On structural conditions for weak persistency and semilinearity of Petri nets

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Abstract

A necessary and sufficient condition for a Petri net to be weakly persistent for every initial marking is obtained. Moreover, a necessary and sufficient condition for reachability is obtainable for this class of Petri nets. As a sufficient condition for a Petri net to have a semilinear reachability set, the notion of sinklessness has been proposed, where a marked Petri net is said to be sinkless if the total number of tokens in each minimal circuit is not decreased to 0 by firing transitions. We show that the reachability set is semilinear if the total number of times that sinklessness is violated is finite during each firing, and define a new subclass of Petri nets which have this property for every initial marking.

1. Introduction

Petri nets are a structural model for representing discrete and concurrent systems, which describes how each component of a system communicates with other ones. Classification of Petri nets has been studied for two kinds of properties: one is structural and the other is behavioral. Several classes of Petri nets have been proposed as structural properties, such as marked graphs, state machines, free-choice nets, conflict-free nets [3], trap-circuit nets, deadlock-circuit nets [1] and so on. On the other hand, liveness, fairness, persistency [2] and weak persistency [6] have been
studied as behavioral properties. In this paper we study relations between structural and behavioral properties, which are important in designing and verifying systems.

We first study weak persistency. A necessary condition for weak persistency is discussed in [7]. We show a necessary and sufficient condition for weak persistency, which is described as a structural property. Moreover, a necessary and sufficient condition for reachability is obtainable for this class of Petri nets.

Persistency, weak persistency and sinklessness [7] are introduced to define classes of Petri nets whose reachability sets are effectively computable semilinear. A marked Petri net is said to be sinkless if the total number of tokens in each minimal circuit is not decreased to 0 by firing transitions. We show that the reachability set is semilinear if the total number of times that sinklessness is violated is finite during each firing, and obtain a structural condition for a Petri net to have this property for every initial marking.

2. Definitions and Notation

Let \( Z \) denote the set of integers and let \( N \) denote the set of nonnegative integers. For a nonempty finite set \( S \), \( Z^S \) (or \( N^S \)) denotes the set of all \( |S| \)-dimensional vectors of integers (nonnegative integers). A function \( f: S \rightarrow Z \) can be identified with an element of \( Z^S \) by introducing a total order on \( S \). Moreover, for nonempty finite sets \( S_1 \) and \( S_2 \), a function \( S_1 \times S_2 \rightarrow Z \) can be identified with a \( |S_1| \times |S_2| \) matrix of integers. For a vector \( x \) in \( Z^S \) and a subset \( S' \) of \( S \), \( x \restriction S' \) denotes the projection of \( x \) onto \( S' \). For vectors \( x \) and \( y \) and an integer \( k \), the addition \( x + y \), the scalar product \( kx \), and the partial order \( x \leq y \) are defined componentwise as usual. For vectors \( x \) and \( y \), \( \langle x, y \rangle \) denotes the concatenation of \( x \) and \( y \), e.g. \( \langle [1, 0, 4], [2, 1, 3] \rangle = [1, 0, 4, 2, 1, 3] \), and similarly for \( \langle x, y, z, \ldots \rangle \).

A Petri net is a triple \( C = (P, T, A) \), where \( P \) is a finite set of places, \( T \) is a finite set of transitions, and \( A: (P \times T) \cup (T \times P) \rightarrow N \) is a function which represents the incidence relation between \( P \) and \( T \). In this paper we restrict the function \( A \) to \((P \times T) \cup (T \times P) \rightarrow \{0, 1\}\). This class of Petri nets is called single arc. \( M = (C, m^0) = (P, T, A, m^0) \) is called a marked Petri net, where \( m^0: P \rightarrow N \) is the initial marking. If \( A(t, p) = 1 \) for a transition \( t \) and a place \( p \), \( p \) is called an input place of \( t \) and \( t \) is called an output transition of \( p \), similarly for an output place and an input transition.

A transition \( t \) is fireable in a marking \( m \) if \( m(p) \geq A(t, p) \) for each place \( p \) in \( P \). The firing of \( t \) yields the marking \( m' = m + t \), where \( t \) is defined by

\[
(t)(p) = A(t, p) - A(p, t) \quad \text{for each } p \in P.
\]

A sequence of transitions is called a firing sequence. Let \( \emptyset \) denote the empty firing sequence. For firing sequences \( x, \beta \) in \( T^* \), \( x\beta \) denotes their concatenation, and \( x, \beta \) denotes the sequence in \( (T^*)^* \). We simply write \( (x)^n \) to indicate \( x x \ldots x \) (\( n \) times). For a firing sequence \( \sigma = s_1 s_2 \ldots s_r \), \( s_i \in T \) (\( i = 1, \ldots, r \)), \( \sigma_i = (1 \leq i \leq j \leq r) \) denotes the subsequence \( s_i s_{i+1} \ldots s_j \) and \( \sigma_i \) denotes \( s_i \).
We can extend the notion of firing to sequences of transitions:
(i) $\lambda$ is firable in every marking and $\lambda=0$,
(ii) $t\sigma$ is firable in a marking $m$ and $t\sigma=t+\sigma$ if a transition $t$ is firable in $m$ and a firing sequence $\sigma$ is firable in $m+t$.

We write $m[\sigma]>m'$ to indicate that a firing sequence $\sigma$ is firable in a marking $m$ and the firing of $\sigma$ yields the marking $m'$. When we do not need to specify the resulting marking, we simply write $m[\sigma]$, and similarly for $m[^m\sigma]$.

Let $B_p: \mathbb{P} \times T \rightarrow \mathbb{Z}$ denote the incidence matrix of a Petri net $C=(P, T, A)$, which is defined by $B_t(p, t)=t(p)$. $\psi: T^* \rightarrow \mathbb{N}^T$ is called the Parikh mapping. For a firing sequence $\sigma$ in $T^*$, $\psi(\sigma)(t)$ denotes the number of occurrences of a transition $t$ in $\sigma$. Since $\sigma=B_t(\psi(\sigma))$, we have the following lemma.

**Lemma 2.1.** Let $M=(C, m^0)=(P, T, A, m_0)$ be a marked Petri net. If $m^0[\sigma]>m$, then there exists a vector $x$ in $\mathbb{N}^T$ such that $m-m^0=B_t(x)$.

Let $C=(P, T, A)$ be a Petri net. A word $w_1w_2...w_n$ in $(P \cup T)^*$ is called a path if $A(w_i, w_{i+1})=1$ for $i=1, ..., n$. A path $w_1w_2...w_n$ is called a circuit if $w_1=w_n$. Let $P_u$ and $T_u$ denote the set of places and the set of transitions, respectively, occurring in a circuit $u$. A circuit $u$ is called simple if $u$ does not contain any other circuits. A simple circuit $u$ is called minimal if the set of places in $u$ does not properly include the set of places in any other circuit.

We define the following for a marked Petri net $M=(C, m^0)=(P, T, A, m_0)$:

\[\Sigma(M) = \Sigma(C, m^0) = \{\sigma | \sigma \in T^* \land m^0[\sigma]\}\]
\[R(M) = R(C, m^0) = \{m^0 + \sigma | \sigma \in \Sigma(M)\}\]
\[ER(M) = ER(C, m^0) = \{\langle m^0 + \sigma, \psi(\sigma) \rangle | \sigma \in \Sigma(M)\}\]

$R(M)$ is called the reachability set of $M$, and $ER(M)$ is called the extended reachability set of $M$.

Restrictions of a Petri net $C=(P, T, A)$ and a marked Petri net $M=(C, m^0)$ are defined as follows.
(i) **Restriction of places:** Let $S$ be a given subset of $P$. The restriction of $C$ to $S$ is defined by $C|S=(S, T|S, A|S)$, where

\[T|S = \{t | \exists p \in S: A(p, t)=1 \lor A(t, p)=1\}\]

and $A|S$ is the restriction of $A$ to $(S, T|S)$. The restriction of $M$ to $S$ is defined by $M|S=(C|S, m^0|S)$, where $m^0|S$ is the restriction of $m^0$ to $S$.

(ii) **Restriction of transitions:** Let $H$ be a given subset of $T$. The restriction of $C$ to $H$ is defined by $C|H=(P|H, H, A|H)$, where

\[P|H = \{p | \exists t \in H: A(p, t)=1 \lor A(t, p)=1\}\]
and \(A|H\) is the restriction of \(A\) to \((P|H, H)\). The restriction of \(M\) to \(H\) is defined by 
\[
M|H = (C|H, m^0|H) \mid (P|H) \).
\]

(iii) Restriction to a vector in \(N^T\): Let \(x\) be a given vector in \(N^T\). The restriction of \(T\) to \(x\) is defined by 
\[
T|x = \{ t \mid t \in T \land \psi(t) \leq x \}.
\]
We simply write \(C|x\) for \(C|(T|x)\) and 
\[
M|x = (C|x, m^0|x)\) for \(M|(T|x)\).
\]

(iv) Restriction to a circuit: We simply write \(C|u\) and \(M|u\) to indicate the Petri net \(C\) and the marked Petri net \(M\) restricted to places and transitions in a circuit \(u\), respectively.

Let \(S\) be a set of places in a Petri net \(C = (P, T, A)\). We define \(\tau(S), \tau^+(S)\) and \(\tau^-(S)\) by
\[
\tau(S) = \{ t \mid \exists p \in S: A(p, t) = 1 \land \exists q \in S: A(t, q) = 1 \},
\]
\[
\tau^+(S) = \{ t \mid \exists p \in S: A(p, t) = 1 \land \forall q \in S: A(q, t) = 0 \},
\]
\[
\tau^-(S) = \{ t \mid \exists p \in S: A(p, t) = 1 \land \forall q \in S: A(t, q) = 0 \}.
\]

\(S\) is called a deadlock if \(\tau^+(S) = \emptyset\), and is called a trap if \(\tau^-(S) = \emptyset\). A circuit \(u\) is called a deadlock circuit if \(P_u\) is a deadlock, and is called a trap circuit if \(P_u\) is a trap. A place \(p\) is called a source place if \(p\) has no input transitions. For a deadlock \(D\) without source places, there exists a deadlock circuit \(u\) such that \(P_u \subseteq D\). A deadlock \(D\) is called a token-free deadlock (TFD) in a marking \(m\) if \(m|D = 0\). A deadlock circuit \(u\) is called a token-free deadlock circuit (TFDC) in a marking \(m\) if \(m|P_u = 0\).

**Lemma 2.2** (Yamasaki [5]). If a marked Petri net \(M\) has no firable transitions, then there exists a TFD in \(M\).

**Lemma 2.3.** Let \(M = (C, m^0) = (P, T, A, m^0)\) be a marked Petri net. If \(m + B_x \succeq 0\) for a marking \(m\) and a vector \(x\) in \(N^T\), then \(M|x\) has no token-free source places in \(m\).

**Proof.** If \(M|x\) has a token-free source place \(p\) in \(m\), then the component of \(m + B_x\) corresponding to the place \(p\) is negative.

**Lemma 2.4.** Let \(M\) be a marked Petri net. If a firing sequence \(\sigma\) is firable in a marking \(m\), then \(M|\psi(\sigma)\) has no TFDs in \(m\).

**Proof.** Assume that \(M|\psi(\sigma)\) has a TFD \(D\) in \(m\). Then a transition in \(M|\psi(\sigma)\) which has an input place in \(D\) is not firable in every marking reachable from \(m\).

**Lemma 2.5.** For each circuit \(u\) in a Petri net \(C = (P, T, A)\), there exists a circuit \(u'\) such that \(T_u = \tau(P_u)\) and \(P_u = P_u\).

**Proof.** Let \(u = p_1 t_1 \ldots p_n t_n p_1\). Clearly, \(\tau(P_u) \gtrless T_u\) holds. If there exists a transition \(t\) in \(\tau(P_u) \setminus T_u\) such that \(A(p_i, t) = A(t, p_j) = 1\) \((1 \leq i, j \leq n)\), then we have the circuit
By repeating this process, we can obtain a circuit $u'$ such that $T_u = \tau(P_u)$ and $P_{u'} = P_u$. 

A circuit $u$ is called complete if $T_u = \tau(P_u)$.

3. Weak persistency

Let $M = (P, T, A, m^0)$ be a marked Petri net. $M$ is said to be weakly persistent if $M$ satisfies the following condition [6]: for every firing sequences $\alpha$ and $\beta$ such that $m^0[\alpha] \land m^0[\beta] \land \psi(\alpha) \leq \psi(\beta)$, there exists a firing sequence $\gamma$ such that $m^0[\alpha \gamma] \land \psi(\alpha \gamma) = \psi(\beta)$. In this section we study a necessary and sufficient condition for a Petri net to be weakly persistent for every initial marking. We first define the following conditions on $M$, which will be used to obtain the necessary and sufficient condition.

**Condition 3.1.** Let $M = (C, m^0) = (P, T, A, m^0)$ be a marked Petri net and let $x$ be a vector in $N^T$ such that

(i) $m^0 + B_x \geq 0$,

(ii) $M | x$ has no TFDCs in $m^0$.

Then for every firing sequence $\alpha$ in $\Sigma(M)$ such that $\psi(\alpha) \leq x$, $M | x - \psi(\alpha)$ has no TFDCs in $m^0 + x$.

**Condition 3.2.** Let $C = (P, T, A)$ be a Petri net. For each circuit $u$ in $C$, there exists a nonempty set of places $S$ such that

(i) $S \subset P_u$,

(ii) $\tau^-(S) = \emptyset$, i.e. $S$ is a trap in $C$.

(iii) $\tau^+(S) \cap \tau(P_u) = \emptyset$.

We can obtain the following lemmas, which will be proved later.

**Lemma 3.3.** If a marked Petri net $M = (P, T, A, m^0)$ satisfies Condition 3.1, then $M$ is weakly persistent.

**Lemma 3.4.** If a Petri net $C = (P, T, A)$ satisfies Condition 3.2, then the marked Petri net $M = (C, m^0)$ satisfies Condition 3.1 for every initial marking $m^0$.

**Lemma 3.5.** Let $C = (P, T, A)$ be a Petri net. If the marked Petri net $M = (C, m^0)$ is weakly persistent for every initial marking $m^0$, then the Petri net $C$ satisfies Condition 3.2.

By Lemmas 3.3–3.5, we have the following theorem.
Theorem 3.6. A Petri net \( C = (P, T, A) \) satisfies Condition 3.2 if and only if the marked Petri net \( M = (C, m^0) \) is weakly persistent for every initial marking \( m^0 \).

**Proof of Lemma 3.3.** Assume that \( x \) and \( \beta \) are firing sequences such that \( m^{\beta}[x] \wedge m^0[\beta] \wedge \psi(x) \leq \psi(\beta) \). Let \( x = \psi(\beta) \), then \( x \) satisfies Condition 3.1 by Lemmas 2.1 and 2.4. \( M|\psi(\beta) - \psi(x) \) has no token-free source places in \( m^n + z \) by Lemma 2.3, and has no TFDC's by Condition 3.1. Therefore, \( M|\psi(\beta) - \psi(x) \) has a firable transition \( t \) in \( m^n + z \) by Lemma 2.2. This is also true for \( x \) and \( \beta \). By repeating this process, we can obtain a firing sequence \( \gamma \) such that \( m^0[x'] \wedge \psi(x') = \psi(\beta) \).

**Proof of Lemma 3.4.** Assume that a Petri net \( C = (P, T, A) \) satisfies Condition 3.2 and a marked Petri net \( M = (C, m^0) \) does not satisfy Condition 3.1. Then there exists a vector \( x \) in \( N^2 \) and a firing sequence \( \gamma \) in \( \Sigma(M) \) that satisfy the following conditions:

(i) \( x \) satisfies Condition 3.1;
(ii) \( \psi(x) \leq x \);
(iii) \( M|x - \psi(x) \) has a TFDC \( u \) in \( m^n + z \).

Let \( S \) be a set of places that satisfies Condition 3.2 for \( u \). It follows that \( \tau^+(S) = \emptyset \) in \( M|x - \psi(x) \) by Condition 3.2(iii). There are two cases:

(a) \( \tau^+(S) = \emptyset \) in \( M|x \): By Condition 3.1, \( S \) is not token-free in \( m^n \). Since \( S \) is a trap in \( M, S \) is not token free in \( m^n + z \).

(b) \( \tau^+(S) \neq \emptyset \) in \( M|x \): In the firing of \( \gamma \), transitions in \( \tau^+(S) \) bring some tokens into \( S \). Since \( S \) is a trap in \( M, S \) is not token-free in \( m^n + z \).

\( S \) is not token-free in both cases. This is a contradiction.

**Proof of Lemma 3.5.** We first show the following two facts, and prove the contrapositive of the statement of Lemma 3.5.

(a) Let \( C = (P, T, A) \) be a Petri net and let \( u \) be a simple circuit in \( C \) such that \( \tau^-(P_u) \neq \emptyset \). If \( C|u \) does not have a nonempty deadlock \( D \) such that \( D \subseteq P_u \), then the marked Petri net \( M = (C, m) \) is not weakly persistent for the initial marking \( m \) such that

\[
m(p) = 1 \quad \text{if } p \in P_u \text{ and } A(p, t) = 1,

m(p) = 0 \quad \text{if } p \in P_u \text{ and } A(p, t) = 0,

m(p) = k \quad \text{otherwise},
\]

where \( t \in \tau^-(P_u) \) and \( k \) is a sufficiently large integer.

Let \( u = p_1 t_1 \ldots p_i t_i P_1 \). Assume that \( A(p_i, t) = 1 \) without loss of generality. Let \( m_1 \) be the marking such that \( m_1(p_i) = 1, m_1(p_i) = 0 \), and \( m_1(p) = k \) (\( p \notin P_u \)). Since \( M | u \) has no TFDCs in \( m_1 \), \( M | u \) has a firable transition by Lemma 2.2. Since \( m_1(p_i) = 0 \), only \( t_i \) is firable in \( M | u \). Let \( m_1(t_i) = m_2 \) and let \( m_2^i \) be the marking such that \( m_2^i(p_2) = 1, m_2^i(p_1) = 0 \), and \( m_2^i(p) = m_2(p) \) (\( p \notin P_u \)). By similar arguments, \( t_2 \) is
firable in \( m' \), and is firable in \( m \), because \( m_2 \geq m' \). Repeating this process, we can show that \( \sigma = t_1 f_2 \ldots f_n \) is firable in \( m_1 \).

Since \( m \geq m_1 \), \( \sigma \) is firable in \( m \). Every place \( p_i \) in \( P_u \) such that \( A(p_i, t) = 1 \) has a token in \( m + \sigma \); hence, \( t \) is firable in \( m + \sigma \). The sequence \( \sigma t \) is firable in \( m \) and no transitions in \( T_u \) are firable in \( m + t \). Therefore, the marked Petri net \( M = (C, m) \) is not weakly persistent.

(b) Let \( C = (P, T, A) \) be a Petri net. Suppose that, for each circuit \( u \) in \( C \) such that \( \tau^-(P_u) \neq \emptyset \), \( C|u \) has a nonempty deadlock \( D \) such that \( D \supseteq P_u \). Then for each circuit \( u \) in \( C \) such that \( \tau^-(P_u) \neq \emptyset \), there exists a nonempty set of places \( S \) that satisfies the following conditions:

\[(i) \quad S \subseteq P_u; \]
\[(ii) \quad \tau^-(S) = \emptyset;\]
\[(iii) \quad \tau^+(S) \cap \tau(P_u) = \emptyset.\]

Assume that \( u \) is complete. If not, we can have a complete circuit \( u_c \) such that \( P_u = P_{u_c} \) by Lemma 2.5. Moreover, if there exists a set of places \( S \) that satisfies the above three conditions for \( u_c \), then \( S \) also satisfies them for \( u \).

From the assumption and the fact that \( u \) is a circuit, i.e. \( C|u \) has no token-free source places, \( C|u \) has a deadlock circuit \( u' \). If \( \tau^-(P_u) \neq \emptyset \) in \( C \), then \( C|u' \) has a deadlock circuit \( u'' \). The circuit \( u'' \) is also a deadlock circuit in \( C|u \). By repeating this process, we can obtain a circuit \( u_c \) such that \( u_c \) is a deadlock circuit in \( C|u \) and \( \tau^-(P_u) = \emptyset \) in \( C \). Since \( u \) is complete, it follows that \( \tau^+(P_u) \cap \tau(P_u) = \emptyset \).

If \( \tau^-(P_u) = \emptyset \) for a circuit \( u \), then Condition 3.2 holds for \( S = P_u \). Hence, the facts (a) and (b) imply the following: if a Petri net \( C \) does not satisfy Condition 3.2, then there exists a marking \( m \) such that the marked Petri net \( M = (C, m) \) is not weakly persistent.

A Petri net that satisfies Condition 3.2 is said to be a structurally weakly persistent Petri net (SWPN). A necessary and sufficient condition for reachability can be obtained for an SWPN.

**Theorem 3.7.** Let \( M = (C, m^0) \) be a marked Petri net such that the Petri net \( C = (P, T, A) \) is an SWPN. A marking \( m \) is reachable from \( m^0 \) if and only if \( M \) satisfies the following:

\[(i) \quad \text{There exists a vector } x \text{ in } N^T \text{ such that } m - m^0 = B_c x; \]
\[(ii) \quad \text{\( C|x \) has no TFDCs in } m^0. \]

**Proof.** "Only if" part: It is clear by Lemmas 2.1 and 2.4.

"If" part: Since \( M \) satisfies Condition 3.1, we can obtain a firing sequence \( \sigma \) such that \( m^0[\sigma]m \) by the following procedure:

1. Set \( z := \lambda \).
2. \( M|z - \psi(z) \) has no token-free source places by Lemma 2.3, and has no TFDCs by Condition 3.1. Therefore, \( M|z - \psi(z) \) has at least one firable transition by Lemma 2.2. Choose a firable transition \( t \) and fire it.
3. Set \( z := zt \) and \( x := x - \psi(t) \). If \( x = 0 \), then \( \sigma := z \). Otherwise, go to (2). \( \square \)
4. Semilinearity of the reachability set

The reachability set of a weakly persistent marked Petri net is effectively computable semilinear [6]. In this section we consider structural conditions for a Petri net to have a semilinear reachability set for every initial marking. We first define the notion of linear sets and semilinear sets. For a positive integer \( r \), \( \mathbb{N}^r \) denotes the set of all \( r \)-dimensional vectors of nonnegative integers. For given subsets \( V, W \) of \( \mathbb{N}^r \), \( L(V, W) \) denotes the set of all \( x \) in \( \mathbb{N}^r \) which can be represented in the form

\[
x = x_0 + x_1 + \cdots + x_n
\]

with \( x_0 \) in \( V \) and \( x_1, \ldots, x_n \) a (possibly empty) finite sequence of elements of \( W \). A subset \( L \) of \( \mathbb{N}^r \) is said to be linear if there exists an element \( r \) in \( \mathbb{N}^r \) and a finite subset \( W \) of \( \mathbb{N}^r \) such that \( L = L([r], W) \). A subset of \( \mathbb{N}^r \) is said to be semilinear if it is a finite union of linear sets.

**Lemma 4.1.** Let \( M \) be a marked Petri net. If \( ER(M) \) is semilinear, then \( R(M) \) is semilinear.

**Proof.** It is immediately obtained by the fact that a projection of a semilinear set is also semilinear.

The converse of Lemma 4.1 is not always true. The notion of sinklessness is proposed as a sufficient condition for the reachability set to be semilinear [7]. A circuit \( u \) in a marked Petri net \( M = (C, m^0) = (P, T, A, m^0) \) is said to be sinkless if the total number of tokens in \( u \) is not decreased to 0 by firing transitions, i.e. \( m|P_u \neq 0 \) implies \( m + \sigma|P_u \neq 0 \) for every marking \( m \) in \( R(M) \) and every firing sequence \( \sigma \) in \( \Sigma(C, m) \). A marked Petri net \( M = (C, m^0) \) is said to be sinkless if every minimal circuit in \( M \) is sinkless.

Let \( F(C, m) \) denote the set of all token-free minimal circuits in a Petri net \( C \) with a marking \( m \). Then a marked Petri net \( M = (C, m^0) \) is sinkless if \( F(C, m) \supset F(C, m + \sigma) \) for every marking \( m \) in \( R(M) \) and every firing sequence \( \sigma \) in \( \Sigma(C, m) \).

**Theorem 4.2.** (Yamasaki [7]). If a marked Petri net \( M \) is sinkless, then we can effectively compute the reachability set in the form of a semilinear set.

A Petri net is normal if each minimal circuit is a trap circuit [7]. Normality is the structural property for sinklessness.

**Theorem 4.3.** (Yamasaki [7]). A Petri net \( C \) is normal if and only if the marked Petri net \( M = (C, m^0) \) is sinkless for every initial marking \( m^0 \).
Theorem 4.4. Every SWPN is normal.

Proof. Let \( C = (P, T, A) \) be an SWPN. Since \( C \) satisfies Condition 3.2, every circuit in \( C \) contains a trap circuit. This implies that each minimal circuit in \( C \) is a trap circuit. \( \square \)

Lemma 4.5. Let \( M = (C, m^0) = (P, T, A, m^0) \) be a marked Petri net. Let \( \alpha \) and \( \beta \) be firing sequences in \( T^* \) such that \( m^0[\alpha \land m^0[\beta \land \psi(\alpha) \leq \psi(\beta). \) If \( M[\psi(\beta) - \psi(\alpha) \) is sinkless and \( F(C, m^0 + \alpha) = F(C, m^0 + \beta) \), then there exists a firing sequence \( \gamma \) such that \( m^0[\gamma \land \psi(\gamma) = \psi(\beta). \)

Proof. Set \( x = \psi(\beta) - \psi(\alpha). \) If \( x = 0, \) then \( \gamma = \lambda. \) We consider the case \( x \geq 0. \) We first show that \( M[x \) has a firable transition in \( m^0 + x. \) Assuming the contrary, we can show that \( M[x \) has a TFDC \( u \) in \( m^0 + x \) by Lemmas 2.2 and 2.3. There exists a minimal circuit \( u' \) in \( M[x \) such that \( P_{u'} \subseteq P_u. \) It is necessary for \( u' \) to have tokens in firings of a transition which has an input place in \( u'. \) Since \( M[x \) is sinkless, \( u' \) still has tokens in \( m^0 + \beta. \) This contradicts \( F(C, m^0 + \alpha) = F(C, m^0 + \beta). \) Choosing a firable transition \( t, \) similar arguments are applicable for \( xt \) and \( \beta. \) By repeating this process, we can obtain a firing sequence \( \gamma \) such that \( m^0[\gamma \land \psi(\gamma) = \psi(\beta). \) \( \square \)

For a Petri net \( C = (P, T, A), \) the set \( \{x \in \mathbb{N}^T \land x \neq 0 \land B_x \geq 0\} \) becomes a semigroup and its generator \( T_x \) is a finite set, because \( T_x \) is a projection of \( \text{Min}\{\langle B_x, x, x \rangle \in \mathbb{N}^T \land x \neq 0 \land B_x \geq 0\}, \) where \( \text{Min} \) denotes the set of all minimal elements in a subset \( V \) of \( \mathbb{N}^T. \) We define the subset \( T_x \) of \( T \) by \( T_x = \{t \mid \exists x \in T_x, t \in T|x\}. \)

In the next theorem, we construct the extended reachability set of a sinkless marked Petri net.

Theorem 4.6. Let \( M = (C, m^0) = (P, T, A, m^0) \) be a marked Petri net. If \( M[T_x \) is sinkless, then \( ER(M) \) is semilinear.

Proof. Since \( F = \{F(C, m) \mid m \in R(M)\} \) is finite, we set \( F = \{F_1, \ldots, F_n\}. \) Let

\[
\Delta'(m) = \text{Min}\{\langle \sigma, \psi(\sigma) \rangle \in m[\sigma] \land \sigma \geq 0 \land \sigma \neq \lambda \land F(C, m') = F_i\},
\]

\[
\Delta^i = \{\Delta'(m) \mid m \in R(M) \land F(C, m) = F^i\},
\]

\[
\bigcup_{m \in R(M)} \Delta'(m) \text{ is finite because it is a projection of}
\]

\[
\text{Min}\{\langle \sigma, \psi(\sigma) \rangle \in m \in R(M) \land m[\sigma] \land \sigma \geq 0 \land \sigma \neq \lambda \land F(C, m') = F_i\}.
\]

Therefore, \( \Delta^i \) is finite and we set \( \Delta^i = \{\Delta^{i1}, \ldots, \Delta^{in}\}. \) Let

\[
ER^{ij} = \{\langle m^0 + \sigma, \psi(\sigma) \rangle \mid \sigma \in \Sigma(M) \land F(C, m^0 + \sigma) = F_i \land \Delta'(m^0 + \sigma) = \Delta^{ij}\}.
\]
Since $M|T_1$ is sinkless, $ER^{(i)} \subseteq L(\text{Min } ER^{(i)}, A^{(i)})$ holds by Lemma 4.5. Then we have

$$ER(M) = \bigcup_{F \in F} \bigcup_{\psi \in \mathcal{L}} ER^{(i)}$$

$$\subseteq \bigcup_{F \in F} \bigcup_{\psi \in \mathcal{L}} L(\text{Min } ER^{(i)}, A^{(i)})$$

$$\subseteq ER(M).$$

Hence, $ER(M)$ is semilinear.

Now we show that the reachability set of a marked Petri net is semilinear if the number of times that sinklessness is violated is finite during each firing of transitions. Let $M = (C, m^0) = (P, T, A, m^0)$ be a marked Petri net. We define the decomposition of a firing sequence $\sigma$ in $\Sigma(M)$, which decomposes $\sigma$ into "sinkless" subsequences.

```
procedure decompose(C, m^0, \sigma, \delta)
    (Input: C = (P, T, A), m^0 and \sigma; Output: \delta)
    begin
        if \sigma = \lambda then \delta := \lambda.
        else begin
            \Upsilon := the length of \sigma;
            \Pi := the set of all \(k\) (1 \leq k \leq \Upsilon) s.t. \((C, m^0)i(\{T_1, . . . , T_k\})\) is sinkless;
            if \Upsilon = 0 then begin \delta_0 := \lambda; \Upsilon := \iota; end
            else begin \Upsilon := \text{Max } k \text{ s.t. } k \in \Pi; \delta_0 := \sigma_{1, r}; \end;
            if \Upsilon = \Pi then \delta := \delta_0.
            else begin
                \delta := \delta_0; \sigma_{r + 1}. . . \sigma_\iota.
                end
            end
        end
    end
```

For a firing sequence $\sigma$ in $\Sigma(M)$, $\tilde{\sigma}$ denotes the sequence $\delta$ obtained by the execution of decompose($C, m^0, \sigma, \delta$). It is a sequence of $(T^*)^*$ and has the form $\tilde{\sigma} = \sigma_0, \sigma_1, \sigma_2, . . . , \sigma_{r^*}, . . . , \sigma_\iota$, where $\sigma_0, \sigma_i \in T^*$ and $\sigma_i \in T$ ($i = 1, 2, . . .$). Let $\sigma[i]$ denote the sequence $s_1, s_2, s_3, . . . , s_i, \sigma_i$, and let $\tilde{\sigma}[i]$ denote the sequence $\sigma_0, \sigma_1, s_2, s_3, . . . , s_i, \sigma_i$.

**Lemma 4.7.** Let $M = (C, m^0) = (P, T, A, m^0)$ be a marked Petri net. Let $\alpha$ and $\beta$ be sequences in $\Sigma(M)$ such that

1. $\alpha' = \beta'; = s_1, s_2, . . . , s_r$;
2. $\psi(\alpha_i) \leq \psi(\beta_i)$ for $i = 0, 1, . . . , r$, where $\alpha = \alpha_0, \alpha_1, \alpha_2, . . . , \alpha_r$, and $\beta = \beta_0, \beta_1, . . . , \beta_2, . . . , \beta_r$;
(iii) \( \tilde{\beta}[i] \leq \tilde{\beta}[i] \) for \( i = 0, 1, \ldots, r \);
(iv) \( F(C, m^0 + \tilde{\beta}[i]) = F(C, m^0 + \tilde{\beta}[i]) \) for \( i = 0, 1, \ldots, r \).

Then there exists a firing sequence \( \gamma \) in \( \Sigma(M) \) that satisfies the following conditions:
(v) \( \gamma^t = x^t = \beta^t \);
(vi) \( \psi(x_i \tilde{\delta}_i) = \psi(\beta_i) \) for \( i = 0, 1, \ldots, r \), where \( \gamma^t = x_0 \tilde{\delta}_0.s_1.\tilde{\delta}_1.s_2 \ldots s_r.\tilde{\delta}_r.\)

**Proof.** We prove the lemma by induction on the length of \( x^t \). It is obvious in the case \( r = 0 \), i.e. \( x^t = x^t \), by Lemma 4.5. Assume that the lemma holds for \( r < k \) and we consider the case \( r = k + 1 \). Then there exists a firing sequence \( \gamma' \) in \( \Sigma(M) \) such that \( \gamma' = x_0 \tilde{\delta}_0.s_1.\tilde{\delta}_1.s_2 \ldots s_k.\tilde{\delta}_k.\psi(x_0 \tilde{\delta}_0.\) for \( i = 0, 1, \ldots, k \). Let \( m = m^0 + \gamma' s_{k+1} \).

Since \( \tilde{\beta}[k] \leq \tilde{\beta}[k] = \gamma' \), we have \( m[x_{k+1}] \land m[\beta_{k+1}] \land \psi(x_{k+1}) \leq \psi(\beta_{k+1}) \) and \( F(C, m + \gamma' s_{k+1}) = F(C, m + \beta_{k+1}) \). Since \( \beta - x \geq 0 \), it follows that \( T|\psi(\beta_{k+1}) - \psi(x_{k+1})| < T. \) Therefore, \( (C, m)|\psi(\beta_{k+1}) - \psi(x_{k+1}) \) is sinkless from the decomposition procedure. By Lemma 4.5, there exists a firing sequence \( \delta_{k+1} \) such that \( m[x_{k+1}] \land \psi(x_{k+1}) = \psi(\beta_{k+1}). \)

Let \( M \) be a marked Petri net, let \( \sigma \) be a firing sequence in \( \Sigma(M) \) and let \( \tilde{\delta} = \sigma_0.s_1.\sigma_1.s_2 \ldots s_r.\sigma_r.\) For a firing sequence \( \gamma \) in \( \Sigma(M) \) such that \( \gamma^t = \sigma^t \), we write \( \sigma \uparrow \gamma \) if
(i) \( \gamma \neq \sigma; \)
(ii) \( \gamma^t = \sigma_0.s_1.\sigma_1.s_2 \ldots s_r.\sigma_r.\)
(iii) \( \sum_{j=0}^r \tilde{\delta}_j \geq 0 \) for \( i = 0, 1, \ldots, r.\)

**Lemma 4.8.** Let \( M \) be a marked Petri net. Let \( \sigma, \alpha \) and \( \beta \) be firing sequences in \( \Sigma(M) \) such that \( \sigma^t = \alpha^t = \beta^t. \) If \( \sigma \uparrow \alpha \) and \( \sigma \uparrow \beta \), then there exists a firing sequence \( \gamma \) in \( \Sigma(M) \) such that \( \psi(\gamma) = (\psi(\gamma) - \psi(\sigma)) + (\psi(\beta) - \psi(\sigma)) + \psi(\sigma). \)

**Proof.** Let \( \tilde{x} = \sigma_0.s_0.\sigma_1.s_1 \ldots s_r.\sigma_r \) and \( \tilde{\beta} = \sigma_0.\beta_0.s_1.\beta_1.s_2 \ldots s_r.\sigma_r.\) Then \( \gamma = \sigma_0.\alpha_0.s_0.\gamma_1.s_1.\gamma_2 \ldots s_r.\gamma_r.\) is in \( \Sigma(M). \)

**Lemma 4.9.** Let \( M = (C, m^0) \) be a marked Petri net. If the length of \( \sigma^t \) is finite for every firing sequence \( \sigma \) in \( \Sigma(M) \), then \( E \Sigma(M) \) is semilinear.

**Proof.** Since \( \Sigma = \{ \sigma | \sigma \in \Sigma(M) \} \) is finite, we set \( \Sigma = \{ u^1, \ldots, u^n \}. \) For each \( u^i = s_1 \ldots s_r \), let \( \Sigma^i = \{ \sigma | \sigma \in \Sigma(M) \} \land \sigma^t = u^i. \) For each firing sequence \( \sigma \) in \( \Sigma^i \), let \( F^i(\sigma) = \sigma(\sigma + \tilde{\delta}[0], \ldots, \sigma(\sigma + \tilde{\delta}[r]) \}, \)

where \( f(m) \) denotes the vector whose dimension is equal to the number of minimal circuits in \( M \), and each element of \( f(m) \) is 0 if the corresponding minimal circuit is token-free in \( m \), otherwise 1. Since \( F^i = \{ F^i(\sigma) | \sigma \in \Sigma^i \} \) is finite, we set
$F^i = F^{i_1}, ..., F^{i_n}$. Let $\bar{\sigma} = \sigma_0, \sigma_1, \sigma_2, ..., \sigma_r$. Then we define the following:

$$\mu^i(\sigma) = \langle m^0 + \bar{\sigma}[0], ..., m^0 + \bar{\sigma}[r], \psi(\sigma_0), ..., \psi(\sigma_r) \rangle.$$  

$$Z^{ij}(\sigma) = \min \{ \mu^i(\cdot) - \mu^j(\cdot) : \sigma \in \Sigma^i \land \sigma \uparrow \gamma \land F^i(\gamma) = F^j(\gamma) \}.$$  

$$Z^{ij} = \{ Z^{ij}(\sigma) : \sigma \in \Sigma^i \land F^i(\sigma) = F^j(\sigma) \}.$$  

$\bigcup_{\gamma \in \Sigma^i} Z^{ij}(\sigma)$ is finite because it is a projection of

$$\min \{ \langle \mu^i(\cdot) - \mu^j(\cdot) : m^0 + \bar{\sigma}[0], ..., m^0 + \bar{\sigma}[r], \psi(\sigma_0), ..., \psi(\sigma_r) \rangle : \sigma \in \Sigma^i \land \sigma \uparrow \gamma \land F^i(\gamma) = F^j(\gamma) \}.$$  

Therefore, $Z^{ij}$ is finite and we set $Z^{ij} = \{ Z^{ij}(\cdot) : \sigma \in \Sigma^i \land F^i(\sigma) = F^j(\sigma) \}$. Let

$$\mu^{ijk} = \{ \mu^i(\sigma) : \sigma \in \Sigma^i \land F^i(\sigma) = F^j \land Z^{ij}(\sigma) = Z^{jk} \}.$$  

It follows that $\mu^{ijk} \subseteq L(\min \mu^{ijk}, Z^{ijk})$ by Lemma 4.7. Let

$$ER^{ijk} = \{ \langle m^0 + \bar{\sigma}, \psi(\sigma) \rangle : \sigma \in \Sigma^i \land F^i(\sigma) = F^j \land Z^{ij}(\sigma) = Z^{jk} \}.$$  

$$A^{ijk} = \min \{ \langle \sigma, \psi(\sigma) \rangle : \sigma \in \Sigma^i \land \sigma \uparrow \gamma \land F^i(\sigma) = F^j \} \land Z^{ij}(\sigma) = Z^{jk}.$$  

then we have $ER^{ijk} \subseteq L(\min ER^{ijk}, A^{ijk})$. By Lemma 4.8, it follows that $L(\min ER^{ijk}, A^{ijk}) \subseteq ER(\Sigma)$. Therefore,

$$ER(M) = \bigcup_{\sigma \in \Sigma} \bigcup_{F^i(\sigma) \neq F^j(\sigma)} \bigcup_{Z^{ijk}(\sigma) \neq \sigma} ER^{ijk}$$  

$$= \bigcup_{\sigma \in \Sigma} \bigcup_{F^i(\sigma) \neq F^j(\sigma)} \bigcup_{Z^{ijk}(\sigma) \neq \sigma} L(\min ER^{ijk}, A^{ijk})$$  

$$\subseteq ER(\Sigma).$$

Hence, $ER(M)$ is semilinear.

A marked Petri net $M$ is called finally sinkless if the following holds for every firing sequence $\sigma$ in $\Sigma(M)$ and every minimal circuit $u$ in $M$: the number of times that $u$ becomes token-free during the firing of $\sigma$ is finite.

**Theorem 4.10.** Let $C = (P, T, A)$ be a Petri net. The marked Petri net $M = (C, m^0)$ is finally sinkless for every initial marking $m^0$ if and only if $C$ satisfies Condition 4.11.

**Condition 4.11.** Let $C = (P, T, A)$ be a Petri net. The following holds for every minimal circuit $u$ in $C$ and every $x$ in $T^u$: if $B \cdot x \cap \tau \cdot (P_u) = \emptyset$, then $T \cdot \tau \cdot \tau \cdot (P_u) = \emptyset$.

A Petri net that satisfies Condition 4.11 is called a structurally finally sinkless Petri net (SFSN). To prove the theorem, we show several lemmas. Lemma 4.12 is immediately obtained by the definition of minimal circuits.
Lemma 4.12. Let \( C = (P, T, A) \) be a Petri net and let \( u = p_1t_1 \cdots p_nt_n \) be a minimal circuit in \( C \). If \( A(p_i, t) = A(t, p_j) = 1 \) \((1 \leq i, j \leq n)\), then \( j = i + 1 \).

By Lemma 4.12, the number of tokens in a minimal circuit \( u \) increases only by firings of \( \tau^+ (P_u) \), and decreases only by firings of \( \tau^- (P_u) \). This implies that the following condition is equivalent to Condition 4.11.

**Condition 4.13.** Let \( C = (P, T, A) \) be a Petri net. The following holds for every minimal circuit \( u \) in \( C \) and every \( x \) in \( \Gamma_c \): if \( B_c x | P_u = 0 \), then \( T | x \cap \tau^- (P_u) = \emptyset \).

**Lemma 4.14 (Peterson [4]).** Every infinite sequence of nonnegative integer vectors contains an infinite nondecreasing subsequence.

**Lemma 4.15.** Let \( C = (P, T, A) \) be a Petri net and let \( u \) be a minimal circuit in \( C \). If \( B_c x | P_u = 0 \land T | x \cap \tau^+ (P_u) \neq \emptyset \) for a vector \( x \) in \( N^T \), then there exists a marking \( m \) and a firing sequence \( \sigma \) such that \( m[\sigma] \wedge m | P_u = 0 \wedge \psi (\sigma) = x \).

**Proof.** Firability of transitions is considered in \( C | P_u \) because we can set enough tokens in each place other than \( P_u \). Let \( x^+ \) be the maximum vector in \( N^T \) such that \( x^+ \leq x \wedge T | x^+ = \tau^+ (P_u) \); let \( x^- \) be the maximum vector in \( N^T \) such that \( x^- \leq x \wedge T | x^- = \tau^- (P_u) \). Let \( m \) be a marking such that \( m | P_u = 0 \), and let \( m_1 = m + B_x x^+ \), \( m_2 = m_1 + B_x x_y \) and \( m_3 = m_2 + B_x x^- \). It follows that \( m_1 | P_u \geq 0 \), \( m_2 | P_u \geq 0 \) and \( m_3 | P_u = 0 \). Obviously, \( m[> m_1 \wedge m_2 ] > m_3 \). We show \( m_1 [> m_2 \). Since \( u \) is minimal, \( P_u = P_u \) holds for every circuit \( u' \) in \( C | P_u \). Therefore, every circuit in \( (C | y) | P_u \) is a trap and deadlock circuit, i.e. \( (C | y) | P_u \) is an SWPN. Since \( x^+ \neq 0 \), \( P_u \) has some tokens in \( m_1 \). This implies that \( (C | y) | P_u \) has no TFDCs in \( m_1 \). Hence, \( m_1 [> m_2 \) by Theorem 3.7.

**Proof of Theorem 4.10.** "If" part: Assume that \( M \) is not finally sinkless. Then there exists a minimal circuit \( u \) and an infinite firing sequence \( \sigma_0 \sigma_1 \sigma_2 \cdots \) in \( \Sigma (M) \) such that

1. \( m_i | P_u = 0 \) for \( i = 0, 1, \ldots \), where \( m_i = m^0 + \sum_{j=0}^{i} \sigma_j \);
2. \( T | \psi (\sigma_i) \cap \tau^+ (P_u) \neq \emptyset \) for \( i = 1, 2, \ldots \).

The infinite sequence \( m_0 m_1 m_2 \cdots \) contains an infinite nondecreasing subsequence by Lemma 4.14. Therefore, there exist \( i \) and \( j \) \((i < j)\) such that \( \delta \geq 0 \wedge \delta | P_u = 0 \wedge T | \psi (\delta) \cap \tau^+ (P_u) \neq \emptyset \) for \( \delta = \sigma_1 \cdots \sigma_j \). This contradicts Condition 4.11.

"Only if" part: Assume that there exists a vector \( x \) in \( \Gamma_c \), that does not satisfy Condition 4.11 for a minimal circuit \( u \). By Lemma 4.15, there exists a marking \( m \) and a firing sequence \( \sigma \) such that \( m[\sigma] \wedge m | P_u = 0 \wedge \psi (\sigma) = x \). Since \( \sigma x \in \Sigma (M) \) holds for every positive integer \( n \), the marked Petri net \( M = (C, m) \) is not finally sinkless.

**Lemma 4.16.** Let \( M = (C, m^0) = (P, T, A, m^0) \) be a marked Petri net and let \( \sigma \) be a firing sequence in \( \Sigma (M) \). If the length of \( \sigma / \) is infinite, then there exists an infinite sequence
Proof. Let \( \tilde{\sigma} = \sigma_0, \sigma_1, \sigma_2, \ldots \) Existence of a tuple \((k_1, k_2, \ldots, u)\) satisfying (i) is immediately obtained from the decomposition procedure. Let \( U \) be the set of all tuples satisfying (i). Assume that every tuple in \( U \) does not satisfy (ii). Then there exists an infinite subsequence \( \sigma' = \sigma_{r_1+1}, \sigma_{r_2+1}, \sigma_{r_3+2}, \ldots \) of \( \sigma \) such that \( T | \psi(\tilde{\sigma}[k_{i+1}]) - \psi(\tilde{\sigma}[k_i]) \cap \tau^{-1}(P_u) \neq \emptyset \) for \( i = 1, 2, \ldots \)

\begin{lemma}
Let \( M = (C, m^0) = (P, T, A, m^0) \) be a marked Petri net. If \( C | T_+ \) is an SFSN, then the length of \( \sigma \) is finite for every firing sequence \( \sigma \) in \( \Sigma(M) \).
\end{lemma}

Proof. Assume that the length of \( \sigma \) is infinite for a firing sequence \( \sigma \) in \( \Sigma(M) \). Then there exists an infinite sequence \( k_1, k_2, \ldots \) and a minimal circuit \( u \) in \( M \) such that satisfy Lemma 4.16(i), (ii). There exists a firing sequence \( \delta_i \) such that \( m^0 + \delta[k_i] \delta_i \neq m_i \wedge m_i | P_u = 0 \) for \( i = 1, 2, \ldots \). Let \( e_j = (m^0 + \delta[k_i] \delta_i, \psi(\tilde{\sigma}[k_i] \delta_i) \) for \( i = 1, 2, \ldots \). By Lemma 4.14, the infinite sequence \( e_1, e_2, \ldots \) contains an infinite nondecreasing subsequence, and so there exist \( i \) and \( j (i < j) \) such that \( e_i \geq e_j \). It follows that \( B_i, \psi \geq 0 \wedge B_i, \psi | P_u = 0 \wedge T | \psi(\tilde{\sigma}[k_i] \delta_i) \neq \emptyset \) for \( \psi = \psi(\tilde{\sigma}[k_i] \delta_i) - \psi(\tilde{\sigma}[k_i] \delta_j) \). This contradicts Condition 4.13.

By Lemmas 4.1, 4.9 and 4.7, we have the following theorem.

\begin{theorem}
Let \( C = (P, T, A) \) be a Petri net. If \( C | T_+ \) is an SFSN, then the reachability set of the marked Petri net \( M = (C, m^0) \) is semilinear for every initial marking \( m^0 \).
\end{theorem}

5. Illustrative examples

Figure 1 shows an example of SWPN. Every circuit has a set of places that satisfies Condition 3.2. For example, the circuit \( p_1, p_2, p_3, t_1, p_1 \) has \( S = \{ p_1, t_1 \} \).
Figure 2 shows an example of SFSN. The circuit $u = p_1 t_1 p_2 t_2 p_1$ is not sinkless in a marking $[0, 0, n, 1]$. But the number of times $u$ becomes token-free is bounded by $n$.

References