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# New Distance Regular Graphs Arising from Dimensional Dual Hyperovals

ANTONIO PASINI AND SATOSHI YOSHIARA

In [4] we have studied the semibiplanes $\Sigma^e_{m,h} = Af(\mathcal{S}^e_{m,h})$ obtained as affine expansions of the $d$-dimensional dual hyperovals of Yoshiara [6]. We continue that investigation here, but from a graph theoretic point of view. Denoting by $\Gamma^e_{m,h}$ the incidence graph of (the point-block system of) $\Sigma^e_{m,h}$, we prove that $\Gamma^e_{m,h}$ is distance regular if and only if either $m + h = e$ or $(m + h, e) = 1$. In the latter case, $\Gamma^e_{m,h}$ has the same array as the coset graph $\mathcal{K}^e_h$ of the extended binary Kasami code $K(2^e, 2^h)$ but, as we prove in this paper, we have $\Gamma^e_{m,h} \cong \mathcal{K}^e_h$ if and only if $m = h$. Finally, by exploiting some information obtained on $\Gamma^e_{m,h}$, we prove that if $e \leq 13$ and $m \neq h$ with $(m + h, e) = 1$, then $\Sigma^e_{m,h}$ is simply connected.



## 1. INTRODUCTION

In this Introduction we first recall a few definitions and known results (Sections 1.1–1.3). Then we state our main results (Section 1.4). Finally, we discuss a conjecture concerning the simple connectedness of some of the semibiplanes considered in this paper (Section 1.5).

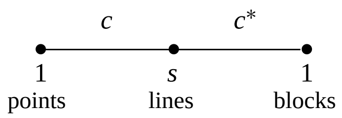
The rest of the paper is organized as follows. In Section 2 we recall some results taken from [4], to be used in Sections 3 and 4. The main theorems of this paper are proved in Section 3. In Section 4 we collect some evidence for the conjecture discussed in Section 1.5.

*1.1. Semibiplanes and d-dual hyperovals.* We refer to [3] for the few notions of diagram geometry used in this paper. We recall here that a *semibiplane* of *order s* is a connected finite incidence structure $\Sigma = (\mathcal{P}, \mathcal{B})$, where $\mathcal{P}$ and $\mathcal{B}$ are the set of *points* and the set of *blocks* such that:

(S1) any two distinct points (blocks) are incident with either zero or two common blocks (points) and
(S2) every block (point) is incident to exactly $s + 2$ points (blocks).

If $A$, $B$ are distinct blocks with non-trivial intersection and $\{a, b\} = A \cap B$ (see (S1)), then the pair $(\{a, b\}, \{A, B\})$ is called a *line*, with the convention that $a$ and $b$ (respectively $A$ and $B$) are the points (blocks) incident to it. In this way, $\Sigma$ is viewed as a rank three geometry with diagram and orders as follows:

$(c.c^*)$

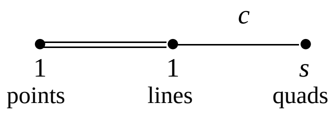



If $(\{a, b\}, \{A, B\})$ is a line then, by (S1), either of the pairs $\{a, b\}$ or $\{A, B\}$ uniquely determines the other one. Accordingly, lines may also be regarded as pairs of points (or blocks) belonging to the same block (respectively with non-trivial intersection).

The *folding* $\Phi = Fld(\Sigma)$ of a semibiplane $\Sigma$ is the rank three geometry of which the elements are the points and the blocks of $\Sigma$ (called *points* of $\Phi$), the point-block flags of $\Sigma$ (called *lines* of $\Phi$) and the lines of $\Sigma$ (called *quads*), with the incidence relation inherited from $\Sigma$. The diagram and the order of $\Phi$ are as follows:

$(C_2.c)$




Clearly, the points and the lines of $\Phi$ are the vertices and the edges of the incidence graph of (the point-block system of) $\Sigma$ and, in view of (S1), every quadrangle of that graph belongs to a quad. Furthermore, the universal cover of $\Phi$ is the folding of the universal cover of $\Sigma$ (Rinauro [5]). Hence, by [3, Theorem 12.64], we have the following.

PROPOSITION 1.1. *A semibiplane $\Sigma$ is simply connected as a rank three geometry if and only if, regarding $\Sigma$ as a rank two geometry of points and blocks, every closed path of its incidence graph splits into quadrangles.*

Halved hypercubes and projective (elation, homology and Baer) semibiplanes are the best known examples of semibiplanes, but we are not going to recall their definitions here. The reader may see [4] for these. We only recall a construction of semibiplanes from dimensional dual hyperovals.

A *$d$-dimensional dual hyperoval* of $PG(n, 2)$ (a *$d$-dual hyperoval*, for short) is a family $\mathcal{S}$ of $d$-dimensional subspaces of $PG(n, 2)$ such that:

(H1) every point of $PG(n, 2)$ belongs to either no or just two members of $\mathcal{S}$,
(H2) any two members of $\mathcal{S}$ have just one point in common and
(H3) the set $S_0 := \bigcup_{X \in \mathcal{S}} X$ spans $PG(n, 2)$.

Given a $d$-dual hyperoval $\mathcal{S}$ of $PG(n, 2)$ and regarding $PG(n, 2)$ as the geometry at infinity of $AG(n + 1, 2)$, the *affine expansion* $Af(\mathcal{S})$ of $\mathcal{S}$ is the rank three geometry defined as follows. The points of $Af(\mathcal{S})$ are the points of $AG(n + 1, 2)$ and the blocks of $Af(\mathcal{S})$ are the $(d + 1)$-subspaces of $AG(n + 1, 2)$ having a member of $\mathcal{S}$ as the space at infinity. The lines of $Af(\mathcal{S})$ are the lines of $AG(n + 1, 2)$ with point at infinity belonging to $S_0$. The incidence relation is the natural one, inherited from $AG(n + 1, 2)$.

The connectedness of $Af(\mathcal{S})$ follows from (H3). Furthermore, by (H1) and (H2), the pair $(\mathcal{S}, S_0)$ is a complete graph with $2^{d+1}$ vertices. Hence $Af(\mathcal{S})$ is a semibiplane of order $s = 2^{d+1} - 2$.

Two $d$-dual hyperovals $\mathcal{S}$ and $\mathcal{S}'$ of $PG(n, 2)$ are said to be *isomorphic* (and we write $\mathcal{S} \cong \mathcal{S}'$) if $\mathcal{S}' = \varphi(\mathcal{S})$ for some $\varphi \in L_{n+1}(2)$ $(= \mathrm{Aut}(PG(n, 2)))$. The *automorphism group* $\mathrm{Aut}(\mathcal{S})$ of $\mathcal{S}$ is the stabilizer of $\mathcal{S}$ in $L_{n+1}(2)$.

Regarding $L := \mathrm{Aut}(\mathcal{S})$ as a subgroup of the stabilizer in $A = A\Gamma L_{n+1}(q)$ of a distinguished point of $AG(n + 1, 2)$, we can consider the extension $A_{\mathcal{S}} := TL$ of $L$ by the translation group $T = O_2(A)$ of $AG(n + 1, 2)$. Clearly, $A_{\mathcal{S}}$ is a subgroup of $\mathrm{Aut}(Af(\mathcal{S}))$. It is flag transitive on $Af(\mathcal{S})$ if and only if $L$ is two transitive on $\mathcal{S}$. We call $A_{\mathcal{S}}$ the *affine automorphism group* of $Af(\mathcal{S})$.

*1.2. The semibiplane $Af(\mathcal{S}^e_{m,h})$ and the graph $\Gamma^e_{m,h}$.* The semibiplanes considered in this paper are affine expansions of the $d$-dual hyperovals of Yoshiara [6]. The latter are defined as follows.

Let $q = 2^e$ with $e \geq 2$ and regard $GF(q)$ as an $e$-dimensional vector space over $GF(2)$. Accordingly, the set $V := GF(q) \times GF(q)$ has the structure of a $2e$-dimensional vector space over $GF(2)$. Given two positive integers $m, h < e$, relatively prime with $e$, let

$$X(t) := \{(x, x^{2^m} t + t^{2^h} x)\}_{x \in GF(q)} \qquad (\text{for } t \in GF(q))$$

and $\mathcal{S}^e_{m,h} := \{X(t)\}_{t\in GF(q)}$. Clearly, $X(t)$ is an $e$-dimensional subspace of $V$, namely an $(e-1)$-dimensional subspace of $PG(V) \cong PG(2e-1, 2)$. As proved by Yoshiara [6], the family $\mathcal{S}^e_{m,h}$ is an $(e-1)$-dimensional dual hyperoval of the span $\langle \mathcal{S}^e_{m,h} \rangle$ of $\bigcup_{t\in GF(q)} X(t)$ in $PG(V)$. Furthermore, we have the following.

PROPOSITION 1.2 (Yoshiara [6]). *If $m+h=e$, then $\langle \mathcal{S}^e_{m,h}\rangle$ is a hyperplane of $PG(V)$, otherwise $\langle \mathcal{S}^e_{m,h}\rangle = PG(V)$.*

In any case, the affine expansion $Af(\mathcal{S}^e_{m,h})$ of $\mathcal{S}^e_{m,h}$ is a semibiplane of order $2^e-2$. When $m+h=e$, that semibiplane has $2^{2e-1}$ points; otherwise, it has $2^{2e}$ points. The following has also been proved by Yoshiara [6].

PROPOSITION 1.3. *Let $m, n, h, k$ be positive integers less than $e$ and relatively prime with $e$.*

(1) *If $m+h=n+k=e$, then $\mathcal{S}^e_{m,h} \cong \mathcal{S}^e_{n,k}$.*
(2) *Suppose $m+h \neq e \neq n+k$. Then $\mathcal{S}^e_{m,h} \cong \mathcal{S}^e_{n,k}$ if and only if either $(m,h)=(n,k)$ or $m+n=h+k=e$.*

The following are proved in [4].

PROPOSITION 1.4. *The universal cover of $Af(\mathcal{S}^e_{m,h})$ is a halved hypercube if and only if $m=h$.*

PROPOSITION 1.5. *If $m+h=e$, then $Af(\mathcal{S}^e_{m,h})$ is an elation semibiplane.*

Therefore, as elation semibiplanes are simply connected (Baumeister and Pasechnik [1]), we have the following.

COROLLARY 1.6. *If $m+h=e$, then $Af(\mathcal{S}^e_{m,h})$ is simply connected.*

In this paper, we are mainly interested in the incidence graph $\Gamma^e_{m,h}$ of $Af(\mathcal{S}^e_{m,h})$, where $Af(\mathcal{S}^e_{m,h})$ is regarded as a point-block structure. The main properties of $\Gamma^e_{m,h}$ will be stated in Section 1.4. Here, we only mention that, when $m+h$ is relatively prime with $e$, the graph $\Gamma^e_{m,h}$ has the same array as the coset graph $\mathcal{K}^e_h$ of the extended binary Kasami code $K(2^e, 2^h)$ (see Brouwer, Cohen and Neumaier [2, 11.2]). We describe this graph in the next subsection.

*1.3. The graph $\mathcal{K}^e_h$.* Given an odd positive integer $e$ and a positive integer $h < e$ coprime to $e$, let $q := 2^e$ and $F := GF(q)$. Consider the space $GF(2)^F$ of row vectors with entries indexed by $F$, and identify a vector $v = (v_x)_{x\in F}$ of $GF(2)^F$ with its support $\{x \in F \mid v_x = 1\}$. Then the set

$$\left\{ S \subseteq F \mid |S| \text{ even}, \sum_{x\in S} x = 0, \sum_{x\in S} x^{2^h+1} = 0 \right\}$$

is a subspace of $GF(2)^F$, called the *extended binary Kasami code* $K(2^e, 2^h)$ (notation as in [2, 11.2]). This subspace is the kernel of the $GF(2)$-linear map

$$\begin{aligned} f : GF(2)^F &\mapsto GF(2) \times F \times F \\ (v_x)_{x\in F} &\mapsto \left( \sum_{x\in F} v_x, \sum_{x\in F} v_x x, \sum_{x\in F} v_x x^{2^h+1} \right) \end{aligned}$$

(where $GF(2) \times F \times F$ is regarded as a $(2e+1)$-dimensional vector space over $GF(2)$). The vertices of $\mathcal{K}_h^e$ are the cosets of $K := K(2^e, 2^h)$ in $GF(2)^F$, two such cosets $X + K$ and $Y + K$ ($X, Y \subseteq F$) being adjacent precisely when there are vectors $X' \in X + K$ and $Y' \in Y + K$ such that either $X' \subset Y'$ and $|X'| = |Y'| + 1$ or $Y' \subset X'$ and $|Y'| = |X'| + 1$.

It is well known (Brouwer, Cohen and Neumaier [2, 11.2]) that $\mathcal{K}_h^e$ is a bipartite distance-regular graph of diameter four with intersection array as follows:

$$\left\{ \begin{matrix} q & q-1 & q-2 & (q/2)+1 & * \\ * & 1 & 2 & (q/2)-1 & q \end{matrix} \right\}.$$

Clearly, the vertices of $\mathcal{K}_h^e$ bijectively correspond to the values of the function $f$. In particular, $K$ corresponds to $(0, 0, 0)$ and we can construct a copy of $\mathcal{K}_h^e$ on the image $\mathrm{Im}(f)$ of $f$ as follows:

($*$) two distinct elements $(i, x, y)$, $(j, z, t)$ of $\mathrm{Im}(f)$ are adjacent as vertices of $\mathcal{K}_h^e$ precisely when $i + j = 1$ and $(x + z)^{2^h+1} = y + t$.

The following is now straightforward:

PROPOSITION 1.7. *For $i = 1, 2, 3, 4$, let $\mathcal{K}_i$ be the $i$-neighborhood of $(0, 0, 0)$ in $\mathcal{K}_h^e$. Then,*

$$\begin{aligned}
\mathcal{K}_1 &= \{(1, x, x^{2^h+1})\}_{x \in F}, \\
\mathcal{K}_2 &= \{(0, x, x^{2^h+1} + xy^{2^h} + x^{2^h} y) \mid x, y \in F,\ x \neq 0\}, \\
\mathcal{K}_3 &= \{(1, x, y) \mid (1, x, y) \in \mathrm{Im}(f) - \mathcal{K}_1\}, \\
\mathcal{K}_4 &= \{(0, x, y) \mid (0, x, y) \in \mathrm{Im}(f) - (\mathcal{K}_2 \cup \{(0, 0, 0)\})\}.
\end{aligned}$$

*1.4. Main results.* Clearly $\Gamma_{m,h}^e$, being the incidence graph of a rank two geometry, is bipartite. When $m + h = e$, it easily follows from Proposition 1.5 that $\Gamma_{m,h}^e$ is distance regular of diameter four, with intersection array as follows, where $q = 2^e$:

$$\left\{ \begin{matrix} q & q-1 & q-2 & 1 & * \\ * & 1 & 2 & q-1 & q \end{matrix} \right\}.$$

The following will be proved in Section 3.

THEOREM 1.8. *Assume $m + h \neq e$. Then $\Gamma_{m,h}^e$ has diameter four. Furthermore, $\Gamma_{m,h}^e$ is distance regular if and only if $e$ is coprime to $m + h$. If that is the case, then $\Gamma_{m,h}^e$ has the same array as $\mathcal{K}_h^e$.*

(Note that, as $m, h$ are coprime to $e$, if $e$ is also coprime to $m + h$ then it is odd, as required for $\mathcal{K}_h^e$.) The following will also be proved in Section 3.

THEOREM 1.9. *Let $e > 2$. Then $\Gamma_{m,h}^e \cong \mathcal{K}_h^e$ if and only if $m = h$.*

The next corollary immediately follows from Theorem 1.9 and Proposition 1.4.

COROLLARY 1.10. *The graph $\mathcal{K}_h^e$ is covered by the (collinearity graph of the) $2^e$-dimensional hypercube.*

Theorems 1.8 and 1.9 also imply the following.

COROLLARY 1.11. *Given an odd positive integer $e$ and a positive integer $h < e$ coprime to $e$, suppose there is a positive integer $m < e$, different from $h$ and such that $e$ is coprime to both $m$ and $m + h$. Then there exists a distance-regular graph of diameter 4, with the same array as $\mathcal{K}_h^e$ but not isomorphic to $\mathcal{K}_h^e$.*

*1.5. On the universal cover of $Af(\mathcal{S}^e_{m,h})$.* In view of Corollary 1.6, the $c.c^*$-geometry $Af(\mathcal{S}^e_{m,e-m})$ is simply connected for any positive integer $m < e$ coprime to $e$. On the other hand, by Proposition 1.4, the universal cover of $Af(\mathcal{S}^e_{m,m})$ is the $2^e$-dimensional halved hypercube. Hence $Af(\mathcal{S}^e_{m,m})$ is not simply connected when $e > 2$. (Note that $Af(\mathcal{S}^2_{1,1})$ is a copy of the four-dimensional halved hypercube.)

As noticed in [4] (final remark of Section 1), when $m + h \neq e$, $m \neq h$ and $e = 5$ or 7, then $Af(\mathcal{S}^e_{m,h})$ is simply connected. (Note that, when $e = 2, 3, 4$ or 6, no pair $(m, h)$ exists with $(m, e) = (h, e) = 1$, $m \neq h$ and $m + h \neq e$.) The above result for the cases of $e = 5$ and 7 has been obtained by applying coset enumeration to the amalgam of element stabilizers in the affine automorphism group of $Af(\mathcal{S}^e_{m,h})$. Regretfully, computing times with that method seem too long when $e > 7$.

We continue that investigation in this paper (Section 4), but with a different method. In view of Proposition 1.1, in order to prove that $Af(\mathcal{S}^e_{m,h})$ is simply connected, we only need to show that every closed path of $\Gamma^e_{m,h}$ splits into quadrangles. Exploiting the information obtained on $\Gamma^e_{m,h}$ in Section 3, but under the additional assumption that $(m + h, e) = 1$ (which forces $e$ to be odd), we shall prove that every closed path of $\Gamma^e_{m,h}$ splits into quadrangles and hexagons. To finish, we should also prove that every hexagon of $\Gamma^e_{m,h}$ splits into quadrangles. Computer aided calculations show that this is indeed the case when $e \leq 14$ for all pairs $(m, h)$ with $m \neq h$ and $(m + h, e) = 1$. (Once again, we recall that the condition $(m + h, e) = 1$ forces $e$ to be odd.) On the basis of the above, we dare to propose the following.

CONJECTURE. *If $m \neq h$ and $(m + h, e) = 1$, then $Af(\mathcal{S}^e_{m,h})$ is simply connected.*

## 2. MORE INFORMATION ON $Af(\mathcal{S}^e_{m,h})$

In this section we recall some definitions and results of [4], to be used in Section 3.

Henceforth, $e$ is an integer greater than unity and $h$ and $m$ are positive integers less than $e$ and relatively prime with $e$. As Proposition 1.5 completely settles the case of $m + h = e$, we also assume $m + h \neq e$. Hence $e > 2$.

We set $q := 2^e$ and $V := GF(q) \times GF(q)$, regarded as a $2e$-dimensional vector space over $GF(2)$.

The members of $\mathcal{S}^e_{m,h}$ are distinguished $e$-dimensional linear subspaces of $V$ and, as $m+h \neq e$ by assumption, they span $V$ (Proposition 1.2). The points (blocks) of $Af(\mathcal{S}^e_{m,h})$ are the vectors of $V$ (the cosets in $V$ of the members of $\mathcal{S}^e_{m,h}$). However, another description of $Af(\mathcal{S}^e_{m,h})$, more suited to our purposes in this paper, is given in [4].

Let $\Sigma^e_{m,h} := (H_1, H_0, *)$ be the incidence structure with $H_1$ as the set points, $H_0$ as the set of blocks and $*$ as the incidence relation, where

$$H_0 := \{(0; x, y) | (x, y) \in V\}, \qquad H_1 := \{(1; x, y) \mid (x, y) \in V\} \qquad \text{and}$$
$$(1; x, y) * (0; t, z) \qquad \text{iff} \quad y + z = x^{2^m} t + x t^{2^h}.$$

Then $\Sigma^e_{m,h}$ is a semibiplane, and the function sending every element $(1; x, y) \in H_1$ to $(x, y)$ and $(0; t, z) \in H_0$ to the block $(0, z) + X(t)$ of $Af(\mathcal{S}^e_{m,h})$ is an isomorphism from $\Sigma^e_{m,h}$ to $Af(\mathcal{S}^e_{m,h})$, the latter being now regarded as a point-block structure [4, Proposition 3.4].

The affine automorphism group $G$ of $Af(\mathcal{S}^e_{m,h})$ is described in [4, Subsection 3.3]. It is a subgroup of the automorphism group $ASL(2e, 2)$ of the geometry $AG(V)$ of affine varieties of $V$ and contains the translation group $T$ of $AG(V)$. The stabilizer in $G$ of a point of $AG(V)$ is a copy of $\mathrm{Aut}(\mathcal{S}^e_{m,h})$. When $e > 3$ or $e = 3$ but $m \neq h$, then $\mathrm{Aut}(\mathcal{S}^e_{m,h}) = T_0MS \cong A\Gamma L_1(q)$, where $T_0$ is elementary abelian, of order $q = 2^e$, and $M$ and $S$ are cyclic, of order

$q-1$ and $e$, respectively. The elements of $T_0$ are translations $\tau_a$ (for $a \in GF(q)$) acting as follows on $V$ and on the members of $\mathcal{S}^e_{m,h}$:

$$\tau_a : \begin{cases} (x, y) \mapsto (x, x^{2^m}a + a^{2^h}x + y), \\ X(t) \mapsto X(t + a). \end{cases}$$

In order to describe the action of $M$, we need to state some notation. Note first that, as $m$ and $h$ are coprime to $e$, the functions

$$\gamma : x \mapsto x^{2^h-1}, \qquad \delta : x \mapsto x^{2^m-1}, \qquad (x \in GF(q)^\times)$$

are automomorphisms of the multiplicative group $GF(q)^\times$ of $GF(q)$. We denote by $\varepsilon$ the composition of $\gamma$ with the inverse $1/\delta$ of $\delta$, and $1/\varepsilon$ is the inverse of $\varepsilon$:

$$\varepsilon : x \mapsto x^{(2^h-1)/(2^m-1)}, \qquad 1/\varepsilon : x \mapsto x^{(2^m-1)/(2^h-1)}, \qquad (x \in GF(q)^\times).$$

Furthermore, we define a mapping $\eta : GF(q)^\times \mapsto GF(q)^\times$ as follows:

$$\eta : x \mapsto x^{(2^{m+h}-1)/(2^h-1)}.$$

We can now describe the action of $M$. The elements of $M$ are dilatations $\mu_b$ (for $b \in GF(q)^\times$), acting as follows:

$$\mu_b : \begin{cases} (x, y) \mapsto (xb, yb^\eta), \\ X(t) \mapsto X(b^{1/\varepsilon}t). \end{cases}$$

Finally, the elements of $S$ are field automorphisms $\sigma \in \mathrm{Aut}(GF(q))$, acting as follows:

$$\sigma : \begin{cases} (x, y) \mapsto (x^\sigma, y^\sigma), \\ X(t) \mapsto X(t^\sigma). \end{cases}$$

When $e = 3$ and $m = h$, then $\mathrm{Aut}(\mathcal{S}^e_{m,h}) \cong ASL_e(2)$, but $\mathrm{Aut}(\mathcal{S}^e_{m,h})$ still contains a subgroup $T_0MS$ as above.

The action of $G$ on $\Sigma^e_{m,h}$ is easy to describe. The group $T$ acts as follows: for every $v = (a, b) \in V$, the translation $t_v \in T$ associated with $v$ sends $(1; x, y)$ and $(0; t, z)$ to $(1; x + a, y + b)$ and $(0; t + a, z + b)$ respectively. Turning to $\mathrm{Aut}(\mathcal{S}^e_{m,h})$, the elements of $T_0$, $M$ and $S$ act as follows:

$$\tau_a : \begin{cases} (1; x, y) \mapsto (1; x, x^{2^m}a + a^{2^h}x + y), \\ (0; t, z) \mapsto (0; t + a, z) \end{cases}$$
$$\mu_b : \begin{cases} (1; x, y) \mapsto (1; xb, yb^\eta), \\ (0; t, z) \mapsto (0; b^{1/\varepsilon}t, b^\eta z) \end{cases}$$
$$\sigma : \begin{cases} (1; x, y) \mapsto (1; x^\sigma, y^\sigma), \\ \sigma : (0; t, z) \mapsto (0; t^\sigma, z^\sigma). \end{cases}$$

When $e = 3$ and $m = h$, we should also say how the elements of $\mathrm{Aut}(\mathcal{S}^e_{m,h})$ not belonging to $T_0MS$ act on $\Sigma^e_{m,h}$, but we do not need this information for the following.

## 3. Proofs of Theorems 1.8 and 1.9

Henceforth we assume $m + h \neq e$, as in Section 2. Hence $e \geq 3$ and $Af(\mathcal{S}^e_{m,h})$ is not an elation semibiplane.

As the semibiplane $\Sigma^e_{m,h} = (H_1, H_0, *)$ defined in Section 2 is isomorphic to the point-block system of $Af(\mathcal{S}^e_{m,h})$, we may regard $\Gamma^e_{m,h}$ as the incidence graph of $\Sigma^e_{m,h}$. Accordingly, $H_1 \cup H_0$ is the set of vertices of $\Gamma^e_{m,h}$ and two distinct vertices $(1; x, y)$ and $(0; t, z)$ of $\Gamma^e_{m,h}$ form an edge of $\Gamma^e_{m,h}$ if and only if $y + z = x^{2^m} t + x t^{2^h}$.

The graph $\Gamma^e_{m,h}$ is bipartite with parts $H_1$ and $H_0$ and each of $H_1$ and $H_0$ is a regular orbit of the group $T$ (Section 2). As in Section 2, in the following $T_0$ and $M$ are the subgroups of $\mathrm{Aut}(\mathcal{S}^e_{m,h})$ consisting of the translations and the dilatations, respectively.

*3.1. Lemmas.* Two points $(1; x, y)$ and $(1; x', y')$ of $\Sigma^e_{m,h}$ have distance two in $\Gamma^e_{m,h}$ if and only if there is a block $(0; t, z)$ incident with both of them. This occurs exactly when $y + z = x^{2^m} t + x t^{2^h}$ and $y' + z = (x')^{2^m} t + x' t^{2^h}$ for some $t, z \in GF(q)$. The last condition is equivalent to saying that $x \neq x'$ and the following equation has a solution in $GF(q)$:

$$(*) \qquad t^{2^h} + (x + x')^{(2^m - 1)} t + \frac{y + y'}{x + x'} = 0.$$

We recall that, denoting by $\mathrm{Tr}(x)$ the trace over $GF(2)$ of an element $x \in GF(q)$, an equation $t^{2^h} + at + b = 0$ with $a, b \in GF(q)^\times$ has a solution in $GF(q)$ if and only if $\mathrm{Tr}(b/a^{2^h/(2^h-1)}) = 0$ (see [6, Proof of Lemma 2]). By this criterion applied to $(*)$, we obtain the following.

LEMMA 3.1. *Two points $(1; x, y)$ and $(1; x', y')$ of $\Sigma^e_{m,h}$ have distance two in $\Gamma^e_{m,h}$ if and only if $x \neq x'$ and*

$$\mathrm{Tr}((y + y')/(x + x')^{(2^{m+h}-1)/(2^h-1)}) = 0.$$

*Similarly, two blocks $(0; t, z)$ and $(0; t', z')$ of $\Sigma^e_{m,h}$ have distance two in $\Gamma^e_{m,h}$ if and only if $t \neq t'$ and*

$$\mathrm{Tr}((z + z')/(t + t')^{(2^{m+h}-1)/(2^m-1)}) = 0.$$

Given a vertex $v$ of $\Gamma^e_{m,h}$, for $i = 1, 2, 3, 4$ we denote by $\Gamma_i(v)$ its $i$-neighborhood in $\Gamma^e_{m,h}$. When $v = (1; 0, 0)$, we briefly write $\Gamma_i$ for $\Gamma_i(1; 0, 0)$.

LEMMA 3.2. *The graph $\Gamma^e_{m,h}$ has diameter four and $\Gamma_1$, $\Gamma_2$, $\Gamma_3$ and $\Gamma_4$ satisfy the following.*

(1) *The elements of $\Gamma_1$ are the $q$ blocks $(0; t, 0)$. They form a single orbit under $T_0$.*
(2) *The elements of $\Gamma_2$ are the $q(q-1)/2$ points $(1; x, y)$ where $x \neq 0$ and $\mathrm{Tr}(y/x^{(2^{m+h}-1)/(2^h-1)}) = 0$. They form an orbit of $T_0 M$.*
(3) *The elements of $\Gamma_3$ are the $q(q-1)$ blocks $(0; t, z)$ with $z \neq 0$. These blocks form $2^{(e,m+h)} - 1$ distinct orbits under $T_0 M$ with representatives $(0; 0, \zeta^j)$ $(j = 0, \ldots, 2^{(e,m+h)} - 2$ and $\zeta$ a generator of $GF(q)^\times)$.*
(4) *The elements of $\Gamma_4$ are the $q(q-1)/2$ points $(1; x, y)$ where $x \neq 0$ and $\mathrm{Tr}(y/x^{(2^{m+h}-1)/(2^h-1)}) = 1$, as well as the $q - 1$ points $(1; 0, y)$ with $y \neq 0$. The former points form a single orbit of $T_0 M$, while the latter split into $2^{(e,m+h)} - 1$ orbits under $T_0 M$ with representatives $(1; 0, \zeta^j)$ $(j = 0, \ldots, 2^{(e,m+h)} - 2$ and $\zeta$ as above).*

PROOF. A block $(0; t, z)$ of $\Sigma^e_{m,h}$ is incident to $(1; 0, 0)$ if and only if $z = 0$. Suppose that no point $(1; x, y)$ incident to a block $(0; t, z)$ with $z \neq 0$ has distance two from $(1; 0, 0)$. Then, for any $x, y \in GF(q)$ with $y + z = x^{2^m} t + x t^{2^h}$, either $x = 0$ or $x \neq 0$ and $\mathrm{Tr}(y/x^{(2^{m+h}-1)/(2^h-1)}) = 1$. The element

$$\begin{aligned}&(x^{2^m} t + x t^{2^h})/x^{(2^{m+h}-1)/(2^h-1)} \\ &\quad = t/x^{(2^m-1)/(2^h-1)} + (t/x^{(2^m-1)/(2^h-1)})^{2^h}\end{aligned}$$

has trace 0. Hence $\mathrm{Tr}(z/x^{(2^{m+h}-1)/(2^h-1)}) = 1$ for every $x \neq 0$. However, the elements of the form $x^{(2^{m+h}-1)/(2^h-1)}$ with $x \neq 0$ form a subgroup of $GF(q)^\times$ of index $(2^{m+h}-1, 2^e-1) = 2^{(e,m+h)}-1$. Let $\zeta_1$ be a generator of that (cyclic) subgroup. As $e \neq m+h$, we have $\zeta_1 \neq 1$. Then $\sum_{j=0}^{k-1} \zeta_1^j = 0$ and hence $\sum_{j=0}^{k-1} \mathrm{Tr}(z\zeta_1^j) = 0$, where $k = (2^e-1)/(2^{(e,m+h)}-1)$ is the order of $\zeta_1$. However, as $\mathrm{Tr}(z\zeta_1^j) = 1$ for every $j = 0, 1, \ldots, k-1$ and $k$ is odd, we obtain $\sum_{j=0}^{k-1} \mathrm{Tr}(z\zeta_1^j) = 1$, which is a contradiction. Hence we have proved that every block has distance at most three from the point $(1; 0, 0)$. Since every point is incident to a block, the diameter of $\Gamma_{m,h}^e$ is at most four.

The above argument also shows that $\Gamma_1$ consists of the $q$ blocks $(0; t, 0)$ and that $\Gamma_3$ consists of the $q^2-q$ remaining blocks $(0; t, z)$ with $z \neq 0$. The group $T_0$ acts regularly on $\Gamma_1$ and each $T_0$-orbit on $\Gamma_3$ contains a unique block of the shape $(0; 0, z)$. Now applying dilatations, it is easy to see that the latter block is sent to exactly one of $(0; 0, \zeta^j)$ for $j = 0, \ldots, 2^{(e,m+h)}-2$ and $\zeta$ a generator of $GF(q)^\times$.

The previous remark shows that $\Gamma_2$ consists of the points $(1; x, y)$ with $x \neq 0$ and $\mathrm{Tr}(y/x^{(2^{m+h}-1)/(2^h-1)}) = 0$. Since there are exactly $q/2$ elements of $GF(q)$ with trace 0, for a given $x \in GF(q)^\times$ there are exactly $q/2$ elements $y \in GF(q)$ with $\mathrm{Tr}(y/x^{(2^{m+h}-1)/(2^h-1)}) = 0$ and hence $\Gamma_2$ consists of $q(q-1)/2$ points. By applying a suitable dilatation, every point of $\Gamma_2$ is sent to a point of the form $(0; 1, y)$. As this point has distance two from $(0; 0, 0)$, we have $\mathrm{Tr}(y) = 0$, and so $y = a^{2^h} + a$ for some $a \in GF(q)$. Then, by applying the translation $\tau_a$, the point $(0; 1, y)$ is sent to $(0; 1, 0)$. Thus $\Gamma_2$ is an orbit of $T_0M$.

The set $\Gamma_4$ of the remaining points consists of the $q(q-1)/2$ points of the form $(1; x, y)$ with $x \neq 0$ and $\mathrm{Tr}(y/x^{(2^{m+h}-1)/(2^h-1)}) = 1$ and the $q-1$ points of the form $(1; 0, y)$ with $y \neq 0$. Each of the latter points is fixed by $T_0$ and sent by $M$ to exactly one of $(1; 0, \zeta^j)$ $(j = 0, \ldots, 2^{(e,m+h)}-2)$. On the other hand, the argument in the previous paragraph shows that the points of the former shape are sent by $T_0M$ to a point $(1; 1, \zeta_0)$, where $\zeta_0$ is a given element of $GF(q)$ of trace unity. □

LEMMA 3.3. *Let* $A := \{x \in GF(q) | \mathrm{Tr}(x) = 0\}$ *and, for* $u, v \in GF(q)$, *define* $uA := \{ua\}_{a\in A}$ *and* $v + uA := \{v + ua\}_{a \in A}$. *Then for any two distinct elements* $u, v \in GF(q) - GF(2)$ *and every* $w \in GF(q)$, *we have*

$$|uA \cap A| = |uA \cap vA| = |uA \cap (w + vA)| = q/4.$$

PROOF. First note that $uA$ contains an element of trace unity, for otherwise $uA = A$ and so $uA' = A'$ where $A' := \{x \in GF(q) | \mathrm{Tr}(x) = 1\}$. This implies that the element $u$ acts fixed-point freely on $A'$, as $A'$ does not contain zero. However, $u$ has odd order $>1$ while $|A'| = q/2$, which is a contradiction. Hence $uA \cap A' \neq \emptyset$.

Given an element $ua \in uA$ with $\mathrm{Tr}(ua) = 1$, the map $ua' \mapsto ua + ua' = u(a + a')$ induces a bijection from $uA \cap A$ to $uA \cap A'$. As $uA$ is the disjoint union of $uA \cap A$ and $uA \cap A'$, we have $|uA \cap A| = |uA \cap A'| = |uA|/2 = q/4$.

Then $|uA \cap vA| = |A \cap u^{-1}vA| = q/4$, as $u^{-1}v \neq 0, 1$. Furthermore, as $v^{-1}w + A$ coincides with $A$ or $A'$ according to whether $\mathrm{Tr}(v^{-1}w) = 0$ or 1, we have $|uv^{-1}A \cap (wv^{-1} + A)| = |uv^{-1}A \cap A|$ or $|uv^{-1}A \cap A'|$. In either case, $|uA \cap (w + vA)| = |uv^{-1}A \cap (wv^{-1} + A)| = q/4$. □

3.2. *Proof of Theorem 1.8.* The graph $\Gamma_{m,h}^e$ has diameter four, by the first claim of Lemma 3.2. It remains to prove that $\Gamma_{m,h}^e$ is distance regular if and only if $(m+h, e) = 1$ and that, if $(m+h, e) = 1$, then $\Gamma_{m,h}^e$ has the same array as $\mathcal{K}_h^e$.

Let $d := (m+h, e)$ and assume first that $d = 1$. Then Lemma 3.2 implies that, for each $i = 1, 2, 3$, the $i$-neighborhood $\Gamma_i$ of $(1; 0, 0)$ is an orbit of $T_0 M$. Thus, for $i = 1, 2, 3$, the number of vertices in $\Gamma_{i-1}$ (resp. $\Gamma_{i+1}$) adjacent to a vertex $x \in \Gamma_i$ does not depend on the particular choice of $x$. As the diameter of $\Gamma^e_{m,h}$ is four, there are exactly $q$ vertices of $\Gamma_3$ adjacent to every vertex of $\Gamma_4$. Hence $\Gamma^e_{m,h}$ is distance regular. It is not difficult to check that its array is the same as that of $\mathcal{K}^e_h$.

Conversely, let $d > 1$. Then $GF(2^d)$ is a subfield of $GF(q)$ properly containing $GF(2)$. Thus, if $w$ is a generator of $GF(2^d)^\times$, we have $w \neq 1$ and $w^{2^{m+h}-1} = 1$. Therefore, the dilatation $\mu_w$ fixes the point $(1; 0, y)$ for every $y \neq 0$ and acts on the set of points $\Gamma_2 \cap \Gamma_2((1; 0, y))$ at distance two from both $(1; 0, 0)$ and $(1; 0, y)$. As a point of $\Gamma_2$ has the form $(1; a, b)$ with $a \neq 0$, every non-trivial element of $\langle \mu_w \rangle$ moves every point of $\Gamma_2 \cap \Gamma_2((1; 0, y))$. In particular, for $(1; 0, y) \in \Gamma_4$, the cardinal number $|\Gamma_2 \cap \Gamma_2((1; 0, y))|$ is a multiple of the order of $w$. On the other hand, it follows from Lemma 3.1 that, for another point $(1; 1, \zeta_0) \in \Gamma_4$, the set $\Gamma_2 \cap \Gamma_2((1; 1, \zeta_0))$ consists of points $(1; x, y)$ with $x \neq 0, 1$ and

$$\mathrm{Tr}\left(\frac{y}{x^{(2^{m+h}-1)/(2^h-1)}}\right) = \mathrm{Tr}\left(\frac{y+\zeta_0}{(x+1)^{(2^{m+h}-1)/(2^h-1)}}\right) = 0.$$

With $A$ as in Lemma 3.3, the above condition is equivalent to the following:

$$y \in (\zeta_0 + (x+1)^{(2^{m+h}-1)/(2^h-1)} A) \cap x^{(2^{m+h}-1)/(2^h-1)} A$$

(for all $x \in GF(q) - GF(2)$). Now it follows from Lemma 3.3 that

$$|(\zeta_0 + (x+1)^{(2^{m+h}-1)/(2^h-1)} A) \cap x^{(2^{m+h}-1)/(2^h-1)} A| = q/4.$$

Hence $|\Gamma_2 \cap \Gamma_2((1; 1, \zeta_0))| = (q-2)q/4$. However, the order of $w$, being a divisor of $|GF(q)^\times| = q-1$, is prime to $(q-2)q/4$. Thus $|\Gamma_2 \cap \Gamma_2((1; 1, \zeta_0))| \neq |\Gamma_2 \cap \Gamma_2((1; 0, y))|$. This implies that the number of points at distance two from $(1; 0, 0)$ and from a point $v \in \Gamma_4$ does depend on the choice of $v$, and hence $\Gamma^e_{m,h}$ is not distance regular. □

*3.3. Proof of Theorem 1.9.* With $\mathcal{K}_i$ as in Proposition 1.7, let $\rho$ be the map from $\mathcal{K}^e_h$ to $\Gamma^e_{h,h}$ that, for $i = 1, \ldots, 4$ and $(k, a, b) \in \mathcal{K}_i$, sends $(k, a, b)$ to the vertex $(k+1; a, b + a^{2^h+1})$ of $\Gamma^e_{m,h}$. Clearly, $\rho$ sends $(0, 0, 0)$ to $(1; 0, 0)$ and induces a bijection from $\mathcal{K}_1$ to the one-neighborhood $\Gamma_1$ of $(1; 0, 0)$ in $\Gamma^e_{m,h}$. It also maps $\mathcal{K}_2$ into $\Gamma_2$. Indeed, according to Proposition 1.7, if $(0, a, b) \in \mathcal{K}_2$ then $a \neq 0$ and $b = a^{2^h+1} + ac^{2^h} + a^{2^h}c$ for some $c \in GF(q)$. Therefore $\rho(0, a, b) = (1; a, ac^{2^h} + a^{2^h}c)$ and

$$\frac{ac^{2h} + a^{2^h}c}{a^{2^h+1}} \;=\; (c/a)^{2^h} + c/a$$

has trace zero. Comparing sizes, we conclude that $\rho$ induces a bijection from $\mathcal{K}_2$ to $\Gamma_2$.

Let $(1, a, b) \in \mathcal{K}_3$. Then $\rho(1, a, b) = (0; a, b + a^{2^h+1})$ belongs to either $\Gamma_3$ or $\Gamma_1$. In the latter case we have $b + a^{2^h+1} = 0$, hence $b = a^{2^h+1}$ and $(1, a, b) \in \mathcal{K}_1$: contradiction. Therefore, $\rho$ maps $\mathcal{K}_3$ into $\Gamma_3$. Comparing sizes, we see that $\rho$ induces a bijection from $\mathcal{K}_3$ to $\Gamma_3$.

Finally, let $(0, a, b) \in \mathcal{K}_4$. Hence $\rho(0, a, b) = (1; a, b + a^{2^h+1})$ belongs to either $\Gamma_4$ or $\Gamma_2$. Suppose it belongs to $\Gamma_2$. Then, as $\rho$ induces a bijection from $\mathcal{K}_2$ to $\Gamma_2$, there are $c, d \in GF(q)$ such that $a = c$ and $b + a^{2^h+1} = cd^{2^h} + c^{2^h}d$. Hence $b = a^{2^h+1} + ad^{2^h} + a^{2^h}d$ and, according to Proposition 1.7, this forces $(0, a, b) \in \mathcal{K}_2$: contradiction. Hence $\rho(0, a, b) \in \Gamma_4$. As $\mathcal{K}_4$ and $\Gamma_4$ have the same size, $\rho$ induces a bijection from $\mathcal{K}_4$ to $\Gamma_4$.

According to Section 1.3, $(*)$, two vertices $(i, a, b)$ and $(j, c, d)$ of $\mathcal{K}_h^e$ are adjacent if and only if $i + j = 1$ and $(a + c)^{2^h+1} = b + d$, namely

$$i + 1 \neq j + 1 \qquad \text{and} \qquad (b + a^{2^h+1}) + (d + c^{2^h+1}) = a^{2^h}c + ac^{2^h}.$$

The latter says that $\rho(i, a, b)$ and $\rho(j, c, d)$ are incident in $\Sigma_{h,h}^e$. Hence $\rho$ is an isomorphism from $\mathcal{K}_h^e$ to $\Gamma_{h,h}^e$.

Conversely, let $m \neq h$. An isomorphism $\rho$ between $\Gamma_{m,h}^e$ and $\Gamma_{h,h}^e$, if any, naturally extends to an isomorphism between the $c.c^*$-geometries $Af(\mathcal{S}_{m,h}^e)$ and $Af(\mathcal{S}_{h,h}^e)$. By Proposition 1.4, the semibiplanes $Af(\mathcal{S}_{h,h}^e)$ and $Af(\mathcal{S}_{m,h}^e)$ have different universal covers. Hence $Af(\mathcal{S}_{m,h}^e) \ncong Af(\mathcal{S}_{h,h}^e)$. Consequently, $\Gamma_{h,h}^e \ncong \Gamma_{m,h}^e$ and, as $\mathcal{K}_h^e \cong \Gamma_{h,h}^e$, $\mathcal{K}_h^e \ncong \Gamma_{m,h}^e$. □

## 4. On the Final Conjecture of Section 1

Suppose $m \neq h$. As remarked in Section 1.5, in order to prove that $Af(\mathcal{S}_{m,h}^e)$ is simply connected, we must show that every closed path of $\Gamma_{m,h}^e$ splits into quadrangles. As $\Gamma_{m,h}^e$ is bipartite of diameter four (Theorem 1.8), every closed path of $\Gamma_{m,h}^e$ splits into octagons, hexagons or quadrangles. Hence, we only need to prove that all octagons and hexagons of $\Gamma_{m,h}^e$ split into quadrangles.

Needless to say, the above might be very hard to prove (possibly false) if $\Gamma_{m,h}^e$ is not distance regular. Thus, in view of Theorem 1.8, henceforth we assume the following.

$(*)$ $(e, m + h) = 1$ (hence $e$ is odd and $m \neq h$, as already assumed above).

Accordingly, $\Gamma_{m,h}^e$ is distance regular, with the same array as $\mathcal{K}_h^e$. In the following, as in Section 3, the vertices of $\Gamma_{m,h}^e$ are regarded as elements of $H_1 \cup H_0$ and, given a vertex $v$ of $\Gamma_{m,h}^e$, we denote by $\Gamma_i(v)$ the set of vertices of $\Gamma_{m,h}^e$ at distance $i \leq 4$ from $v$. We set $v_0 := (1; 0, 0)$.

Lemma 4.1. *Every octagon of $\Gamma_{m,h}^e$ is a sum of hexagons and quadrangles.*

Proof. If all points of $Af(\mathcal{S}_{m,h}^e)$ belonging to an octagon containing $v_0$ have distance two from $v_0$, then every such octagon splits into hexagons and there is nothing to prove. Thus, we assume that there are points at distance four from $v_0$ that belong to some octagons containing $v_0$. Let $v$ be any of these points.

Define a graph $B(v_0, v)$ on $\Gamma_3(v_0) \cap \Gamma_1(v)$ by declaring that two blocks $B, B' \in \Gamma_3(v_0) \cap \Gamma_1(v)$ are adjacent when $\Gamma_1(v_0)$ contains a block at distance two from both $B$ and $B'$. If $B$ and $B'$ are adjacent in $B(v_0, v)$, every octagon containing $v_0$, $B$, $v$ and $B'$ splits into hexagons and, possibly, quadrangles. Therefore, if the graph $B(v_0, v)$ is connected, then every octagon containing $v_0$ and $v$ is a sum of hexagons and, possibly, quadrangles.

Thus, we only need to show that $B(v_0, v)$ is connected. We will show that every block of $\Gamma_3(v_0) \cap \Gamma_1(v)$ is adjacent in $B(v_0, v)$ to at least $q/2$ blocks. As $\Gamma_3(v_0) \cap \Gamma_1(v)$ consists of $q$ blocks, the above implies that every two non-adjacent blocks have an adjacent block in common, whence $B(v_0, v)$ is connected.

Lemma 3.2 implies that the stabilizer of $v_0$ in $\mathrm{Aut}(\Gamma_{m,h}^e)$ is transitive on $\Gamma_i(v_0)$ for $i = 1, 2, 3$. Hence we may assume that $v$ is incident with the block $(0; 0, 1) \in \Gamma_3(P_0)$. Accordingly, $v = (1; x, 1)$ for some $x \in GF(q)$.

Take a block $B = (0; c, d)$ of $\Gamma_3(v_0) \cap \Gamma_1(v)$. As $B$ is incident to $v = (1; x, 1)$, we have $d = 1 + x^{2^m}c + xc^{2^h}$. Moreover, $d \neq 0$, as $B$ is not incident to $v_0$. A block $(0; t, 0)$ of $\Gamma_1(v_0)$ has distance two from $B$ if and only if $t \neq c$ and $\mathrm{Tr}(d/(c + t)^k) = 0$, where $k = (2^{m+h} - 1)/(2^m - 1)$.

Let $B' = (0; c', d')$ be a block of $\Gamma_3(v_0) \cap \Gamma_1(v)$ distinct from $B$ and not adjacent to $B$. Then $c \neq c'$, for otherwise $d = d' = 1 + x^{2^m} c + xc^{2^h}$ and $B = B'$. The definition of the adjacency in $B(v_0, v)$ implies the following:

(a) for every block $(0; t, 0) \in \Gamma_1(v_0)$ with $c \neq t$ and $\mathrm{Tr}(d/(c+t)^k) = 0$, either $t = c'$ or $t \neq c'$ and $\mathrm{Tr}(d'/(c'+t)^k) = 1$.

Let $f$ be the function sending $d/(c+t)^k$ to $d'/(c'+t)^k$. Note that the equation $s = d/(c+t)^k$ for $c \neq t$ is equivalent to $t = (d/s)^{1/k} + c$ for $s \neq 0$, since $(e, m+h) = 1$. Thus $f$ works as follows:

$$f(s) = \frac{d'}{((d/s)^{1/k} + c + c')^k} = \frac{d's}{(d^{1/k} + (c+c')s^{1/k})^k}$$

where $s \in GF(q) - \{0, d/(c+c')^k\}$, in order to avoid null denominators in the above expressions. As $f(s) = d'/(c+c')^k$ implies $d/s = 0$, but $d \neq 0$, the image of $f$ does not contain $d'/(c+c')^k$. Then $f$ is a bijection from $GF(q) - \{0, d/(c+c')^k\}$ to $GF(q) - \{0, d'/(c+c')^k\}$ and (a) can be rephrased as follows:

(b) if $\mathrm{Tr}(s) = 0$ and $s \notin \{0, d/(c+c')^k\}$, then $\mathrm{Tr}(f(s)) = 1$ (whence $f(s) \neq s$).

The equation $f(s) = s$ has a unique solution

$$s_0 = \left(\frac{d^{1/k} + (d')^{1/k}}{c + c'}\right)^k,$$

which is different from either 0 or $d'/(c+c')^k$. By (b), $\mathrm{Tr}(s_0) = 1$.

We shall show that $\mathrm{Tr}(d/(c+c')^k) = 0$. Suppose $\mathrm{Tr}(d/(c+c')^k) = 1$. Then there are $q/2 - 1$ elements of trace 0 in the domain $GF(q) - \{0, d/(c+c')^k\}$ of $f$. Their images by $f$ have trace unity. As $s_0$ is not the image of any element of trace zero, there are at least $q/2$ elements of trace unity in the image of $f$. As there are exactly $q/2$ elements of $GF(q)$ of trace unity, all of them belong to the image of $f$. In particular, $\mathrm{Tr}(d'/(c+c')^k) = 0$. Then

$$\mathrm{Tr}(d+d')/(c+c')^k) = \mathrm{Tr}(d/(c+c')^k) + \mathrm{Tr}(d'/(c+c')^k) = 1 + 0 = 1,$$

which contradicts the assumption that $B = (0; c, d)$ and $B' = (0; c', d')$ are at distance two. We conclude that if $B' = (0; c', d')$ is a block of $\Gamma_3(v_0) \cap \Gamma_1(v)$ distinct from $B = (0; c, d)$ and not adjacent to $B$, then $c \neq c'$ and $\mathrm{Tr}(d/(c+c')^k) = 0$. As $(e, m+h) = 1$, the function sending $c' \in GF(q) - \{c\}$ to $d/(c+c')^k \in GF(q) - \{0\}$ is a bijection.

Thus, for a given $B = (0; c, d) \in \Gamma_3(v_0) \cap \Gamma_1(v)$, there are exactly $q/2 - 1$ elements $c' \neq c$ with $\mathrm{Tr}(d/(c+c')^k) = 0$. As $d' = 1 + x^{2^m} c + xc^{2^h}$ is uniquely determined by $c'$, there are at most $(q/2) - 1$ blocks $B' \in \Gamma_3(v_0) \cap \Gamma_1(v)$ distinct from $B$ and not adjacent to $B$ in $B(v_0, v)$. Hence there are at least $q/2$ blocks of $\Gamma_3(v_0) \cap \Gamma_1(v)$ distinct from $B$ and adjacent to $B$. The connectivity of $B(v_0, v)$ follows. □

So far, we are reduced to seeing whether every hexagon of $\Gamma^e_{m,h}$ is a sum of quadrangles. In view of this, given a point-block pair $(v_0, B_0)$ at distance three in $\Gamma^e_{m,h}$, we consider another graph, which we denote $E^e_{m,h}(v_0, B_0)$ (also $E^e_{m,h}$, for short, when no confusion arises). Its vertices are the incident block-point pairs $(B, v)$ with $B \in \Gamma_1(v_0) \cap \Gamma_2(B_0)$ and $v \in \Gamma_2(v_0) \cap \Gamma_1(B_0)$. Two distinct vertices $(B, v)$ and $(B', v')$ of $E^e_{m,h}$ are declared to be adjacent when either $B = B'$ or $v = v'$. It is easy to see that, if $E^e_{m,h}$ is connected, then every hexagon containing $v_0$ and $B_0$ is a sum of quadrangles.

Still with $v_0 := (1; 0, 0)$, let $B_0 := (0; 0, a)$ ($\in \Gamma_3(v_0)$) for some $a \in GF(q)^\times$. Then the following hold:

$$\Gamma_1(v_0) \cap \Gamma_2(B_0) = \{(0; t, 0) \mid t \in GF(q)^\times, \mathrm{Tr}(a/t^k) = 0\},$$
$$\Gamma_2(v_0) \cap \Gamma_1(B_0) = \{(1; x, a) \mid x \in GF(q)^\times, \mathrm{Tr}(x/t^{k'}) = 0\},$$

where $k = (2^{m+h} - 1)/(2^m - 1)$ and $k' = (2^{m+h} - 1)/(2^h - 1)$. As $(e, m+h) = 1$, the maps $x \mapsto 1/x^k$ and $x \mapsto 1/x^{k'}$ are bijections on $GF(q)^\times$. Thus

$$|\Gamma_1(v_0) \cap \Gamma_2(B_0)| = |\Gamma_2(v_0) \cap \Gamma_1(B_0)| = (q/2) - 1.$$

For every block $(0; t, 0)$ of $\Gamma_1(v_0) \cap \Gamma_2(B_0)$ there are exactly two points incident with both $(0; t, 0)$ and $B_0 = (0; 0, a)$, namely $(1; x, a)$ and $(1; x + t^\varepsilon, a)$, where $\varepsilon$ has the meaning stated in Section 2 and $x \in GF(q)^\times$ is a solution of the following equation:

$$t^{2^h} x + t x^{2^m} = a. \tag{1}$$

Note that such a solution exists because $t \in GF(q)^\times$ and $\mathrm{Tr}(a/t^k) = 0$. Similarly, for every point $(1; x, a) \in \Gamma_2(v_0) \cap \Gamma_1(B_0)$, there are exactly two blocks incident with both $(1; x, a)$ and $v_0$, namely $(0; t, 0)$ and $(0; t + x^{1/\varepsilon}, 0)$, with $t \in GF(q)^\times$ a solution of equation (1) (where $t$ is now regarded as the unknown of (1)). In particular, $E^e_{m,h}$ has $2((q/2) - 1)$ vertices.

Therefore, we may identify the vertices $((1; t, 0), (0; x, a))$ of $E^e_{m,h}$ with the solutions $(t, x)$ of the equation (1) (the latter being now regarded as an equation in two unknowns). Accordingly, the connected component of $E^e_{m,h}$ containing a given solution $(t_0, x_0)$ of (1) is obtained inductively as follows:

(2) define $t_{i+1} := t_i + x_i^{1/\varepsilon}$ and $x_{i+1} := x_i + t_i^\varepsilon$ for $i = 0, \ldots, n-1$, until we obtain $(t_n, x_n) = (t_0, x_0)$. Then the pairs $(t_i, x_i)$, $(t_{i+1}, x_i)$, $(t_i, x_{i+1})$ (for $i = 0, \ldots, n-1$) form the connected component of $E^e_{m,h}$ containing $(t_0, x_0)$.

The above is suitable for a computer, once explicit values for $h$, $m$ and $a$ are given. Here are the results for $e = 5, 7, 11, 13$, obtained via GAP. We only list the pairs $(h, m)$ for which $E^e_{m,h}$ is connected. We also assume $m \le e/2$, in view of the isomorphism $\mathcal{S}^e_{m,h} \cong \mathcal{S}^e_{e-m,e-h}$.

| $e$ | $(h, m)$ |
|---|---|
| 5 | (2, 1), (1, 2) |
| 7 | (2, 1), (3, 1), (4, 1), (1, 2), (4, 2), (1, 3), (5, 3), (6, 3) |
| 11 | (4, 1), (3, 2), (5, 2), (8, 2), (2, 3), (1, 4), (2, 5) |
| 13 | none |

If $E^e_{m,h}(v_0, B_0)$ is disconnected, we shall find two solutions of the equation (1) which connect distinct components 'homotopically'. We start with a solution $(t_0, x_0)$ of (1) and produce other solutions $(t_i, x_i)$ according to the recursive rule (2). At the same time, we add other solutions obtained in the following way. Consider the hexagon

$$H := (v_0, (0; t_0, 0), (1; x_0, a), B_0, (1; x_i, a), (0; t_i, 0), v_0).$$

As the edges $\{(0; t_0, 0), (1; x_0, a)\}$ and $\{(1; x_i, a), (0; t_i, 0), v_0)\}$ of $H$ belong to the same connected component of $E^e_{m,h}$, the hexagon $H$ splits into quadrangles. Take a block $B_0^{(i)} = (0; s_i, w_i) \neq B_0$ incident to $(1, x_0, a)$ and $(1; x_i, a)$, and choose an automorphism $\sigma_i$ in the stabilizer of $v_0$ which sends $B^{(i)}$ to $B_0$. Then $(t_0^{\sigma_i}, x_0^{\sigma_i})$ and $(t_i^{\sigma_i}, x_i^{\sigma_i})$ are solutions of (1). They yield two paths of length three joining $v_0$ to $B_0$, which form a hexagon $H'$. The hexagon $H'$ is the image by $\sigma_i$ of the hexagon

$$(v_0, (0; t_0, 0), (1; x_0, a), B_0^{(i)}, (1; x_i, a), (0; t_i, 0), v_0).$$

The latter is a sum of $H$ and a quadrangle $((1; x_0, a), B_0^{(i)}, (1; x_i, a), B_0, (1; x_0, a))$. Thus $H'$ splits into quadrangles.

Now we define a graph $\mathcal{C}^e_{m,h}(v_0, B_0)$ on the set of connected components of $E^e_{m,h}(v_0, B_0)$ by declaring two components to be adjacent when each of them contains one of the solutions $(t_0^{\sigma_i}, x_0^{\sigma_i})$ and $(t_i^{\sigma_i}, x_i^{\sigma_i})$ of (1), for some $(t_0, x_0)$ and $(t_i, x_i)$ and $\sigma_i$ as above. The previous observation implies that, if the graph $\mathcal{C}^e_{m,h}(v_0, B_0)$ is connected, then every hexagon of $\Gamma^e_{h,m}$ is a sum of quadrangles, whence $Af(\mathcal{S}^e_{h,m})$ is simply connected.

The condition for $B_0^{(i)} = (0; s_i, w_i)$ to be incident to both $(1; x_0, a)$ and $(1; x_i, a)$ forces

$$s_i = (x_0 + x_i)^{1/\varepsilon}, \qquad w_i = a + (x_0 + x_i)^{(1/\varepsilon)-1}(x_0^{2^m} x_i + x_0 x_i^{2^m}).$$

Then we may choose $\mu_{b_i^\varepsilon}\tau_{s_i}$ as $\sigma_i$, where

$$b_i := \left(\frac{a}{a + (x_0 + x_i)^{(2^m-2^h)/(2^h-1)}(x_0^{2^m} x_i + x_0 x_i^{2^m})}\right)^{(2^m-1)/(2^{m+h}-1)}. \tag{3}$$

Accordingly, $(b_i(t_0+(x_0+x_i)^{1/\varepsilon}), b_i^\varepsilon x_0)$ and $(b_i(t_i+(x_0+x_i)^{1/\varepsilon}), b_i^\varepsilon x_i)$ are the new solutions. Summarizing, the following is proved.

LEMMA 4.2. *Let $a \in GF(q)^\times$. Declare two connected components $C$ and $D$ of $E^e_{h,m}$ to be adjacent when there is a connected component containing the solutions $(t_0, x_0)$ and $(t_i, x_i)$ of the equation* (1) (*with $t_i$ and $x_i$ defined as in* (2))*, such that $C$ contains one of the following two solutions of* (1) *and $D$ contains the other one:*

$$(b_i(t_0 + (x_0 + x_i)^{1/\varepsilon}), b_i^\varepsilon x_0), \qquad (b_i(t_i + (x_0 + x_i)^{1/\varepsilon}), b_i^\varepsilon x_i)$$

*where $b_i$ is defined as in* (3)*. If the resulting graph is connected, then every hexagon of $\Gamma^e_{m,h}$ is a sum of quadrangles.*

The following is an another easy but useful observation.

LEMMA 4.3. *The connected components of the graph $E^e_{m,h}(v_0, B_0)$ bijectively correspond to those of $E^e_{h,m}(v_0, B_0)$, via the map sending $(t, x) \in E^e_{m,h}$ to $(x, t) \in E^e_{m,h}$. In particular, $E^e_{m,h}$ is connected if and only if $E^e_{h,m}$ is connected.*

In view of this, we may restrict ourselves to the case of $m < h$, also assuming $m \leq e/2$ in view of the isomorphism $\mathcal{S}^e_{m,h} \cong \mathcal{S}^e_{e-m,e-h}$ (Proposition 1.3). Relying on Lemma 4.2 and with the above restrictions on $(h, m)$, the following is verified with the aid of GAP [7].

RESULT 4.4. Let $e \leq 14$, $h \neq m$ and $(m + h, e) = 1$. Then $Af(\mathcal{S}^h_{m,e})$ is simply connected.

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ANTONIO PASINI
*Department of Mathematics,*
*University of Siena,*
*via del Capitano 15,*
*Siena, 53100*
*Italy*
*E-mail: pasini@unisi.it*
AND

SATOSHI YOSHIARA
*Division of Mathematical Sciences,*
*Osaka Kyoiku University,*
*Kashiwara,*
*Osaka 582-8582,*
*Japan*
*E-mail: yoshiara@cc.osaka-kyoiku.ac.jp*