JOURNAL OF COMPUTER AND SYSTEM SCIENCES: 3, 93–118 (1969)

Time-Varying Linear Sequential Machines. I

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Received October 14, 1968

ABSTRACT

A time-varying linear sequential machine (TVLSM) model is considered which has the property that the dimensionality of the system is a function of time. General response formulas for TVLSM's are developed and an effective procedure is described for minimizing such systems with respect to dimensionality. It is shown that every TVLSM has a minimal form unique up to isomorphism. "Equivalent" fixed sequential machines, referred to as "fixed analogs," may be constructed for certain periodic TVLSM's.

INTRODUCTION AND SUMMARY

A time-varying sequential machine is a more general model of a system exhibiting behavior that may be characterized in terms of a sequential machine [*1*, *2*].[1] Although any time-varying sequential machine of practical interest may be represented by an equivalent fixed (non-time-varying) sequential machine [*3*], the greater flexibility and reduced memory requirements of the time-varying machines present some real advantages.

Linear sequential machines (abbreviated LSM's—the mathematical characterization of linear sequential circuits or LSC's) have important applications in the areas of digital computation, control and communication systems and have been studied in detail [*4*]. Many of the well-known concepts of linear system theory and the theory of sequential machines have been utilized in the development of the theory of LSM's.

Certain practical systems, however, unavoidably contain time-varying components. Also, if an attempt is made to implement coding schemes for multiple error correction using LSM's it is seen that a time-varying coder is required. The bibliography of

* The research reported herein was supported by the Air Force Office of Scientific Research, Office of Aerospace Research, U. S. Air Force, under Grant AF-AFOSR-639-65.

[1] Numbers in brackets refer to the references appended to this paper.

Ref. [5] lists some articles describing time-varying coders for multiple error correction.

The concept of a time-varying linear sequential machine (TVLSM) is more than just a simple extension of that of an LSM. It will be shown later that, in any case where a TVLSM model is justifiably used, no equivalent LSM exists.

Work has been done on TVLSM's which are characterized by triangular transmission matrices, utilizing matrix algebra to evaluate system performance [5-10]. Methods for obtaining transforms for binary systems of this sort have been devised [9].

In this paper, a new and more general model for TVLSM's is employed in which the system is described in terms of time-varying characterizing matrices. The most obvious model of a TVLSM would be a system similar to an LSM with the exception that the scalar multipliers would be time-varying. This fixed dimensionality approach would not be in keeping with the primary purpose for studying time-varying sequential machines which is the reduction of required memory capacity at the expense of increased logical complexity. Thus the dimensionality of the system will be allowed to vary with time. A large time-varying linear system may be composed of several independent parts. The concept of time-varying dimensionality allows for time-sharing of the memory components of the system. This also holds for computer simulation of TVLSM's since a dynamic data allocation scheme may be utilized.

The formal definition of a TVLSM appears in Section 1, along with other essential definitions and preliminaries. Many of the terms that will be used throughout the paper appear in this section. Section 2 develops the general response formulas for TVLSM's which allow the computation of machine response to input sequences directly from the characterizing matrices rather than by an iterative approach. Since practical systems will be periodic in nature, Section 3 defines the terms that will be used to describe the periodicity of TVLSM's and discusses situations under which TVLSM's may or may not be eventually periodic. The various notions of state and machine equivalence for TVLSM's are defined in Section 4. Conditions for equivalence are developed in this section which will later be utilized in Sections 7, 8, and 9 for minimizing TVLSM's with respect to dimensionality. Section 5 contains procedures for constructing finite state "fixed analogs" of TVLSM's which enable one to construct state-output diagrams. The effects of coordinate transformations on TVLSM's are discussed in Section 6.

Section 7 describes the construction of finite dimensional t-diagnostic matrices which provide a direct test for t-equivalence of states. These diagnostic matrices are used as tools in Section 8 which contains procedures for the minimization of TVLSM's and it is shown that the minimal form is unique. An illustrative example of minimization appears in this section. Section 10 links the notions of equivalence and minimality with the concept of matrix-equivalence which appears in Section 6.

Much of the material that follows is analogous to that for LSM's which appears in Gill's recent book [4].

1. Definitions and Preliminaries

DEFINITION 1.1. A *sequential machine* [1, 2] is a 5-tuple $M = \langle \Sigma, \Delta, S, f, g \rangle$

(i) Σ is a nonempty set (*input alphabet*).

(ii) Δ is a nonempty set (*output alphabet*).

(iii) S is a nonempty set (*state set*).

(iv) f is a function from $S \times \Sigma$ into S (*direct transition function*).

(v) g is a function from $S \times \Sigma$ into Δ (*output function*).

Functions f and g are extended to mappings from $S \times \Sigma^{*}$[2] into S and Δ respectively in the following manner.

For each $s \in S$, $\sigma \in \Sigma$, $x \in \Sigma^*$ define

$$f(s, \Lambda) = s \qquad f(s, x\sigma) = f(f(s, x), \sigma) \tag{1.1}$$

$$g(s, \Lambda) = \Lambda \qquad g(s, x\sigma) = g(f(s, x), \sigma). \tag{1.2}$$

Define the length preserving function $\hat{g}$ as

$$\hat{g}(s, \Lambda) = \Lambda \qquad \hat{g}(s, \sigma x) = g(s, \sigma)\,\hat{g}(f(s, \sigma), x). \tag{1.3}$$

The reader is urged to familiarize himself with time-varying sequential machines [3] and LSM's [4] since many of the concepts are essential to the understanding of the material that follows.

A class of time-varying sequential machines with very interesting properties which are examined in detail in this paper is formed by the TVLSM's. In particular, to simplify notation, we shall restrict our attention to two-terminal TVLSM's noticing that most results hold for the more general multi-terminal case. Thus a TVLSM is defined as having common input and output alphabets over an arbitrary field F.

The formal definition of a TVLSM which follows differs with respect to that of an LSM in that the scalar multipliers and the dimensionality of the system are allowed to vary with time. Unfortunately, because of the intricate nature of time-varying systems, the notation will become quite complex.

DEFINITION 1.2. *A time-varying linear sequential machine* (TVLSM) is a 9-tuple

$$M = \langle F, n(\cdot), V, \mathbf{A}(\cdot), \mathbf{B}(\cdot), \mathbf{C}(\cdot), \mathbf{D}(\cdot), f, g \rangle$$

[2] If A and B are sets of words, then the *concatenation* of A and B, written AB, is the set of words $\{ab \mid a \in A, b \in B\}$. If A is a set of words, then $A^* = \bigcup_{n=0}^{\infty} A^n$, where $A^0 = \{\Lambda\}$ (Λ is the *null word*) and $A^{i+1} = A^i A$ for $i \geqslant 0$. $A^+ = A^* - \{\Lambda\}$. $\varnothing$ is used to designate the *empty set*.

(i) F is a field forming the common *input and output alphabets.*

(ii) $n(\cdot)$ is a mapping from the natural numbers N into N. $n(t)$ is the *dimensionality of M at time t.*

(iii) $V = \bigcup_{t=0}^{\infty} V_t$ is the *state set* of M. For each t, V_t is a t-subscripted, row-vector space of dimensionality $n(t)$ over F. State $\mathbf{s}(t) \in V_t$ will be written

$$\mathbf{s}(t) = (s_1, s_2, \ldots, s_{n(t)})_t .$$

(iv) $\mathbf{A}(\cdot) = [a_{ij}(\cdot)]_{n(\cdot)\times n(\cdot)'}$ is a function mapping N into the set of all matrices of finite dimensionality over F.

$$\mathbf{A}(t) = [a_{ij}(t)]_{n(t)\times n(t)'}$$

an $n(t) \times n(t)'$ matrix, where $n(t)' = n(t+1)$.

(v) $\mathbf{B}(\cdot) = [b_i(\cdot)]_{1\times n(\cdot)'}$ is a function from N into the set of all finite dimensional row-vectors over F.

$$\mathbf{B}(t) = [b_i(t)]_{1\times n(t)'} \text{ an } 1 \times n(t+1) \text{ matrix.}$$

(vi) $\mathbf{C}(\cdot) = [c_j(\cdot)]_{n(\cdot)\times 1}$ is a function from N into the set of all finite dimensional column-vectors over F.

$$\mathbf{C}(t) = [c_j(t)]_{n(t)\times 1} \text{ an } n(t) \times 1 \text{ matrix.}$$

(vii) $\mathbf{D}(\cdot) = [d(\cdot)]_{1\times 1}$ is a function from N into F.

$$\mathbf{D}(t) = [d(t)]_{1\times 1} \text{ a constant, forming a } 1 \times 1 \text{ matrix.}$$

(viii) f is a mapping from $V \times F$ into V defined by

$$(\forall t)_N\, (\forall \mathbf{s}(t))_{V_t}\, (\forall x)_F\, f(\mathbf{s}(t), x) = \mathbf{s}(t)\,\mathbf{A}(t) + x\mathbf{B}(t). \tag{1.4}$$

(ix) g is a mapping from $V \times F$ into F defined by

$$(\forall t)_N\, (\forall \mathbf{s}(t))_{V_t}\, (\forall x)_F\, g(\mathbf{s}(t), x) = \mathbf{s}(t)\,\mathbf{C}(t) + x\,\mathbf{D}(t). \tag{1.5}$$

Matrices $\mathbf{A}(t)$, $\mathbf{B}(t)$, $\mathbf{C}(t)$ and $\mathbf{D}(t)$ are called the *t-characterizing matrices* of M. The term *t-characteristic matrix* will refer to $\mathbf{A}(t)$.

If there exists an integer n such that $n(t) \leqslant n$ for all t, then M is said to be a *bounded* TVLSM. The least such integer n is called the *maximum dimensionality of M*. Clearly, any system of practical interest is bounded and therefore we shall, unless otherwise noted, consider only bounded TVLSM's.

A TVLSM is completely specified by the field F and the characterizing matrices since they imply the dimensionality of the system and the f and g functions. For this reason a shorter notation, $M = \langle F, \mathbf{A}(\cdot), \mathbf{B}(\cdot), \mathbf{C}(\cdot), \mathbf{D}(\cdot)\rangle$, will often be used.

$M(t)$ will refer to a TVLSM M' obtained from a TVLSM M by advancing M t units of time, e.g., at time t', $n'(t') = n(t' + t)$ and $\mathbf{A}'(t') = \mathbf{A}(t' + t)$.[3]

2. General Response Formulas

For sequential machines the response formulas are given by Eqs. (1.1) and (1.2). Given the characterizing matrices, initial state and input sequence x applied to a TVLSM in that state we may recursively compute the successor state and resulting output sequence from Eqs. (1.4) and (1.5). The following general response formulas allow us to determine the results directly from the characterizing matrices.

Proposition 2.1. *For a* TVLSM $M = \langle F, \mathbf{A}(\cdot), \mathbf{B}(\cdot), \mathbf{C}(\cdot), \mathbf{D}(\cdot)\rangle$, *the generalized transition function is*

$$(\forall t)_N\, (\forall \mathbf{s}(t))_{V_t}\, (\forall x)_{F^+}\; lg(x) = p,^4 \qquad x = x_0 x_1 \cdots x_{p-1}$$

$$f(\mathbf{s}(t), x) = \mathbf{s}(t) \prod_{i=0}^{p-1} \mathbf{A}(t+i) + \sum_{i=0}^{p-2} x_i \mathbf{B}(t+i) \prod_{j=i+1}^{p-1} \mathbf{A}(t+j) + x_{p-1}\mathbf{B}(t+p-1) \tag{2.1}$$

Proof. The proof is by induction on p.

Basis. $p = 1$.

Equation (2.1) becomes $f(\mathbf{s}(t), x) = \mathbf{s}(t)\mathbf{A}(t) + x_0\mathbf{B}(t)$ which is in agreement with (1.4).

Induction Step.

For all $\mathbf{s}(t) \in V_t$, $x \in F^p$, $x_p \in F$

$$f(\mathbf{s}(t), xx_p) = f(f(s(t), x), x_p).$$

By the inductive hypothesis we have

$$\begin{aligned} f(f(\mathbf{s}(t), x), x_p) &= \Big[\mathbf{s}(t) \prod_{i=0}^{p-1} \mathbf{A}(t+i) + \sum_{i=0}^{p-2} x_i \mathbf{B}(t+i) \prod_{j=i+1}^{p-1} \mathbf{A}(t+j) \\ &\qquad + x_{p-1}\mathbf{B}(t+p-1)\Big] \mathbf{A}(t+p) + x_p\mathbf{B}(t+p) \\ &= \mathbf{s}(t) \prod_{i=0}^{p} \mathbf{A}(t+i) + \sum_{i=0}^{p-1} x_i \mathbf{B}(t+i) \prod_{j=i+1}^{p} \mathbf{A}(t+j) + x_p\mathbf{B}(t+p). \end{aligned}$$

[3] For a TVLSM M, such symbols as V for the state set of M will be used as standard notation. When several TVLSM's are being referred to simultaneously then a notation such as $n'(t)$, for example, will denote the dimensionality of machine M' at time t.

[4] $lg(x)$ is the length of the sequence x, where $\lg(\Lambda) = 0$.

This is exactly Eq. (2.1) for $lg(x) = p + 1$. By induction, therefore, proposition 2.1 holds.

PROPOSITION 2.2. *The generalized output function of a* TVLSM

$$M = \langle F, \mathbf{A}(\cdot), \mathbf{B}(\cdot), \mathbf{C}(\cdot), \mathbf{D}(\cdot)\rangle$$

is given by

$$(\forall t)_N\, (\forall \mathbf{s}(t))_{V_t}\, (\forall x)_{F^+}\, lg(x) = p, \qquad x = x_0 x_1 \cdots x_{p-1}$$

$$g(\mathbf{s}(t), x) = \mathbf{s}(t) \left[\prod_{i=0}^{p-2} \mathbf{A}(t+i)\right] \mathbf{C}(t+p-1)$$
$$+ \sum_{i=0}^{p-3} x_i \mathbf{B}(t+i) \left[\prod_{j=i+1}^{p-2} \mathbf{A}(t+j)\right] \mathbf{C}(t+p-1)$$
$$+ x_{p-2}\mathbf{B}(t+p-2)\,\mathbf{C}(t+p-1) + x_{p-1}\,\mathbf{D}(t+p-1). \qquad (2.2)$$

Proof. From (1.2), setting $x = x' x_{p-1}$, $x' \in F^{p-1}$, $x_{p-1} \in F$

$$g(\mathbf{s}(t), x' x_{p-1}) = g(f(\mathbf{s}(t), x'), x_{p-1}).$$

Applying Eqs. (2.1) and (1.5)

$$g(\mathbf{s}(t), x) = \left[\mathbf{s}(t) \prod_{i=0}^{p-2} \mathbf{A}(t+i)\right.$$
$$\left. + \sum_{i=0}^{p-3} x_i \mathbf{B}(t+i) \prod_{j=i+1}^{p-2} \mathbf{A}(t+j) + x_{p-2}\mathbf{B}(t+p-2)\right] \mathbf{C}(t+p-1)$$
$$+ x_{p-1}\,\mathbf{D}(t+p-1) = \mathbf{s}(t) \left[\prod_{i=0}^{p-2} \mathbf{A}(t+i)\right] \mathbf{C}(t+p-1)$$
$$+ \sum_{i=0}^{p-3} x_i \mathbf{B}(t+i) \left[\prod_{j=i+1}^{p-2} \mathbf{A}(t+j)\right] \mathbf{C}(t+p-1)$$
$$+ x_{p-2}\mathbf{B}(t+p-2)\,\mathbf{C}(t+p-1) + x_{p-1}\,\mathbf{D}(t+p-1)$$

which proves proposition 2.2.

If the characterizing matrices $\mathbf{A}(\cdot)$, $\mathbf{B}(\cdot)$, $\mathbf{C}(\cdot)$, $\mathbf{D}(\cdot)$ are constant functions with values **A**, **B**, **C**, and **D** respectively, (2.1) and (2.2) reduce to the following general response formulas for a linear sequential machine (LSM)

$$(\forall t)_N\,(\forall \mathbf{s}(t))_{V_t}\,(\forall x)_{F^+}\; lg(x) = p, \qquad x = x_0x_1 \cdots x_{p-1}$$

$$f(\mathbf{s}(t), x) = \mathbf{s}(t)\,\mathbf{A}^p + \sum_{i=0}^{p-1} x_i\mathbf{B}\mathbf{A}^{p-i-1} \tag{2.3}$$

$$g(\mathbf{s}(t), x) = \mathbf{s}(t)\,\mathbf{A}^{p-1}\mathbf{C} + \sum_{i=0}^{p-2} x_i\mathbf{B}\mathbf{A}^{p-i-2}\mathbf{C} + x_{p-1}\,\mathbf{D}. \tag{2.4}$$

Note that this does not coincide exactly with the definition of an LSM since the present state is still dependent on t. In the second part of this paper it will be shown that for such machines all V_t's are, roughly speaking, "equivalent." Hence, the t subscript may be dropped and V becomes a time-invariant vector space of dimensionality n over F (**A** is an $n \times n$ matrix). The resulting LSM is a "fixed analog" [3] of the original TVLSM with constant matrices.

PROPOSITION 2.3. *Let* $M = \langle F, \mathbf{A}(\cdot), \mathbf{B}(\cdot), \mathbf{C}(\cdot), \mathbf{D}(\cdot)\rangle$ *be a* TVLSM. *The following properties of functions f and g hold.*

$$(\forall t)_N\,(\forall \mathbf{s}_1(t), \mathbf{s}_2(t))_{V_t}\,(\forall c)_F\,(\forall x)_{F^+}\; lg(x) = p$$

$$f(\mathbf{s}_1(t) \pm c\mathbf{s}_2(t), x) = f(\mathbf{s}_1(t), x) \pm cf(\mathbf{s}_2(t), 0^p) \tag{2.5}$$

$$g(\mathbf{s}_1(t) \pm c\mathbf{s}_2(t), x) = g(\mathbf{s}_1(t), x) \pm cg(\mathbf{s}_2(t), 0^p) \tag{2.6}$$

$$\hat{g}(\mathbf{s}_1(t) \pm c\mathbf{s}_2(t), x) = \hat{g}(\mathbf{s}_1(t), x) \pm c\hat{g}(\mathbf{s}_2(t), 0^p). \tag{2.7}$$

Proof. Immediate from (2.1), (2.2), (1.1), (1.2) and (1.3).

PROPOSITION 2.4. *The response of a* TVLSM $M = \langle F, \mathbf{A}(\cdot), \mathbf{B}(\cdot), \mathbf{C}(\cdot), \mathbf{D}(\cdot)\rangle$ *in state* $\mathbf{s}(t)$ *to a sequence* x *of length* p *may be written as a sum of the zero-input response (free response)* $g(\mathbf{s}(t), 0^p)$ *and the t-zero-state response* $g(\mathbf{0}_t, x)$[5] *(forced response) in the following manner*

$$(\forall t)_N\,(\forall \mathbf{s}(t))_{V_t}\,(\forall x)_{F^+}\; lg(x) = p$$

$$g(\mathbf{s}(t), x) = g(\mathbf{s}(t), 0^p) + g(\mathbf{0}_t, x). \tag{2.8}$$

Proof. $\mathbf{s}(t)$ may be written as $\mathbf{0}_t + \mathbf{s}(t)$, thus from (2.6)

$$g(\mathbf{s}(t), x) = g(\mathbf{0}_t + \mathbf{s}(t), x) = g(\mathbf{0}_t, x) + g(\mathbf{s}(t), 0^p).$$

[5] $\mathbf{0}_t$ is used to signify the zero-state at time t.

For a fixed value of t, (2.8) is known as the *t-decomposition property* of M. Time-varying machines exhibiting this property for all t are said to have the *decomposition property*. This property plays an important role in the discussion of equivalence of states and TVLSM's which allows us to consider the minimization of TVLSM's with respect to dimensionality. It also permits the independent analysis of the effects of the initial state and the applied input sequence on the response of the system.

3. Periodicity of a TVLSM

We have so far assumed that time-varying scalar multipliers are freely available for use in a system and not questioned how we should implement such a device. One possible method would be to use an autonomous finite-state control for each multiplier. This of course leads to the fact that the time-varying scalar multiplier would be an eventually periodic function of time. We assume therefore that most systems of practical interest will be eventually periodic thus these will play a large role in the forthcoming discussion.

However, the time-varying element may be introduced because the system is unrealiable or under operator control. In these and similar circumstances we may have non-periodic systems which nevertheless yield to much of the following analysis. We now proceed to formalize these concepts.

Definition 3.1. A TVLSM M is called *eventually periodic* if there exist positive integers τ and T such that for all $t \geqslant \tau$

$$\begin{aligned} \mathbf{A}(t) &= \mathbf{A}(t+T) \\ \mathbf{B}(t) &= \mathbf{B}(t+T) \\ \mathbf{C}(t) &= \mathbf{C}(t+T) \\ \mathbf{D}(t) &= \mathbf{D}(t+T). \end{aligned}$$

The *phase* of M at time $t \geqslant \tau$ is given by

$$\varphi(t) \equiv (t-\tau) \pmod{T}, \qquad 0 \leqslant \varphi(t) < T$$

T is called the *period* and τ the *transient* of M. A machine for which $\tau = 0$ is said to be *strictly periodic*. The least values of τ and T satisfying the above conditions are called the *minimal transient* and *minimal period* of M respectively. Clearly if the characterizing matrices are each eventually periodic then M is eventually periodic since we may choose τ as the maximum value of the transients and T as the least common multiple of the periods of the individual matrices.

4. Quasi-Equivalence, t-Equivalence and Equivalence of TVLSM's

The concepts of state and machine equivalence for non-time-varying machines are well-known [1]. Because of the importance of the time at which a particular state is observed, Gill [3] has introduced the notions of quasi-equivalence and the more restrictive t-equivalence of time-varying states and sequential machines. In the simplest terms, quasi-equivalence for time-varying machines is congruent to equivalence for non-time-varying machines. To be t-equivalent two states or machines must be observed simultaneously and found to be quasi-equivalent. More precise definitions follow.

Let M and M' be TVLSM's in states $\mathrm{s}(t)$ and $\mathrm{s}'(t')$ respectively, then s(t) and s(t') are said to be *quasi-equivalent* if the response of M in state $\mathrm{s}(t)$ to any input sequence $x \in F^*$ is identical to the response of M' in state $\mathrm{s}'(t')$ to the same sequence x. Symbolically

$$(\forall t, t')_N\, (\forall \mathrm{s}(t))_{V_t}\, (\forall \mathrm{s}'(t'))_{V'_{t'}}\, (\forall x)_{F^*}$$

$$\mathrm{s}(t) \overset{Q}{\equiv} \mathrm{s}'(t') \Leftrightarrow g(\mathrm{s}(t), x) = g'(\mathrm{s}'(t'), x). \tag{4.1}$$

If $t = t'$, quasi-equivalent states $\mathrm{s}(t)$ and $\mathrm{s}'(t')$ are said to be *t-equivalent*. In symbols

$$(\forall t, t')_N\, (\forall \mathrm{s}(t))_{V_t}\, (\forall \mathrm{s}'(t'))_{V'_{t'}}\, (\forall x)_{F^*}$$

$$\mathrm{s}(t) \overset{t}{\equiv} \mathrm{s}'(t') \Leftrightarrow (\mathrm{s}(t) \overset{Q}{\equiv} \mathrm{s}'(t')) \wedge (t = t'). \tag{4.2}$$

For each time t, t-equivalence defines a *t-equivalence partition* on the union of the state sets V_t and V'_t of TVLSM's M and M'. Two states are in the same class if and only if they are t-equivalent.

In the above discussion, M and M' may refer to the same machine.

For machines M and M' subspaces $W_t \subseteq V_t$ and $W'_{t'} \subseteq V'_{t'}$ are called *quasi-equivalent* if for each state $\mathrm{s}(t) \in W_t$ there corresponds at least one quasi-equivalent state $\mathrm{s}'(t') \in W'_{t'}$ and vice-versa. If W_t and $W'_{t'}$ are quasi-equivalent and $t = t'$ then W_t and $W'_{t'}$ are said to be *t-equivalent* subspaces.

Machines M and M' are said to be t-equivalent if V_t and V'_t are t-equivalent. M and M' are called *equivalent* if they are t-equivalent for all t. Symbolically

$$M \equiv M' \Leftrightarrow (\forall t)_N\, M \overset{t}{\equiv} M'.$$

The following proposition gives two properties of t-equivalent states that will be useful for the minimization of TVLSM's which is discussed beginning in Section 8.

PROPOSITION 4.1. *States* $\mathbf{s}_1(t)$, $\mathbf{s}_2(t) \in V$, *the state set of a* TVLSM M, *are t-equivalent iff for all* $p \geqslant 0$

$$\text{(i)} \qquad g(\mathbf{s}_1(t), 0^p) = g(\mathbf{s}_2(t), 0^p) \tag{4.3}$$

or

$$\text{(ii)} \qquad (\mathbf{s}_1(t) - \mathbf{s}_2(t)) \stackrel{t}{\equiv} \mathbf{0}_t . \tag{4.4}$$

Proof. (i) $(\forall x)_{F^+}\, lg(x) = p$, $g(\mathbf{s}_1(t), x) = g(\mathbf{s}_2(t), x)$ hence by the t-decomposition property, for all $p \geqslant 0$

$$g(\mathbf{s}_1(t), 0^p) + g(\mathbf{0}_t, x) = g(\mathbf{s}_2(t), 0^p) + g(\mathbf{0}_t, x)$$

which yields (4.3). The converse is trivial.

(ii) From (i)

$$(\mathbf{s}_1(t) - \mathbf{s}_2(t)) \stackrel{t}{\equiv} \mathbf{0}_t \qquad \text{iff for all} \qquad p \geqslant 0$$
$$g(\mathbf{s}_1(t) - \mathbf{s}_2(t), 0^p) = g(\mathbf{0}_t, 0^p) = 0.$$

Applying equation (2.6) we have $g(\mathbf{s}_1(t), 0^p) - g(\mathbf{s}_2(t), 0^p) = 0$ iff by part (i) $\mathbf{s}_1(t) \stackrel{t}{\equiv} \mathbf{s}_2(t)$.

The importance of these notions is clear. If we have two systems and desire that they respond in exactly the same manner as far as terminal behavior is concerned when each is observed beginning at time t then we only require that they be t-equivalent. The astute reader will notice that t-equivalence does not necessarily imply t'-equivalence for any other time t'. t-equivalence plays a role in the minimization of TVLSM's analogous to equivalence in the minimization of LSM's.

Quasi-equivalence resulting from the periodicity of the characterizing matrices in machines over finite fields as described below allows us to construct finite state "fixed analogs" (discussed in Section 5).

If M is an eventually periodic TVLSM with transient τ and period T, then for each $t \geqslant \tau$, $M(t)$ is indistinguishable from $M(t + T)$. This follows from the periodicity of the characterizing matrices of M. Proposition 4.2, as an immediate consequence of this fact, is stated without proof.

PROPOSITION 4.2. *If* M *is an eventually periodic* TVLSM *with transient* τ *and period* T, *then*

$$(\forall t)_N\, (t \geqslant \tau) \Rightarrow M(t) \approx M(t + kT) \qquad \textit{for} \qquad k = 1, 2, \ldots$$

under the correspondence

$$\mathbf{s}(t) = (s_1, s_2, \ldots, s_{n(t)})_t \stackrel{Q}{\equiv} \mathbf{s}(t + kT)$$
$$= (s_1, s_2, \ldots, s_{n(t+kT)})_{(t+kT)}$$

where $n(t) = n(t + kT)$.

In other words, $s(t)$ and $s(t + kT)$ differ only with respect to subscripts. Such quasi-equivalent states are called *phase-equivalent.*

5. Fixed Analogs and State-Output Diagrams

A fixed analog [3] of a time-varying sequential machine M is a non-time-varying sequential machine $\tilde{M}$ with the property that every state in the state set M has a quasi-equivalent counterpart in the state set of $\tilde{M}$ and vice-versa. If $\tilde{M}$ is such that no other fixed analog for M has fewer states, then $\tilde{M}$ is called the *minimal fixed analog* of M.

The following assertion describes an effective procedure for constructing a, not always minimal, fixed analog for any eventually periodic TVLSM.

Proposition 5.1. *An eventually periodic* TVLSM M, *with transient* τ *and period* T, *has a fixed analog*

$$\tilde{M} = \langle F, F, V', f', g' \rangle$$

where

$$V' = \bigcup_{t=0}^{\tau+T-1} V_t, \qquad V_t \subseteq V, \qquad t = 0, 1, \ldots, \tau + T - 1$$

where f' *and* g' *are functions from* $V' \times F$ *into* V' *and* F *respectively defined by*

$$(\forall s(t))_{V_t} (\forall x)_F f'(s(t), x) = f(s(t), x) \qquad \text{for} \qquad 0 \leqslant t \leqslant \tau + T - 2$$

$$(\forall x)_F f'(s(\tau + T - 1), x) = s(\tau) \overset{Q}{\equiv} f(s(\tau + T - 1), x) = s(\tau + T)$$

i.e., $s(\tau + T)$ *and* $s(\tau)$ *differ only with respect to subscripts and by periodicity are phase-equivalent.*

$$(\forall s(t))_{V_t} (\forall x)_F g'(s(t), x) = g(s(t), x) \qquad \text{for} \qquad 0 \leqslant t \leqslant \tau + T - 1.$$

Proof. Proposition 5.1 is an immediate consequence of proposition 4.2.

State-output diagrams are often used as a characterization of sequential machines. It is assumed that the reader is familiar with the construction of such diagrams [*4*]. Since the state sets of TVLSM's are by definition countably infinite we cannot represent them in diagram form. However, the fixed analog, given by proposition 5.1, of an eventually periodic TVLSM of finite dimensionality over a finite field F may be represented by a state-output diagram. This diagram will be called the state-output diagram for the corresponding TVLSM.

Example

Consider the following TVLSM M over Z_2, the integers modulo 2.

$$M\colon t \equiv 0 \pmod 2 \qquad \mathbf{A}(t) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \mathbf{B}(t) = [11]$$

$$\mathbf{C}(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \mathbf{D}(t) = [1]$$

$$t \equiv 1 \pmod 2 \qquad \mathbf{A}(t) = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \qquad \mathbf{B}(t) = [01]$$

$$\mathbf{C}(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \mathbf{D}(t) = [1].$$

The state-output diagram for this machine is illustrated in Fig. 1. The phase-equivalence classes[6] will be represented by the member bearing the subscript denoting the phase. For example

$$(00)_0 \qquad \text{represents} \qquad [(00)_t \mid t \equiv 0 \pmod 2)].$$

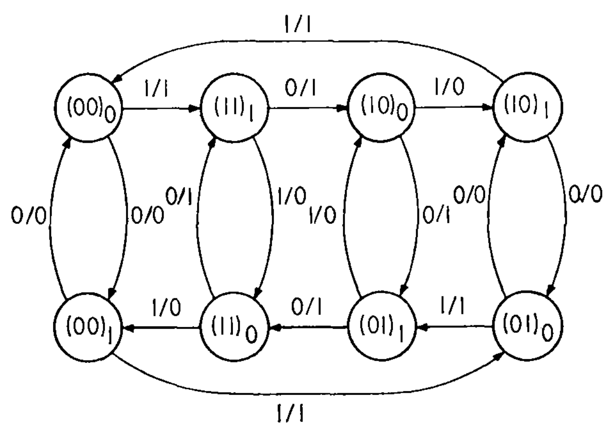

FIG. 1. State-output diagram of a TVLSM over Z_2.

[6] For each phase, the set of all phase-equivalent states forms a phase-equivalence class.

6. Matrix-Equivalent TVLSM's

LSM's are based in fixed coordinate systems. A transformation of coordinates results in a new machine equivalent and isomorphic to the original one. Such LSM's are called *similar* and it is known that two minimal LSM's are similar if and only if they are equivalent [4].

Considering a TVLSM in a fixed coordinate system, for each time t we may apply a different coordinate transformation forming a new TVLSM. In this section, a TVLSM M and the machine resulting from M through coordinate transformations are referred to as "matrix equivalent" and it is shown that such machines are equivalent.

Section 10 contains additional results for minimal TVLSM's. The mathematical prerequisites and definitions follow.

Definition 6.1. An $m \times n$ matrix $\mathbf{A}_1$ is said to be *equivalent* to an $m \times n$ matrix $\mathbf{A}_2$ if there exist nonsingular $m \times m$ and $n \times n$ matrices $\mathbf{P}$ and $\mathbf{Q}$ respectively such that

$$\mathbf{A}_1 = \mathbf{P}\mathbf{A}_2\mathbf{Q}.$$

If, for some time t, there exists an infinite sequence of nonsingular matrices $\mathbf{P}(t)$, $\mathbf{P}(t+1)$, $\mathbf{P}(t+2)$,... such that the characterizing matrices of TVLSM's M and M' are related by

$$i \geqslant 0 \begin{cases} \mathbf{A}'(t+i) = \mathbf{P}(t+i)\,\mathbf{A}(t+i)\,\mathbf{P}^{-1}(t+i+1) \\ \mathbf{B}'(t+i) = \mathbf{B}(t+i)\,\mathbf{P}^{-1}(t+i+1) \\ \mathbf{C}'(t+i) = \mathbf{P}(t+i)\,\mathbf{C}(t+i) \\ \mathbf{D}'(t+i) = \mathbf{D}(t+i) \end{cases} \tag{6.1}$$

then M and M' are said to be *t-matrix equivalent*. For the case

$$\mathbf{P}(t) = \mathbf{P}(t+1) = \mathbf{P}(t+2) = \cdots,$$

M and M' are said to be *t-similar*.

Lemma 6.2. t-matrix equivalence implies t-equivalence under the correspondence

$$\mathbf{s}(t) \stackrel{t}{\equiv} \mathbf{s}(t)\,\mathbf{P}^{-1}(t) = \mathbf{s}'(t).$$

Proof. For any input $x = x_0 x_1 \cdots x_{p-1} \in F^+$, applied to machine M in state $\mathbf{s}(t)$, the response is given by (2.2) as

$$g(\mathbf{s}(t), x)$$

$$= \mathbf{s}(t)\left[\prod_{i=0}^{p-2} \mathbf{A}(t+i)\right]\mathbf{C}(t+p-1) + \sum_{i=0}^{p-3} x_i \mathbf{B}(t+i)\cdot\left[\prod_{j=i+1}^{p-2}\mathbf{A}(t+j)\right]\mathbf{C}(t+p-1)$$

$$+ x_{p-2}\mathbf{B}(t+p-2)\,\mathbf{C}(t+p-1)$$

$$+ x_{p-1}\,\mathbf{D}(t+p-1)$$

$$= \mathbf{s}(t)\,\mathbf{P}^{-1}(t)\left[\prod_{i=0}^{p-2}\mathbf{P}(t+i)\,\mathbf{A}(t+i)\,\mathbf{P}^{-1}(t+i+1)\right]\mathbf{P}(t+p-1)\,\mathbf{C}(t+p-1)$$

$$+ \sum_{i=0}^{p-3} x_i\mathbf{B}(t+i)\,\mathbf{P}^{-1}(t+i+1)\left[\prod_{j=i+1}^{p-2}\mathbf{P}(t+j)\,\mathbf{A}(t+j)\,\mathbf{P}^{-1}(t+j+1)\right]\mathbf{P}(t+p-1)$$

$$\cdot\,\mathbf{C}(t+p-1) + x_{p-2}\mathbf{B}(t+p-2)\,\mathbf{P}^{-1}(t+p-1)\,\mathbf{P}(t+p-1)\,\mathbf{C}(t+p-1)$$

$$+ x_{p-1}\,\mathbf{D}(t+p-1)$$

$$= \mathbf{s}'(t)\left[\prod_{i=0}^{p-2}\mathbf{A}'(t+i)\right]\mathbf{C}'(t+p-1)$$

$$+ \sum_{i=0}^{p-3} x_i\mathbf{B}'(t+i)\cdot\left[\prod_{j=i+1}^{p-2}\mathbf{A}'(t+j)\right]\mathbf{C}'(t+p-1)$$

$$+ x_{p-2}\mathbf{B}'(t+p-2)\,\mathbf{C}'(t+p-1) + x_{p-1}\,\mathbf{D}'(t+p-1) = g'(\mathbf{s}'(t), x).$$

LEMMA 6.3. *If* TVLSM's *M and M' are t-matrix equivalent then they are* $t + i$ *matrix equivalent for* $i \geqslant 0$.

Proof. Obvious from the definition of matrix equivalence.

THEOREM 6.4. *Let M and M' be t-matrix equivalent* TVLSM's *then*

$$M(t) \equiv M'(t).$$

Proof. Follows immediately from lemmas 6.2 and 6.3.

0-matrix equivalent TVLSM's are called *matrix-equivalent* and are clearly equivalent from theorem 6.4.

7. Method for Determining t-Equivalence Classes

State equivalence tests for finite-state machines are well-known [1]. For finite dimensional, eventually periodic TVLSM's over finite fields constructing a fixed analog provides a method for testing both quasi- and t-equivalence of states. In general, however, for non-periodic TVLSM's or machines over arbitrary fields proposition 4.1 (i) yields the only method for establishing whether or not two states are equivalent. Since this requires comparison over all possible length zero sequences it does not provide a finite test. *A diagnostic matrix* [4].

$$\mathbf{K} = [\mathbf{C}, \mathbf{AC}, \mathbf{A}^2\mathbf{C}, \dots, \mathbf{A}^{n-1}\mathbf{C}]$$

may be constructed for n-dimensional LSM's which has the property that states $\mathbf{s}_1$ and $\mathbf{s}_2$ are equivalent iff $\mathbf{s}_1\mathbf{K} = \mathbf{s}_2\mathbf{K}$. The derivation of this matrix utilizes the property that all $n \times n$ matrices satisfy their own minimal polynomials which are of degree less than or equal to n. Hence for $0 \leqslant j \leqslant \infty$ we may express $\mathbf{A}^j$ as a linear combination of $\mathbf{A}^0 = \mathbf{I}, \mathbf{A}^1, \mathbf{A}^2, \dots, \mathbf{A}^{n-1}$.

No simple relationship of this sort holds for TVLSM's because of the time-varying matrices and dimensionality of the system. It is necessary, in order to construct a diagnostic matrix for time-varying systems analogous to the one previously described for LSM's, to express a product of characterizing matrices of the form $\prod_{i=0}^{j} \mathbf{A}(t+i)$, for any j, in terms of a set $\mathbf{A}(t), \mathbf{A}(t+1), \dots, \mathbf{A}(t+k)$ for a fixed k. This is the purpose of the next lemma.

Lemma 7.1. *For $t \geqslant \tau$ and any $m \geqslant 0$ there exists a set of coefficients $c_0, c_1, \dots, c_{n(t)-1} \in F$ such that*

$$\left[\prod_{i=0}^{T-1} \mathbf{A}(t+i)\right]^m = \mathbf{\Pi}(t)^m = \sum_{i=0}^{n(t)-1} c_i \mathbf{\Pi}(t)^i.$$

Proof. $\mathbf{\Pi}(t)$ is an $n(t) \times n(t)$ matrix. Thus the degree of the minimal polynomial of $\mathbf{\Pi}(t)$ is at most $n(t)$ from which it follows that for any m, $\mathbf{\Pi}(t)^m$ may be written as a linear combination of $\mathbf{\Pi}(t)^i$, $0 \leqslant i \leqslant n(t) - 1$.

We now proceed to define diagnostic matrices for TVLSM's and prove that they have the desired properties.

Definition 7.2. The t-diagnostic matrix $\mathbf{K}(t)$ of an eventually periodic TVLSM M having minimal transient τ and minimal period T, is an $n(t) \times k(t)$ matrix defined by

$$\mathbf{K}(t) = \left[\mathbf{C}(t), \mathbf{A}(t)\,\mathbf{C}(t+1), \dots, \left[\prod_{i=0}^{k(t)-2} \mathbf{A}(t+i)\right] \mathbf{C}(t+k(t)-1)\right] \quad \text{for} \quad t = 0, 1, \dots, \tau + T - 1$$

$$\mathbf{K}(t) = \mathbf{K}((t-\tau)\,(\text{mod } T) + \tau) \quad \text{for} \quad t \geqslant \tau + T \tag{7.1}$$

where, for $t = 0, 1, \ldots, \tau + T - 1$

$$k(t) = t' - t + n_{\min} \cdot T$$
$$n_{\min} = \min\{n(t) \mid t = \tau, \tau + 1, \ldots, \tau + T - 1\} \tag{7.2}$$

and t' is the earliest time after t in the first or second period when minimum dimensionality occurs. This choice is made to keep $k(t)$ as small as possible without having to find the degree of minimal polynomials of any matrices.

LEMMA 7.3.

$$\mathbf{s}(t)\,\mathbf{K}(t) = 0 \qquad \text{iff} \qquad \mathbf{s}(t)\left[\prod_{i=0}^{j-1} \mathbf{A}(t+i)\right]\mathbf{C}(t+j) = 0$$

for $j \geqslant 0$.

Proof. By the definition of $\mathbf{K}(t)$ the lemma holds for $0 \leqslant j < k(t)$ where $k(t)$ is given by (7.2). For any $j \geqslant k(t)$, we may write[7]

$$\left[\prod_{i=0}^{j-1} \mathbf{A}(t+i)\right]\mathbf{C}(t+j) = \left[\prod_{i=0}^{t'-t-1} \mathbf{A}(t+i)\right] \cdot \left[\prod_{i=0}^{T-1} \mathbf{A}(t'+i)\right]^m \cdot \left[\prod_{i=t'}^{t''-1} \mathbf{A}(i)\right]\mathbf{C}(t'')$$
$$= \mathbf{\Pi}_1(t, t')\,\mathbf{\Pi}(t')^m\,\mathbf{\Pi}_2(t', t'') \tag{7.3}$$

where

$$m = \operatorname{ent}\left[\frac{(j + t - t')}{T}\right]^{8}$$

and

$$t'' = (t + j - \tau)\,(\operatorname{mod} T) + \tau, \qquad 0 \leqslant t'' \leqslant \tau + T - 1$$

$$\mathbf{\Pi}(t') = \prod_{i=0}^{T-1} \mathbf{A}(t'+i) \qquad \text{is an} \qquad n_{\min} \times n_{\min} \text{ matrix.}$$

Thus the minimal polynomial of $\mathbf{\Pi}(t')$ is of degree at most $n_{\min}$. By lemma 7.1 we have that $\mathbf{\Pi}(t')^m$ may be expressed as a linear combination of $\mathbf{\Pi}(t')^p$, $0 \leqslant p \leqslant n_{\min} - 1$.

It follows that (7.3) may be written as a linear combination of

$$\mathbf{\Pi}_1(t, t')\,\mathbf{\Pi}(t')^p\,\mathbf{\Pi}_2(t', t'') \quad 0 \leqslant p \leqslant n_{\min} - 1.$$

[7] t' has the same meaning as it does in (7.2) and will be used as standard notation throughout this section.

[8] ent[x] is the greatest integer less than or equal to x (entier of x).

Setting p to the upper limit, $n_{\min} - 1$, the number of characteristic matrices in the above product is

$$\pi_p = t' - t + (n_{\min} - 1) \cdot T + t'' - t'$$

which is maximal when j is such that $t'' - t' = T - 1$ yielding

$$\pi_{\max} = t' - t + n_{\min} \cdot T - 1 = k(t) - 1.$$

This means that (7.3) can be expressed as a linear combination of columns of $\mathbf{K}(t)$. Thus if $\mathrm{s}(t)\,\mathbf{K}(t) = \mathbf{0}$ it follows that

$$\mathrm{s}(t)\left[\prod_{i=0}^{j-1} \mathbf{A}(t+i)\right]\mathbf{C}(t+j) = 0$$

for $j \geqslant 0$. The converse is trivial.

We can now prove the main theorem which verifies that the t-diagnostic matrix indeed has the desired properties.

THEOREM 7.4. *States* $\mathrm{s}_1(t), \mathrm{s}_2(t) \in V$ *the state set of a* TVLSM M, *are t-equivalent iff*

$$\mathrm{s}_1(t)\,\mathbf{K}(t) = \mathrm{s}_2(t)\,\mathbf{K}(t). \tag{7.4}$$

Proof. Equation (7.4) holds iff

$$(\mathrm{s}_1(t) - \mathrm{s}_2(t))\,\mathbf{K}(t) = 0$$

which is true by the previous lemma iff

$$(\mathrm{s}_1(t) - \mathrm{s}_2(t)) \prod_{i=0}^{j-1} \mathbf{A}(t+i)\,\mathbf{C}(t+j) = 0 \qquad \text{for all} \qquad j \geqslant 0$$

iff by proposition 4.1

$$\mathrm{s}_1(t) \overset{t}{\equiv} \mathrm{s}_2(t).$$

PROPOSITION 7.5. *Let* $\mathrm{s}'(t) \in V$ *the state set of a* TVLSM M, *be t-equivalent to* $\mathbf{0}_t$. *Then for each state* $\mathrm{s}(t) \in V$, $c \in F$ *we have*

$$\mathrm{s}(t) \pm c\mathrm{s}'(t) \overset{t}{\equiv} \mathrm{s}(t).$$

Proof. Property (2.6) yields for each $x \in F^+$, $lg(x) = p$

$$g(\mathbf{s}(t) \pm c\mathbf{s}'(t), x) = g(\mathbf{s}(t), x) \pm cg(\mathbf{s}'(t), 0^p) = g(\mathbf{s}(t), x) \pm cg(\mathbf{0}_t, 0^p) = g(\mathbf{s}(t), x)$$

which proves the proposition.

The set of all states in V_t t-equivalent to $\mathbf{0}_t$ will be denoted as E_t. Symbolically

$$E_t = \{\mathbf{s}_i(t) \mid \mathbf{s}_i(t) \in V_t, \mathbf{s}_i(t) \stackrel{t}{\equiv} \mathbf{0}_t\}.$$

Clearly E_t forms a subspace of V_t by the previous proposition. It is easily verified that E_t is the null space of $\mathbf{K}(t)$.

V_t is an additive abellian group, hence E_t is a subgroup of V_t. This leads to the following proposition.

PROPOSITION 7.6. *E_t induces a coset partition on V_t which is the t-equivalence partition.*

Proof. Clearly E_t induces a coset partition on V_t. States $\mathbf{s}_1(t)$ and $\mathbf{s}_2(t)$ are in the same class iff

$$(\mathbf{s}_1(t) - \mathbf{s}_2(t)) \in E_t$$

iff

$$(\mathbf{s}_1(t) - \mathbf{s}_2(t)) \stackrel{t}{\equiv} \mathbf{0}_t$$

iff

$$\mathbf{s}_1(t) \stackrel{t}{\equiv} \mathbf{s}_2(t) \qquad \text{by proposition 4.1(ii).}$$

The t-equivalence classes are easily found, therefore, since they are just the cosets of the subgroup E_t in V_t.

The t-diagnostic matrix $\mathbf{K}(t)$ will often contain columns which are not linearly independent. Suppose that the rank of $\mathbf{K}(t)$ is $r(t)$. Note that $r(t)$ is less than or equal to $n(t)$ which is the column dimension of $\mathbf{K}(t)$. Let $\tilde{\mathbf{K}}(t)$ denote a matrix consisting of $r(t)$ linearly independent columns of $\mathbf{K}(t)$. $\tilde{\mathbf{K}}(t)$ will be called a *reduced t-diagnostic matrix* of the machine in question. Clearly $\tilde{\mathbf{K}}(t)$ has all the properties previously attributed to $\mathbf{K}(t)$ with respect to determination of state equivalence, null space, etc.

8. t-MINIMIZATION OF TVLSM'S

A TVLSM M is said to be *t-minimal* if no two states in V_t are t-equivalent.

Suppose that the rank of $\mathbf{K}(t)$, the t-diagnostic matrix of M, is $r(t)$. Let $\tilde{\mathbf{K}}(t)$ be an $n(t) \times r(t)$ reduced t-diagnostic matrix of M. An effective procedure will now be described for constructing a TVLSM $\overset{t}{\check{M}}$ with dimensionality $r(t)$ at time t which is equivalent to M and t-minimal.

Define the characterizing matrices of $\overset{t}{\check{M}}$ to be identical to those of M with the following exceptions

$$\overset{t}{\check{\mathbf{A}}}(t-1) = \mathbf{A}(t-1)\,\tilde{\mathbf{K}}(t)$$

$$\overset{t}{\check{\mathbf{A}}}(t) = \mathbf{L}(t)\,\mathbf{A}(t)$$

$$\overset{t}{\check{\mathbf{B}}}(t-1) = \mathbf{B}(t-1)\,\tilde{\mathbf{K}}(t)$$

$$\overset{t}{\check{\mathbf{C}}}(t) = \mathbf{L}(t)\,\mathbf{C}(t)$$

where $\mathbf{L}(t)$ is an $r(t) \times n(t)$ left inverse of $\tilde{\mathbf{K}}(t)$.

LEMMA 8.1. TVLSM's *M and $\overset{t}{\check{M}}$ are t-equivalent under the following correspondence: state* $\mathbf{s}(t)$ *of M is t-equivalent to state* $\overset{t}{\check{\mathbf{s}}}(t) = \mathbf{s}(t)\,\tilde{\mathbf{K}}(t)$ *of $\overset{t}{\check{M}}$. Conversely, state* $\overset{t}{\check{\mathbf{s}}}(t)$ *of $\overset{t}{\check{M}}$ is t-equivalent to state* $\overset{t}{\check{\mathbf{s}}}(t)\,\mathbf{L}(t)$ *of M.*

Proof. Since $\mathbf{L}(t)$ is a left inverse of $\tilde{\mathbf{K}}(t)$ we have

$$\mathbf{s}(t)\,\tilde{\mathbf{K}}(t) = \mathbf{s}(t)\,\tilde{\mathbf{K}}(t)\,(\mathbf{L}(t)\,\tilde{\mathbf{K}}(t)) = (\mathbf{s}(t)\,\tilde{\mathbf{K}}(t)\,\mathbf{L}(t))\,\tilde{\mathbf{K}}(t)$$

from which it follows that $\mathbf{s}(t) \overset{t}{\equiv} \mathbf{s}(t)\,\tilde{\mathbf{K}}(t)\,\mathbf{L}(t)$ by the definition of $\tilde{\mathbf{K}}(t)$ and theorem 7.4. For all $x \in F$, therefore

$$\mathbf{s}(t)\,\mathbf{A}(t) + x\mathbf{B}(t) \overset{t+1}{\equiv} \mathbf{s}(t)\,\tilde{\mathbf{K}}(t)\,\mathbf{L}(t)\,\mathbf{A}(t) + x\mathbf{B}(t) = \overset{t}{\check{\mathbf{s}}}(t)\,\overset{t}{\check{\mathbf{A}}}(t) + x\overset{t}{\check{\mathbf{B}}}(t)$$

and

$$\mathbf{s}(t)\,\mathbf{C}(t) + x\mathbf{D}(t) = \mathbf{s}(t)\,\tilde{\mathbf{K}}(t)\,\mathbf{L}(t)\,\mathbf{C}(t) + x\mathbf{D}(t) = \overset{t}{\check{\mathbf{s}}}(t)\,\overset{t}{\check{\mathbf{C}}}(t) + x\overset{t}{\check{\mathbf{D}}}(t).$$

This establishes that $\mathbf{s}(t) \overset{t}{\equiv} \overset{t}{\check{\mathbf{s}}}(t) = \mathbf{s}(t)\,\tilde{\mathbf{K}}(t)$ and it is easily seen that state $\overset{t}{\check{\mathbf{s}}}(t)\,\mathbf{L}(t)$ of M is t-equivalent to state $\overset{t}{\check{\mathbf{s}}}(t)\,\mathbf{L}(t)\,\tilde{\mathbf{K}}(t) = \overset{t}{\check{\mathbf{s}}}(t)$ of $\overset{t}{\check{M}}$.

LEMMA 8.2. *$\overset{t}{\check{M}}$ is a t-minimal* TVLSM.

Proof. The t-diagnostic matrix of $\overset{t}{\check{M}}$ is

$$\begin{aligned}
\overset{t}{\check{\mathbf{K}}}(t) &= [\overset{t}{\check{\mathbf{C}}}(t), \overset{t}{\check{\mathbf{A}}}(t)\,\overset{t}{\check{\mathbf{C}}}(t+1), \ldots] \\
&= [\mathbf{L}(t)\,\mathbf{C}(t), \mathbf{L}(t)\,\mathbf{A}(t)\,\mathbf{C}(t+1), \ldots] \\
&= \mathbf{L}(t)\,\mathbf{K}(t)
\end{aligned}$$

States $s_1(t)$ and $s_2(t)$ of $\overset{t}{\check{M}}$ are t-equivalent iff

$$s_1(t)\,\overset{t}{\check{\mathbf{K}}}(t) = s_2(t)\,\overset{t}{\check{\mathbf{K}}}(t)$$

which holds by substitution iff $s_1(t)\,\mathbf{L}(t)\,\mathbf{K}(t) = s_2(t)\,\mathbf{L}(t)\,\mathbf{K}(t)$ iff by the previous lemma $s_1(t)\,\mathbf{L}(t) \overset{t}{\doteq} s_2(t)\,\mathbf{L}(t)$ iff by definition of a reduced t-diagnostic matrix $s_1(t)\,\mathbf{L}(t)\,\tilde{\mathbf{K}}(t) = s_2(t)\,\mathbf{L}(t)\,\tilde{\mathbf{K}}(t)$ iff $s_1(t) = s_2(t)$ since $\mathbf{L}(t)$ is a left inverse of $\mathbf{K}(t)$.

THEOREM 8.3. *$\overset{t}{\check{M}}$ is a t-minimal* TVLSM, *equivalent to the* TVLSM M, *of dimensionality $r(t)$ at time t where $r(t)$ is the rank of* $\mathbf{K}(t)$ *the t-diagnostic matrix of M.*

Proof. The previous lemma establishes that $\overset{t}{\check{M}}$ is t-minimal. From the definition of $\overset{t}{\check{M}}$ we note that $M(t+1)$ and $\overset{t}{\check{M}}(t+1)$ are identical TVLSM's. Thus it is clear that

$$M \overset{(t+j)}{\equiv} \overset{t}{\check{M}} \qquad \text{for} \qquad j \geqslant 1.$$

Lemma 8.1 yields that M and $\overset{t}{\check{M}}$ are t-equivalent. At time $t-1$, by lemma 8.1 for any $x \in F$

$$\begin{aligned} s(t-1)\,\mathbf{A}(t-1) + x\mathbf{B}(t-1) &\overset{t}{\doteq} (s(t-1)\,\mathbf{A}(t-1) + x\mathbf{B}(t-1)\,\tilde{\mathbf{K}}(t))\,\tilde{\mathbf{K}}(t) \\ &= s(t-1)\,\mathbf{A}(t-1)\,\tilde{\mathbf{K}}(t) + x\mathbf{B}(t-1)\,\tilde{\mathbf{K}}(t) \\ &= s(t-1)\,\overset{t}{\check{\mathbf{A}}}(t-1) + x\overset{t}{\check{\mathbf{B}}}(t-1) \end{aligned}$$

and

$$s(t-1)\,\mathbf{C}(t-1) + x\mathbf{D}(t-1) = s(t-1)\,\overset{t}{\check{\mathbf{C}}}(t-1) + x\overset{t}{\check{\mathbf{D}}}(t-1)$$

which demonstrates that M and $\overset{t}{\check{M}}$ are $(t-1)$-equivalent under the correspondence of $(t-1)$-equivalence of identical state vectors. Since M and $\overset{t}{\check{M}}$ have identical characterizing matrices for times prior to time $t-1$ it follows from the above discussion that M and $\overset{t}{\check{M}}$ are t'-equivalent for all $t' \geqslant 0$, hence equivalent. The definition of $\overset{t}{\check{M}}$ establishes that $\overset{t}{\check{n}}(t) = r(t)$.

Theorem 8.3 is the basic tool used in the construction of a minimal form for a TVLSM M.

9. MINIMAL TVLSM'S

DEFINITION 9.1. A TVLSM M is *minimal* if it is t-minimal for all t.

It will be shown in the second part of this paper that no minimal TVLSM is equivalent to a TVLSM having lesser dimensionality at any time t. Since the cost of constructing a circuit for a given TVLSM is certainly in rough proportion to its dimensionality, or the number of delays required to realize it, we shall associate minimal TVLSM's with minimum costs.

The following lemma establishes an important connection between t-equivalent state spaces of minimal machines.

LEMMA 9.2. *Let M and M' be minimal* TVLSM's. *If $W_t \subseteq V_t$ and $W'_t \subseteq V'_t$ are t-equivalent state spaces then there exists a linear isomorphism, $h : W_t \to W'_t$.*

Proof. For every state $s(t) \in W_t$, define $h(s(t))$ to be the unique state $s'(t) \in W'_t$ t-equivalent to $s(t)$. Minimality and t-equivalence of W_t and W'_t yield that h is one-to-one and onto.

Hence, for states $s_1(t)$, $s_2(t) \in W_t$ there exist respective t-equivalent states $s'_1(t)$, $s'_2(t) \in W'_t$. With the aid of Eq. (2.7) we see that

$$(\forall c)_F (\forall x)_{F^+} \hat{g}(s_1(t) \pm cs_2(t), x) = \hat{g}'(s'_1(t) \pm cs'_2(t), x)$$

thus

$$(\forall c)_F\, s_1(t) \pm cs_2(t) \stackrel{t}{\equiv} s'_1(t) \pm cs'_2(t)$$

which implies that

$$(\forall s_1(t), s_2(t))_{W_t} (\forall c)_F\, h(s_1(t) \pm cs_2(t)) = h(s_1(t)) \pm ch(s_2(t)) \tag{9.1}$$

and h is a linear isomorphism.

DEFINITION 9.3. TVLSM's M and M' are said to be *isomorphic* if there exists an isomorphism $h : V \to V'$ such that

$$(\forall t)_N (\forall s(t))_{V_t} (\forall x)_F\, h(f(s(t), x)) = f'(h(s(t)), x) \tag{9.2}$$

$$(\forall t)_N (\forall s(t))_{V_t} (\forall x)_F\, g(s(t), x) = g'(h(s(t)), x). \tag{9.3}$$

THEOREM 9.4. *Equivalent minimal* TVLSM's *are isomorphic.*

Proof. If M and M' are equivalent minimal TVLSM's, then, for each t, V_t and V'_t are t-equivalent state spaces. Lemma 9.2 yields that a linear isomorphism exists which maps V_t onto V'_t. Define h so that h restricted to V_t is this isomorphism (refer to the proof of Lemma 9.2). Successors of t-equivalent states are $(t + 1)$-equivalent which gives Eq. (9.2), and Eq. (9.3) follows directly from the fact that $s(t)$ and $h(s(t))$ are t-equivalent.

Given a TVLSM M we may effectively construct a minimal TVLSM $\check{M}$ that is equivalent to M. If $\mathbf{A}(\cdot)$, $\mathbf{B}(\cdot)$, $\mathbf{C}(\cdot)$ and $\mathbf{D}(\cdot)$ are the characterizing matrices of M, then define

$$(\forall t)_N \begin{cases} \check{\mathbf{A}}(t) = \mathbf{L}(t)\,\mathbf{A}(t)\,\tilde{\mathbf{K}}(t+1) \\ \check{\mathbf{B}}(t) = \mathbf{B}(t)\,\tilde{\mathbf{K}}(t+1) \\ \check{\mathbf{C}}(t) = \mathbf{L}(t)\,\mathbf{C}(t) \\ \check{\mathbf{D}}(t) = \mathbf{D}(t). \end{cases} \tag{9.4}$$

THEOREM 9.5. *For every* TVLSM M *there exists a* TVLSM $\check{M}$, *which may be effectively constructed from* M, *such that* $\check{M}$ *is minimal, equivalent to* M *and unique up to isomorphism.*

Proof. The definition of $\check{M}$ in terms of its characterizing matrices by (9.4) and theorem 8.3 yields that M is t-minimal for all $t \geqslant 0$, hence minimal, equivalent to M and effectively constructable from M. Theorem 9.4 guarantees that the minimal form of M is unique up to isomorphism.

Example. Let the TVLSM M be given by the following characterizing matrices over the field of integers modulo 3.

$$\mathbf{A}(0) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 2 & 2 & 0 \end{bmatrix} \qquad \mathbf{B}(0) = [0 \quad 1 \quad 0]$$

$$\mathbf{C}(0) = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \qquad \mathbf{D}(0) = [1]$$

$t \equiv 1$ *(mod 2)*

$$\mathbf{A}(t) = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{B}(t) = [1 \quad 2 \quad 0]$$

$$\mathbf{C}(t) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \mathbf{D}(t) = [0]$$

$t \equiv 0$ *(mod 2)*, $t > 1$

$$\mathbf{A}(t) = \begin{bmatrix} 0 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \qquad \mathbf{B}(t) = [2 \quad 2 \quad 1]$$

$$\mathbf{C}(t) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad \mathbf{D}(t) = [1]$$

M has transient 1 and period 2. The minimal form of M, $\check{M}$, will now be found.

$t = 0$

$$E_0 = \begin{Bmatrix} (0\ 0\ 0) & (1\ 0\ 1) & (2\ 0\ 2) \\ (0\ 1\ 0) & (1\ 1\ 1) & (2\ 1\ 2) \\ (0\ 2\ 0) & (1\ 2\ 1) & (2\ 2\ 2) \end{Bmatrix}$$

$$\mathbf{K}(0) = \begin{bmatrix} 2 & 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 2 & 0 & 0 \end{bmatrix}.$$

From $\mathbf{K}(0)$ we see that $r(0) = 1$ so we may choose

$$\tilde{\mathbf{K}}(0) = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{L}(0) = [0 \quad 0 \quad 1]$$

$t \equiv 1\,(mod\ 2)$

$$E_t = \{(0\ 0\ 0)\}$$

Hence M is t-reduced for these values of t and

$$\tilde{\mathbf{K}}(t) = \mathbf{L}(t) = \mathbf{I} \qquad \text{a } 3 \times 3 \text{ identity matrix.}$$

$t \equiv 0\ (mod\ 2),\ t > 1$

$$E_t = \{(0\ 0\ 0),\quad (1\ 2\ 2),\quad (2\ 1\ 1)\}$$

$$\mathbf{K}(t) = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 \end{bmatrix}$$

which has rank 2 ($r(t) = 2$), thus let

$$\tilde{\mathbf{K}}(t) = \begin{bmatrix} 1 & 0 \\ 1 & 2 \\ 0 & 1 \end{bmatrix} \qquad \text{and} \qquad L(t) = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

The characterizing matrices of $\check{M}$ may now be computed.

$t = 0$

$$\check{\mathbf{A}}(0) = \mathbf{L}(0)\,\mathbf{A}(0)\,\mathbf{K}(1) = [0 \quad 0 \quad 1] \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 2 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= [2 \quad 2 \quad 0]$$

$$\check{\mathbf{B}}(0) = \mathbf{B}(0)\,\mathbf{K}(1) = \mathbf{B}(0) = [0 \quad 1 \quad 0]$$

$$\check{\mathbf{C}}(0) = \mathbf{L}(0)\,\mathbf{C}(0) = [0 \quad 0 \quad 1] \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = [1]$$

$$\check{\mathbf{D}}(0) = \mathbf{D}(0) = [1]$$

$t \equiv 1$ *(mod 2)*

$$\check{\mathbf{A}}(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\check{\mathbf{B}}(t) = [1 \quad 2 \quad 0] \begin{bmatrix} 1 & 0 \\ 1 & 2 \\ 0 & 1 \end{bmatrix} = [0 \quad 1]$$

$$\check{\mathbf{C}}(t) = \mathbf{C}(t) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\check{\mathbf{D}}(t) = \mathbf{D}(t) = [0]$$

$t \equiv 0$ *(mod 2)*, $t > 1$

$$\check{\mathbf{A}}(t) = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\check{\mathbf{B}}(t) = \mathbf{B}(t) = [2 \quad 2 \quad 1]$$

$$\check{\mathbf{C}}(t) = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\check{\mathbf{D}}(t) = \mathbf{D}(t) = [1].$$

10. Minimal TVLSM's and Matrix-Equivalence

Theorem 10.1. *If M and M' are equivalent minimal* TVLSM's, *then M and M' are matrix-equivalent.*

Proof. Since M and M' are minimal and equivalent, by lemma 9.2 for each time t there exists a linear isomorphism mapping V_t onto V'_t. For each t, therefore, this mapping may be represented as a nonsingular matrix $\mathbf{P}(t)$, such that for every state $\mathbf{s}(t) \in V_t$ there exists one and only one equivalent state $\mathbf{s}(t)\,\mathbf{P}(t) = \mathbf{s}'(t) \in V'_t$.

Claim. For all t

a. $\mathbf{A}'(t) = \mathbf{P}^{-1}(t)\,\mathbf{A}(t)\,\mathbf{P}(t+1)$

b. $\mathbf{B}'(t) = \mathbf{B}(t)\,\mathbf{P}(t+1)$

c. $\mathbf{C}'(t) = \mathbf{P}^{-1}(t)\,\mathbf{C}(t)$

d. $\mathbf{D}'(t) = \mathbf{D}(t)$.

a. *and* b.

Since the successors of t-equivalent states are $(t+1)$-equivalent

$$(\forall t)_N\,(\forall \mathbf{s}(t))_{V_t}\,(\forall x)_F\,(\mathbf{s}(t)\,\mathbf{A}(t) + x\mathbf{B}(t))\,\mathbf{P}(t+1) = \mathbf{s}'(t)\,\mathbf{A}'(t) + x\mathbf{B}'(t)$$
$$= \mathbf{s}(t)\,\mathbf{P}(t)\,\mathbf{A}'(t) + x\mathbf{B}'(t)$$

which implies that

$$\mathbf{A}(t)\,\mathbf{P}(t+1) = \mathbf{P}(t)\,\mathbf{A}'(t)$$

or

$$\mathbf{A}'(t) = \mathbf{P}^{-1}(t)\,\mathbf{A}(t)\,\mathbf{P}(t+1) \qquad \text{and} \qquad \mathbf{B}'(t) = \mathbf{B}(t)\,\mathbf{P}(t+1).$$

c. *and* d.

$$(\forall t)_N\,(\forall \mathbf{s}(t))_{V_t}\,(\forall x)_F\,\mathbf{s}(t)\,\mathbf{C}(t) + x\mathbf{D}(t) = \mathbf{s}'(t)\,\mathbf{C}'(t) + x\mathbf{D}'(t)$$
$$= \mathbf{s}(t)\,\mathbf{P}(t)\,\mathbf{C}'(t) + x\mathbf{D}'(t)$$

which yields

$$\mathbf{C}'(t) = \mathbf{P}^{-1}(t)\,\mathbf{C}(t) \qquad \text{and} \qquad \mathbf{D}'(t) = \mathbf{D}(t).$$

This completes the proof of theorem 10.1.

Acknowledgment

The guidance of Professor M. A. Harrison has been a major factor in the research reported here. Discussions with Professors A. Gill and P. Varaiya have contributed significantly to this paper.

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Printed in Belgium