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Fundamental structures of M(brane) theory

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ABSTRACT

A dynamical symmetry, as well as special diffeomorphism algebras generalizing the Witt–Virasoro algebra, related to Poincaré invariance and crucial with regard to quantization, questions of integrability, and M(atrix) theory, are found to exist in the theory of relativistic extended objects of any dimension.

The simplicity of classical string theory, and decades of presenting it from one and the same point of view, have made it difficult to realize some of the central features of relativistic extended objects (described below in the light-cone gauge), namely:

- relativistic invariance implying the existence of a dynamical symmetry (irrespective of the dimension of the extended object), and
- the Virasoro algebra being just the simplest example of certain (extended) infinite-dimensional diffeomorphism algebras reappearing, after gauge fixing (and on the constrained phase space), in the reconstruction of x^-.

For the purpose of this Letter, I will restrict myself to the purely bosonic theory [1,2], i.e. (analogous results for the supersymmetrized theory will easily follow)

$$H = \frac{1}{2\eta}\int_{\Sigma_0} \frac{\vec{p}^2 + g}{\rho}\, d^M\varphi = H[\vec{x}, \vec{p}; \eta, \zeta_0] = \int \mathcal{H}\, d^M\varphi,$$

$$g = \det\left(\frac{\partial \vec{x}}{\partial \varphi^a}\cdot\frac{\partial \vec{x}}{\partial \varphi^b}\right)_{a,b=1,\ldots,M}, \tag{1}$$

with $\rho(\varphi)$ being a non-dynamical density of weight one (i.e. $\int_{\Sigma_0}\rho(\varphi)\, d^M\varphi = 1$), x_i and p_j $(i, j = 1, \ldots, d = D-2)$ canonically conjugate fields satisfying

$$\int f^a \vec{p}\cdot\partial_a\vec{x}\, d^M\varphi = 0 \quad \text{whenever } \nabla_a f^a = 0 \tag{2}$$

for the consistency of

$$\eta\partial_a\zeta = \frac{\vec{p}}{\rho}\cdot\partial_a\vec{x} \tag{3}$$

which, together with

$$2\eta^2\dot{\zeta} = \frac{\vec{p}^2 + g}{\rho^2} \tag{4}$$

(that actually can also be thought of as defining η and ρ in terms of the initial parametrized shape, and the velocity, of the time-dependent M-dimensional extended object moving in D-dimensional Minkowski space), determine ζ (usually called x^-) up to $\zeta_0 = \int \zeta\rho\, d^M\varphi$; the time independent positive degree of freedom η (usually called P^+) is canonically conjugate to $-\zeta_0$.

In the mid-eighties, Goldstone (when proving that the above description is fully Poincaré-invariant [2]) solved (3) (assuming (2)) in the form

$$\zeta(\varphi) = \zeta_0 + \frac{1}{\eta}\int G(\varphi,\tilde{\varphi})\tilde{\nabla}^a\left(\frac{\vec{p}}{\rho}\cdot\tilde{\nabla}_a\vec{x}\right)(\tilde{\varphi})\rho(\tilde{\varphi})\, d^M\tilde{\varphi} \tag{5}$$

with (∇_a being the covariant derivative, and Δ the Laplacian on Σ_0)

$$\int G(\varphi,\tilde{\varphi})\rho(\varphi)\, d^M\varphi = 0, \qquad \Delta_{\tilde{\varphi}} G(\varphi,\tilde{\varphi}) = \frac{\delta(\varphi,\tilde{\varphi})}{\rho(\varphi)} - 1. \tag{6}$$

Later it will turn out to be useful to slightly (though, with regard to a variety of aspects: crucially) rewrite (5) as

$$\zeta_0 + \frac{\vec{p}\cdot\vec{x}}{2\eta\rho} - \int \frac{\vec{p}\cdot\vec{x}}{2\eta\rho} + \frac{1}{2}\int G(\varphi,\tilde{\varphi})\left(\frac{\vec{p}}{\eta\rho}\cdot\Delta\vec{x} - \vec{x}\cdot\Delta\frac{\vec{p}}{\eta\rho}\right)(\tilde{\varphi})\rho(\tilde{\varphi})\, d^M\tilde{\varphi} \tag{7}$$

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(splitting $\zeta - \zeta_0$ into parts symmetric resp. antisymmetric with regard to interchanging $\vec{x}$ and $\vec{p}$, and involving only the invariant Laplace operator). I can now present the two key features that I announced at the beginning of this Letter:

Dynamical symmetry

Separate the zero-modes

$$\zeta_0, \quad \eta, \quad X_i = \int x_i \rho \, d^M\varphi, \quad P_i = \int p_i \, d^M\varphi \tag{8}$$

from the internal degrees of freedom,

$$x_{i\alpha} := \int Y_\alpha(\varphi) x_i(\varphi)\rho(\varphi)\, d^M\varphi, \qquad p_{i\alpha} := \int Y_\alpha(\varphi) p_i(\varphi)\, d^M\varphi, \tag{9}$$

letting $\{Y_\alpha\}_{\alpha=1}^\infty$ be a (together with $Y_0 = 1$) complete orthonormal set of eigenfunctions on Σ_0 (conveniently chosen as eigenfunctions of Δ),

$$\int Y_\alpha Y_\beta \rho\, d^M\varphi = \delta_{\alpha\beta}, \qquad \sum_{\alpha=1}^\infty Y_\alpha(\varphi) Y_\alpha(\tilde{\varphi}) = \frac{\delta(\varphi, \tilde{\varphi})}{\rho(\varphi)} - 1,$$
$$\Delta Y_\alpha = -\mu_\alpha Y_\alpha. \tag{10}$$

The Lorentz invariance of the theory, in particular implying that

$$M_{i-} := \int (x_i \mathcal{H} - \zeta p_i)\, d^M\varphi \tag{11}$$

satisfies

$$\{M_{i-}, M_{j-}\} = 0, \tag{12}$$

necessitates that the purely internal contributions

$$2\eta \mathbb{M}_{i-} := \int (x_i \tilde{\mathcal{H}} - \tilde{\zeta} p_i)\, d^M\varphi = x_{i\alpha}\tilde{\mathcal{H}}_\alpha - \tilde{\zeta}_\alpha p_{i\alpha}, \tag{13}$$

with

$$\tilde{\mathcal{H}}_\alpha := \vec{p}_\beta \cdot \vec{p}_\gamma \int Y_\alpha Y_\beta Y_\gamma \rho\, d^M\varphi + \int Y_\alpha \frac{g}{\rho}\, d^M\varphi$$
$$=: \vec{p}_\beta \cdot \vec{p}_\gamma d_{\alpha\beta\gamma} + W_\alpha,$$
$$\tilde{\zeta}_\alpha := 2\eta\zeta_\alpha - 2\vec{P}\cdot\vec{x}_\alpha, \qquad \zeta_\alpha := \int Y_\alpha \zeta \rho\, d^M\varphi, \tag{14}$$

satisfy

$$\{\eta\mathbb{M}_{i-}, \eta\mathbb{M}_{j-}\} = \mathbb{M}^2 \mathbb{M}_{ij}, \quad i, j = 1, \ldots, d. \tag{15}$$

Here

$$\mathbb{M}_{ij} := x_{i\alpha} p_{j\alpha} - x_{j\alpha} p_{i\alpha} \tag{16}$$

are the generators of internal transverse rotations, and

$$\mathbb{M}^2 = 2\eta H - \vec{P}^2 \tag{17}$$

is the square of the relativistically invariant 'internal mass', commuting with M_{i-}, M_{ij}, H, $\vec{P}$, and η, as well as $\eta\zeta_0$ and ηX_i. (15) is a simple (but crucial) consequence of (12), as the parts of M_{i-} that do involve the zero-modes satisfy

$$\left\{X_i H - \zeta_0 P_i + \frac{\mathbb{M}_{ik}}{\eta} P_k,\ X_j H - \zeta_0 P_j + \frac{\mathbb{M}_{jl}}{\eta} P_l\right\} = -\frac{\mathbb{M}^2}{\eta^2}\mathbb{M}_{ij}, \tag{18}$$

which easily follows from $\{H, \mathbb{M}_{ik}\} = 0$, $\{\zeta_0, \eta\} = -1$, $\{X_i, P_j\} = \delta_{ij}$, and

$$\{\mathbb{M}_{ik}, \mathbb{M}_{jl}\} = -\delta_{kj}\mathbb{M}_{il} \pm 3 \text{ more}, \tag{19}$$

and (17). Finally, one checks that

$$\{\eta\mathbb{M}_{k-}, \mathbb{M}_{ij}\} = -\eta\delta_{ki}\mathbb{M}_{j-} + \eta\delta_{kj}\mathbb{M}_{i-}, \tag{20}$$

and that $\mathbb{M}^2$ commutes with $\eta\mathbb{M}_{i-}$ (and $\mathbb{M}_{ij}$).

This 'sign of integrability/dynamical symmetry' ($\mathbb{M}^2$ appearing in the structure constants of a symmetry algebra of itself) should be extremely useful for the further understanding of relativistic extended objects. E.g. if it is possible to promote (15), (19), (20) to commutation relations for corresponding quantum operators (commuting with $\hat{\mathbb{M}}^2$), one may be able to calculate the spectrum of $\hat{\mathbb{M}}^2$ purely algebraically in terms of the Casimirs of the algebra spanned by $L_{ij} := \mathbb{M}_{ij}$ and $L_{i,d+1} := \frac{\eta\mathbb{M}_{i-}}{\sqrt{\mathbb{M}^2}}$, just as in the case of the d-dimensional Hydrogen atom, which is actually *very* close to the relations that I just derived; the difference lying in the explicit a priori relations between the angular momentum $\mathbb{M}_{ij}$ and the generalized Laplace–Runge–Lenz vector (cp. Note added). One possibility to find such additional relations is to understand the interplay of the different diffeomorphism subalgebras involving the totally symmetric structure constants $d_{\alpha\beta\gamma}$ (cp. (14)),

$$e_{\alpha\beta\gamma} := \frac{\mu_\beta - \mu_\gamma}{\mu_\alpha} d_{\alpha\beta\gamma} \tag{21}$$

and

$$g_{\alpha\alpha_1\ldots\alpha_M} := \int_{\Sigma_0} Y_\alpha \epsilon^{a_1\ldots a_M} \frac{\partial Y_{\alpha_1}}{\partial\varphi^{a_1}} \cdots \frac{\partial Y_{\alpha_M}}{\partial\varphi^{a_M}}\, d^M\varphi, \tag{22}$$

part of which I will now come to.

Reconstruction algebras

To directly verify (15) (just using (13)) is a very instructive, but complicated, calculation; in particular one finds that the modes of ζ (times η) close under Poisson brackets (on the constrained phase-space, i.e. assuming (2)),

$$\{\eta\zeta_\alpha, \eta\zeta_{\alpha'}\} = f^\epsilon_{\alpha\alpha'}\eta\zeta_\epsilon \tag{23}$$

(whose simplest, $M = 1$, example leads to the Virasoro algebra). Let me calculate the structure constants and identify the generators as special diffeomorphisms of Σ_0:

Using (3)/(5) and (6)/(10) (implying $G = \sum_{\alpha=1}^\infty \frac{-1}{\mu_\alpha} Y_\alpha(\varphi) Y_\alpha(\tilde{\varphi})$) one has (corresponding to a vectorfield whose divergence is $\nabla_a f^a = -Y_\alpha$)

$$L_\alpha := \eta\zeta_\alpha := \int Y_\alpha \zeta \rho\, d^M\varphi$$
$$= \frac{1}{\mu_\alpha}\int (\nabla^a Y_\alpha)\vec{p}\cdot\partial_a\vec{x} = \int f^a_\alpha \vec{p}\cdot\partial_a\vec{x}; \tag{24}$$

hence ($\approx$ indicating the use of (2), i.e. equal, modulo volume-preserving diffeomorphisms)

$$\mu_\alpha\mu_{\alpha'}\{\eta\zeta_\alpha, \eta\zeta_{\alpha'}\}$$
$$= \left\{\int \nabla^a Y_\alpha \vec{p}\cdot\partial_a\vec{x}\, d^M\varphi, \int \nabla^{a'} Y_{\alpha'}\vec{p}\cdot\partial_{a'}\vec{x}\, d^M\varphi'\right\}$$
$$= \int \left(\nabla^b Y_\alpha \nabla_b\left(\nabla^a Y_{\alpha'}\right) - \nabla^b Y_{\alpha'}\nabla_b\left(\nabla^a Y_\alpha\right)\right)\vec{p}\cdot\partial_a\vec{x}\, d^M\varphi$$
$$\approx -\int \left(\nabla^b Y_\alpha \nabla_a \nabla_b \nabla^a Y_{\alpha'} - (\alpha \leftrightarrow \alpha')\right)\eta\zeta\rho\, d^M\varphi, \tag{25}$$

so that

$$f_{\alpha\alpha'}^{\epsilon} = \frac{\mu_{\alpha'} - \mu_{\alpha}}{2\mu_{\alpha}\mu_{\alpha'}}(\mu_{\alpha} + \mu_{\alpha'} - \mu_{\epsilon})d_{\alpha\alpha'\epsilon} = e_{[\alpha,\alpha']\epsilon}. \tag{26}$$

For $M = 1$ the combination of eigenvalues gives $\frac{m^2-n^2}{mn}$ which indeed (multiplying, in accordance with the standard oscillator expansions, the generators L_m by m) gives $(m - n)$.

Consequences of the dynamical symmetry, Lorentz invariance in matrix models, generalizations to the supersymmetric theories, and properties of the various algebras of local fields arising from $d_{\alpha\beta\gamma}$ and $e_{\alpha\beta\gamma}$ (cp. (21)) will be discussed in forthcoming papers.

Note added

As suggested by the referee, let me make some further comments about the mass-squared appearing in commutation relations as a structure-constant, in particular concerning the analogy with the hydrogen atom: The relations (15), (19), (20) say that $L_{ij} := \mathbb{M}_{ij}$ and $L_{i,d+1} := \frac{\eta\mathbb{M}_{i-}}{\sqrt{\mathbb{M}^2}}$ form a (basis of a) representation of so(D-1). While this is already useful by itself (string-theorists, when discussing the light-cone gauge, usually point out how non-trivial it is that the massive states combine to form representations of so(D-1); due to (15), (19), (20) so(D-1)is 'manifest'), the more important issue is whether such relations could possibly be used to calculate the spectrum of relativistic extended objects algebraically. This could be done (assuming (15), (19), (20) to hold in a quantized theory as well; in fact, with respect to the difficulty of quantizing a non-linear theory with infinitely many degrees of freedom, the requirement that these relations should hold, could be taken as a guide-line) if one knew how these infinite-dimensional representations of so(D-1) decompose into finite-dimensional irreducible ones. In the case of the hydrogen atom (cp. [3]) this can be done due to the Hamiltonian, the Laplace–Runge–Lenz vector, and the angular momentum vector, satisfying additional relations that make it possible to identify the occurring representations. In the case of relativistic extended objects (note that neither the dimension M nor other topological data enter explicitly) this information must also be encoded in the particular expressions for the generators (of course vastly more difficult to extract explicitly; on the other hand: what else could one expect from a relativistically invariant theory of extended objects? The mass spectrum should precisely depend on those topological data; and nothing else).

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