Invariant Zeros of Linear Systems Connected in Series

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Abstract. In this paper, relationships are presented between the invariant zeros of two linear time-invariant systems to be connected in series, and those of the series connection of them.

INTRODUCTION

Several authors have defined and studied zeros of linear multivariable time-invariant systems, pointed out their role in both the analysis and the control of such systems and made comparisons among the different notions of zero (see, e.g., [1-5] and the references quoted in [3,5]). Recently, the notions of zero were extended to linear periodic discrete-time systems [6-8] and to nonlinear systems [9,10]. Analyses of the zeros of linear composite systems were presented in [11-13]. Specifically, in [13], cascade systems are considered and their transmission zeros (namely, the zeros of the numerator polynomials in the Smith-McMillan form of the transfer matrix of the system) are investigated in terms of the component subsystems.

In this paper, a similar contribution is presented on invariant zeros, i.e., the zeros of the invariant polynomials in the Smith form of the Rosenbrock’s system matrix [1]. Relations are given between the invariant zeros of two systems and those of the series connection of them, which holds also for non-reachable and/or non-observable systems, i.e., also, when the set of transmission zeros is a proper subset of the set of invariant zeros, and, therefore, the relations in [13] cannot be applied to invariant zeros.

NOTATIONS AND PRELIMINARIES

For the linear time-invariant system $\Sigma$ described by

$$
\begin{align*}
\Delta x(t) &= Ax(t) + Bu(t), \\
y(t) &= Cx(t) + Du(t),
\end{align*}
$$

where $t \in T$, $\Delta$ denotes either differentiation ($T = \mathbb{R}$) or the one step forward shift operator ($T = \mathbb{Z}$), $x(t) \in \mathbb{R}^n =: X$, $u(t) \in \mathbb{R}^r =: U$, $y(t) \in \mathbb{R}^q =: Y$, and $A, B, C, D$ are matrices with real entries, consider the corresponding transfer matrix $W(s) := C(sI_n - A)^{-1}B + D$, and system matrix $P(s)$ [1] defined as follows

$$
P(s) := \begin{bmatrix} A - sI_n & B \\ C & D \end{bmatrix},
$$

where $I_n$ denotes the identity matrix of dimension $n$. Let $r$ be the normal rank of $W(s)$, and denote with $\varepsilon_1(s), \ldots, \varepsilon_r(s)$ the $r$ numerator polynomials in the Smith-McMillan form of $W(s)$, in such an order that $\varepsilon_j(s)$ divides $\varepsilon_{j+1}(s)$ ($j = 1, \ldots, r-1$). It is well-known that the

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(normal) rank of $P(s)$ is $n + r$ [1]. Denote with $\alpha_1(s), \ldots, \alpha_{n+r}(s)$ the $n + r$ invariant polynomials in the Smith form of $P(s)$, in such an order that $\alpha_j(s)$ divides $\alpha_{j+1}(s)$ ($j = 1, \ldots, n + r - 1$). Write $Q_1(s) \triangleq Q_2(s)$ for polynomial matrices $Q_1(s)$ and $Q_2(s)$ with the same dimensions and the same Smith form (i.e., $Q_1(s) = L(s)Q_2(s)R(s)$, with $L(s)$ and $R(s)$ unimodular).

The zeros of the polynomials $\delta(s) := \prod_{j=1}^{n+r} \alpha_j(s)$ and $\eta(s) := \prod_{j=1}^{n+r} \epsilon_j(s)$ are called here, respectively, the invariant zeros of $\Sigma$ and the transmission zeros of $\Sigma$ (see, e.g., [5]), with multiplicities equal to their algebraic multiplicities as zeros of $\delta(s)$ and $\eta(s)$ respectively.

Without loss of generality, assume the basis of $X$ to be canonical with respect to reachability and unobservability [14], i.e.,

$$A = \begin{bmatrix} A_{aa} & A_{ab} & A_{ac} & A_{ad} \\ 0 & A_{bb} & 0 & A_{bd} \\ 0 & 0 & A_{cc} & A_{cd} \\ 0 & 0 & 0 & A_{dd} \end{bmatrix}, \quad B = \begin{bmatrix} B_a \\ B_b \\ 0 \\ 0 \end{bmatrix}, \quad C = [0 \ C_b \ 0 \ C_d],$$

with $(A_{ab}, B_b, C_b)$ reachable and observable, and denote with $n_a$, $n_b$, $n_c$, and $n_d$ the dimensions of $A_{aa}$, $A_{bb}$, $A_{cc}$, and $A_{dd}$, respectively. Lastly, denote with $\psi_1(s), \ldots, \psi_{n+n_e}(s)$ ($\xi_1(s), \ldots, \xi_{n+n_e}(s)$) the invariant polynomials of the non-reachable (unobservable) part of $\Sigma$, namely those of the matrix

$$\begin{bmatrix} sI_{n_a} - A_{cc} & -A_{cd} \\ 0 & sI_{n_e} - A_{dd} \end{bmatrix},$$

in such an order that, for each $j = 1, \ldots, n_e + n_d - 1$ ($j = 1, \ldots, n_a + n_c - 1$), $\psi_j(s)$ divides $\psi_{j+1}(s)$ ($\xi_j(s)$ divides $\xi_{j+1}(s)$). A similar notation for the invariant polynomials of $sI_{n_a} - A_{aa}$ or $sI_{n_e} - A_{dd}$ is omitted; however, they will be assumed to have been ordered in a similar way as the $\psi_j(s)$’s and $\xi_j(s)$’s have been; therefore the first $m$ of these invariant polynomials will be referred to as the $m$ invariant polynomials of the lower degrees.

As regards the relation between invariant zeros and transmission zeros, it is well-known that, if $\Sigma$ is reachable and observable, then $\alpha_j(s) = 1$ ($j = 1, \ldots, n$) and $\epsilon_j(s) = \alpha_{n+j}(s)$ ($j = 1, \ldots, r$) [1]; in general, the polynomial $\prod_{j=1}^{n+r} \alpha_j(s)$ divides the polynomial $\prod_{j=1}^{n+r} \alpha_j(s)$ [15], and, in particular, $\eta(s)$ divides $\delta(s)$. The following result gives some further information about the relations among $\delta(s)$, $\eta(s)$ and the characteristic polynomials of $A_{aa}$, $A_{cc}$, and $A_{dd}$.

**THEOREM 1.** The following relation holds:

$$\delta(s) = \eta(s) \det(sI_{n_e} - A_{cc}) g_a(s) g_d(s),$$

where $g_a(s)$ ($g_d(s)$) divides $\det(sI_{n_e} - A_{aa})$ ($\det(sI_{n_e} - A_{dd})$), and, if $n_a > p - r$ ($n_d > q - r$), then $g_a(s)$ ($g_d(s)$) is divided by the product of the $n_a - p + r$ ($n_d - q + r$) invariant polynomials of the lower degrees of $sI_{n_e} - A_{aa}$ ($sI_{n_e} - A_{dd}$).

**COROLLARY 1.** [4] If $r = p$, then $g_a(s) = \det(sI_{n_e} - A_{aa})$. Dually, if $r = q$, then $g_d(s) = \det(sI_{n_e} - A_{dd})$.

**PROOF OF THEOREM 1:** Theorem 4.1 in [1], together with (2) and (3), yields

$$P(s) \triangleq \begin{bmatrix} Z_1(s) & sI_{n_a} - A_{aa} & Z_2(s) & -A_{ac} & -A_{ad} \\ 0 & 0 & 0 & Z_3(s) \\ 0 & 0 & 0 & sI_{n_e} - A_{cc} & -A_{cd} \\ 0 & 0 & 0 & 0 & sI_{n_e} - A_{dd} \\ 0 & 0 & 0 & 0 & Z_4(s) \end{bmatrix} =: P_0(s),$$

where

$$P_0(s)$$

is a polynomial matrix with the same dimensions and Smith form as $P(s)$, and $Z_1(s)$, $Z_2(s)$, $Z_3(s)$, and $Z_4(s)$ are such that $Z_1(s) = L(s)Z_2(s)R(s)$, with $L(s)$ and $R(s)$ unimodular.
and $Z_j(s)$ ($j = 1, \ldots, 4$) are polynomial matrices. Now, it is readily seen that all non-zero $(n + r) \times (n + r)$ minors of $P_0(s)$ consist of all the rows of the three first block-rows of $P_0(s)$ and a number of rows of its two last block-rows equal to $\dim(A_{dd})$, and all the columns of the three last block-columns and a number of columns of its two first block-columns equal to $\dim(A_{dd})$. Since $S(s)$ is the greatest common divisor of all such minors, (5) follows, where $\gamma_a(s)$ ($\gamma_d(s)$) is the greatest common divisor of all non-zero minors of order $n_a$ ($n_d$) of the $n_a \times (p - r + n_a)$ $((q - r + n_d) \times n_d)$ matrix $[Z_1(s) \ s I_{n_a} - A_{aa}] \ ((s I_{n_d} - A_{dd}) \ Z_d(s))]$. Therefore, the first property of $g(s)$ ($\gamma(s)$) is obvious. The second one is obtained by considering a Laplace expansion of any non-zero $n_a \times n_a$ ($n_d \times n_d$) minor of the mentioned matrix, after partitioning such a minor into two blocks of $p - j$ and $n_a - p + r$ ($q - r$ and $n_d - q + r$) columns (rows), the latters belonging to $s I_{n_a} - A_{aa}$ ($s I_{n_d} - A_{dd}$): hence, such a minor is divided by the greatest common divisor of all non-zero minors of $s I_{n_a} - A_{aa}$ of order $n_a - p + r$ (of $s I_{n_d} - A_{dd}$ of order $n_d - q + r$).

In addition, note that the characterization of reachability (observability) in terms of relatively left (right) primeness of the pair $(s I_n - A, B)$ ($(s I_n - A, C)$) [1], together with the canonical decomposition (3), imply directly that

$$
\begin{align}
[A - s I_n & B] \aleq [\text{diag}(I_n, \psi_1(s), \ldots, \psi_{n_a+n_a}(s)) \ 0], \\
[A' - s I_n & C'] \aleq [\text{diag}(I_n, \xi_1(s), \ldots, \xi_{n_a+n_a}(s)) \ 0].
\end{align}
$$

(8a) (8b)

Now, assume $\Sigma$ to be the series connection of two systems $\Sigma_1$ and $\Sigma_2$ described by

$$
\Delta x_1(t) = A_1 x_1(t) + B_1 u_1(t),
$$

$$
y_1(t) = C_1 x_1(t) + D_1 u_1(t),
$$

(9a) (9b)

with $x_1(t) \in \mathbb{R}^{n_1}$, $u_1(t) \in \mathbb{R}^{p_1}$, $y_1(t) \in \mathbb{R}^{q_1}$, $A_1, B_1, C_1, D_1$ real $(i = 1, 2)$, $n_1 + n_2 = n$, $p_1 = p$, $q_1 = p_2$, $q_2 = q$. Namely, assume the representation of $\Sigma$ to be obtained from (9) and

$$
u(t) = u_1(t), \quad u_2(t) = y_1(t), \quad y(t) = y_2(t), \quad x(t) = [x_1(t) \ y_2(t)]'.
$$

(10)

Let symbols $W_i(s), P_i(s), r_i, \alpha_i(s), \ldots, \alpha_i, n_i, r_i(s), A_{i,aa}, A_{i,bb}, A_{i,cc}, A_{i,dd}, n_{i,a}, n_{i,b}, n_{i,c}, n_{i,d}, \psi_i(s), \xi_i(s) (i = 1, 2)$ be defined for $\Sigma_1$ in a similar way as $W(s), P(s), r, \alpha_1(s), \ldots, \alpha_n(s), A_{aa}, A_{bb}, A_{cc}, A_{dd}, n_a, n_b, n_c, n_d, \psi(s), \xi(s)$ have been for $\Sigma$. Therefore, the invariant zeros of $\Sigma_1$ are the zeros of $\delta_i(s) := \prod_{j=1}^{n_i+r_i} \alpha_{i,j}(s) (i = 1, 2)$, with multiplicities equal to their algebraic multiplicities in this polynomial.

**Main Results**

Since $r = \text{rank} W_i(s)$ $(i = 1, 2)$ and $r = \text{rank} (W_2(s)W_1(s))$, in general, $r \leq r_i (i = 1, 2)$. By this reason, some transmission zeros of $\Sigma_1 (i = 1, 2)$ may disappear in $\Sigma$, i.e., not be transmission zeros of it [13]. A similar situation may occur in the case of invariant zeros. The following theorem and its corollary point out some relations between $\delta_i(s)$ and $\delta_i(s)$ $(i = 1, 2)$, and therefore between the invariant zeros of $\Sigma$ and those of $\Sigma_1$ and $\Sigma_2$.

**Theorem 2.**

(a) If $r = p$, the least common multiple of the polynomials $h_{12}(s) := \delta_1(s) \ \det(s I_{n_a} - A_{2aa}) \ \det(s I_{n_d} - A_{2cc})$ and $\varphi_{21}(s) := (\prod_{j=1}^{n_{1c}} \alpha_{21}(s)) ((\prod_{j=1}^{n_{1d}} \alpha_{21}(s)) (\prod_{j=1}^{r - n_{1c} + n_{1d}} \psi_{1}(s)))$ (where the latter factor in $\varphi_{21}(s)$ vanishes if $n_{1c} + n_{1d} \leq q - r$) divides the polynomial
\( \delta(s) \). Dually, if \( r = q \), the least common multiple of the polynomials \( h_{21}(s) := \delta_2(s) \det(sI_{n_1} - A_{2cc}) \det(sI_{n_2} - A_{1cd}) \) and \( \varphi_{12}(s) := (\prod_{j=1}^{n_1} \alpha_{1j}(s)) \prod_{j=1}^{p - r - n_2 - n_{2c}} \xi_{2j}(s) \) (where the latter factor in \( \varphi_{12}(s) \) vanishes if \( n_{2a} + n_{2c} \leq p - r \)) divides the polynomial \( \delta(s) \).

(b) In general, the least common multiple of the polynomials \( \varphi_{12}(s) \) and \( \varphi_{21}(s) \) divides the polynomial \( \delta(s) \).

Corollary 2. If \( p = q = r \), the least common multiple of the polynomials \( h_{12}(s) \) and \( h_{21}(s) \) divides the polynomial \( \delta(s) \).

Proof of Theorem 2: First, note that, in general,

\[
P(s) \equiv \begin{bmatrix}
0 \\
A_2 - sI_{n_2} & \begin{pmatrix} I_{n_1} & 0 \\
0 & B_2 \\
0 & D_2 \end{pmatrix} P_1(s) \\
C_2
\end{bmatrix},
\]

(11a)

\[
P(s) \equiv \begin{bmatrix}
0 \\
A_1 - sI_{n_1} & B_1 \\
0 & P_2(s) \begin{pmatrix} I_{n_2} & 0 \\
0 & C_1 \\
0 & D_1 \end{pmatrix}
\end{bmatrix}.
\]

(11b)

Now, if \( r = p \), \( P(s) \) has full column-rank, and \( r_1 = p_1 = r \) (since \( p_1 = p = r \leq r_1 \leq p_1 \)), i.e., \( P_1(s) \) has full column-rank, while \( r \leq r_2 \leq \min(p_2, q_2) \). By (11a) and the Laplace expansion of a determinant, all non-zero minors of \( P(s) \) of maximal order (i.e., of order \( n_1 + n_2 + p_1 \)) can be expressed as a sum of products of a minor of order \( n_2 \) of the first block-column in the right-hand side of (11a) by a minor of order \( n_1 + p_1 \) of the second block-column in the right-hand side of (11a). Therefore, \( \delta(s) \), which is the greatest common divisor of all the non-zero minors of maximal order of \( P(s) \), is divided by the product of the greatest common divisor of all non-zero minors of \( \det\left( A_2' - sI_{n_2} \right) \) of order \( n_2 \) (i.e., by \( (8b) \)), the characteristic polynomial \( \det(sI_{n_2} - A_{2cc}) \) of the unobservable part of \( \Sigma_2 \), by the greatest common divisor of all non-zero minors of \( P_1(s) \) of order \( n_1 + p_1 \) (i.e., \( \delta_1(s) \)). This proves that \( h_{12}(s) \) divides \( \delta(s) \). The proof that \( \varphi_{21}(s) \) divides \( \delta(s) \) can be similarly based on (11b) and the Laplace expansion of the non-zero minors of \( P(s) \) of order \( n_1 + n_2 + r \), after partitioning such minors into two block-rows containing, respectively, \( n_1 \) and \( n_2 + r \) rows, and taking into account \( (8a) \) and that, in general, \( r \leq q_2 = q \). The second statement of part (a) is obtained by duality. The same kind of reasoning proves (b) in general.

Remark: A comparison between Theorem 2(a) and Corollary 1 (or, in the general case, between Theorem 2(b) and Theorem 1) shows that they provide some complementary information on \( \delta(s) \), i.e., on the invariant zeros of \( \Sigma \). Although Corollary 1 and Theorem 1 state that the set of invariant zeros of \( \Sigma \) contains some input decoupling zeros and some output decoupling zeros of \( \Sigma \) (together with its transmission zeros), Theorem 2 gives some information on how some invariant zeros of \( \Sigma_1 \) and \( \Sigma_2 \) are retained as invariant zeros of \( \Sigma \), together with some output decoupling zeros and/or input decoupling zeros of \( \Sigma_1 \) and \( \Sigma_2 \). For example, Corollary 1 guarantees that, if \( r = q \), \( \delta(s) \) contains the product of the characteristic polynomials of the non-reachable parts of \( \Sigma_1 \) and \( \Sigma_2 \) as a factor, while Theorem 2(a), under the same hypothesis, guarantees that \( \delta(s) \) contains, as a factor, the product of the characteristic polynomial of the non-reachable part of \( \Sigma_1 \) by \( \delta_2(s) \), which, in turn, contains as a factor the characteristic polynomial of the non-reachable part of \( \Sigma_2 \) by the same Corollary 1; Theorem 2(a) guarantees also that \( \delta(s) \) contains \( \varphi_{12}(s) \) as a factor, i.e., that the invariant zeros of \( \Sigma_1 \), which are roots of \( \alpha_{1j}(s) \) from \( j = 1 \) up to \( j = n_1 + r \), are invariant zeros of \( \Sigma \), together with the output decoupling zeros of \( \Sigma_2 \), which are roots of \( \xi_{2j}(s) \) from \( j = 1 \) up to \( j = n_{2a} + n_{2c} - p + r \) (i.e., the roots of the last \( r_1 - r \) polynomials \( \alpha_{1j}(s) \) and those of the last \( p - r \) polynomials \( \xi_{2j}(s) \) may be cut off and not be invariant zeros of \( \Sigma \)). Notice that, in general, \( \varphi_{12}(s) \) and \( \varphi_{21}(s) \) divide, respectively, \( h_{12}(s) \) and \( h_{21}(s) \), and
not all the invariant zeros of $\Sigma_1$ and $\Sigma_2$ are guaranteed to be invariant zeros of $\Sigma$. This is guaranteed by Corollary 2 only when $r = p = q$, what implies $r_1 = p_1 = r$ and $r_2 = q_2 = r$. Finally, notice that it is a well-known implication of the solution of the robust tracking and regulation problem [2,16] that each invariant zero of $\Sigma_2$ is an invariant zero of $\Sigma$ whenever $r_2 = q_2$ and $r = q$ (see the second statement of Theorem 2(a)).

**Concluding Remarks**

The analysis of the invariant zeros of $\Sigma$ presented here holds for a linear periodic discrete-time system too, possibly constituted by the series connection of two linear periodic discrete-time systems characterized by the same period. In fact, for this class of systems the notions of invariant zero, transmission zero, input (output) decoupling zero, eigenvalue and pole were introduced in [7-8], through a time-invariant associated system, and have the same meaning, properties and relationships as in the time-invariant case. It seems useful to mention that, although these notions were defined to be periodic in principle, since the time-invariant associated system depends on the initial time actually considered for the periodic system, nevertheless they were shown to be time-invariant (together with their multiplicities), except the null ones [7-8].

**References**


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