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A FUZZY SYSTEMS APPROACH IN DATA ENVELOPMENT ANALYSIS

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Abstract—The use of fuzzy set-theoretic measures is explored here in the context of data envelopment analysis, which utilizes a nonparametric approach to measure efficiency. Three types of fuzzy statics are employed here e.g., fuzzy mathematical programme, fuzzy regression and fuzzy entropy, to illustrate the types of decisions and solutions that are achievable, when the data are vague and prior information is inexact and imprecise.

1. INTRODUCTION

The measurement of productive efficiency of systems using various inputs to produce various outputs is an important area of current research in management science and operations research. The new tool of data envelopment analysis (DEA) originally developed by Charnes, Cooper and his coworkers [1,2] has provided a challenging new nonparametric method of measuring the relative efficiency of a set of decision-making units (DMUs) by stipulating that a given DMU is not efficient in producing its outputs from given inputs, if some other DMU or combination of DMUs can produce more of some output without producing less of any other output and without utilizing more of any input. This technique, which has been recently extended by a number of authors [3,4] has been found to be most useful in public enterprises e.g., public schools, hospitals, where the concept of profit is not easily definable since many of the inputs and outputs do not have observed market prices.

The fuzzy systems approach pioneered by Zadeh [5,6] and applied in systems engineering by number of researchers [7,8] has many features which are particularly suitable for the theory and practice of DEA models. First of all, the DEA model tests for the efficiency of the observed vector points by means of a sequence of linear programming (LP) models. For heterogeneous input-output data sets the efficiency facets characterized by the sequence of LP models need be classified into relatively homogeneous subsets. The principles of fuzzy classification and fuzzy regression are particularly applicable here. Secondly, the observed input output data are usually subject to stochastic generating mechanisms and hence the efficiency frontier in DEA model is probabilistic; however the assumption of any specific error distribution is not very realistic and the normal distribution is not appropriate here due to nonnegativity restrictions on the input-output space. Also the sample sizes are generally small. It is more robust in this situation to apply the methods of fuzzy mathematical programming and thereby determine an optimal solution under conditions of inadequate knowledge. Finally, the econometric theory of the production frontier [9,10] shows that the DEA model can be closely related to the minimization of the L_1-norm rather than the L_2-norm of errors associated with the output equation and in case of stochastic data this may lead to a minimax formulation [11,12] of the efficiency frontier. This minimax formulation can be suitably generalized by incorporation maximum entropy estimation with inexact information i.e, by using the concept of entropy of fuzzy events.

Our objects here is to explore the use of fuzzy measures and fuzzy mathematical programs in the DEA models, where the observed data set provides vague and imprecise knowledge about the generating process. Specifically, we consider three aspects of generalizing the DEA model as follows: (1) a fuzzy LP model approach based on memberships functions, (2) a fuzzy clustering

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX and regression approach to analyze heterogeneity, and (3) a minimax entropy characterization of the efficiency surface. Each of these approaches define nonparametric and robust procedures and they open up new lines of research for applying decision criteria under conditions of vague data.

2. FUZZY PROGRAMS IN DEA MODELS

The DEA model provides a nonparametric measurement of efficiency of a set of DMUs each using some common inputs to produce one or more outputs and this technique raises some important issues when the input-output data are subject to incomplete knowledge. Even when such data are assumed to be generated by some stochastic generating mechanism, several problems in characterizing efficiency remain. These are related e.g., to the lack of robustness of the efficiency frontier and the probabilistic feasibility of the inequality constraints of the DEA model. A fuzzy LP transformation may be most suitable in such situations as an alternative viable approach.

Let $D = (X, y)$ be the input-output data set, where $X = (x_{ji})$ is the quantity of input i, $i \in I_m = \{1, 2, \dots, m\}$ for the j^{th} DMU $j \in I_n = \{1, 2, \dots, n\}$ producing a single output y_j. The DEA efficiency frontier is then specified by the following LP model:

$$\begin{aligned} &\min_{\beta} g_k = \sum_{i=1}^{m} x_{ki}\beta_i, \\ &\text{subject to} \qquad \sum_{i=1}^{m} x_{ji}\beta \geq y_j, \qquad j = 1, 2, \dots, n, \\ &\qquad\qquad\qquad \beta \geq 0; \qquad i = 1, 2, \dots, m, \end{aligned} \tag{1}$$

when the inputs of the reference unit k is used in the objective function. For a fixed k, where k belongs to the set I_n let $\beta^*(k)$ be the optimal solution of the above LP model. Then the unit k is efficient if it holds that $\sum_{i=1}^{m} x_{ki}\beta_i^*(k) = y_k$, and $s_k = y_k^* - y_k = 0$, where s_k is the slack variable representing the excess of potential output $y_k^* = \sum_{i=1}^{m} x_{ki}\beta_i^*(k)$ over actual output y_k. By varying k over the set I_n we can generate the whole efficiency surface and the associated values of the production parameters: $\{\beta^*(k),\ k \in S\}$, where S is a subset of I_n containing only the efficient units.

When the data set $D = (X, y)$ has stochastic components, one may take two viewpoints in characterizing the subset of efficient units. One is the estimation viewpoints, where the objective is to estimate the parameters β from the distribution of errors $e_j(\beta) = y_j - \sum_{i=1}^{m} x_{ji}\beta_i$. Thus in ordinary least squares (OLS) we would minimize $\sum_{j=1}^{n} e_j{}^2(\beta)$ subject to $\beta \geq 0$, thus yielding a quadratic programming (QP) estimate. However since the errors $e_j(\beta)$ are nonpositive for all j due to the production frontier being an envelope, this estimate may not be very appropriate. More over since the errors are one-sided and the parameter estimates are required to be nonnegative, the usual notions of an unbiased estimate and/or the class of best linear unbiased estimates fail to apply. A alternative approach is to adopt the least absolute value (LAV) of errors criterion, by which we minimize the loss function: $\sum_{i=1}^{n} |e_j(\beta)|$ subject to $\beta \geq 0$. But since the errors are one-sided, this leads to the equivalent LP model:

$$\begin{aligned} &\min \bar{g} = \bar{x}'\beta \\ &\text{s.t. } X\beta \geq y; \qquad \beta \geq 0, \end{aligned} \tag{2}$$

where $\bar{x} = (\bar{x}_i)$ is an m-element column vector of mean inputs i.e., $\bar{x}_i = \sum_{j=1}^{n} x_{ji}/n$.

This is the method followed by Timmer [10] in his empirical applications to U.S. agricultural production functions. One objection to this formulation is the truncation method followed by Timmer, by which he discards, e.g., two, three or five percent of the efficient observations in order to obtain estimates of β which are closest to 'average practice' rather than 'best practice' situations. Clearly such truncation of efficient observations tend to produce inferior estimates away from the best production frontier. However the real issue is the robustness of an estimate $\bar{\beta}^*$ of the mean-input model (2) against specified types of truncation.

A second viewpoint in characterizing efficiency in the LP model (1) under a stochastic data set $D = (X, y)$ is to adopt a chance-constrained programming approach [13,14], whereby each constraint is feasible at a probability level α or higher $(0 < \alpha \leq 1)$. This yields, e.g., a nonlinear programming (NLP) model for determining the efficiency parameters β as follows:

$$\begin{aligned} \text{Find } \beta \quad \text{s.t. } & \Pr.(X\beta \geq y) \geq \alpha e \\ & \Pr.(X_k'\beta \leq g_k) \geq \alpha \qquad \beta \geq 0 \end{aligned} \tag{3}$$

where e is a column vector with each element unity. For a fixed α, this specifies a stochastic efficiency surface if one varies k over the index set I_n. Note that other types of NLP formulations are also possible (see, e.g., [15]).

Although these stochastic transformations of the DEA model provide interesting generalizations of the deterministic DEA model, they are deficient in several aspects. First of all, one has to assume specific forms of distributions, e.g., normal or exponential to derive specific results in e.g., the chance-constrained model. And on *a priori* basis there is very little empirical evidence to choose one type of distribution except on grounds of mathematical convenience. Secondly, the solutions of the stochastic DEA models (2) and (3) always emphasize point solutions rather than interval solutions, although in terms of data sensitivity the latter may be more preferable. Finally, the heterogeneity of the data structure in terms of diverse clusters is almost ignored. Hence the need for a fuzzy systems approach which provides partial if not complete answers to the problems above.

Some basic motivations for applying the fuzzy systems approach in this context are as follows. First of all, one could apply the "principle of incompatibility" of the fuzzy set theory here, which has the ability to arrive at decisions based on qualitative data in contrast to formal mathematics and precise quantitative data. As Zadeh [5] has explained this principle thus: "The closer one looks at a 'real world' problem, the fuzzier becomes its solution. Stated informally, the essence of this principle is that as the complexity of system increases, our ability to make precise and yet significant statements about its behavior diminishes until a threshold is reached beyond which precision and significance (or relevance) become almost mutually exclusive characteristics." Secondly, the methods of fuzzy mathematical programming [16,18] which transform the standard LP versions of the DEA model can be easily applied here so as to incorporate the elements of imprecision and uncertainty. Finally, by using suitable membership functions the stochastic transformations such as (3) of the DEA model may be given a fuzzy programming interpretation, which may be more nonparametric and also more robust in suitable cases.

For simplicity we consider only two versions of the fuzzy mathematical programming model in the framework of data envelopment analysis. One uses a linear membership function and the other nonlinear. In the linear case we write the DEA model (1) as:

$$\underset{\sim}{\min}\, X_0'\beta, \quad \text{s.t. } X_j'\beta \underset{\sim}{\geq} y_j,\ j \in I_n \quad \beta \geq 0, \tag{4}$$

by setting $k = 0$ in the objective function of the reference unit and using the notation $\sim$ to indicate that both the objective function and the n constraints are fuzzy. By making the constraints fuzzy we accept tolerances in their realization, e.g., by fixing maximal levels of tolerance violations $d_j, j \in I_n$ we can define linear membership functions

$$\mu_j(\beta) = 1 - \frac{y_j - X_j'\beta}{d_j}, \quad j \in I_n. \tag{5a}$$

Similarly the membership function for the fuzzy objective function can be related to a predetermined aspiration level g_0 with

$$\mu(\beta) \leq 1 - \frac{X_0'\beta - g_0}{d_0},$$

where d_0 is the maximal level of tolerance violation. This yields

$$d_0\mu_0(\beta) + X_0'\beta \leq g_0 + d_0. \tag{5b}$$

The decision problem (4) is then to find a fuzzy solution vector β where the membership function is given by:

$$\lambda = \min_{j=0}^{n} [\mu_j(\beta)], \qquad \beta \in R^m. \tag{5c}$$

This solution can be interpreted in two ways. One is to reformulate it as an LP model:

$$\begin{aligned} &\max \lambda \\ &\text{s.t.} \quad \lambda d_0 + X_0'\beta \le g_0 + d_0, \\ &\qquad \lambda d_j \le d_j + X_j'\beta - y_j, \qquad j \in I_n, \\ &\qquad 0 \le \lambda \le 1, \qquad \beta \ge 0, \end{aligned} \tag{5d}$$

where the sensitivity of the optimal $\beta^* = \beta^*(d_0, d_1, \dots, d_n)$ to tolerance variations can be parametrically analyzed. It is clear that there will always exist an optimal solution $\beta^*(d)$ for some tolerance vector d, furthermore if the tolerance variations can be generated by a known stochastic generating mechanism, then the sensitivity coefficients $\frac{\partial \beta^*(d)}{\partial d}$ can be given a statistical interpretation. A second interpretation of (5c) is that it defines a set R of the elements of an m-dimensional nonnegative orthant such that

$$R = \left\{ \beta \mid \max_{\beta} \min_{j=1}^{n} [\mu_j(\beta)] \right\}, \tag{6a}$$

i.e., R is the set of m-dimensional vector points in the nonnegative orthant which minimizes $X_0'\beta$ in a fuzzy sense and has the greatest degree of membership to fuzzy set solution $\underset{\sim}{D}$ of the n fuzzy constraints

$$\mu_{\underset{\sim}{D}}(\beta) = \min_{j=1}^{n} [\mu_j(\beta)]. \tag{6b}$$

Note that the LP model (6) defines only the first stage of the DEA model, where the reference unit in the objective function has to be varied in the second stage over $k \in I_n$, where $k = 0$, is the notation used for the objective function. This second stage variation may be handled by the same method of linear fuzzy membership function as developed before; also the concept of nondominated solutions for multi-objective LP models can be easily applied here.

Nonlinear membership functions provide a more general transformation than the linear functions and these are particularly suitable for problems of stochastic LP models. Although several standard forms of nonlinear membership functions [19] have been applied in fuzzy systems theory, the following two are closely related to the exponential and normal distributions which have been frequently applied [20] in stochastic DEA models

$$\begin{aligned} \mu_j(\beta) &= 1 - \exp\left\{-k_j\left[X_j'\beta - y_j\right]\right\}, \qquad X_j'\beta \ge y_j, \qquad k_j \ge 1; \qquad j \in I_n \\ \mu_0(\beta) &= 1 - \exp\left\{-k_0\left[g_0 - X_0'\beta\right]\right\}, \qquad g_0 \ge X_0'\beta, \qquad k_0 \ge 1, \end{aligned} \tag{7a}$$

for the exponential case and,

$$\begin{aligned} \mu_j(\beta) &= 1 - \exp\left\{-k_j\left[X_j'\beta - y_j\right]^2\right\}, \qquad k_j \ge 1; \qquad j \in I_n \\ \mu_0(\beta) &= 1 - \exp\left\{-k_0\left[g_0 - X_0'\beta\right]^2\right\}, \qquad k_0 \ge 1, \end{aligned} \tag{7b}$$

for the normal case. Some comments are in order about these nonlinear membership functions which can be directly substituted into the formula (5a) to derive a fuzzy optimal solution. First of all, they generally lead to nonlinear programming formulations, which have far less restrictive assumptions than the LP model such as (5d). In applied studies [20] of DEA models it has been consistently found that the nonlinear models out perform the linear ones. Secondly, the nonlinear membership functions (7a) can be closely related to some of the stochastic formulations of the DEA model, where the transformed model tends to be linear. Finally, the decision maker may choose between a class of nonlinear membership functions by applying the criterion of minimal predictive error [20].

3. FUZZY REGRESSION IN DEA MODELS

The programming approach to efficiency measurement in the DEA models raises two major issues against the application of standard regression theory. One is due to the heterogeneity of the input output data set over which the DEA model is applied. The regression application to estimate production frontiers for U.S. agriculture, Timmer [10] found striking differences between the two approaches and he tried to correct them by *ad hoc* procedures of rejection some observations in a certain way. The second reason is that the errors $e_j = y_j \sum_{j=a}^{m} x_{ji}\beta_i$ in the DEA model are one-sided (i.e., $e_j \leq 0$) and nonsymmetric, where minimizing the L_1-norm (i.e., the LAV criterion) may be more appropriate than the L_2-norm (i.e., the least squares criterion).

The fuzzy regression approach in DEA framework is most suitable in relaxing the two objections above. The standard theory of linear fuzzy regression [22] first classifies the set of observations into homogeneous clusters and then minimizes for the k^{th} fuzzy cluster ($k = 1, 2, \ldots, K$) the loss functions

$$L_k = \sum_{j=1}^{n} f_{jk}\, e_{jk}^2 = (y - Xa_k)' F_k (y - Xa_k),$$

where

$$y = Xa_k + e_k, \tag{8a}$$

$F = (f_{jk}), f_{jk}$ is the membership grade of object 0_j to the k^{th} fuzzy cluster,

$$\text{such that } 0 \leq f_{jk} \leq 1, \sum_{k=1}^{K} f_{jk} = 1,$$

by applying the generalized least squares approach one easily obtains the vector $\hat{a}_k$ of linear fuzzy regression coefficients

$$\hat{a}_k = (X' F_k X)^{-1} X' F_k y.$$

This approach however is not directly applicable to the DEA model, since the object 0_j characterized by the output and input variables $(y, x_1, \ldots, x_m)$ are not randomly determined. We may adopt however an alternative two-stage approach. In the first stage we solve the n LP models (1) and determine for each unit j turning out to the efficient, then we order the units as $p_{(1)} \geq p_{(2)} \geq \cdots \geq p_{(n)}$. Clearly $0 \leq p_{(j)} \leq 1$. Then in the second stage we form clusters on the basis of these ordered frequencies $p_{(j)}$. For example cluster one may contain those units which have $p_{(j)} \geq 0.90$, the second cluster may have $0.79 \leq p_{(j)} < 0.90$ and so on. For each cluster one could then estimate linear of nonlinear regressions. This method has been applied [4,21] in several empirical studies of DEA models, where it has out-performed the ordinary regression approach. We may note two advantages of this approach. One is that it is completely data-based, since the estimates of the probabilities $p_{(j)}$ have to be derived directly from the sequence of n LP models of the DEA system. These estimates would be more reliable if the number n is large. Secondly, the role of those units which are most often efficient i.e., they have the highest frequency of retaining their efficiency label may be empirically tested by a dummy variable regression procedure. Some empirical results are reported elsewhere [21]. It is clear that once we have the clusters determined by the estimated probabilities $p_{(j)}$ above, it is easy to apply the least squares principle in the form (8a), which can be rewritten in terms of the DEA notation as

$$y = X\beta_k - e_k. \tag{8b}$$

But since e_k is nonnegative with a certain probability, the LAV method of estimation can also be applied. Since the probabilities p_j may be more appropriately viewed as fuzzy measures (i.e., "possibilities" in fuzzy set theory [23]) due to the inadequate and imprecise knowledge about the data generating process, the above estimation procedure is based on fuzzy statistics.

4. MAXIMUM ENTROPY ESTIMATION

One of the major motivations of exploring the use of fuzzy set-theoretic measures in DEA models is the lack of exact prior information about the stochastic generating process. To see this very clearly one could consider the mean LP model (2) proposed by Timmer and transform it as a two-person noncooperative game model

$$\begin{aligned} &\min_{\beta\in C(\beta)} \max_{\gamma\in C(\gamma)} \phi(\beta,\gamma) = (\beta'X' - y')\gamma, \\ \text{s.t. } &C(\beta) = \{\beta \mid X\beta - y \geq 0,\, \beta' e_m = 1,\, \beta \geq 0\}, \\ &C(\gamma) = \{\gamma \mid \gamma' e_n = 1,\, \gamma \geq 0\}. \end{aligned} \tag{9a}$$

Here player I is the decision maker who has to estimate the vector β, where the second player is Nature who selects the vector γ. In the language of the theory of statistical games, player II maximizes the loss of player I which is proportional to $e' = \beta'X' - y'$, whereas player I chooses β to minimize the loss. This can be seen more clearly by dropping the term $y'\gamma$ and rewriting the payoff function as $\phi(\beta,\gamma) = \gamma' X\beta$, which is the expected loss or cost for player I, if γ is known or given. Clearly if $\gamma_j = 1/n$ for all $j \in I_n$ we get back the mean LP model (2). But this means that either the second player is passive or player I lacks adequate information to statistically estimate the value of γ_j in any other way i.e., the principle of insufficient reasons. The minimax principle specified in (9a) provides more caution as a safety-first decision rule for player I, since for each of his strategies he considers first the worst risk (i.e., most unfavorable distribution of γ_j's) and then selects the best of the worst. Note that the strategy vectors β and γ may now be treated as mixed strategies and also the n data points may be grouped into r subsets ($r < n$) with γ_j as the probability of state of nature belonging to the j^{th} subset, $j = 1, 2, \ldots, r$. Now the decision problem is that there is lack of exact information on γ, although there may be some inadequate or vague knowledge. Two approaches are available in this framework. One is based on the principle of maximum entropy and the other on the principle of "possibility distribution" [24] as defined in fuzzy systems theory. Following the first approach one would modify the payoff function of (9a) as follows:

$$\min_{\beta\in C(\beta)} \max_{\gamma\in C(\gamma)} \phi(\beta,\gamma) = \sum_{i=1}^{m}\sum_{j=1}^{r} \gamma_j (x_{ji}\beta_i - y_j) - \sum_{j=1}^{r} \gamma_j \ln \gamma_j, \tag{9b}$$

to incorporate the entropy measure $H(\gamma) = -\sum_{j=1}^{r} \gamma_j \ln \gamma_j$ of uncertainty in the Shannon sense. This modified payoff function (9b) has two interesting interpretations. One is that it provides a nonparametric method of introducing the so-called least informative prior distribution in Bayesian language. Secondly, it is compatible with the concept of a fuzzy entropy in the form

$$H(\mu,\gamma) = -\sum_{j=1}^{r} \mu_j \gamma_j \ln \gamma_j, \tag{9c}$$

first introduced by Zadeh [23] in fuzzy set theory, where μ_j denotes a membership function discussed before. Maximizing Shannon entropy $H(\gamma)$ is considered to be the most objective way of assigning the probabilities γ_j of states of nature. Likewise the maximization of the fuzzy entropy measure (9c) may be the best way of assigning γ_j in the presence of uncertainly of data and inexact information [25].

Some theoretical results may be useful to emphasize here.

THEOREM 1. *If the mean LP model (2) has an optimal solution for given data set $D = (X, y)$, then there must exist a Nash equilibrium point (β^0, γ^0) for the model* (9b) *which is defined as a pair of strategies satisfying*

$$\begin{aligned} \phi(\beta^0,\gamma^0) &= \max_{\gamma\in C(\gamma)} \phi(\beta^0,\gamma^0) \\ \phi(\beta^0,\gamma^0) &= \min_{\beta\in C(\beta)} \phi(\beta,\gamma^0) \end{aligned} \tag{9d}$$

Proof. Since the mean LP model (2) has an optimal solution, there exists a vector $\tilde{\gamma} \in C(\gamma)$, for which the function $\phi(\beta, \tilde{\gamma})$ is minimized with respect to $\beta \in C(\beta)$. But the conditional payoff function $\phi(\beta, \gamma \mid \gamma = \tilde{\gamma})$ is continuous over the compact set $C(\beta)$ for every vector γ in $C(\gamma)$. Hence by the theorem of bimatrix games, there exists a Nash equilibrium point satisfying (9d). ∎

Theorem 2. *A necessary and sufficient condition that the point (β^0, γ^0) defines a Nash equilibrium point is that there exist scalar values u^0, v^0 and a vector λ^0 such that they satisfy the following conditions*

$$\begin{aligned}
&\text{(i)} \ X'\lambda^0 + u^0 e_m \le X'\gamma^0 \\
&\text{(ii)} \ \gamma^{0\prime} X\beta^0 - \lambda^0 X\beta^0 - u^0 = 0 \\
&\text{(iii)} \ X\beta^0 - \ln \gamma^0 - (1 + v^0) e_n \le y \\
&\text{(iv)} \ \gamma^0(X\beta^0 - y) - \gamma^{0\prime} \ln \gamma^0 - (1 + v^0) = 0 \\
&\text{(v)} \ \beta^0 \in C(\beta), \gamma^0 \in C(\gamma) \\
&\text{(vi)} \ u^0 \ge 0, \ v^0 \ge 0, \ \lambda^0 \ge 0
\end{aligned} \tag{9e}$$

Proof. The Lagrangean function for the problem

$$L(\beta, \gamma) = \gamma'(X\beta - y) + \lambda'(y - X\beta) + u(1 - \beta' e_m) + v(1 - \gamma' e_n) - \gamma' \ln \gamma$$

is a continuous differentiable function over the compact set $C(\beta) \cap C(\gamma)$. Furthermore, for any given $\beta \in C(\beta)$, it is strictly concave and twice differentiable in γ and for any given $\gamma \in C(\gamma)$ it is linear and convex in β. Hence the constraint qualification for the application of the Kuhn-Tucker theorem on nonlinear programming holds. The proof follows by a direct application of the Kuhn-Tucker theorem.

Two implications of these theorems are of some importance in economic applications. First of all, the set of γ_j's may be interpreted in a Bayesian framework as providing maximal average data information relative to the information in the prior distribution, with information being represented by Shannon's entropy measure. This aspect has been analyzed by Zellner [26] to derive operational results for the problem of selecting 'diffuse' prior density functions. clearly this line of reasoning provides a more logical basis than the principle of insufficient reason which sets $\gamma_j = 1/r$ for all j. Secondly, this formulation can easily incorporate fuzzy measures by replacing the term $H(\gamma)$ in (9b) by the fuzzy entropy defined by (9c).

Note however that there is an alternative way of introducing fuzzy measures by means of "possibility distribution" [23,25] particularly when the available data are vague and inexact. Assume that the state of nature is indexed by a variable S which can take the value $s_1, s_2, \ldots, s_r$ with probabilities $\gamma_1, \gamma_2, \ldots, \gamma_r$ and possibilities $\pi_1, \pi_2, \ldots, \pi_r$. To estimate over given observations, the probabilities γ_j's with respect to the prior information set $\{\pi_j\}$, we need to introduce first a relationship between the two distributions. An intuitively appealing principle advocated in fuzzy systems theory is the possibility-probability consistency principle

$$\sum_{j=1}^{r} \gamma_j \pi_j = \alpha, \alpha \text{ preassigned}, \tag{9f}$$

which represents the appealing notion that lessening of the possibility of a random event tends to lessen its probability. Introducing (9f) as separate constraint in the minimax model (9b), one could compute an optimal distribution $\gamma^{*\prime} = (\gamma_1^*, \gamma_2^*, \ldots, \gamma_r^*)$ which generates a confluence of the possibility and probability distributions. Furthermore, the consistency principle stated in (9f) may itself be represented as a fuzzy subset defined, e.g., by the following membership function:

$$\mu(\gamma, \pi) = \exp\left[-\left\{\sum_j \gamma_j \pi_j(s_j) - \alpha\right\}^t\right]$$

where t is a positive number. Thus the maximum entropy principle is made applicable through the fuzzification of the moment constraint, although the exact numerical value of the expectation parameter α is not known.

5. CONCLUSION

Three methods based on fuzzy set statistics are proposed here to generalize the scope of applicability of the technique of data envelopment analysis. These methods employ fuzzy transformations of the linear programming models used in DEA theory, and also fuzzy regression and fuzzy entropy. These methods are most suitable when the available data are inexact, vague and fail to satisfy the usual conditions required for random variables.

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