the Baire theorem there exists $k \in \mathbb{N}$ such that $\operatorname{int}_A(F_k \cap A)$ is nonempty. Taking $V = \operatorname{int}_A(F_k \cap A)$ we proved our corollary. □

## 3. The main theorem

**Lemma 3.1.** *For every positive integer $n$ we have $C_p(\boldsymbol{I}^n) \geqslant C_p(X, X^{[1,n]})$.*

**Proof.** Let $(U_i)_{i\in\mathbb{N}}$ be a family of open sets in $X$ such that:

(i) $U_i \supset \operatorname{cl} U_{i+1}$;
(ii) $\bigcap_{i=0}^{\infty} U_i = X^{[1,n]}$;
(iii) $U_0 = X$.

Now, let us consider the closed set $A = \bigcup_{i=0}^{\infty} \operatorname{bd} U_i \cup X^{[1,n]}$. From Proposition 2.13 we know that

$$C_p\bigl(X, X^{[1,n]}\bigr) \simeq C_p(X, A) \times C_p\bigl(A, X^{[1,n]}\bigr).$$

It is obvious that $\Phi : C_p(X, A) \to \prod_{i\in\mathbb{N}}^{*} C_p(\operatorname{cl} U_i \setminus U_{i+1}, \operatorname{bd} U_{i+1} \cup \operatorname{bd} U_i)$ defined as $\Phi(f)_i = f|\operatorname{cl} U_i \setminus U_{i+1}$ is the linear homeomorphism such that $\|\Phi(f)\| = \|f\|$. Because $\operatorname{bd} U_i \cap \operatorname{bd} U_j = \emptyset$, for $i \neq j$, we have $C_p(A, X^{[1,n]}) \simeq \prod_{i\in\mathbb{N}}^{*} C_p(\operatorname{bd} U_i)$. Thus we get

$$C_p\bigl(X, X^{[1,n]}\bigr) \simeq \prod_{i\in\mathbb{N}}^{*} C_p(\operatorname{cl} U_i \setminus U_{i+1}, \operatorname{bd} U_{i+1} \cup \operatorname{bd} U_i) \times \prod_{i\in\mathbb{N}}^{*} C_p(\operatorname{bd} U_i).$$

It is obvious that $C_p(X, X^{[1,n]}) \leqslant C_p(X, X^{[1,n]}) \times \prod_{i\in\mathbb{N}}^{*} C_p(\operatorname{bd} U_i)$. Since $\operatorname{bd} U_0 = \emptyset$ $(\prod_{i\in\mathbb{N}}^{*} C_p(\operatorname{bd} U_i))^2 \simeq \prod_{i\in\mathbb{N}}^{*} C_p(\operatorname{bd} U_i \cup \operatorname{bd} U_{i+1})$ and we get

$$\begin{aligned} C_p\bigl(X, X^{[1,n]}\bigr) \leqslant{}& \prod_{i\in\mathbb{N}}^{*} C_p(\operatorname{cl} U_i \setminus U_{i+1}, \operatorname{bd} U_{i+1} \cup \operatorname{bd} U_i) \\ &\times \prod_{i\in\mathbb{N}}^{*} C_p(\operatorname{bd} U_i \cup \operatorname{bd} U_{i+1}). \end{aligned}$$

Reassuming $C_p(X, X^{[1,n]}) \leqslant \prod_{i\in\mathbb{N}}^{*} C_p(\operatorname{cl} U_i \setminus U_{i+1})$. According to Lemma 2.20 we have that, for each $i \in \mathbb{N}$, $C_p(\operatorname{cl} U_i \setminus U_{i+1}) \leqslant C_p(\boldsymbol{I}^n)$. Now using Facts 2.9 and 2.16 we get the final result, that is $C_p(X, X^{[1,n]}) \leqslant C_p(\boldsymbol{I}^n)$. □

**Proposition 3.2.** *For every ordinal $\alpha$ and positive integer $n$*

$$C_p\bigl(\boldsymbol{I}^n\bigr) \geqslant C_p\bigl(X, X^{[\alpha,n]}\bigr).$$

**Proof.** We will prove this proposition by the induction on $\alpha$. For $\alpha = 1$ it follows from the previous lemma. Let as assume that $\alpha = \beta + 1$ and that for $\beta$ proposition is true. Then we have $C_p(X, X^{[\alpha,n]}) \simeq C_p(X, X^{[\beta,n]}) \times C_p(X^{[\beta,n]}, X^{[\alpha,n]})$. Using the inductive assumption, Fact 2.8, Corollary 2.17 and Lemma 3.1 we proved our proposition for $\alpha = \beta + 1$. Now, let us assume that, for every $\beta < \alpha$, $C_p(X, X^{[\beta,n]}) \leqslant C_p(\boldsymbol{I}^n)$, where $\alpha \in \mathbf{Lim}$. It is obvious that if $X^{[\alpha,n]} = \emptyset$ then there exists $\beta < \alpha$ such that $X^{[\beta,n]} = \emptyset$ so without loss of generality we can assume that $X^{[\alpha,n]} \neq \emptyset$. Denote by $(\beta_i)_{i\in\mathbb{N}}$ strictly increasing sequence of ordinals converging to $\alpha$. It is clear that there exists sequence $(U_i)_{i\in\mathbb{N}}$ of the open subsets of $X$ satisfying the following conditions:

(i) $U_i \supset X^{[\beta_i,n]}$;
(ii) $U_i \supset \operatorname{cl} U_{i+1}$;
(iii) $\bigcap_{i=0}^{\infty} U_i = X^{[\alpha,n]}$;
(iv) $U_0 = X$.

Now let us consider the closed set $A = \bigcup_{i=0}^{\infty} \operatorname{bd} U_i \cup X^{[\alpha,n]}$. Repeating the reasoning from Lemma 3.1 we get $C_p(X, X^{[\alpha,n]}) \leqslant \prod_{i\in\mathbb{N}}^{*} C_p(\operatorname{cl} U_i \setminus U_{i+1})$. From Fact 2.4 we have that $(\operatorname{cl} U_i \setminus U_{i+1})^{[\beta_{i+1},n]} = \emptyset$ so by the inductive assumption $C_p(\operatorname{cl} U_i \setminus U_{i+1}) \leqslant C_p(\boldsymbol{I}^n)$ for every $i \in \mathbb{N}$. Therefore by Facts 2.16 and 2.9 we get $C_p(X, X^{[\alpha,n]}) \leqslant C_p(\boldsymbol{I}^n)$. □

Let us observe that: $\operatorname{ed}_n(X) = \emptyset \iff \forall\, A \subseteq X$ where $\operatorname{cl}_X(A) = A \neq \emptyset \exists U \subseteq A$ where $\operatorname{int}_A(U) = U \neq \emptyset$ such that $U$ is embeddable into the $n$-cube $\boldsymbol{I}^n$.

Thus we can formulate the main theorem using the concept of $\operatorname{ed}_n$:

**Main Theorem 2.** *For every positive integer $n$ the following equivalence holds*:

$$X \sim^u \boldsymbol{I}^n \iff \operatorname{ed}_n(X) = \emptyset \quad \textit{and} \quad \dim X = n.$$

**Proof.** "⇒" By Theorem 2.21 we get $\dim X = n$. If $\operatorname{ed}_n(X) \neq \emptyset$ then every nonempty open subset $U$ of $\operatorname{ed}_n(X)$ cannot be embedded into $\boldsymbol{I}^n$. This, by Corollary 2.22, gives us a contradiction.

"⇐" From Proposition 3.2 we get $C_p(X) \leqslant C_p(\boldsymbol{I}^n)$. Let $\alpha < \omega_1$ be such that $X^{[\alpha,n]} = \emptyset$. Obviously $X = \bigcup_{\beta<\alpha} X^{[\beta,n]} \setminus X^{[\beta+1,n]}$. Since, for every $\beta < \alpha$, the set $X^{[\beta,n]} \setminus X^{[\beta+1,n]}$ is $\sigma$-compact there exists $\gamma < \alpha$ such that $X^{[\gamma,n]} \setminus X^{[\gamma+1,n]}$ is $n$-dimensional (see [6]). From the definition of the embedding derivative it follows that $X^{[\gamma,n]} \setminus X^{[\gamma+1,n]}$ is a countable union of open subset (thus $\sigma$-compact) $U_k$ which can be embedded into $\boldsymbol{I}^n$. Hence one of the sets $U_k$ must be $n$ dimensional and as a subset of $\boldsymbol{I}^n$ contains a copy of $\boldsymbol{I}^n$ (see [6]). Therefore, by Corollary 2.14 we get $C_p(X) \geqslant C_p(\boldsymbol{I}^n)$. Using Corollary 2.19 we obtain $C_p(X) \simeq C_p(\boldsymbol{I}^n)$. □

## 4. Linear case

This section is devoted to the problem of Arkhangel'skiĭ (see [3, Problem 30]):

**Problem 4.1.** Give an inner classification of compacta which are $l$-equivalent to the $n$-cube.

By simple modification of the above reasoning we are able to prove the following:

**Theorem 4.2.** *For the space $X$ such that $X^{[i,n]} = X^{[i+1,n]}$ (for some $n > 0$ and $i \in \mathbb{N}$) the following conditions are equivalent*:

(i) $X \sim^l \boldsymbol{I}^n$;
(ii) $X^{[\omega,n]} = \emptyset$ (*or, by compactness argument,* $X^{[i,n]} = \emptyset$) *and* $\dim X = n$.

Now it is natural to formulate the hypothesis which could be an answer to the question of Arkhangel'skiĭ:

**Hypothesis 4.3.** $X \sim^l \boldsymbol{I}^n \iff X^{[\omega,n]} = \emptyset, \dim X = n.$

Arkhangel'skiĭ in [2] introduced the notion of Euclidean-resolvable spaces. We set $p(X) = 0$ if $X$ either is zero-dimensional, locally compact, separable, and metrizable, or $X$ is homeomorphic to an open subspace of Euclidean space $E^n$ for some $n \in \mathbb{N}$. Inductively we define $p(X) = n$ if for no $i \in \{0, \ldots, n-1\}$ $p(X) = i$ and there exists an open subspace $Y \subset X$ such that $p(Y) = 0$ and $p(X \setminus Y) = n-1$. A space $X$ with $p(X) = n$ for some $n \in \mathbb{N}$ is said to be Euclidean-resolvable. The following theorem holds:

**Theorem 4.4** (A.V. Arkhangel'skiĭ [2]). *If a compactum $X$ of dimension $n \geqslant 1$ is Euclidean-resolvable, then $X$ is l-equivalent to the Euclidean cube $\boldsymbol{I}^n$.*

It is not too difficult to prove the following:

**Fact 4.5.** *If a compactum is Euclidean-resolvable, then $X^{[\omega,n]} = \emptyset$ for $n = \dim X \geqslant 1$.*

**Proof.** It is easy to observe that every Euclidean-resolvable compactum is finite-dimensional, therefore $n$ given by the equality $n = \dim X$ is well defined. By induction on $k$ we can easily prove that if $p(X) = k$ then $X^{[k,n]} = \emptyset$ which finishes the proof. □

However it is not difficult to show that these conditions are not equivalent. It is enough to consider the discrete union of the square $\boldsymbol{I}^2$ and the Cantor fan (i.e., the cone over the Cantor set). Thus Theorem 4.2 is more general than Theorem 4.4.

At the end of this paper let us formulate the following problem which is the special case of Problem 4.1:

**Problem 4.6.** Are the spaces $\boldsymbol{I}$ and $\boldsymbol{I} \times [1, \omega^\omega]$ $l$-equivalent?

$[1, \omega^\omega]$ is the closed interval of ordinals with the standard order topology. It is easy to see that $(\boldsymbol{I} \times [1, \omega^\omega])^{[\omega,1]} \neq \emptyset$.

## Acknowledgements

I am very grateful to Witold Marciszewski for many valuable remarks and suggestions.

## References

[1] A.V. Arkhangel'skiĭ, Structure and classification of topological spaces and their cardinal invariants, Uspekhi Mat. Nauk 33 (6(204)) (1978) 29–84.
[2] A.V. Arkhangel'skiĭ, On linear topological classification of spaces of continuous function in the topology of pointwise convergence, Math. USSR Sb. 70 (1991) 129–142.
[3] A.V. Arkhangel'skiĭ, $C_p$-Theory Recent Progress in General Topology, Elsevier, Amsterdam, 1992, pp. 1–56.
[4] J. Baars, J. de Groot, On Topological and Linear Equivalence of Certain Function Spaces, in: CWI Tract, Vol. 86, Stichting Mathematisch Centrum, Centrum voor Wiskunde en Informatica, Amsterdam, 1992.
[5] J. Dugundji, An extension of Tietze's theorem, Pacific J. Math. 1 (1951) 353–367.

[6] R. Engelking, Theory of Dimension, Finite and Infinite, Heldermann, Berlin, 1995.
[7] S.P. Gul'ko, The space $C_p(X)$ for countable infinite compact $X$ is uniformly homeomorphic to $c_0$, Bull. Acad. Pol. Sci. 36 (1988) 391–396.
[8] S.P. Gul'ko, On uniform homeomorphisms of spaces of continuous functions, Proc. Steklov Inst. Math. 3 (1993) 87–93.
[9] K. Kawamura, K. Morishita, Linear topological classification of certain function spaces on manifolds and CW complexes, Topology Appl. 69 (1996) 265–282.
[10] A. Koyama, T. Okada, On compacta which are $l$-equivalent to $\boldsymbol{I}^n$, Tsukuba J. Math. 11 (1987) 147–156.
[11] K. Morishita, On spaces that are $l$-equivalent to a disk, Topology Appl. 99 (1999) 111–116.
[12] V.V. Pavlovskiĭ, On spaces of continuous functions, Soviet Math. Dokl. 22 (1980) 34–37.
[13] V.V. Uspenskiĭ, A characterization of compactness in terms of uniform structure in a function space, Uspekhi Matem. Nauk 37 (3(193)) (1982) 183–184.
[14] V.M. Valov, Linear topological classification of certain function spaces, Trans. Amer. Math. Soc. 327 (1991) 583–600.
[15] P. Wojtaszczyk, Banach Spaces for Analysts, Cambridge University Press, Cambridge, 1996.