1. Introduction

Recall that for monomials $u = x_1^{a_1} \cdots x_n^{a_n}$ and $v = x_1^{b_1} \cdots x_n^{b_n}$, the lexicographic ordering specifies that $u > v$ if either $\deg u > \deg v$ or $\deg u = \deg v$ and $a_i - b_i > 0$ the first time it is non-zero. An ideal $I$ of $R = k[x_1, \ldots, x_n]$ is said to be a \textit{lex-segment ideal} of degree $d$ if it is generated by monomials of degree $d$ and if $u, v \in I$ are monomials of degree $d$ and $u \geq m \geq v$, then $m \in I$. More generally, $I$ is called a (completely) lex-segment ideal if whenever $u, v \in I$ are monomials of equal degree and $u \geq m \geq v$, then $m \in I$. An \textit{initial lex-segment ideal} is a lex-segment ideal such that in each degree $q$ for which $I_q \neq 0$, $I_q$ contains the monomial $x_1^q$. A \textit{final lex-segment ideal} is a lex-segment ideal which contains $x_n^q$ in each degree $q$ for which $I_q \neq 0$.

Initial lex-segment ideals have been well-studied. Macaulay used initial lex-segment ideals in 1927 in [8] to find an upper bound on the possible Hilbert functions of a cyclic graded module. This was the first discovery of several “extrema” properties possessed by lex-segment ideals. In [6], Eliahou and Kervaire gave an explicit resolution for a class of ideals called stable ideals (which include the initial lex-segment ideals) and consequently obtained a formula for their Betti numbers. Bigatti and Hulett both used this formula to show another extremal property, namely that among ideals with a given Hilbert function, the Betti numbers of initial lex-segment ideals are maximal, see [3] and [7]. These results have been used in several other papers, see, for instance, [9]. On the other hand, Deery [5] has shown that certain final lex-segment ideals have minimal Betti numbers for given Hilbert functions. In [1], Aramova and Herzog used a different and more comprehensive technique to derive, among other things, the Eliahou–Kervaire resolution. Sub-

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sequently, Aramova, Herzog and Hibi considered a natural generalization to squarefree (initial) lex-segment ideals in [2].

In this paper, we will follow the methods of [1] and [2] closely by using what could be considered a modified notion of a stable ideal. We give an explicit description of the basis elements of the graded Koszul homology modules of arbitrary lex-segment ideals and also a formula for the graded Betti numbers. In contrast with the initial lex-segment case, the basis elements of lex-segment ideals are less well-behaved, so our computations become somewhat more complex. We circumvent some of the difficulties by computing the Koszul homology and Betti numbers in each degree separately, and then putting them all together. As one application, we are able to calculate the depth of $R/I$ for certain lex-segment ideals.

We also give criteria for an arbitrary lex-segment ideal $I$ to have a linear resolution and more generally, we compute the Castelnuovo–Mumford regularity of lex-segment ideals, which turns out to be surprisingly well-behaved.

2. Koszul homology and Betti numbers

In this section we prove our main theorem for this paper which is a description of vector space bases for the graded components of the Koszul homology modules associated to a lex-segment ideal.

Throughout this paper, $R = k[x_1, \ldots, x_n]$ is a polynomial ring over a field $k$ of arbitrary characteristic and all modules are assumed to be graded. If $M$ is a graded $R$-module, we will denote the $i$th graded component of $M$ by $M_i$. For an ideal $I$, let $\varepsilon : R \to R/I$ denote the canonical homomorphism.

We denote the Koszul complex over $R/I$ with respect to the regular sequence $\bar{x} = x_1, \ldots, x_n$ by $K(\bar{x}; R/I)$. This is a complex of free $R/I$-modules, $K_i(\bar{x}; R/I)$, with standard basis elements $e_\sigma$, where $\sigma \subset \{1, \ldots, n\}$, $|\sigma| = i$ and $e_\sigma = e_{j_1} \wedge e_{j_2} \wedge \cdots \wedge e_{j_i}$ if $\sigma = (j_1, \ldots, j_i)$. Recall that the differential map of $K(\bar{x}; R/I)$ is $\partial(e_\sigma) = \sum_{s=1}^i (-1)^{s+1} x_s e_{\sigma \setminus j_s}$, where $\sigma = (j_1, \ldots, j_i)$ and $j_1 < j_2 < \cdots < j_i$. If $M$ is a graded $R/I$-module, we will denote the Koszul complex on $M$ by $K(\bar{x}; M) = K(\bar{x}; R/I) \otimes M$. Since $K(\bar{x}; M)$ is a graded complex, the corresponding homology modules $H_i(\bar{x}; M)$ are graded modules. It follows from the fact that $\bar{x}$ is a regular sequence on $R$ that if $d' < d$, then $H_i(\bar{x}; R/I) = 0$ for $t < d + i - 1$. Moreover, it is easy to see that if $M_t = 0$, then $H_i(\bar{x}; M)_{x_i + i} = 0$.

We will also need to use the following well-known facts about Koszul complexes. For more details, see, for instance, [4]. First, it follows from the mapping-cone construction of the Koszul complexes that for each $j = 1, \ldots, n - 1$, there is a graded long exact sequence on homology

$$\cdots \to H_{i+1}(x_j, \ldots, x_n; R/I) \to H_i(x_{j+1}, \ldots, x_n; R/I) \xrightarrow{\pm e_j} H_i(x_j+1, \ldots, x_n; R/I) \to \cdots$$

Also, if there is a graded short exact sequence of graded $R/I$-modules

$$0 \to M \to N \to Q \to 0,$$
then there is a graded long exact sequence on Koszul homology

$$\cdots \rightarrow H_i(x_1, \ldots, x_n; M) \rightarrow H_i(x_1, \ldots, x_n; N) \rightarrow H_i(x_1, \ldots, x_n; Q) \rightarrow \cdots$$

There is also an isomorphism of graded $k$-vector spaces $H_i(\bar{x}; R/I) \cong \text{Tor}_i^R(R/I, k)$, and hence the graded Betti numbers of $R/I$ are $\beta_{ij}(R/I) = \dim_k H_i(x_1, \ldots, x_n; R/I)$.

Finally, if $u$ is a monomial in $R$, we set $m(u) = i$ if $x_i$ divides $u$, but $x_j$ does not divide $u$ for any $j > i$. Also, we put $u' = u/x_{m(u)}$. We order indexing permutations $\sigma = (i_1, \ldots, i_k)$ and $\rho = (j_1, \ldots, j_k)$ lexicographically by saying $\sigma < \rho$ if the first non-zero entry in $(j_1 - i_1, \ldots, j_k - i_k)$ is positive. If $r = \sum u_\sigma e_\sigma$ is an element of $K_i(\bar{x}; M)$, we put $\min r = \min \{ \sigma : u_\sigma \neq 0 \}$.

The following theorem is the main computational tool we use in this paper.

**Theorem 2.1.** Let $I$ be a lex-segment ideal in $R$ of degree $d$, such that $I_{d'} = 0$ for $d' < d$. For $i \geq 1$ and $j = 1, \ldots, n$, the $k$-vector space $H_i(x_1, \ldots, x_n; R/I)_{d+i-1}$ has as a basis the elements

$$\varepsilon(u')e_\sigma \land e_{m(u)} + r(u)$$

which satisfy the following conditions, with $\sigma = (k_1, \ldots, k_{i-1})$:

(i) $u \in I_d$, $j \leq k_1$, $\min r(u) > \sigma \cup \{m(u)\}$,

(ii) For some $\ell$ with $k_1 < \ell \leq m(u)$ and $\ell \notin \sigma$, $x_\ell$ divides $u$ and

$$\frac{x_{k_1}u'x_s}{x_\ell} \in I$$

for $s \in \sigma \cup \{m(u)\}$ minimal such that $\ell \leq s$.

Recall that a monomial ideal $I$ is called a stable ideal if whenever $u$ is a monomial in $I$, then $x_ju' \in I$ for all $j < m(u)$. While initial lex-segment ideals are always stable, general lex-segment ideals are not. Condition (ii) above gives a modified version of stability that will provide just what we need.

**Proof.** We prove the theorem by using induction on $n - j$. If $n - j = 1$, then $H_1(x_n; R/I)_d$ clearly has as a basis all the elements $\varepsilon(u')e_n$, where $u = u'x_n \in I$ has degree $d$. The remaining conditions are vacuous.

Suppose $n - j > 1$. First, we will show the theorem holds for $H_1(x_1, \ldots, x_n; R/I)_d$. We have the graded long exact sequence on Koszul homology

$$H_1(x_{j+1}, \ldots, x_n; R/I)_{d-1} \xrightarrow{\pm x_j} H_1(x_1, \ldots, x_n; R/I)_d \rightarrow H_1(x_1, \ldots, x_n; R/I)_d$$

$$\rightarrow [R/(I, x_{j+1}, \ldots, x_n)]_{d-1} \xrightarrow{\pm x_j} [R/(I, x_{j+1}, \ldots, x_n)]_d.$$ 

By degree considerations, the leftmost module is zero. Furthermore, it is an easy computation to see that the kernel of the rightmost mapping has as a basis the images of
the monomials $u^i \in \mathbb{k}[x_1, \ldots, x_n]_{d-1}$ for which $u = u^i x_j \in I$. Using this and the inductive hypothesis, we see that $H_1(x_j, \ldots, x_n; R/I)_d$ has as a basis the elements

$$\varepsilon(u^i)e_s + r(u),$$

where $j \leq s \leq n$, $u^i x_s \in I_d$ and $\min r(u) > s$. Thus all the relevant conditions are satisfied.

Now, for $i > 1$, we again have the graded long exact sequence on Koszul homology:

$$\begin{align*}
H_i(\tilde{x}_{j+1}; R/I)_d + 2 &\longrightarrow H_i(\tilde{x}_{j+1}; R/I)_{d+i-2} \rightarrow H_i(\tilde{x}_{j+1}; R/I)_{d+i-1} \\
&\longrightarrow H_{i-1}(\tilde{x}_{j+1}; R/I)_{d+i-2} \longrightarrow H_{i-1}(\tilde{x}_{j+1}; R/I)_{d+i-1},
\end{align*}$$

where here we use $\tilde{x}_{j+1}$ and $\tilde{x}_j$ to denote the sequences $x_{j+1}, \ldots, x_n$ and $x_j, \ldots, x_n$, respectively. Again, by degree considerations, the leftmost module is zero, and since we know by the inductive hypothesis a basis for the module $H_i(\tilde{x}_{j+1}; R/I)_{d+i-1}$, we need only compute a basis for the kernel of multiplication by $x_j$ on the module $H_{i-1}(\tilde{x}_{j+1}; R/I)_{d+i-2}$.

Let $K$ be the kernel of multiplication on $H_{i-1}(\tilde{x}_{j+1}; R/I)_{d+i-2}$ by $x_j$. We claim that a basis of $K$ consists of the following elements, where we write $\sigma = (k_1, \ldots, k_{i-1})$ with $j < k_1 < \cdots < k_{i-1}$:

\(\varepsilon(u^i)e_\sigma\),

where $u = u^i x_{k_{i-1}} \in I_d$ and $x_j u^i \in I_d$,

\[-(-1)^{i-1} \varepsilon(u^i)e_\sigma + \sum_{k_i \in \sigma, k_i \leq \ell} (-1)^{i+1} \varepsilon(u^i x_{k_i}/x_\ell)e_{(\sigma \setminus k_i, \ell)},\]

where $u = u^i x_{k_{i-1}} \in I_d$, $k_p < \ell < k_{p+1}$ for some $p = 1, \ldots, i-2$, $x_\ell$ divides $u^i$, and

$$x_j u^i x_{k_{p+1}}/x_\ell \in I_d.$$

First note that each of these elements does in fact represent an element in the homology $H_{i-1}(\tilde{x}_{j+1}; R/I)_{d+i-2}$. Indeed, in case (i),

$$\partial(\varepsilon(u^i)e_\sigma) = \sum_{k_i \in \sigma} (-1)^{i+1} \varepsilon(x_{k_i} u^i)e_{(\sigma \setminus k_i, \ell)},$$

and since $j < k_1$, then $x_j u^i > x_{k_1} u^i \geq u$ for each $t$, and therefore $x_{k_1} u^i \in I_d$ by the lex-segment property. Thus we see that $\partial(\varepsilon(u^i)e_\sigma) = 0$.

For the elements in case (ii), each summand is a cycle: using (1) and the fact that $x_j u^i x_{k_{p+1}}/x_\ell > x_{k_1} u^i \geq u$, we see that $\partial(\varepsilon(u^i)e_\sigma) = 0$. For the other summands, if $k_i \in \sigma$ with $k_\ell < \ell$, then again we have $x_j u^i x_{k_{p+1}}/x_\ell > x_{k_1} u^i x_{k_\ell}/x_\ell > u$ for each $k_\ell \in (\sigma \setminus k_1, \ell)$. Thus by (1) and the lex-segment property, each of the coefficients appearing in the expansion of $\partial(\varepsilon(u^i x_{k_\ell}/x_\ell)^{e_{(\sigma \setminus k_1, \ell)}})$ is in $I_d$, which shows that this summand is a cycle.

Next, we show that each of the given elements is an element of $K$. In case (i), $x_j u^i \in I_d$ so elements of the type $\varepsilon(u^i)e_\sigma$ are in $K$. For elements of the second type, we note that $\ell \notin \sigma$, and thus we can write
\[ \partial \left( \varepsilon \left( x_j u' x_k / x_\ell \right) e(\sigma \setminus k_\ell) \right) = \sum_{k_\ell < \ell} (-1)^{t+1} \varepsilon \left( x_j u' x_k / x_\ell \right) e(\sigma \setminus k_\ell) \]
\[ + (-1)^p \varepsilon \left( x_j u' \right) e_{\sigma} + \sum_{k_\ell > \ell} (-1)^{t+2} \varepsilon \left( x_j u' x_k / x_\ell \right) e(\sigma \setminus k_\ell). \]

By (1) and the lex segment property, each of the monomials \( x_j u' x_k / x_\ell \) is in \( Id \) for \( \ell < k_\ell \), and so each term in the second summation vanishes. This shows that \( x_j \) multiplies each of the elements given in (i) and (ii) above to a boundary, and therefore the elements are all in \( K \).

Since the given elements are clearly independent, it suffices to show that every element in \( K \) can be written as a linear combination of the given elements. Let \( z \in K \) represent a homology class and write \( z \) in terms of the basis of \( H_{d-i}(\tilde{x}_{j+1}; R/I)_{d+i-2} \), which is known by the inductive hypothesis. After collecting terms if necessary, we may assume that

\[ z = \left( \sum_{g} \varepsilon (u'_g) \right) e_{\sigma} + T, \]

where \( \sigma = (k_1, \ldots, k_{i-1}) \), \( u'_g x_{k_{i-1}} \in I_d \), and \( \sigma < \tau \) for each permutation \( \tau \) appearing in the term \( T \). Since \( x_j \) kills the homology class of \( z \), there must exist elements \( m_\gamma \in [R/I]_{d-1} \), \( |\gamma| = i \), such that

\[ x_j z = \partial \left( \sum_{\gamma} \varepsilon (m_\gamma e_\gamma) \right) = \varepsilon \left( \sum_{\gamma} x_\ell m_\gamma \right) e_{\sigma} + S, \tag{2} \]

where the summation in the second line is taken over those \( \gamma \) and those \( \ell \in \gamma \) such that \( \gamma \setminus \ell = \sigma \), and where \( S \) is a sum of other terms not involving \( e_\sigma \). Thus, we have an equality \( \sum_{g} \varepsilon (x_j u'_g) = \sum_{\gamma} \varepsilon (x_\ell m_\gamma) \). Since \( I \) is a monomial ideal, this shows that for each \( g \), either \( x_j u'_g \in I_d \) or for some particular \( \gamma \) and \( \ell \in \gamma \), we have \( x_j u'_g = x_\ell m_\gamma \).

The first case gives rise to a term as in case (i) above. For the second case, let \( s \in \sigma \) be minimal such that \( \ell < s \) and note that the term \( \varepsilon (x_\ell m_\gamma) e_{\gamma \setminus s} \) necessarily occurs as part of the summation \( S \). Thus in order for the equality (2) to hold, this term must either cancel with one from \( x_j T \), or we must have \( x_\ell m_\gamma \in I \). However, \( \gamma \setminus \ell < \gamma \setminus s = \sigma \), and since every permutation in \( T \) is strictly larger than \( \sigma \), we see that \( \varepsilon (x_\ell m_\gamma) e_{\gamma \setminus s} \) cannot cancel with anything from \( T \). Thus we have \( x_j u'_g x_\ell / x_\ell = x_\ell m_\gamma \in I_d \).

We have shown, therefore, that each of the monomials appearing in the leading term of \( z \) has one of the forms in (i) or (ii) above, and so appears as the leading term of one of the proposed basis elements of \( K \). Letting \( w \) be the sum of the corresponding basis elements, we see that \( z - w \in K \), and the leading permutation in \( z - w \) is strictly larger than the leading permutation in \( z \). By an induction argument on the leading permutation, it is clear that \( z \) can be written as a combination of the proposed basis elements, which is what we wanted to show.

To finish the proof of the theorem, we only need to pull this basis back to basis elements of \( H_i(\tilde{x}_j; R/I)_{d+i-1} \). A term like \( \varepsilon (u') e_\sigma + r \), where \( u' \) satisfies the conditions
given above pulls back to a term like \( \varepsilon(u')e_{i(\sigma)} + r_{ij} + r' \), where \( r_{ij} \) means to add \( j \) onto the beginning of each indexing permutation occurring in \( r \), and where \( r' \) is an additional “error term” none of whose indexing permutations involve \( j \). This is necessarily a basis element for \( H_i(\tilde{x}_j; R/I)_{d+i-1} \) and it satisfies the conditions in the statement of the theorem. Finally, basis elements of \( H_i(\tilde{x}_j; R/I)_{d+i-1} \) are sent via the identity mapping into \( H_i(\tilde{x}_j; R/I)_{d+i-1} \) and, using the inductive hypothesis, it is clear that a basis for \( H_i(\tilde{x}_j; R/I)_{d+i-1} \) has the form specified. \( \square \)

**Example 2.2.** In \( R = k[x_1, x_2, x_3] \), the ideal
\[
I = \langle x_1^2x_2^3, x_1^2x_2x_3^2, x_1^3x_3, x_1x_2^4, x_1x_2x_3, x_1x_2^2x_3^2 \rangle
\]
is a lex-segment ideal of degree 5. In \( H_2(\tilde{x}; R/I)_6 \), the monomial \( u = x_1^2x_3^2 \) gives rise to the two basis elements \( (x_1x_2^2)e_2 \wedge e_3 \) and \( (x_1x_2) e_1 \wedge e_3 \), since \( x_2x_1x_2^3x_3^2x_3 / I \) and \( x_1(x_1x_2^2)x_3/x_2 \in I \), respectively. On the other hand, none of the first four monomials give rise to a basis element of the form \( u' e_1 \wedge e_m(u) \), since we would have to have \( x_1u'x_3/x_2 \in I \), and this fails for the first three monomials because of the degree of \( x_3 \), and for the fourth monomial because of the degree of \( x_2 \). In fact, it is easy to check that the first and fourth monomials give rise to no basis elements, the second and third monomials give rise to one basis element each, and the fifth and sixth monomials each give two basis elements. Hence \( \dim H_2(\tilde{x}; R/I)_6 = 6 \).

The theorem allows us to write a formula for the lowest degree Betti numbers for a lex-segment ideal \( I \). For each \( u \) in \( I_d \), define two integers \( \ell_i(u) \) and \( s_i(u) \) as follows: if there exists a pair of integers \( \ell \) and \( s \) such that \( i < \ell \leq s \leq m(u) \) and \( x_iu'x_j/x_\ell \in I_d \), then let \( \ell_1 \) be the maximal such \( \ell \) and \( s_1 \) the minimal such \( s \) for that particular choice of \( \ell \). If no such pair exists, then let \( \ell_1(u) = i \) and \( s_1(u) = m(u) \).

It follows from the theorem that if \( j \geq 2 \), and \( |\sigma| = j - 2 \), then an element \( \varepsilon(u')e_i \wedge e_\sigma \wedge e_{m(u)} + r(u) \) represents a basis element of \( H_j(\tilde{x}; R/I)_{d+j-1} \) if and only if \( \min(\sigma) > i \), \( \max(\sigma) < m(u) \), and \( \sigma \) does not include any integers between \( \ell_i(u) \) and \( s_i(u) - 1 \). Thus, the total number of times \( u' \) appears as a coefficient is
\[
\sum_{i=1}^{m(u)-j+1} \binom{m(u)-i-s_i(u)+\ell_i(u)-1}{j-2},
\]
where we use the convention that \( \binom{a}{b} = 0 \) for \( a < b \). Taking the sum over all \( u \) in \( I_d \) we get the Betti number \( \beta_{j,d+j-1} \) for \( j \geq 2 
\]
\[
\beta_{j,d+j-1}(R/I) = \sum_{u \in I_d} \sum_{i=1}^{m(u)-j+1} \binom{m(u)-i-s_i(u)+\ell_i(u)-1}{j-2}.
\]
Note that if \( I \) is an initial lex-segment ideal, then \( \ell_1(u) = s_1(u) = m(u) \) for each \( u \in I_d \), so the formula simplifies to \( \beta_{j,d+j-1}(R/I) = \sum_{u \in I_d} \binom{m(u)-1}{j-1} \), which is the same as in [6].
We can now use standard techniques to compute the higher degree Betti numbers for a lex-segment ideal. Suppose that \( I \) is a lex-segment ideal (in each degree) and for each \( j \geq 1 \), define \( I(j) \) to be the lex-segment ideal generated by the monomials in \( I \) of degree \( d + j \). Then we have a collection of short exact sequences

\[
0 \to I(j)/I(j+1) \to R/I(j+1) \to R/I(j) \to 0. \tag{3}
\]

For example, in \( k[x_1, \ldots, x_6] \), if \( I = (x_1^3 x_3 x_5, x_1^3 x_3 x_6, \ldots, x_1^3 x_3^2 x_6, x_1^3 x_3 x_4^2) \), then \( R/I \) has depth 1, since \( u = x_1^3 x_3^2 x_6 \), and \( x_2 u' \in I \) but \( x_1 u'/I \not\in I \).

**Remark 2.3.** The condition on the Betti numbers is necessary, since for the ideal \( I = (ac, b^2) \) in \( k[a, b, c] \), we have \( t = 3 \), but depth \( R/I = 1 \neq 2 = t - 1 \). We remark that generally speaking, lex-segment ideals which do not have this condition on Betti numbers tend to have very small numbers of generators. Certainly lex-segment ideals with linear resolutions satisfy this condition; more generally, it would be nice to find larger classes of lex-segment ideals which also work.

### 3. Completely lex-segment ideals

In this section, we examine the condition that an ideal generated by a lex-segment of degree \( d \) be completely lex-segment. Of course, this is a non-trivial condition; for example, the ideal \( I = (ac^2, b^3) \) in \( k[a, b, c] \) is missing the monomial \( ab^2 c \) in degree 4. On the other hand, it is clear that ideals generated by initial lex-segments are completely lex-segment, and Deery showed in [5] that a final lex-segment ideal of degree \( d \), say, is completely lex-segment if and only if it has \( x_1^d \) as a generator.

Our first theorem is an attempt to combine these two results, by giving an easy condition on the generators. We will subsequently use this condition to give a criterion for when a completely lex-segment has a linear resolution, and to compute the Castelnuovo–Mumford regularity of completely lex-segment ideals.

**Theorem 3.1.** Suppose \( I = (u_1, \ldots, u_k) \) is generated in degree \( d \) by a lex-segment, and assume that \( x_1 \) divides \( u_1 \). Set \( v_1 = u_1/x_1 \) and \( v_k = u_k/x_1 \). If \( v_1 \geq v_k \), then \( I \) is lex-segment in degree \( d + 1 \).

**Proof.** Note that the initial and final monomials in \( I_{d+1} \) are \( x_1 u_1 \) and \( u_k x_n \), respectively, so we need to show that if \( w \) is a monomial of degree \( d + 1 \), then \( w \in I_{d+1} \) if and only if \( x_1 u_1 \geq w \geq u_k x_n \). Clearly, if \( w \in I_{d+1} \), then \( x_1 u_1 \geq w \geq u_k x_n \). So, suppose \( x_1 u_1 \geq w \geq u_k x_n \), but that \( w \notin I_{d+1} \). First note that if \( x_1 \) divides \( w \), then \( u_1 \geq w/x_1 \), and if \( x_1 \) does not divide \( w \), then \( u_1 \geq w/x_j \) for all \( x_j \) dividing \( w \), since \( x_1 \) divides \( u_1 \). On the other hand, if \( x_j = m(w) \), then \( w/x_j \geq u_k \) for, if \( j = n \), this is immediate, and if \( j < n \), it
follows by degree considerations. Thus, if \( w \in \text{Id} + 1 \), there exist integers \( s > t \) such that \( x_s \) and \( x_t \) both divide \( w \), and for \( s > i > t \), \( x_i \) does not divide \( w \), and \( w/x_s > u_t \geq u_k > w/x_t \).

Writing \( u_1 = x_1^{q_1} \cdots x_n^{q_n}, u_k = x_1^{q_1} \cdots x_n^{q_n} \), and \( w = x_1^{r_1} \cdots x_n^{r_n} \), the inequalities above say that the \( n \)-tuple

\[
(p_1 - r_1, \ldots, p_t - r_t - 1, \ldots, p_n - r_n)
\]

is non-zero, with its first non-zero component negative and that the \( n \)-tuple

\[
(q_1 - r_1, \ldots, q_s - r_s - 1, \ldots, q_n - r_n)
\]

is non-zero, with its first non-zero component positive. Subtracting, we see that the \( n \)-tuple

\[
(p_1 - q_1, \ldots, p_s - q_s + 1, \ldots, p_t - q_t - 1, \ldots, p_n - q_n)
\]

is non-zero, with first non-zero component negative. But this contradicts the assumption on \( v_1 \) and \( v_k \) that says

\[
(p_1 - q_1 - 1, \ldots, p_m(u_k) - q_m(u_k) + 1, \ldots, p_n)
\]

is either zero, or has its first component positive. □

**Remark 3.2.** The condition in the theorem that \( x_1 \) divide \( u_1 \) is necessary. If we defined \( v_1 = u_1 / \min(u_1) \), then every final lex-segment ideal would have \( v_1 \geq v_k \); but Deery’s result [5, Corollary 2.13] implies that not every final lex-segment ideal is completely lex-segment. On the other hand, this is not an difficult restriction, since to compute Betti numbers we can use a change of rings to reduce to the case in the theorem.

The next result is an immediate corollary. It reduces the question of whether an ideal is completely lex-segment to just examining a finite number of degrees.

**Corollary 3.3.** If \( I \) is generated in degrees \( d_1 \) through \( d_2 \) with \( d_1 < d_2 \), then \( I \) is completely lex-segment if and only if \( I \) is lex-segment in degree \( d \) for each \( d_1 \leq d \leq d_2 \).

**Proof.** Only the sufficiency needs a proof. Suppose the monomials in \( I \) of degree \( d_2 \) are \( u_1, \ldots, u_k \) and in degree \( d_2 - 1 \) are \( w_1, \ldots, w_\ell \). Then since \( I \) is lex-segment in degree \( d_2 \) we have

\[
u_1 \geq x_1 w_1 \geq \cdots \geq w_\ell x_n \geq u_k.
\]

Putting \( v_1 = u_1/x_1 \) and \( v_k = u_k/x_1 \) we see that

\[
v_1 \geq w_1 \geq \cdots \geq w_\ell \geq v_k.
\]

Thus by Theorem 3.1, \( I \) is lex-segment in degree \( d_2 + 1 \). Continuing inductively, we see that \( I \) is lex-segment in each degree. □
Theorem 3.4. Suppose $I = (u_1, \ldots, u_k)$ is a lex-segment ideal generated in a single degree $d$, and assume that $x_1$ divides $u_1$. With the notation as above, if $v_1 \geq v_k$, then $R/I$ has a linear resolution.

Proof. It is enough to show that $\beta_i, d+1(R/I) = 0$ for all $i$ (that is, there is no strand in the resolution directly following the linear strand). If this is true, then it follows from the exact sequence (4) that $\beta_i, d+i+j(R/I) = \beta_i, d+j+i+1(R/I(j)) = 0$, since $I(j)$ is generated in degree $d + j$ and obviously satisfies the hypotheses of the theorem.

We will show that each $H_j(R/I)_{d+j}$ has no non-zero basis elements. The basis elements of $H_j(R/I)_{d+j}$ come from the basis elements of $H_j(R/I(1))_{d+j}$ which do not map to zero in $H_j(R/I_{d+j})$. More specifically, suppose $z \in K_j(R/I(1))_{d+j}$ represents a basis element in $H_j(R/I(1))_{d+j}$. Write $z = \epsilon(u')e_i \wedge e_\sigma \wedge e_{m(u)} + r(u)$. Since $z$ represents a basis element in $R/I(1)$, we know that $x_1u' x_{j}/x_{\ell} \in I(1)$ for certain $\ell$ and $s$. We claim that $z$ is a boundary in $K_j(R/I)_{d+j}$. This would show that $z$ does not give rise to a basis element in $H_j(R/I)_{d+j}$.

We argue by cases. First, if $i = 1$, then the condition says that $x_1u' x_{j}/x_{\ell} \in I(1)$, which means $x_1u_1 \geq x_1u' x_{j}/x_{\ell}$, and hence $u_1 \geq u'$. Since we always have $u' \geq u_k$, this implies $u' \in I$, and hence $z = 0$ in $K_j(R/I)_{d+j}$. Second, suppose $x_1$ does not divide $u'$. Then the assumption that $x_1$ divides $u_1$ implies we again have $u_1 \geq u' \geq u_k$. Thus $u' \in I$ and $z = 0$ in $K_j(R/I)_{d+j}$.

Thus we may assume that $i > 1$ and $x_1$ divides $u'$. Then

$$
\partial \left( \epsilon \left( \frac{u'}{x_1} \right) e_i \wedge e_\sigma \wedge e_{m(u)} \right) = \epsilon(u')e_i \wedge e_\sigma \wedge e_{m(u)} + \text{other terms},
$$

(4)

where the other terms all have coefficients like $\epsilon(x_1u'/x_1)$ for $i \leq j \leq m(u)$. Note that $u' > \frac{x_1u'}{x_1} > \frac{u}{x_1}$ for each $i \leq j \leq m(u)$.

Now write

$$
u_1 = x_1^{p_1} \cdots x_n^{p_n}, \quad u_k = x_1^{q_1} \cdots x_n^{q_n}
$$

and $u' = x_1^{r_1} \cdots x_n^{r_n}$. Since $x_1u_1 \geq u_k x_{n}$, we must have $p_1 + 1 \geq r_1$. If $p_1 > r_1$, then $u_1 > u' \geq u_k$, so that $\epsilon(u')e_i \wedge e_\sigma \wedge e_{m(u)} = 0$ in $K_j(R/I)_{d+j}$ and so is trivially a boundary.

If $p_1 = r_1$ or $p_1 = r_1 - 1$, then clearly $u_1 > \frac{x_1u'}{x_1}$ for each $i \leq j \leq m(u)$. By the previous result, however, we must have $u = w x_{\sigma}$ for some $w \in I_d$ with $u_1 \geq w \geq u_k$. Write $w = x_1^{p_1} \cdots x_n^{p_n-1} \cdots x_n^{r_n}$. If $w = m(u)$, then $u' = w \in I_d$ and so again $\epsilon(u')e_i \wedge e_\sigma \wedge e_{m(u)} = 0$ in $K_j(R/I)_{d+j}$. If $w < m(u)$, then $u' \notin I$. But since we always have $u' \geq u_k$, this implies that $u' > u_1$, which in turn implies that $r_1 > p_1$ for the first index where they are non-equal. This, together with the fact that $w \geq u_k$ and the assumption that $u_1 \geq u_k$, implies that $\frac{u_1}{x_1} \geq u_k$.

Thus we have $u_1 \geq \frac{x_1u'}{x_1} \geq u_k$ for $i \leq j \leq m(u)$, which shows that the other terms in (5) are zero in $K_j(R/I)_{d+j}$. Hence $\epsilon(u')e_i \wedge e_\sigma \wedge e_{m(u)}$ is a boundary in $K_j(R/I)_{d+j}$, which finishes the proof. \qed

Corollary 3.5. Suppose $I$ is a lex-segment ideal which is generated in the degrees $d_1 \leq \cdots \leq d_j$. If $d = d_1 = d_j$, then $\beta_{i, d+i+j}(R/I) = 0$ for all $i$ and all $j \geq 2$. If $d_1 < d_2$, then $\beta_{i, d_2+i+j}(R/I) = 0$ for all $j \geq 1$. 
Proof. For this, we just have to note that for $j \geq 1$, $I(j)$ is a lex-segment ideal which satisfies the conditions of the above theorem. In the first stated case, as in the proof above, the exact sequence (4) implies that $\beta_{i,d+i+j}(R/I) = \beta_{i,d+i+j}(R/I(1))$ for all $j \geq 2$. Since $R/I(1)$ has a linear resolution by the above theorem, this implies that the resolution of $R/I$ has at most 2 strands. In the second stated case, we have $\beta_{i,d_2+i+j}(R/I) = \beta_{i,d_2+i+j-1}(R/I(d_2 - 1))$, and $R/I(d_2 - 1)$ has a linear resolution by the above theorem. Thus, the resolution of $R/I$ has no strands beyond degree $d_2$. □

Recall that the Castelnuovo–Mumford regularity of a graded module $M$ can be defined as

$$\text{reg}(M) = \max\{j - i : j \in \mathbb{Z} \text{ and } \beta_{i,j} \neq 0\}.$$  

That is, $\text{reg}(M)$ is determined by the highest strand in the resolution of $M$. Consequently, we can restate the above corollary:

**Corollary 3.6.** If $I$ is a lex-segment ideal which is generated in degrees $d_1 \leq \cdots \leq d_2$, then $\text{reg}(R/I) \leq d_1 + 1$ if $d_1 = d_2$, and $\text{reg}(R/I) = d_2$ if $d_1 < d_2$.

References