On structural conditions for weak persistency and semilinearity of Petri nets

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Abstract


A necessary and sufficient condition for a Petri net to be weakly persistent for every initial marking is obtained. Moreover, a necessary and sufficient condition for reachability is obtainable for this class of Petri nets. As a sufficient condition for a Petri net to have a semilinear reachability set, the notion of sinklessness has been proposed, where a marked Petri net is said to be sinkless if the total number of tokens in each minimal circuit is not decreased to 0 by firing transitions. We show that the reachability set is semilinear if the total number of times that sinklessness is violated is finite during each firing, and define a new subclass of Petri nets which have this property for every initial marking.

1. Introduction

Petri nets are a structural model for representing discrete and concurrent systems, which describes how each component of a system communicates with other ones. Classification of Petri nets has been studied for two kinds of properties: one is structural and the other is behavioral. Several classes of Petri nets have been proposed as structural properties, such as marked graphs, state machines, free-choice nets, conflict-free nets [3], trap-circuit nets, deadlock-circuit nets [1] and so on. On the other hand, liveness, fairness, persistency [2] and weak persistency [6] have been
studied as behavioral properties. In this paper we study relations between structural
and behavioral properties, which are important in designing and verifying systems.

We first study weak persistency. A necessary condition for weak persistency is
discussed in [7]. We show a necessary and sufficient condition for weak persistency,
which is described as a structural property. Moreover, a necessary and sufficient
condition for reachability is obtainable for this class of Petri nets.

Persistency, weak persistency and sinklessness [7] are introduced to define classes
of Petri nets whose reachability sets are effectively computable semilinear. A marked
Petri net is said to be sinkless if the total number of tokens in each minimal circuit is
not decreased to 0 by firing transitions. We show that the reachability set is semilinear
if the total number of times that sinklessness is violated is finite during each firing, and
obtain a structural condition for a Petri net to have this property for every initial
marking.

2. Definitions and Notation

Let \( \mathbb{Z} \) denote the set of integers and let \( \mathbb{N} \) denote the set of nonnegative integers. For
a nonempty finite set \( S \), \( \mathbb{Z}^S \) \((\mathbb{N}^S)\) denotes the set of all \(|S|\)-dimensional vectors of
integers (nonnegative integers). A function \( f: S \rightarrow \mathbb{Z} \) can be identified with an element
of \( \mathbb{Z}^S \) by introducing a total order on \( S \). Moreover, for nonempty finite sets \( S_1 \) and \( S_2 \),
a function \( S_1 \times S_2 \rightarrow \mathbb{Z} \) can be identified with a \(|S_1| \times |S_2|\) matrix of integers. For
a vector \( x \) in \( \mathbb{Z}^S \) and a subset \( S' \) of \( S, x \mid S' \) denotes the projection of \( x \) onto \( S' \). For
vectors \( x \) and \( y \) and an integer \( k \), the addition \( x + y \), the scalar product \( kx \), and the
partial order \( x \leq y \) are defined componentwise as usual. For vectors \( x \) and \( y \), \( \langle x, y \rangle \)
denotes the concatenation of \( x \) and \( y \), e.g. \( \langle [1, 0, 4], [2, 1, 3] \rangle = [1, 0, 4, 2, 1, 3] \), and
similarly for \( \langle x, y, z, \ldots \rangle \).

A Petri net is a triple \( C = (P, T, A) \), where \( P \) is a finite set of places, \( T \) is a finite set
of transitions, and \( A: (P \times T) \cup (T \times P) \rightarrow \mathbb{N} \) is a function which represents the incidence
relation between \( P \) and \( T \). In this paper we restrict the function \( A \) to
\( (P \times T) \cup (T \times P) \rightarrow \{0, 1\} \). This class of Petri nets is called single arc.
\( M = (C, m^0) = (P, T, A, m^0) \) is called a marked Petri net, where \( m^0: P \rightarrow \mathbb{N} \) is the initial
marking. If \( A(t, p) = 1 \) for a transition \( t \) and a place \( p, p \) is called an input place of \( t \) and \( t \)
is called an output transition of \( p \), similarly for an output place and an input transition.

A transition \( t \) is firable in a marking \( m \) if \( m(p) \geq A(t, p) \) for each place \( p \) in \( P \). The
firing of \( t \) yields the marking \( m' = m + t \), where \( t \) is defined by
\[
\ell(t, p) = A(t, p) - A(p, t) \quad \text{for each } p \in P.
\]

A sequence of transitions is called a firing sequence. Let \( \lambda \) denote the empty firing
sequence. For firing sequences \( \alpha, \beta \) in \( T^* \), \( \alpha \beta \) denotes their concatenation, and \( \alpha \cdot \beta \)
denotes the sequence in \((T^*)^*\). We simply write \( (x)^n \) to indicate \( xx \ldots x \) \((n \text{ times})\). For
a firing sequence \( \sigma = s_1s_2 \ldots s_n \), \( s_i \in T \) \((i = 1, \ldots, r) \), \( \sigma_{+j} \) \((1 \leq i \leq j \leq r) \) denotes the sub-
sequence \( s_i s_{i+1} \ldots s_j \), and \( \sigma_{+j} \) denotes \( s_j \).
We can extend the notion of firing to sequences of transitions:
(i) $\lambda$ is firable in every marking and $\lambda = 0$,
(ii) $t\sigma$ is firable in a marking $m$ and $t\sigma = t + \sigma$ if a transition $t$ is firable in $m$ and a firing sequence $\sigma$ is firable in $m + t$.

We write $m[\sigma]m'$ to indicate that a firing sequence $\sigma$ is firable in a marking $m$ and the firing of $\sigma$ yields the marking $m'$. When we do not need to specify the resulting marking, we simply write $m[\sigma]$. Similarly for $m[m']$.

Let $B : P \times T \to \mathbb{Z}$ denote the incidence matrix of a Petri net $C = (P, T, A)$, which is defined by $B(p, t) = t(p)$. $\psi : T^* \to \mathbb{N}^T$ is called the Parikh mapping. For a firing sequence $\sigma$ in $T^*$, $\psi(\sigma)(t)$ denotes the number of occurrences of a transition $t$ in $\sigma$. Since $\sigma = B, \psi(\sigma)$, we have the following lemma.

**Lemma 2.1.** Let $M = (C, m^0) = (P, T, A, m')$ be a marked Petri net. If $m^0[\sigma] m'$, then there exists a vector $x$ in $N^T$ such that $m - m^0 = B, x$.

Let $C = (P, T, A)$ be a Petri net. A word $w_1 w_2 \ldots w_n$ in $(P \cup T)^*$ is called a path if $A(w_i, w_{i+1}) = 1$ for $i = 1, \ldots, n$. A path $w_1 w_2 \ldots w_n$ is called a circuit if $w_1 = w_n$. Let $P_u$ and $T_u$ denote the set of places and the set of transitions, respectively, occurring in a circuit $u$. A circuit $u$ is called simple if $u$ does not contain any other circuits. A simple circuit $u$ is called minimal if the set of places in $u$ does not properly include the set of places in any other circuit.

We define the following for a marked Petri net $M = (C, m^0) = (P, T, A, m')$:

$$\Sigma(M) = \Sigma(C, m^0) = \{ \sigma | \sigma \in T^* \land m^0[\sigma] \},$$

$$R(M) = R(C, m^0) = \{ m^0 + \sigma | \sigma \in \Sigma(M) \},$$

$$ER(M) = ER(C, m^0) = \{ \langle m^0 + \sigma, \psi(\sigma) \rangle | \sigma \in \Sigma(M) \}.$$

$R(M)$ is called the reachability set of $M$, and $ER(M)$ is called the extended reachability set of $M$.

Restrictions of a Petri net $C = (P, T, A)$ and a marked Petri net $M = (C, m^0)$ are defined as follows.

(i) **Restriction of places:** Let $S$ be a given subset of $P$. The restriction of $C$ to $S$ is defined by $C|S = (S, T|S, A|S)$, where

$$T|S = \{ t | \exists p \in S: A(p, t) = 1 \lor A(t, p) = 1 \},$$

and $A|S$ is the restriction of $A$ to $(S, T|S)$. The restriction of $M$ to $S$ is defined by $M|S = (C|S, m^0|S)$, where $m^0|S$ is the restriction of $m^0$ to $S$.

(ii) **Restriction of transitions:** Let $H$ be a given subset of $T$. The restriction of $C$ to $H$ is defined by $C|H = (P|H, H, A|H)$, where

$$P|H = \{ p | \exists t \in H: A(p, t) = 1 \lor A(t, p) = 1 \}.$$
and $A|H$ is the restriction of $A$ to $(P|H, H)$. The restriction of $M$ to $H$ is defined by $M|H = (C|H, m^0|H)$.

(iii) **Restriction to a vector in $N^T$:** Let $x$ be a given vector in $N^T$. The restriction of $T$ to $x$ is defined by $T|x = \{t | t \in T \land \psi(t) \leq x \}$. We simply write $C|x$ for $C|(T|x)$ and $M|x = (C|x, m^0|x)$ for $M|(T|x)$.

(iv) **Restriction to a circuit:** We simply write $C|u$ and $M|u$ to indicate the Petri net $C$ and the marked Petri net $M$ restricted to places and transitions in a circuit $u$, respectively.

Let $S$ be a set of places in a Petri net $C = (P, T, A)$. We define $\tau(S)$, $\tau^+(S)$ and $\tau^-(S)$ by

\[
\tau(S) = \{ t | \exists p \in S: A(p, t) = 1 \land \exists q \in S: A(q, t) = 0 \},
\]

\[
\tau^+(S) = \{ t | \exists p \in S: A(p, t) = 1 \land \forall q \in S: A(q, t) = 0 \},
\]

\[
\tau^-(S) = \{ t | \exists p \in S: A(p, t) = 1 \land \forall q \in S: A(q, t) = 0 \}.
\]

$S$ is called a *deadlock* if $\tau^+(S) = \emptyset$, and is called a *trap* if $\tau^-(S) = \emptyset$. A circuit $u$ is called a *deadlock circuit* if $P_u$ is a deadlock, and is called a *trap circuit* if $P_u$ is a trap. A place $p$ is called a *source place* if $p$ has no input transitions. For a deadlock $D$ without source places, there exists a deadlock circuit $u$ such that $P_u \subseteq D$. A deadlock $D$ is called a *token-free deadlock* (TFD) in a marking $m$ if $m|D = 0$. A deadlock circuit $u$ is called a *token-free deadlock circuit* (TFDC) in a marking $m$ if $m|P_u = 0$.

**Lemma 2.2** (Yamasaki [5]). If a marked Petri net $M$ has no firable transitions, then there exists a TFD in $M$.

**Lemma 2.3.** Let $M = (C, m^0) = (P, T, A, m^0)$ be a marked Petri net. If $m + B_x \geq 0$ for a marking $m$ and a vector $x$ in $N^T$, then $M|x$ has no token-free source places in $m$.

**Proof.** If $M|x$ has a token-free source place $p$ in $m$, then the component of $m + B_x$ corresponding to the place $p$ is negative. \(\square\)

**Lemma 2.4.** Let $M$ be a marked Petri net. If a firing sequence $\sigma$ is firable in a marking $m$, then $M|\psi(\sigma)$ has no TFDs in $m$.

**Proof.** Assume that $M|\psi(\sigma)$ has a TFD $D$ in $m$. Then a transition in $M|\psi(\sigma)$ which has an input place in $D$ is not firable in every marking reachable from $m$. \(\square\)

**Lemma 2.5.** For each circuit $u$ in a Petri net $C = (P, T, A)$, there exists a circuit $u'$ such that $T_u = \tau(P_u)$ and $P_u = P_u'$.

**Proof.** Let $u = p_1 t_1 \ldots p_n t_n p_1$. Clearly, $\tau(P_u) \supseteq T_u$ holds. If there exists a transition $t$ in $\tau(P_u) \setminus T_u$ such that $A(p_i, t) = A(t, p_j) = 1$ $(1 \leq i, j \leq n)$, then we have the circuit...
By repeating this process, we can obtain a circuit $u'$ such that $T_u = \tau(P_u)$ and $P_{u'} = P_u$. □

A circuit $u$ is called complete if $T_u = \tau(P_u)$.

3. Weak persistency

Let $M = (P, T, A, m^0)$ be a marked Petri net. $M$ is said to be weakly persistent if $M$ satisfies the following condition [6]: for every firing sequences $x$ and $\beta$ such that $m^0(x) \land m^0(\beta) \land \psi(x) \leq \psi(\beta)$, there exists a firing sequence $\gamma$ such that $m^0(x\gamma) \land \psi(x\gamma) = \psi(\beta)$. In this section we study a necessary and sufficient condition for a Petri net to be weakly persistent for every initial marking. We first define the following conditions on $M$, which will be used to obtain the necessary and sufficient condition.

Condition 3.1. Let $M = (C, m^0) = (P, T, A, m^0)$ be a marked Petri net and let $x$ be a vector in $N^T$ such that
(i) $m^0 + B_r x \geq 0$,
(ii) $M|x$ has no TFDCs in $m^0$.
Then for every firing sequence $\beta$ in $\Sigma(M)$ such that $\psi(\beta) \leq \psi(x)$, $M|x - \psi(\beta)$ has no TFDCs in $m^0 + \beta$.

Condition 3.2. Let $C = (P, T, A)$ be a Petri net. For each circuit $u$ in $C$, there exists a nonempty set of places $S$ such that
(i) $S \subseteq P_u$,
(ii) $\tau^-(S) = \emptyset$, i.e. $S$ is a trap in $C$,
(iii) $\tau^+(S) \cap \tau(P_u) = \emptyset$.
We can obtain the following lemmas, which will be proved later.

Lemma 3.3. If a marked Petri net $M = (P, T, A, m^0)$ satisfies Condition 3.1, then $M$ is weakly persistent.

Lemma 3.4. If a Petri net $C = (P, T, A)$ satisfies Condition 3.2, then the marked Petri net $M = (C, m^0)$ satisfies Condition 3.1 for every initial marking $m^0$.

Lemma 3.5. Let $C = (P, T, A)$ be a Petri net. If the marked Petri net $M = (C, m^0)$ is weakly persistent for every initial marking $m^0$, then the Petri net $C$ satisfies Condition 3.2.

By Lemmas 3.3–3.5, we have the following theorem.
Theorem 3.6. A Petri net \( C = (P, T, A) \) satisfies Condition 3.2 if and only if the marked Petri net \( M = (C, m^0) \) is weakly persistent for every initial marking \( m^0 \).

Proof of Lemma 3.3. Assume that \( \alpha \) and \( \beta \) are firing sequences such that
\[
m^0[\alpha] \wedge m^0[\beta] \wedge \psi(\alpha) \leq \psi(\beta).
\]
Let \( \alpha = \psi(\beta) \), then \( \alpha \) satisfies Condition 3.1 by Lemmas 2.1 and 2.4. \( M | \psi(\beta) - \psi(\alpha) \) has no token-free source places in \( m^0 + \beta \) by Lemma 2.3, and has no TFDCs by Condition 3.1. Therefore, \( M | \psi(\beta) - \psi(\alpha) \) has a firable transition \( t \) in \( m^0 + \alpha \) by Lemma 2.2. This is also true for \( \alpha t \) and \( \beta \). By repeating this process, we can obtain a firing sequence \( \gamma \) such that
\[
m^0[\gamma] \wedge \psi(\alpha) = \psi(\beta).
\]

Proof of Lemma 3.4. Assume that a Petri net \( C = (P, T, A) \) satisfies Condition 3.2 and a marked Petri net \( M = (C, m^0) \) does not satisfy Condition 3.1. Then there exists a vector \( x \in \mathbb{N}^T \) and a firing sequence \( z \in \Sigma(M) \) that satisfy the following conditions:

(i) \( x \) satisfies Condition 3.1;
(ii) \( \psi(z) \triangleq x \);
(iii) \( M | x - \psi(z) \) has a TFDC \( u \) in \( m^0 + z \).

Let \( S \) be a set of places that satisfies Condition 3.2 for \( M \). It follows that \( \tau^+(S) = \emptyset \) in \( M | x - \psi(z) \) by Condition 3.2(iii). There are two cases:

(a) \( \tau^+(S) = \emptyset \) in \( M | x \): By Condition 3.1, \( S \) is not token-free in \( m^0 \). Since \( S \) is a trap in \( M \), \( S \) is not token free in \( m^0 + z \).

(b) \( \tau^+(S) \neq \emptyset \) in \( M | x \): In the firing of \( z \), transitions in \( \tau^+(S) \) bring some tokens into \( S \). Since \( S \) is a trap in \( M \), \( S \) is not token-free in \( m^0 + z \).

\( S \) is not token-free in both cases. This is a contradiction.

Proof of Lemma 3.5. We first show the following two facts, and prove the contrapositive of the statement of Lemma 3.5.

(a) Let \( C = (P, T, A) \) be a Petri net and let \( u \) be a simple circuit in \( C \) such that \( \tau^-(P_u) \neq \emptyset \). If \( C \) does not have a nonempty deadlock \( D \) such that \( D \subseteq P_u \), then the marked Petri net \( M = (C, m) \) is not weakly persistent for the initial marking \( m \) such that
\[
m(p) = 1 \quad \text{if } p \in P_u \quad \text{and} \quad A(p, t) = 1,
\]
\[
m(p) = 0 \quad \text{if } p \in P_u \quad \text{and} \quad A(p, t) = 0,
\]
\[
m(p) = k \quad \text{otherwise,}
\]
where \( t \in \tau^-(P_u) \) and \( k \) is a sufficiently large integer.

Let \( u = p_1t_1 \ldots p_t, p_t \). Assume that \( A(p_t, t) = 1 \) without loss of generality. Let \( m_i \) be the marking such that \( m_i(p_t) = 1 \), \( m_i(p_i) = 0 \) \((i \neq 1)\) and \( m_i(p) = k \) \((p \notin P_u)\). Since \( M | u \) has no TFDs in \( m_i \), \( M | u \) has a firable transition by Lemma 2.2. Since \( m_i(p_i) = 0 \), only \( t_1 \) is firable in \( M | u \). Let \( m_1(t_1) = m_2 \) and let \( m_2 \) be the marking such that \( m_2(p_2) = 1 \), \( m_2(p_1) = 0 \) \((i \neq 2)\) and \( m_2(p) = m_2(p) \) \((p \notin P_u)\). By similar arguments, \( t_2 \) is
firable in \( m'_2 \), and is firable in \( m_2 \) because \( m_2 \geq m'_2 \). Repeating this process, we can show that \( \sigma = t_1 t_2 \ldots t_n \) is firable in \( m_1 \).

Since \( m \geq m_1 \), \( \sigma \) is firable in \( m \). Every place \( p_i \) in \( P_u \) such that \( A(p_i, t) = 1 \) has a token in \( m + \sigma \); hence, \( t \) is firable in \( m + \sigma \). The sequence \( \sigma t \) is firable in \( m \) and no transitions in \( T_0 \) are firable in \( m + t \). Therefore, the marked Petri net \( M = (C, m) \) is not weakly persistent.

(b) Let \( C = (P, T, A) \) be a Petri net. Suppose that, for each circuit \( u \) in \( C \) such that \( \tau^{-1}(P_u) \neq \emptyset \), \( C|u \) has a nonempty deadlock \( D \) such that \( D \nsubseteq P_u \). Then for each circuit \( u \) in \( C \) such that \( \tau^{-1}(P_u) \neq \emptyset \), there exists a nonempty set of places \( S \) that satisfies the following conditions:

(i) \( S \subseteq P_u \);
(ii) \( \tau^{-1}(S) = \emptyset \);
(iii) \( \tau^{-1}(S) \cap \tau(P_u) = \emptyset \).

Assume that \( u \) is complete. If not, we can have a complete circuit \( u' \) such that \( P_{u'} = P_u \) by Lemma 2.5. Moreover, if there exists a set of places \( S \) that satisfies the above three conditions for \( u' \), then \( S \) also satisfies them for \( u \).

From the assumption and the fact that \( u \) is a circuit, i.e. \( C|u \) has no token-free source places, \( C|u \) has a deadlock circuit \( u' \). If \( \tau^{-1}(P_u) \neq \emptyset \) in \( C \), then \( C|u' \) has a deadlock circuit \( u'' \). The circuit \( u'' \) is also a deadlock circuit in \( C|u \). By repeating this process, we can obtain a circuit \( u_1 \) such that \( u_1 \) is a deadlock circuit in \( C|u \) and \( \tau^{-1}(P_{u_1}) = \emptyset \) in \( C \). Since \( u \) is complete, it follows that \( \tau^{-1}(P_{u_1}) \cap \tau(P_u) = \emptyset \).

If \( \tau^{-1}(P_u) = \emptyset \) for a circuit \( u \), then Condition 3.2 holds for \( S = P_u \). Hence, the facts (a) and (b) imply the following: if a Petri net \( C \) does not satisfy Condition 3.2, then there exists a marking \( m \) such that the marked Petri net \( M = (C, m) \) is not weakly persistent. \( \square \)

A Petri net that satisfies Condition 3.2 is said to be a structurally weakly persistent Petri net (SWPN). A necessary and sufficient condition for reachability can be obtained for an SWPN.

**Theorem 3.7.** Let \( M = (C, m^0) \) be a marked Petri net such that the Petri net \( C = (P, T, A) \) is an SWPN. A marking \( m \) is reachable from \( m^0 \) if and only if \( M \) satisfies the following:

(i) There exists a vector \( x \) in \( N^T \) such that \( m - m^0 = B x \).
(ii) \( M|x \) has no TFDCs in \( m^0 \).

**Proof.** “Only if” part: It is clear by Lemmas 2.1 and 2.4.

“If” part: Since \( M \) satisfies Condition 3.1, we can obtain a firing sequence \( \sigma \) such that \( m^0[\sigma]m \) by the following procedure:

1. Set \( z := \lambda \).
2. \( M|z - \psi(z) \) has no token-free source places by Lemma 2.3, and has no TFDCs by Condition 3.1. Therefore, \( M|z - \psi(z) \) has at least one firable transition by Lemma 2.2. Choose a firable transition \( t \) and fire it.
3. Set \( z := zt \) and \( x := x - \psi(t) \). If \( x = 0 \), then \( \sigma := z \). Otherwise, go to (2). \( \square \)
4. Semilinearity of the reachability set

The reachability set of a weakly persistent marked Petri net is effectively computable semilinear [6]. In this section we consider structural conditions for a Petri net to have a semilinear reachability set for every initial marking. We first define the notion of linear sets and semilinear sets. For a positive integer \( r \), \( \mathbb{N}^r \) denotes the set of all \( r \)-dimensional vectors of nonnegative integers. For given subsets \( V, W \) of \( \mathbb{N}^r \), \( L(V, W) \) denotes the set of all \( x \) in \( \mathbb{N}^r \) which can be represented in the form

\[
x = x_0 + x_1 + \cdots + x_n
\]

with \( x_0 \) in \( V \) and \( x_1, \ldots, x_n \) a (possibly empty) finite sequence of elements of \( W \).

A subset \( L \) of \( \mathbb{N}^r \) is said to be linear if there exists an element \( r \) in \( \mathbb{N}^r \) and a finite subset \( W \) of \( \mathbb{N}^r \) such that \( L = L([r], W) \). A subset of \( \mathbb{N}^r \) is said to be semilinear if it is a finite union of linear sets.

**Lemma 4.1.** Let \( M \) be a marked Petri net. If \( ER(M) \) is semilinear, then \( R(M) \) is semilinear.

**Proof.** It is immediately obtained by the fact that a projection of a semilinear set is also semilinear.

The converse of Lemma 4.1 is not always true. The notion of sinklessness is proposed as a sufficient condition for the reachability set to be semilinear [7]. A circuit \( u \) in a marked Petri net \( M = (C, m^0) = (P, T, A, m^0) \) is said to be sinkless if the total number of tokens in \( u \) is not decreased to 0 by firing transitions, i.e. \( m[P_u \neq 0] \) implies \( m + \sigma[P_u \neq 0] \) for every marking \( m \) in \( R(M) \) and every firing sequence \( \sigma \) in \( \Sigma(C, m) \). A marked Petri net \( M = (C, m^0) \) is said to be sinkless if every minimal circuit in \( M \) is sinkless.

Let \( F(C, m) \) denote the set of all token-free minimal circuits in a Petri net \( C \) with a marking \( m \). Then a marked Petri net \( M = (C, m^0) \) is sinkless if \( F(C, m) \supseteq F(C, m + \sigma) \) for every marking \( m \) in \( R(M) \) and every firing sequence \( \sigma \) in \( \Sigma(C, m) \).

**Theorem 4.2.** (Yamasaki [7]). If a marked Petri net \( M \) is sinkless, then we can effectively compute the reachability set in the form of a semilinear set.

A Petri net is normal if each minimal circuit is a trap circuit [7]. Normality is the structural property for sinklessness.

**Theorem 4.3.** (Yamasaki [7]). A Petri net \( C \) is normal if and only if the marked Petri net \( M = (C, m^0) \) is sinkless for every initial marking \( m^0 \).
Theorem 4.4. Every SWPN is normal.

Proof. Let $C = (P, T, A)$ be an SWPN. Since $C$ satisfies Condition 3.2, every circuit in $C$ contains a trap circuit. This implies that each minimal circuit in $C$ is a trap circuit.

Lemma 4.5. Let $M = (C, m') = (P, T, A, m')$ be a marked Petri net. Let $\alpha$ and $\beta$ be firing sequences in $T^*$ such that $m'[\alpha] \land m'[\beta] \land \psi(\alpha) \leq \psi(\beta)$. If $M|\psi(\beta) - \psi(\alpha)$ is sinkless and $F(C, m' + \alpha) = F(C, m' + \beta)$, then there exists a firing sequence $\gamma$ such that $m'[\gamma] \land \psi(\gamma) = \psi(\beta)$.

Proof. Set $x = \psi(\beta) - \psi(\alpha)$. If $x = 0$, then $\gamma = \lambda$. We consider the case $x \geq 0$. We first show that $M|x$ has a firable transition in $m' + x$. Assuming the contrary, we can show that $M|x$ has a TFDC $u$ in $m' + x$ by Lemmas 2.2 and 2.3. There exists a minimal circuit $u'$ in $M|x$ such that $P_u \leftarrow P_{u'}$. It is necessary for $u'$ to have tokens in firings of a transition which has an input place in $u'$. Since $M|x$ is sinkless, $u'$ still has tokens in $m' + x$. This contradicts $F(C, m' + \alpha) = F(C, m' + \beta)$. Choosing a firable transition $t$, similar arguments are applicable for $x_t$ and $\beta$. By repeating this process, we can obtain a firing sequence $\gamma$ such that $m'[\gamma] \land \psi(\gamma) = \psi(\beta)$.

For a Petri net $C = (P, T, A)$, the set $\{x \mid x \in N^T \land x \neq 0 \land B_x \geq 0\}$ becomes a semigroup and its generator $\Gamma_x$ is a finite set, because $\Gamma_x$ is a projection of $\text{Min}\{\langle B_x, x, x \rangle \mid x \in N^T \land x \neq 0 \land B_x \geq 0\}$, where $\text{Min}$ denotes the set of all minimal elements in a subset $V$ of $N^T$. We define the subset $T_x$ of $T$ by $T_x = \{t \mid \exists x \in \Gamma_x : t \in T|x\}$.

In the next theorem, we construct the extended reachability set of a sinkless marked Petri net.

Theorem 4.6. Let $M = (C, m') = (P, T, A, m')$ be a marked Petri net. If $M|T_x$ is sinkless, then $ER(M)$ is semilinear.

Proof. Since $F = \{F(C, m)\mid m \in R(M)\}$ is finite, we set $F = \{F^1, \ldots, F^n\}$. Let

$$A'(m) = \text{Min}\{\langle \sigma, \psi(\sigma) \rangle \mid m[\sigma]^{m'} \land \sigma \geq 0 \land \sigma \neq \lambda \land F(C, m') = F^i\},$$

$$A' = \{A'(m) \mid m \in R(M) \land F(C, m) = F^i\},$$

$$\bigcup_{m \in R(M)} A'(m)$$

is finite because it is a projection of

$$\text{Min}\{\langle \sigma, \psi(\sigma), m \rangle \mid m \in R(M) \land m[\sigma]^{m'} \land \sigma \geq 0 \land \sigma \neq \lambda \land F(C, m') = F^i\}.$$

Therefore, $A'$ is finite and we set $A' = \{A'^1, \ldots, A'^n\}$. Let

$$ER^{ij} = \{\langle m^0 + \sigma, \psi(\sigma) \rangle \mid \sigma \in \Sigma(M) \land F(C, m^0 + \sigma) = F^i \land A'(m^0 + \sigma) = A'^j\}.$$
Since $M \mid T_r$ is sinkless, $\mathcal{E}(M) \subseteq L(\text{Min } E^{(i)})$ holds by Lemma 4.5. Then we have

$$\mathcal{E}(M) = \bigcup_{F \in F} \bigcup_{\psi \in \psi} E^{(i)}$$

$$\subseteq \bigcup_{F \in F} \bigcup_{\psi \in \psi} L(\text{Min } E^{(i)})$$

$$\subseteq \mathcal{E}(M).$$

Hence, $\mathcal{E}(M)$ is semilinear.

Now we show that the reachability set of a marked Petri net is semilinear if the number of times that sinklessness is violated is finite during each firing of transitions. Let $M = (C, m^0) = (P, T, A, m^0)$ be a marked Petri net. We define the decomposition of a firing sequence $\sigma$ in $\Sigma(M)$, which decomposes $\sigma$ into "sinkless" subsequences.

**procedure** \texttt{decompose}(C, $m^0$, $\sigma$, $\delta$)\r
\texttt{\{Input: C = (P, T, A), m^0 and $\sigma$: Output: $\delta$\}}\r
\begin{algorithmic}
\STATE \textbf{begin}
\STATE \textbf{if} $\sigma = \lambda$ \textbf{then} $\delta := \lambda$, \textbf{else} \r
\STATE \textbf{begin}
\STATE $n :=$ the length of $\sigma$;
\STATE $U :=$ the set of all $k ~(1 \leq k \leq n)$ s.t. $(C, m^0)/(T_+ | \psi(\sigma_{-1}, \lambda))$ is sinkless;
\STATE \textbf{if} $U = \emptyset$ \textbf{then} $r := 0$, $\delta_0 := \lambda$, \textbf{end};
\STATE \textbf{else} \textbf{begin} $r := \max k$ s.t. $k \in U$; $\delta_0 := \sigma_{-1}$, $\delta := \lambda$, \textbf{end};
\STATE \textbf{if} $r = n$ \textbf{then} $\delta := \delta_0$ \textbf{else} \r
\STATE \textbf{if} $r = n - 1$ \textbf{then} $\delta := \delta_0, \sigma_{r+1}, \lambda$, \textbf{else} \r
\STATE \textbf{begin}
\STATE $\text{decompose}(C, m^0 + \sigma^1_{-1}, \ldots, \sigma_{r}, \lambda_1)$;
\STATE $\delta := \delta_0, \sigma_{r+1} \ldots \delta_1$;
\STATE \textbf{end};
\STATE \textbf{end};
\STATE \textbf{end};
\STATE \textbf{end}
\end{algorithmic}

For a firing sequence $\sigma$ in $\Sigma(M)$, $\hat{\sigma}$ denotes the sequence $\delta$ obtained by the execution of $\text{decompose}(C, m^0, \sigma, \delta)$. It is a sequence of $(\mathcal{T}^*)^*$ and has the form

$$\hat{\sigma} = \sigma_0, s_1, \sigma_1, s_2, \sigma_2, \ldots,$$

where $\sigma_0,\sigma_i \in \mathcal{T}^*$ and $s_i \in T$ ($i = 1, 2, \ldots$). Let $\sigma^i$ denote the sequence $s_1 s_2 s_3 \ldots$ and let $\hat{\sigma}[i]$ denote the sequence $\sigma_0 \sigma_1 \sigma_2 s_2 \ldots s_i \sigma_i$.

**Lemma 4.7.** Let $M = (C, m^0) = (P, T, A, m^0)$ be a marked Petri net. Let $\alpha$ and $\beta$ be sequences in $\Sigma(M)$ such that

1. $\alpha = \beta = s_1 s_2 \ldots s_r$;
2. $\psi(\alpha) \subseteq \psi(\beta)$ for $i = 0, 1, \ldots, r$, where $\alpha = \alpha_0, \alpha_1, s_1, \alpha_2, \ldots, s_r, \alpha_r$ and $\beta = \beta_0, s_1, \beta_1, s_2, \ldots, s_r, \beta_r$. 

Then

$$\mathcal{E}(M) \subseteq L(\text{Min } E^{(i)}).$$

Hence, $\mathcal{E}(M)$ is semilinear.
Then there exists a firing sequence \( \gamma \) in \( \Sigma(M) \) that satisfies the following conditions:

(iii) \( \beta[i] \leq \bar{\beta}[i] \) for \( i = 0, 1, \ldots, r \);

(iv) \( F(C, m^0 + \bar{\beta}[i]) = F(C, m^0 + \bar{\beta}[i]) \) for \( i = 0, 1, \ldots, r \).

Proof. We prove the lemma by induction on the length of \( \alpha \). It is obvious in the case \( r = 0 \), i.e. \( \bar{\alpha} = \alpha \), by Lemma 4.5. Assume that the lemma holds for \( r < k \) and consider the case \( r = k + 1 \). Then there exists a firing sequence \( \gamma' \) in \( \Sigma(M) \) such that 

\[
\gamma' = z_0 \delta_0, s_1, \delta_1, s_2, \ldots s_r, \delta_r \wedge \psi(z_0 \delta_r) = \psi(\beta) \quad \text{for} \quad i = 0, 1, \ldots, k.
\]

Let \( m = m^0 + \gamma' \delta_{k+1} \). Since \( \bar{\gamma}[k] = \bar{\beta}[k] \), we have 

\[
m[z_{k+1}] \wedge m[\beta_{k+1}] \wedge \psi(z_{k+1}) = \psi(\beta_{k+1}) \quad \text{and} \quad F(C, m + z_{k+1}) = F(C, m + \beta_{k+1}).
\]

Since \( \beta - x \geq 0 \), it follows that 

\[
T | \psi(\beta_{k+1}) - \psi(z_{k+1}) | T_+.
\]

Therefore, \( (C, m)| \psi(\beta_{k+1}) - \psi(z_{k+1}) \) is sinkless from the decomposition procedure. By Lemma 4.5, there exists a firing sequence \( \delta_{k+1} \) such that 

\[
m[z_{k+1}] \wedge \psi(z_{k+1}) = \psi(\beta_{k+1}).
\]

Let \( M \) be a marked Petri net, let \( \sigma \) be a firing sequence in \( \Sigma(M) \) and let 

\[
\delta = \sigma_0, s_1, \sigma_1, s_2, \ldots s_r, \sigma_r.
\]

For a firing sequence \( \gamma \) in \( \Sigma(M) \) such that \( \gamma = \sigma \), we write \( \sigma \uparrow \gamma \) if

(i) \( \gamma \neq \sigma \);

(ii) \( \gamma = \sigma_0, s_1, \sigma_1, s_2, \ldots s_r, \sigma_r \); 

(iii) \( \sum_{j=0}^{r} \delta_j \geq 0 \) for \( i = 0, 1, \ldots, r \).

Lemma 4.8. Let \( M \) be a marked Petri net. Let \( \sigma, \alpha \) and \( \beta \) be firing sequences in \( \Sigma(M) \) such that \( \sigma \uparrow \alpha = \beta \). If \( \sigma \uparrow \alpha \) and \( \sigma \uparrow \beta \), then there exists a firing sequence \( \gamma \) in \( \Sigma(M) \) such that 

\[
\psi(\gamma) = (\psi(\alpha) - \psi(\sigma)) + (\psi(\beta) - \psi(\sigma)) + \psi(\sigma).
\]

Proof. Let \( \bar{\alpha} = \sigma_0, s_1, \sigma_1, s_2, \ldots s_r, \sigma_r \) and \( \bar{\beta} = \sigma_0, \beta_0, s_1, \sigma_1, \beta_1, s_2, \ldots s_r, \sigma_r, \beta_r \). Then \( \gamma = \sigma_0, \beta_0, s_1, \sigma_1, \beta_1, s_2, \ldots s_r, \sigma_r, \beta_r \) is in \( \Sigma(M) \).

Lemma 4.9. Let \( M = (C, m^0) \) be a marked Petri net. If the length of \( \sigma \) is finite for every firing sequence \( \sigma \) in \( \Sigma(M) \), then \( ER(M) \) is semilinear.

Proof. Since \( \Sigma = \{ \sigma | \sigma \in \Sigma(M) \} \) is finite, we set \( \Sigma = \{ u^1, \ldots, u^n \} \). For each \( u^i = s_1 \ldots s_r \), let \( \Sigma^i = \{ \sigma | \sigma \in \Sigma(M) \land \sigma = u^i \} \). For each firing sequence \( \sigma \) in \( \Sigma^i \), let

\[
F^i(\sigma) = \langle f(m^0 + \bar{\delta}[0]), \ldots, f(m^0 + \bar{\delta}[r]) \rangle,
\]

where \( f(m) \) denotes the vector whose dimension is equal to the number of minimal circuits in \( M \), and each element of \( f(m) \) is 0 if the corresponding minimal circuit is token-free in \( m \), otherwise 1. Since \( F^i = \{ F^i(\sigma) | \sigma \in \Sigma^i \} \) is finite, we set
$F^i = F^{i_1}, \ldots, F^{i_n}$. Let $\hat{\sigma} = \sigma_0, \sigma_1, \sigma_2, \ldots, \sigma_r$. Then we define the following:

$$
\mu^i(\sigma) = \langle m^\sigma + \hat{\sigma}, \psi(\sigma) \rangle \in \Sigma^i \land F^i(\sigma) = F^{i_i}.
$$

$$
Z^i(\sigma) = \langle m^\sigma, \psi(\sigma) \rangle \in \Sigma^i \land F^i(\sigma) = F^{i_i}.
$$

Then we define the following:

$$
\mu^i(\sigma) = \langle m^\sigma + \hat{\sigma}, \psi(\sigma) \rangle \in \Sigma^i \land F^i(\sigma) = F^{i_i}.
$$

$$
Z^i(\sigma) = \langle m^\sigma, \psi(\sigma) \rangle \in \Sigma^i \land F^i(\sigma) = F^{i_i}.
$$

Therefore, $Z^i$ is finite and we set $Z^{ij} = \langle Z^{ij_1}, \ldots, Z^{ij_n} \rangle$. Let

$$
\mu^{ij} = \langle \mu^{ij}(\sigma) \rangle \in \Sigma^i \land F^i(\sigma) = F^{ij} \land Z^i(\sigma) = Z^{ij}.
$$

It follows that $\mu^{ij} \subseteq L(M, \mu^{ij}, Z^{ij})$ by Lemma 4.7. Let

$$
ER^{ij} = \langle \psi(\sigma) \rangle \in \Sigma^i \land F^i(\sigma) = F^{ij} \land Z^i(\sigma) = Z^{ij}.
$$

$$
A^{ij} = \langle \psi(\sigma) \rangle \in \Sigma^i \land F^i(\sigma) = F^{ij} \land Z^i(\sigma) = Z^{ij}.
$$

then we have $ER^{ij} \subseteq L(M, ER^{ij}, A^{ij})$. By Lemma 4.8, it follows that $L(M, ER^{ij}, A^{ij}) \subseteq ER(M)$. Therefore,

$$
ER(M) = \bigcup_{\omega \in \Sigma} \bigcup_{F^i, Z^i \in C} ER^{ij}
$$

$$
\subseteq \bigcup_{\omega \in \Sigma} \bigcup_{F^i, Z^i \in C} L(M, ER^{ij}, A^{ij})
$$

$$
\subseteq ER(M).
$$

Hence, $ER(M)$ is semilinear.

A marked Petri net $M$ is called finally sinkless if the following holds for every firing sequence $\sigma$ in $\Sigma(M)$ and every minimal circuit $u$ in $M$: the number of times that $u$ becomes token-free during the firing of $\sigma$ is finite.

**Theorem 4.10.** Let $C = (P, T, A)$ be a Petri net. The marked Petri net $M = (C, m^0)$ is finally sinkless for every initial marking $m^0$ if and only if $C$ satisfies Condition 4.11.

**Condition 4.11.** Let $C = (P, T, A)$ be a Petri net. The following holds for every minimal circuit $u$ in $C$ and every $x$ in $T$: if $B_x \cap P_u = \emptyset$, then $T_{x \cap \tau^- (P_u)} = \emptyset$.

A Petri net that satisfies Condition 4.11 is called a structurally finally sinkless Petri net (SFSN). To prove the theorem, we show several lemmas. Lemma 4.12 is immediately obtained by the definition of minimal circuits.
Lemma 4.12. Let $C = (P, T, A)$ be a Petri net and let $u = p_1 t_1 ... p_n t_n p_1$ be a minimal circuit in $C$. If $A(p_i, t) = A(t, p_j) = 1$ ($1 \leq i, j \leq n$), then $j = i + 1$.

By Lemma 4.12, the number of tokens in a minimal circuit $u$ increases only by firings of $\tau^+(P_u)$, and decreases only by firings of $\tau^-(P_u)$. This implies that the following condition is equivalent to Condition 4.11.

Condition 4.13. Let $C = (P, T, A)$ be a Petri net. The following holds for every minimal circuit $u$ in $C$ and every $x$ in $\Gamma$: if $B_u x | P_u = 0$, then $T | x \cap \tau^- (P_u) = \emptyset$.

Lemma 4.14 (Peterson [4]). Every infinite sequence of nonnegative integer vectors contains an infinite nondecreasing subsequence.

Lemma 4.15. Let $C = (P, T, A)$ be a Petri net and let $u$ be a minimal circuit in $C$. If $B_u x | P_u = 0$ and $T | x \cap \tau^+(P_u) \neq \emptyset$ for a vector $x$ in $N^T$, then there exists a marking $m$ and a firing sequence $\sigma$ such that $m[\sigma] \wedge m | P_u = 0 \wedge \psi(\sigma) = x$.

Proof. Firability of transitions is considered in $C | P_u$ because we can set enough tokens in each place other than $P_u$. Let $x^+$ be the maximum vector in $N^T$ such that $x^+ = x \wedge T | x^+ = \tau^+(P_u)$; let $x^-$ be the maximum vector in $N^T$ such that $x^- = x \wedge T | x^- = \tau^-(P_u)$. Let $m$ be a marking such that $m | P_u = 0$, and let $m_1 = m + B_u x^+$, $m_2 = m_1 + B_u y$ and $m_3 = m_2 + B_u x^-$. It follows that $m_1 | P_u = 0$, $m_2 | P_u = 0$ and $m_3 | P_u = 0$. Obviously, $m(> m_1)$ and $m(> m_2)$. We show $m_1(>) m_2$. Since $u$ is minimal, $P_u = P_u$ holds for every circuit $u'$ in $C | P_u$. Therefore, every circuit in $(C | y) | P_u$ is a trap and deadlock circuit, i.e. $(C | y) | P_u$ is an SWPN. Since $x^+ \neq 0$, $P_u$ has some tokens in $m_1$. This implies that $(C | y) | P_u$ has no TFDCs in $m_1$. Hence, $m_1(>) m_2$ by Theorem 3.7

Proof of Theorem 4.10. “If” part: Assume that $M$ is not finally sinkless. Then there exists a minimal circuit $u$ and an infinite firing sequence $\sigma_0 \sigma_1 \sigma_2 ...$ in $\Sigma(M)$ such that

(i) $m_i | P_u = 0$ for $i = 0, 1, ...$, where $m_i = m^{0} + \sum_{i=0}^{\infty} \sigma_i$;

(ii) $T | \psi(\sigma_i) \wedge \tau^+(P_u) \neq \emptyset$ for $i = 1, 2, ...$

The infinite sequence $m_0, m_1, m_2 ...$ contains an infinite nondecreasing subsequence by Lemma 4.14. Therefore, there exist $i$ and $j$ ($i < j$) such that $\delta \geq 0 \wedge \delta | P_u = 0 \wedge T | \psi(\delta) \wedge \tau^+(P_u) \neq \emptyset$ for $\delta = \sigma_i ... \sigma_j$. This contradicts Condition 4.11.

“Only if” part: Assume that there exists a vector $x$ in $\Gamma$, that does not satisfy Condition 4.11 for a minimal circuit $u$. By Lemma 4.15, there exists a marking $m$ and a firing sequence $\sigma$ such that $m[\sigma] \wedge m | P_u = 0 \wedge \psi(\sigma) = x$. Since $(\sigma)^e \in \Sigma(M)$ holds for every positive integer $n$, the marked Petri net $M = (C, m)$ is not finally sinkless.

Lemma 4.16. Let $M = (C, m^0) = (P, T, A, m^0)$ be a marked Petri net and let $\sigma$ be a firing sequence in $\Sigma(M)$. If the length of $\sigma$ is infinite, then there exists an infinite sequence
Proof. Let $\delta = \sigma_0, \sigma_1, \sigma_2, \ldots$ Existence of a tuple $(k_1, k_2, \ldots, u)$ satisfying (i) is immediately obtained from the decomposition procedure. Let $U$ be the set of all tuples satisfying (i). Assume that every tuple in $U$ does not satisfy (ii). Then there exists an infinite subsequence $\sigma' = \sigma_{r+1}, \sigma_{r+1}, \sigma_{r+2}, \ldots$ of $\sigma$ such that $T|\psi(\delta[k_1]) - \psi(\delta[k_1]) \cap \tau^- \!(\alpha) \neq \emptyset$ for $i = 1, 2, \ldots$

Lemma 4.17. Let $M = (C, m^0) = (P, T, A, m^0)$ be a marked Petri net. If $C \mid T_+$ is an SFSN, then the length of $\sigma$ is finite for every firing sequence $\sigma$ in $\Sigma(M)$.

Proof. Assume that the length of $\sigma$ is infinite for a firing sequence $\sigma$ in $\Sigma(M)$. Then there exists an infinite sequence $k_1, k_2, \ldots$ and a minimal circuit $u$ in $M \mid T_+$ that satisfy Lemma 4.16(i), (ii). There exists a firing sequence $\delta_i$ such that $m^0 + \delta[k_1] \delta_i \geq m_1 \wedge m_2 \mid \alpha = 0$ for $i = 1, 2, \ldots$. Let $c_i = \langle m^0 + \delta[k_1] \delta_i, \psi(\delta[k_1] \delta_i) \rangle$ for $i = 1, 2, \ldots$ By Lemma 4.14, the infinite sequence $c_1, c_2, \ldots$ contains an infinite nondecreasing subsequence, and so there exist $i$ and $j$ ($i > j$) such that $c_i \geq c_j$. It follows that $B, \emptyset \geq 0 \wedge B, \emptyset \mid \beta = 0 \wedge T \mid \chi \cap \tau^- \!(\alpha) \neq \emptyset$ for $\chi = \psi(\delta[k_1] \delta_i) - \psi(\delta[k_1] \delta_j)$. This contradicts Condition 4.13.

By Lemmas 4.1, 4.9 and 4.7, we have the following theorem.

Theorem 4.18. Let $C = (P, T, A)$ be a Petri net. If $C \mid T_+$ is an SFSN, then the reachability set of the marked Petri net $M = (C, m^0)$ is semilinear for every initial marking $m^0$.

5. Illustrative examples

Figure 1 shows an example of SWPN. Every circuit has a set of places that satisfies Condition 3.2. For example, the circuit $p_1l_2p_2l_3p_3l_4p_1$ has $S = \{p_1, l_3\}$. Figure 1.
Figure 2 shows an example of SFSN. The circuit $u=p_1t_1p_2t_2p_3$ is not sinkless in a marking $[0, 0, n, 1]$. But the number of times $u$ becomes token-free is bounded by $n$.

References


