Periodic solutions for some strongly nonlinear oscillations by He’s variational iteration method

Vasile Marinca*, Nicolae Herisanu

Politehnica University of Timisoara, Department of Mechanics and Vibration, Bd. Mihai Viteazu, 1, 300222 Timisoara, Romania

Received 28 September 2006; accepted 20 December 2006

Abstract

In this paper, we implement a new analytical technique, He’s variational iteration method, for some strongly nonlinear oscillations. A correction functional is constructed by using a general Lagrange multiplier, which can be identified via the variational theory. The obtained approximate solutions and periods are compared with exact or numerical results to verify the effectiveness and accuracy of the method.

© 2007 Elsevier Ltd. All rights reserved.

Keywords: Nonlinear oscillations; Variational iteration method; Periodic solution

1. Introduction

It is well known that the perturbation method is one of the commonly used quantitative methods for analysing nonlinear problems [1,2]. The perturbation method is valid, in principle, only for problems containing small parameters. Its basic idea is to transform, by means of small parameters, a nonlinear problem into an infinite number of linear subproblems, or a complicated linear problem into an infinite number of simpler ones. Therefore, the small parameter plays a very important role in the perturbation method. It determines not only the accuracy of the perturbation approximations, but also the validity of the perturbation method itself. However, in science and engineering, there exist many nonlinear problems which do not contain any small parameters, especially those with strong nonlinearity. Thus it is necessary to develop and improve some nonlinear analytical techniques which are independent of small parameters. There exists a wide body of literature dealing with the problem of approximate solutions to nonlinear equations with various different methodologies. Many different approaches have been proposed, such as the perturbation iteration method [3,4], the modified Lindstedt–Poincare method [5], the Adomian’s decomposition method [6], etc. The variational iteration method (VIM) was proposed by Ji-Huan He [7,8], and has been proved by many authors to be a powerful mathematical tool for various kinds of nonlinear problems. A variational principle for nano thin film lubrication was established by the semi-inverse method by He [9]. The variational iteration method is used for solving three types of nonlinear partial differential equations, such as the coupled Schrodinger-KdV, generalized

* Corresponding author. Tel.: +40 256 422364; fax: +40 256 422364.
E-mail addresses: marinca@mec.utt.ro (V. Marinca), herisanu@mec.utt.ro (N. Herisanu).

0898-1221/$ - see front matter © 2007 Elsevier Ltd. All rights reserved.
KdV and shallow water equations by Abdou and Soliman [10], for solving generalized Burger–Fisher and Burger equations by Moghimi and Hejazi [11], and for solving the linear Helmholtz partial differential equation by Momani and Abouad [12]. The amplitude and period of limit cycles for a modified Van der Pol oscillator are calculated by He’s variational method and the Krylov–Bogoliubov–Mitropolsky method by D’Acunto [13]. VIM is used to construct solitary solutions and compacton-like solutions for nonlinear dispersive equations by He and Wu [14]. The combination of a perturbation method, VIM, method of variation of constants and averaging method to establish an approximate solution of one degree of freedom weakly non-linear systems was proposed by Marinca [15]. Draganescu and Capalnasan [16] applied VIM to a non-linear inelastic model describing the acceleration of the relaxation process in the presence of the vibrations.

In this paper, the variational iteration method is applied to a general non-linear problem, which can be initially approximated with unknown constants. The iterative process is constructed by a general Lagrange multiplier, which can be identified optimally via variational theory. This method is effective and accurate for non-linear problems with approximations converging rapidly to accurate solutions.

2. The iteration procedure

We consider the following nonlinear equation:

$$\ddot{x} + \omega^2 x = f(\Omega t, x, \dot{x}) \quad (2.1)$$

where \(\omega\) and \(\Omega\) are positive constants; in general \(f\) is assumed to be nonlinear function of \(\Omega t, x\) and \(\dot{x}\), which may be expanded in a Fourier series, and \(\dot{x} = \frac{dx}{dt}\). We construct the following iteration formula [7,17]:

$$x_n(t) = x_{n-1}(t) + \int_0^t \lambda(\tau, t)[x''_{n-1}(\tau) + \omega^2 x_{n-1}(\tau) - f(\Omega \tau, \dot{x}_{n-1}(\tau), \ddot{x}_{n-1}(\tau))]d\tau \quad (2.2)$$

where \(\lambda(t, \tau)\) is called a general Lagrange multiplier, which can be identified optimally via variational theory; \(\tilde{x}_{n-1}\) is considered as a restricted variation, i.e. \(\delta \tilde{x}_{n-1} = \delta \ddot{x}_{n-1} = 0\) and \(\dot{'} = \frac{d}{d\tau}\).

The Lagrange multiplier can be readily identified:

$$\lambda(t, \tau) = \frac{1}{\omega} \sin \omega(\tau - t). \quad (2.3)$$

On the other hand, by taking into consideration the identity:

$$\int_0^t \sin \omega(\tau - t)[x''_{n-1}(\tau) + \omega^2 x_{n-1}(\tau)]d\tau = -\omega x_{n-1}(t) + \omega x_{n-1}(0) \cos \omega t + \dot{x}_{n-1}(0) \sin \omega t \quad (2.4)$$

it follows that:

$$x_n(t) = x_{n-1}(0) \cos \omega t + \frac{1}{\omega} \dot{x}_{n-1}(0) \sin \omega t + \frac{1}{\omega} \int_0^t \sin \omega(\tau - t) f(\Omega \tau, x_{n-1}(\tau), \dot{x}_{n-1}(\tau))d\tau. \quad (2.5)$$

The Eq. (2.1) may then be written as:

$$\ddot{x} + \Omega^2 x = F(t, \lambda, x, \dot{x}), \quad \lambda = \Omega^2 - \omega^2, \quad F(t, \lambda, x, \dot{x}) = \lambda x + f(\Omega t, x, \dot{x}) \quad (2.6)$$

with the initial conditions:

$$x(0) = A, \quad \dot{x}(0) = 0. \quad (2.7)$$

The left hand side of the Eq. (2.6) has the solution \(x = A \cos(\Omega t + \varphi)\), where \(A\) and \(\varphi\) are constants, and therefore we try the input of starting function as:

$$x_0 = A_0 \cos(\Omega t + \varphi_0). \quad (2.8)$$

According to Ref. [15], we propose the following iteration formula for Eq. (2.6):

$$x_n(t) = A_n \cos(\Omega t + \varphi_n) + \frac{1}{\Omega} \int_0^t \sin \Omega(\tau - t) F(\tau, \lambda, x_{n-1}(\tau), \dot{x}_{n-1}(\tau))d\tau, \quad n = 1, 2, \ldots \quad (2.9)$$

where \(A_n\) and \(\varphi_n\) are constants and \(x_0\) is given by Eq. (2.8).
Expanding $F(t, \lambda, x_{n-1}, x'_{n-1})$ in a Fourier series, we have:

$$F(t, \lambda, x_{n-1}(t), x'_{n-1}(t)) = \sum_{p=0}^{P} a_p^{n-1}(A_{n-1}, \varphi_{n-1}, \Omega, \lambda) \cos p\Omega t + \sum_{r=0}^{R} b_r^{n-1}(A_{n-1}, \varphi_{n-1}, \Omega, \lambda) \sin r\Omega t$$

(2.10)

and therefore, the approximation of $n$th order (2.9) becomes:

$$x_n(t) = A_n \cos(\Omega t + \varphi_n) + \frac{1}{\Omega} \left[ a_0^{n-1} + \frac{1}{2} a_1^{n-1} (t \sin \Omega t) + \frac{1}{2} b_1^{n-1} \left( t \cos \Omega t + \frac{1}{2\Omega} \sin \Omega t \right) \right] + \sum_{p=2}^{P} \frac{a_p^{n-1} (\cos \Omega t - \cos p\Omega t)}{(p^2 - 1)\Omega} + \sum_{r=2}^{R} \frac{b_r^{n-1} (r \sin \Omega t - \sin r\Omega t)}{(r^2 - 1)\Omega}. \tag{2.11}$$

The solution (2.11) is so chosen that it contains no secular terms, which requires that coefficients $a_1^{n-1}$ and $b_1^{n-1}$ in (2.11) disappear, i.e.:

$$a_1^{n-1}(A_{n-1}, \varphi_{n-1}, \Omega, \lambda) = 0, \quad b_1^{n-1}(A_{n-1}, \varphi_{n-1}, \Omega, \lambda) = 0. \tag{2.12}$$

For real systems, the expansion of the function $F(t, \lambda, x_{n-1}, x'_{n-1})$ usually contains only a small number of harmonics.

3. Examples

3.1

We consider the free oscillation of a nonlinear oscillator with quadratic and cubic nonlinearities [2]:

$$\ddot{x} + \omega^2 x + ax^2 + bx^3 = 0, \quad x(0) = A, \quad \dot{x}(0) = 0 \tag{3.1}$$

where $a$ and $b$ are constants.

By introducing the dependent variable change

$$x = u - \frac{a}{3b} \tag{3.2}$$

the quadratic non-linearity in Eq. (3.1) is eliminated to obtain:

$$\ddot{u} + \left( \omega^2 - \frac{a^2}{3b} \right) u + bu^3 - k = 0, \quad k = \frac{\omega^2 a}{3b} - \frac{2a^3}{27b^2}. \tag{3.3}$$

Substituting Eq. (2.6) into Eq. (3.3), we have:

$$\ddot{u} + \Omega^2 u = F(u) \tag{3.4}$$

where

$$F(u) = \lambda u - bu^3 + k, \quad \lambda = \Omega^2 - \omega^2 + \frac{a^2}{3b}. \tag{3.5}$$

The iteration formula for Eq. (3.4) becomes:

$$u_n(t) = B_n \cos(\Omega t + \varphi_n) + \frac{1}{\Omega} \int_0^1 \sin \omega(t - \tau) F(u_{n-1}(\tau))d\tau \tag{3.6}$$

with the input function as (from \(\dot{u}_0(0) = 0\))

$$u_0(t) = B \cos \Omega t \tag{3.7}$$
where $B$ is obtained from (3.1) and (3.2):

$$B = A + \frac{a}{3b}.$$  

(3.8)

Expanding $F(u_0)$ in a Fourier series, we have:

$$F(u_0) = B \left( \lambda - \frac{3}{4} b B^2 \right) \cos \Omega t - \frac{1}{4} b B^3 \cos 3\Omega t + k.$$  

(3.9)

In order to ensure that no secular terms appear in the next iteration, resonance must be avoided. To do so, the coefficient $a_0^1$ of the cos $\Omega t$ into Eq. (3.9) has to be zero, i.e.:

$$\lambda_1 = \frac{3}{4} b B^2$$  

(3.10)

such that $\Omega^2$ becomes from (3.5)_{2}:

$$\Omega^2 = \omega^2 + \frac{3}{4} b B^2 - \frac{a^2}{3b}.$$  

(3.11)

For $n = 1$, we obtain first-order approximate solution:

$$u_1(t) = B \cos \Omega t - \frac{b B^3}{32 \Omega^2} (\cos \Omega t - \cos 3\Omega t) + \frac{k}{\Omega^2} (1 - \cos \Omega t)$$

or

$$u_1(t) = \left( B - \frac{b B^3}{32 \Omega^2} - \frac{k}{\Omega^2} \right) \cos \Omega t + \frac{k}{\Omega^2} + \frac{b B^3}{32 \Omega^2} \cos 3\Omega t$$  

(3.12)

and therefore:

$$F(u_1) = \left( B - \frac{b B^3}{32 \Omega^2} - \frac{k}{\Omega^2} \right) \left[ \lambda - \frac{3}{4} b \left( B - \frac{b B^3}{32 \Omega^2} - \frac{k}{\Omega^2} \right)^2 - \frac{3b k^2}{\Omega^4} - \frac{3b^2 B^3}{128 \Omega^2} \left( B - \frac{b B^3}{32 \Omega^2} - \frac{k}{\Omega^2} \right)^2 - \frac{3b^3 B^6}{2048 \Omega^4} \right] \cos \Omega t + k + \frac{\lambda k}{\Omega^2} - \frac{b k^3}{\Omega^6} \\
- \frac{3kb}{2\Omega^2} \left( B - \frac{b B^3}{32 \Omega^2} - \frac{k}{\Omega^2} \right)^2 - \frac{3kb^3 B^6}{2048 \Omega^6} - \left[ \frac{3kb}{2\Omega^2} \left( B - \frac{b B^3}{32 \Omega^2} - \frac{k}{\Omega^2} \right)^2 \right. \\
+ \frac{3kb^2 B^3}{32 \Omega^4} \left( B - \frac{b B^3}{32 \Omega^2} - \frac{k}{\Omega^2} \right) \cos 2\Omega t + \left[ \frac{\lambda b B^3}{32 \Omega^2} - \frac{b}{4} \left( B - \frac{b B^3}{32 \Omega^2} - \frac{k}{\Omega^2} \right)^3 \\
- \frac{3b^4 B^9}{131072 \Omega^6} - \frac{3b^2 B^3}{64 \Omega^2} \left( B - \frac{b B^3}{32 \Omega^2} - \frac{k}{\Omega^2} \right)^2 - \frac{3k^2 b^2 A^3}{96 \Omega^8} \right] \cos 3\Omega t \\
- \frac{3kb^3 B^6}{32 \Omega^4} \left( B - \frac{b B^3}{32 \Omega^2} - \frac{k}{\Omega^2} \right) \cos 4\Omega t \\
- \frac{3b^2 B^3}{128 \Omega^2} \left( B - \frac{b B^3}{32 \Omega^2} - \frac{k}{\Omega^2} \right)^2 + \frac{3b^3 B^6}{4096 \Omega^4} \left( B - \frac{b B^3}{32 \Omega^2} - \frac{k}{\Omega^2} \right) \cos 5\Omega t \\
- \frac{3kb^3 B^6}{2048 \Omega^6} \cos 6\Omega t - \frac{3b^3 B^6}{4096 \Omega^4} \left( B - \frac{b B^3}{32 \Omega^2} - \frac{k}{\Omega^2} \right) \cos 7\Omega t - \frac{b^4 B^9}{131072 \Omega^6} \cos 9\Omega t.$$  

(3.13)

Avoiding the presence of a secular term needs:

$$\lambda_2 = \frac{3b}{4} \left( B - \frac{b B^3}{32 \Omega^2} - \frac{k}{\Omega^2} \right)^2 + \frac{3b k^2}{\Omega^4} + \frac{3b^2 B^3}{128 \Omega^2} \left( B - \frac{b B^3}{32 \Omega^2} - \frac{k}{\Omega^2} \right) + \frac{3b^3 B^6}{2048 \Omega^4}. $$  

(3.14)
Substituting Eq. (3.14) into Eq. (3.5)_2, we obtain the equation for \( \Omega \):

\[
\Omega^2 - \omega^2 + \frac{a^2}{3b} - \frac{3bB^2}{4} + \frac{3bB(64k^2 + bB^2)}{128\Omega^2} - \frac{3b(2560k^2 + 16kbB^3 + b^2B^6)}{2048\Omega^4} = 0.
\]

(3.15)

3.2

As the last example, let us consider a family of nonlinear differential equations

\[
\ddot{x} + \alpha x + \gamma x^{2n+1} = 0, \quad \alpha \geq 0, \gamma > 0, n = 1, 2, 3, \ldots
\]

(3.16)

with the initial conditions

\[
x(0) = A, \quad \dot{x}(0) = 0.
\]

(3.17)

The corresponding exact period \( T \) is

\[
T_{ex} = 4 \int_0^{\pi/2} \frac{d\theta}{\sqrt{\alpha + \frac{\gamma}{n+1} A^{2n}(1 + \sin^2 \theta + \sin^4 \theta + \cdots + \sin^{2n} \theta)}}.
\]

(3.18)

The Eq. (3.16) can be written as:

\[
\ddot{x} + \Omega^2 x = F(x)
\]

(3.19)

where

\[
F(x) = (\Omega^2 - \alpha)x - \gamma x^{2n+1}.
\]

(3.20)

With the initial conditions (3.17), the input function is

\[
x_0(t) = A \cos \Omega t.
\]

(3.21)

Expanding \( F(x_0) \) in a Fourier series, we have

\[
F(x_0) = A \cos \Omega t [\Omega^2 - \alpha - \gamma A^{2n} \cos^{2n+1} \Omega t].
\]

(3.22)

By using the identity [18]

\[
\cos^{2n+1} \Omega t = \frac{1}{4^n} \sum_{k=0}^{n} \binom{2n+1}{n-k} \cos(2k+1)\Omega t
\]

(3.23)

where

\[
\binom{n}{p} = \frac{n!}{p!(n-p)!}; \quad \binom{n}{0} = 1.
\]

(3.24)

Eq. (3.22) becomes:

\[
F(x_0) = A \left[\Omega^2 - \alpha - \frac{\gamma A^{2n}}{4^n} \binom{2n+1}{n} \right] \cos \Omega t - \frac{\gamma A^{2n+1}}{4^n} \sum_{k=1}^{n} \binom{2n+1}{n-k} \cos(2k+1)\Omega t.
\]

(3.25)

Avoiding the secular term requires that

\[
\Omega_1(n) = \sqrt{\alpha + \frac{\gamma A^{2n}}{4^n} \binom{2n+1}{n}}.
\]

(3.26)

The first-order approximate solution (2.11) of Eq. (3.19) is

\[
x_1(t) = A \cos \Omega_1 t - \frac{\gamma A^{2n+1}}{4^n} \sum_{k=1}^{n} \frac{1}{k(k+1)} \binom{2n+1}{n-k} \left[ \cos \Omega_1 t - \cos(2k+1)\Omega_1 t \right]
\]

(3.27)

with \( \Omega_1 \) is given by Eq. (3.26).
For an integer \( n \), the approximate period in the first-order approximation is
\[
T_{\text{approx}}(n) = \frac{2\pi}{\Omega_1(n)} = 2\pi \left[ \alpha + \frac{\gamma A^{2n}}{4n} \left( \frac{2n + 1}{n} \right) \right]^{-\frac{1}{2}}. \tag{3.28}
\]

Formula (3.28) is valid for any possible amplitude and gives the maximum errors as the dimensionless amplitude \( \gamma A^{2n} \) tends to infinity. Note that even for \( n = 9 \), the maximum error given by Eq. (3.28) is less than four percent for an amplitude \( A \in [0, \infty) \).

\[
\lim_{\gamma A^{18} \to \infty} \frac{T_{\text{approx}}(9)}{\text{Tex}} \approx \frac{16\pi \sqrt{\frac{5}{36189}}}{0.5406369} \approx 0.967341. \tag{3.29}
\]

The same result is obtained in [19] with the modified iteration perturbation method. For \( n = 1 \), we recover the well-known Duffing equation (with \( \alpha = 1, \gamma = \varepsilon \)), and we obtain:
\[
T_{\text{approx}}(1) = \frac{2\pi}{\sqrt{1 + \frac{3}{4} \varepsilon A^2}}. \tag{3.30}
\]

In case \( \varepsilon A^2 \to \infty \), we have
\[
\lim_{\varepsilon A^2 \to \infty} \frac{T_{\text{approx}}(1)}{\text{Tex}} = 1.02223. \tag{3.31}
\]

Therefore, the approximate analytical solution of the frequency of the Duffing equation with respect to the exact solution never exceeds an error of 2.2%.

4. Numerical examples

In order to illustrate the remarkable accuracy of this method, we compare the approximate results given by means of (3.15) with numerical integration results for the following cases:

4.1

For the equation
\[
\ddot{x} + x + 4x^2 + 50x^3 = 0, \quad x(0) = 2, \quad \dot{x}(0) = 0. \tag{4.1}
\]

In this case \( a = 4, b = 50, \omega = 1, A = 2 \) and from Eqs. (3.3) and (3.8) we obtain \( k = 0.02477037, B = 2.0266667, \) and from Eq. (3.15) we have
\[
\Omega = 12.19942839. \tag{4.2}
\]

Therefore, from Eq. (3.13) we obtain
\[
F(u_1) = 0.002608936 - 0.05090743 \cos 2\Omega t - 102.004667 \cos 3\Omega t - 0.004219655 \cos 4\Omega t
- 12.81158791 \cos 5\Omega t - 0.000953433 \cos 6\Omega t - 0.553925714 \cos 7\Omega t - 0.008343994 \cos 9\Omega t.
\tag{4.3}
\]

Setting \( n = 2 \) in Eq. (3.6), we obtain the second-order approximate solution:
\[
u_2(t) = 2.02666667 \cos \Omega t + 0.0001753(1 - \cos \Omega t) - 0.00114019(\cos \Omega t - \cos 2\Omega t)
- 0.085674386(\cos \Omega t - \cos 3\Omega t) - 0.003586846(\cos \Omega t - \cos 5\Omega t)
- 0.00007754(\cos \Omega t - \cos 7\Omega t). \tag{4.4}
\]
From Eqs. (4.4) and (3.2) we obtain the following result:

\[
x(t) = -0.026649136 + 1.937195679 \cos \Omega t + 0.000114019 \cos 2\Omega t \\
+ 0.085694386 \cos 3\Omega t + 0.003586846 \cos 5\Omega t + 0.00007754 \cos 7\Omega t
\]  
(4.5)

where \( \Omega \) is given by Eq. (4.2).

Fig. 1 shows the comparison between the present solution and the numerical integration results obtained by a fourth order Runge–Kutta method.

It can be seen from Fig. 1 that the solution obtained by the present method is nearly identical with that given by the numerical method.

4.2

For the equation

\[
\ddot{x} + x + 6x^2 + 8x^3 = 0, \quad x(0) = 2, \quad \dot{x}(0) = 0, \quad (a = 6, b = 8, \omega = 1, A = 2).
\]  
(4.6)

It is important that \( k = 0 \). It follows that \( B = 2.25 \) and for the input function:

\[
u_0 = B \cos \Omega t
\]  
(4.7)

we have:

\[
F(u_0) = \lambda u_0 - bu_0^3 = B \left( \lambda - \frac{3bB^2}{4} \right) \cos \Omega t - \frac{bB^3}{4} \cos 3\Omega t.
\]  
(4.8)

In order to avoid the secular term, we set:

\[
\lambda_1 = \frac{3}{4}bB^2, \quad \Omega^2 = -\frac{1}{2} + \frac{3bB^2}{4}.
\]  
(4.9)

The first-order approximate solution is given by the relation:

\[
u_1(t) = B \left( 1 - \frac{bB^2}{32\Omega^2} \right) \cos \Omega t + \frac{bB^3}{32\Omega^2} \cos 3\Omega t
\]  
(4.10)

and therefore

\[
F(u_1) = B \left( 1 - \frac{bB^2}{32\Omega^2} \right) \left[ \lambda - \frac{3bB^2}{4} \left( 1 - \frac{bB^2}{32\Omega^2} \right) + \frac{bB^2}{128\Omega^2} \left( 1 - \frac{bB^2}{32\Omega^2} \right) - \frac{3b^3B^6}{2048\Omega^4} \right] \cos \Omega t.
\]
The requirement of no secular term requires that

\[ \lambda = \frac{3}{4} b B^2 \left( 1 - \frac{b B^2}{32 \Omega^2} \right) + \frac{3 b^3 B^6}{2048 \Omega^4}. \]  

(4.12)

From Eqs. (3.5) and (4.12), we obtain the equation for \( \Omega \):

\[ \Omega^2 + \frac{1}{2} - \frac{3}{4} b B^2 + \frac{3 b^2 B^4}{128 \Omega^2} - \frac{3 b^3 B^6}{2048 \Omega^4} = 0. \]  

(4.13)

Substituting Eq. (4.12) into (4.11), we have:

\[
F(u_1) = -\frac{b B^3}{4} \left( 1 + \frac{3 b B^2}{16 \Omega^2} - \frac{3 b^2 B^4}{256 \Omega^4} + \frac{7 b^3 B^6}{1638 \Omega^6} \right) \cos 3 \Omega t + \frac{3 b^2 B^5}{128 \Omega^2} \left( 1 - \frac{b^2 B^5}{128 \Omega^2} \right) \cos 5 \Omega t
- \frac{3 b^3 B^7}{4096 \Omega^4} \left( 1 - \frac{b B^2}{32 \Omega^2} \right) \cos 7 \Omega t - \frac{b^4 B^9}{131072 \Omega^6} \cos 9 \Omega t. \]  

(4.14)

From Eq. (4.13) we obtain

\[ \Omega = 5.462517735 \]  

(4.15)

and from Eqs. (4.14) and (3.6) for \( n = 2 \), we obtain

\[ u_2(t) = 2.124701943 \cos \Omega t + 0.123522681 \cos 3 \Omega t + 0.001815473 \cos 5 \Omega t - 0.00040097 \cos 7 \Omega t. \]  

(4.16)

From Eq. (3.2), we obtain the second-order approximation

\[
x_2(t) = -0.25 + 2.124701943 \cos \Omega t + 0.123522681 \cos 3 \Omega t + 0.001815473 \cos 5 \Omega t
-0.00040097 \cos 7 \Omega t \]  

(4.17)

where \( \Omega \) is given by Eq. (4.15).

As in the previous example, the results obtained are remarkably good when compared to the results obtained numerically using a fourth-order Runge–Kutta method (Fig. 2).
5. Conclusions

The variational iteration method has been proved to be effective, and has some distinct advantages over usual approximation methods in that the approximate solutions obtained in the present paper are valid not only for weakly nonlinear equations, but also for strongly nonlinear ones. This procedure is effective and accurate for non-linear problems with approximations converging rapidly to accurate solutions.

References