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Described by Linear Parabolic Equations

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We prove a version of the maximum principle for linear parabolic distributed parameter systems that avoids some of the usual smoothness assumptions on the target. © 2001 Academic Press

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1. INTRODUCTION

As motivation, consider the control system

$$\frac{\partial y(t,x)}{\partial t} = \frac{\partial^2 y(t,x)}{\partial x^2} + u(t,x), \qquad y(t,0) = y(t,\pi) = 0 \tag{1.1}$$

with initial condition

$$y(0,x) = \zeta(x) \tag{1.2}$$

and control constraint

$$|u(t,x)| \le 1, \tag{1.3}$$

so that the control space is the unit ball of $L^\infty((0,T)\times(0,\pi))$. Controls satisfying (1.3) are called *admissible*, and their corresponding solutions are *trajectories*. This system can be modelled in Banach space as

$$y'(t) = A_c y(t) + u(t), \qquad y(0) = \zeta \tag{1.4}$$

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in the space $E = C_0[0, \pi]$ of all continuous functions $y(\cdot)$ in $0 \le x \le \pi$ with $y(0) = y(\pi) = 0$ equipped with the supremum norm.[1] The operator is $A_c y(x) = y''(x)$ and the domain $D(A_c)$ the set of all twice continuously differentiable $y(x)$ such that $y(\cdot), y''(\cdot) \in C_0[0, \pi]$. Consider, for instance, the problem of driving time optimally the initial condition ζ to a target $\bar{y}$,

$$y(\bar{t}) = \bar{y}. \tag{1.5}$$

Let $S_c(t)$ be the strongly continuous semigroup generated by A_c in $C_0[0, \pi]$. Existing versions of the maximum principle [6, 8] for the time optimal problem require

$$\bar{y} \in D(A_c). \tag{1.6}$$

The maximum principle for a time optimal control is, in operator language,

$$\langle S_c(\bar{t} - t)^* z, \bar{u}(t) \rangle = \max_{\|u\|_{L^\infty(0, T; E)} \le 1} \langle S_c(\bar{t} - t)^* z, u \rangle, \tag{1.7}$$

where the angled brackets indicate the duality between $L^1(0, \pi)$ and $L^\infty(0, \pi)$. The multiplier z is in a space much larger than $L^1(0, \pi)$, although the smoothing properties of $S(t)^*$ pull $S(\bar{t} - t)^* z$ into $L^1[0, \pi]$ for $t < \bar{t}$. The multiplier z is guaranteed to be nonzero; hence, combining the semigroup equation with analyticity,

$$S(\bar{t} - t)^* z \neq 0 \tag{1.8}$$

in $0 \le t < \bar{t}$. It follows that (1.7) gives nontrivial information on $\bar{u}(t)$ for all t. We may rewrite (1.7) in the form

$$\int_0^\pi z(t, x) \bar{u}(t, x)\, dx = \max_{\|u\|_{L^\infty(0, T; E)} \le 1} \int_0^\pi z(t, x) u(x)\, dx, \tag{1.9}$$

where the *costate* $z(t, x)$ is given by

$$z(t, x) = (S_c(\bar{t} - t)^* z)(x), \tag{1.10}$$

and thus solves the *reverse adjoint equation*

$$\frac{\partial z(t, x)}{\partial t} = -\frac{\partial^2 z(t, x)}{\partial^2 x}, \qquad z(t, 0) = z(t, \pi) = 0 \tag{1.11}$$

in $0 \le t < \bar{t}$, with a "final condition"

$$z(\bar{t}, \cdot) = z(\cdot) \tag{1.12}$$

that needs explaining in view of the extreme generality of z (see below).

[1]If $u(\cdot, \cdot) \in L^\infty((0, T) \times (a, b))$ then not only does $t \to u(t, \cdot)$ not take values in $C_0[a, b]$, but it is not in general a strongly measurable $L^\infty(a, b)$-valued function. See Section 2 for the precise interpretation of (1.4).

There is evidence that (1.6) is too strong for applications. For instance, if

$$u(t, x) = h(x - \pi/2) \tag{1.13}$$

with $h(x) = 1\,(x \geq 0)$, $h(x) = -1\,(x < 0)$, it is shown in [10] that

$$y(t, x) = \frac{e^{-|x-\pi/2|}}{2} + v(t, x), \tag{1.14}$$

where $v(t, \cdot)$ is twice continuously differentiable, so that $y(t, \cdot)$ does not belong to $D(A_c)$ for any t; it has a "crease" along the line $x = \pi/2$. Since each $y(t, \cdot)$ belongs to the reachable set of Eq. (1.1) it follows from standard existence results [5, 6] that for each t there exists an optimal trajectory with $\bar{y}(x) = y(t, x)$ as target. However, $\bar{y}(\cdot) \notin D(A_c)$, and we can't apply the maximum principle to this trajectory. The same happens if the control $u(t, x)$ switches from -1 to 1 in a smooth curve joining the base and the top of the rectangle $(0, \bar{t}) \times (0, \pi)$, or a finite set of these curves.

We show in this paper that (1.6) can be relaxed to

$$\bar{y} \in D(A_\infty), \tag{1.15}$$

where $A_\infty y = y''(x)$ and the domain $D(A_\infty)$ is the set of all continuously differentiable $y(\cdot)$ with absolutely continuous derivative, second derivative in $L^\infty[0, \pi]$ and satisfying the boundary conditions $y(0) = y(\pi) = 0$. The result is actually proved for any dimension n (see Section 4 for the general definition of A_∞). The maximum principle, with (1.6) generalized to (1.15), is proved by means of the standard separation argument and, as can be expected, the "final condition" z is constructed from finitely additive measures, as we use the dual of L^∞ rather than that of a space of continuous functions. For earlier uses of the dual of L^∞ in control problems see [2, 3, 17, 18].

There is evidence in [10–12] that (1.6) (or its generalization (1.15)) cannot be thrown away entirely. On the other hand, it is not unlikely that (1.15) admits further generalizations, although in dimension 1 it is, in some sense, "close to necessary." A (purely heuristic) justification goes as follows. Since we are only interested in *optimal* trajectories, we may limit ourselves to trajectories driven by a control satisfying (1.9). This maximum principle implies that a time optimal control $\bar{u}(t, x)$ satisfies

$$\bar{u}(t, x) = \operatorname{sign} z(t, x) \tag{1.16}$$

outside the *nodal set* of $z(t, x)$ where

$$z(t, x) = 0. \tag{1.17}$$

Now, the solution of (1.11) is smooth in every rectangle $[0, \bar{t} - \epsilon] \times [0, \pi]\,(\epsilon > 0)$; hence results on nodal sets in one space variable [1, 16] show that a typical nodal set may consist of finite sets of (possibly coalescing) curves joining the top $\{\bar{t}\} \times [0, \pi]$ of the rectangle to the bottom $\{0\} \times [0, \pi]$ (top to bottom since time in (1.11) is reversed; coalescing because the lap number decreases). If these curves are smooth, then it is possible to show as in the computation of (1.14) (see [10]) that the solution $y(t, x)$, although having creases along these curves, satisfies $y(t, \cdot) \in D(A_\infty)$ in $0 \le t \le \bar{t} - \epsilon$, thus in $0 \le t < \bar{t}$. Note, however, that this does *not* imply that $y(\bar{t}, \cdot) \in D(A_\infty)$; there is no argument, even heuristic, to support this in view of the extreme generality of the final condition z in (1.12). In fact, the characterization of all *reachable states* $y(\bar{t}, \cdot)$ for the heat equation for arbitrary controls $u(t, x)$ in $L^\infty((0, T) \times (0, \pi))$ or even for controls of the form (1.16) is an open problem at the present time.

The situation is even more obscure in higher dimensions, as the structure of nodal sets may be very complicated; see [7, 13–15] and the Miscellaneous Notes to Part 2 of [9].

2. MODELING OF PARABOLIC EQUATIONS

Let $\Omega \in \mathbb{R}^m$ be a bounded C^∞ domain with boundary Γ, A a uniformly elliptic operator $Ay = \sum_{j=1}^m \sum_{k=1}^m \partial^j (a_{jk}(x)\partial^k y) + \sum_{j=1}^m b_j(x)\partial^j y + c(x)\, y$ ($\partial^j = \partial/\partial x_j$), and β a boundary condition on the boundary Γ (of Dirichlet or variational type $\partial^\nu y = \gamma(x)y$, ∂_ν the conormal derivative). For simplicity, we assume C^∞ coefficients for the operator and the boundary condition. Although simpler than the L^∞ case, modeling of the parabolic equation

$$\frac{\partial}{\partial t} y(t, x) = Ay(t, x) + u(t, x), \quad y(0, x) = \zeta(x) \tag{2.1}$$

in $L^p(\Omega)$ only works well for constraints $\|u(t)\|_{L^p(\Omega)} \le c$. Pointwise constraints

$$|u(t, x)| \le 1 \qquad (x \in \Omega) \tag{2.2}$$

are best dealt with combining $L^\infty(\Omega)$ with the space $C(\overline{\Omega})$ of continuous functions in $\overline{\Omega}$ equipped with the supremum norm (for the Dirichlet boundary condition the space $C(\overline{\Omega})$ is replaced by its subspace $C_0(\overline{\Omega}) \subseteq C(\overline{\Omega})$ determined by $y = 0$ on Γ). The abstract model is

$$y'(t) = A_c y(t) + u(t), \quad y(0) = \zeta, \tag{2.3}$$

where A_c is the operator in $C(\overline{\Omega})$ determined by A and β. The operator A_c can be defined in two equivalent ways: (a) (strong) and (b) (weak).

(a) *Strong definition of* A_c. $y \in D(A_c)$ if and only if

$$y \in \bigcap_{p=1}^{\infty} W_\beta^{p,2}(\Omega) \qquad \text{and} \qquad A_c y = Ay \in C(\overline{\Omega}).$$

Here, $W_\beta^{p,2}(\Omega)$ is the space of all $y \in W^{p,2}(\Omega)$ that satisfy β on Γ.

The *formal adjoint* A' of A is defined by $A'y = \sum_{j=1}^m \sum_{k=1}^m \partial^j (a_{jk}(x)\partial^k y) - \sum_{j=1}^m \partial^j (b_j(x)y) + c(x)y$. If β is Dirichlet, the *adjoint boundary condition* is $\beta' = \beta$; otherwise, the adjoint boundary condition β' is $\partial^\nu y(x) = (\gamma(x) + b(x))y(x)$, where $b(x) = \Sigma b_j(x)\eta_j(x)$, $(\eta_1(x), \ldots \eta_n(x))$ the outer normal vector on Γ. Of course, the definitions are such that if $y(\cdot)$, $z(\cdot)$ are smooth in Ω and $y(\cdot)$ (resp. $z(\cdot)$) satisfies β (resp. β') then $\int_\Omega Ay(x)v(x)\,dx = \int_\Omega y(x)A'v(x)\,dx$.

(b) *Weak definition of* A_c. $y \in D(A_c)$ if and only if there exists $z\,(= A_c y)$ in $C(\overline{\Omega})$ such that

$$\int_\Omega y(x)(A'v)(x)\,dx = \int_\Omega z(x)v(x)\,dx$$

for every v in the space $C_{\beta'}^{(2)}(\overline{\Omega})$ of all $v \in C^{(2)}(\overline{\Omega})$ that satisfy β' on Γ.

In both the weak and the strong definition, $C(\overline{\Omega})$ is replaced by $C_0(\overline{\Omega})$ for the Dirichlet boundary condition.

We denote by $S_c(t)$ the strongly continuous semigroup generated by A_c in $C(\overline{\Omega})$ ($C_0(\overline{\Omega})$ for the Dirichlet boundary condition). The semigroup $S_c(t)$ is analytic in $(C(\overline{\Omega}), C(\overline{\Omega}))$ for $t > 0$.[2]

Subtracting a constant from $c(x)$ we may always assume that

$$\|S_c(t)\|_{(C(\overline{\Omega}),C(\overline{\Omega}))} \le Ce^{-\omega t} \quad (t \ge 0). \tag{2.4}$$

This, in particular, implies $A_c^{-1} \in (C(\overline{\Omega}), C(\overline{\Omega}))$.

There are a few simplifications in the case $m = 1$; here $\Omega = (a, b)$ and $D(A_c)$ consists of all $y(\cdot)$ twice continuously differentiable in $[a, b]$ and such that $y(\cdot)$ satisfies the boundary conditions. For the Dirichlet boundary condition, the space is $E = C_0[a, b]$ and we also require that $y''(\cdot) \in C_0[a, b]$.

[2] (Y, Y) is the space of all linear bounded operators from the Banach space Y into itself outfitted with the operator norm.

3. DUALS AND ADJOINTS, I

For $1 \leq p < \infty$, A_p is the semigroup generator in $L^p(\Omega)$ corresponding to A and β, and $S_p(t)$ is the strongly continuous semigroup generated by A_p; $S_p(t)$ is analytic in $(L^p(\Omega), L^p(\Omega))$ for $t > 0$. If $p > 1, D(A_p)$ can be defined in two ways, corresponding to the two definitions of A_c:

(a) *Strong definition of* A_p. $D(A_p) = W_\beta^{p,2}(\Omega)$ and $A_p y = Ay$.

(b) *Weak definition of* A_p. $y \in D(A_p)$ if and only if there exists $z\,(= A_p y)$ in $L^p(\Omega)$ such that

$$\int_\Omega y(x)(A'v)(x)\,dx = \int_\Omega z(x)v(x)\,dx$$

for every $v \in W_{\beta'}^{2,q}(\Omega)$, $1/p + 1/q = 1$.

For $p = 1$ only the weak definition applies:

Weak definition of A_1. $y \in D(A_1)$ if and only if there exists $z\,(= A_1 y)$ in $L^1(\Omega)$ such that

$$\int_\Omega y(x)(A_c v)(x)\,dx = \int_\Omega z(x)v(x)\,dx$$

for every $v \in C_{\beta'}^{(2)}(\overline{\Omega})$.

The operators A'_c, A'_p, A'_1 are defined in a similar way, using the formal adjoint A' and the adjoint boundary condition β'. Subtracting if need be a positive constant from $c(x)$ we may assume that all the $S_p(t)$ $(p \geq 1)$ satisfy a bound like (2.4) so that the inverses A_p^{-1} are everywhere defined and bounded; A_p^{-1} (resp. A_c^{-1}) is compact in $L^p(\Omega)$ (resp. in $C(\overline{\Omega})$).

Note finally that $C_{\beta'}^{(2)}(\overline{\Omega})$ is dense in $D(A_c)$ in the norm $\|y\|_{D(A_c)} = \|A_c y\|_{C(\overline{\Omega})}$, hence we may replace $C_{\beta'}^{(2)}(\overline{\Omega})$ by $D(A'_c)$ in the definition of A_1.

We need a little bit of linear adjoint theory in multivalued generality. Let A, B be arbitrary unbounded operators in a Banach space $X = \{y, z, \ldots\}$ with dual $X^* = \{y^*, z^*, \ldots\}$. If $y^* \in D(B^*A^*)$ then $A^*y^* \in D(B^*)$ and $\langle B^*A^*y^*, y\rangle = \langle A^*y^*, By\rangle$ $(y \in D(B))$. If, in addition, $By \in D(A)$, then $\langle B^*A^*y^*, y\rangle = \langle y^*, ABy\rangle$. Hence,

$$B^*A^* \subseteq (AB)^*. \tag{3.1}$$

Whether $D(A), D(B)$, or $D(AB)$ is dense is immaterial; thus adjoint operators may be multivalued: (3.1) is understood as "$D(B^*A^*) \subseteq D((AB)^*)$ and every value of $B^*A^*y^*$ is a value of $(AB)^*y^*$" (of course, adjoints may not be densely defined even if the operators are).

On the other hand, if A is bounded and everywhere defined then $D(AB) = D(B)$. If $y^* \in D((AB)^*)$ then $\langle y^*, ABy\rangle = \langle A^*y^*, By\rangle$ is a bounded functional of y, hence $A^*y^* \in D(B^*)$, or $y^* \in D(B^*A^*)$. Moreover, $\langle y^*, ABy\rangle = \langle B^*A^*y^*, y\rangle$, so that $(AB)^* \subseteq B^*A^*$, understood in the same way as (3.1). Combining this with (3.1),

$$(AB)^* = B^*A^*, \tag{3.2}$$

again without assuming that $D(B)$ is dense, so that $(AB)^*$ and B^* may be multivalued; (3.2) means equality of domains and equality of the *sets* $(AB)^*y^*$ and $B^*A^*y^*$.

It is clear from the definition that $A \subseteq B$ implies $B^* \subseteq A^*$, the inclusion of multivalued operators understood as in (3.1).

Let A be an operator (not necessarily densely defined) with a bounded, everywhere defined inverse A^{-1}. Applying (3.1) to $AA^{-1} = I$ we obtain

$$(A^{-1})^*A^* \subseteq I. \tag{3.3}$$

Now, we write $A^{-1}A \subseteq I$ in the form $A^{-1}A = I|_{D(A)}$ and use (3.2), obtaining

$$A^*(A^{-1})^*y^* = \left(I|_{D(A)}\right)^* y^* = y^* + \mathcal{N}_A, \tag{3.4}$$

where the bar indicates restriction and

$$\mathcal{N}_A = \{y^*; \langle y^*, y\rangle = 0; y \in D(A)\}; \tag{3.5}$$

the operators on both sides of (3.4) are multivalued if A is not densely defined. If $y \in D(A)$ then $y = A^{-1}z$, so that $y^* \in \mathcal{N}_A$ is equivalent to $\langle y^*, y\rangle = \langle y^*, A^{-1}z\rangle = 0$ for all $z \in E$; this means

$$\mathcal{N}_A = \{y^*; (A^{-1})^*y^* = 0\} \tag{3.6}$$

is the nullspace of $(-A^{-1})^*$. We can condense (3.3) and (3.4) in

$$(A^*)^{-1} = (A^{-1})^*, \tag{3.7}$$

where if $D(A)$ is densely defined (3.4) is interpreted in the standard way $A^*(A^{-1})^* = I$.

Let $E \subseteq F$ be two linear spaces, A (resp. B) an operator with domain $D(A) \subseteq E$ and range in E (resp. $D(B) \subseteq F$ and range in F) with $A \subseteq B$. Assume A (resp. B) has an inverse $A^{-1} : E \to E$ (resp. an inverse $B^{-1} : F \to F$). Then

$$A^{-1} \subseteq B^{-1}, \qquad A^{-1} = B^{-1}|_E. \tag{3.8}$$

In fact, let $y \in D(A)$. Then $y = A^{-1}Ay = A^{-1}By = B^{-1}By$ so that A^{-1} and B^{-1} coincide in E. The equality follows from the inclusion.

The space $\Sigma(\overline{\Omega})$ consists of all bounded regular Borel measures in $\overline{\Omega}$, equipped with the total variation norm; interpreting functions $y(\cdot) \in L^1(\Omega)$

as measures $\mu(e) = \int_e y(x)\,dx$, we obviously have $L^1(\Omega) \hookrightarrow \Sigma(\overline{\Omega})$ with isometric imbedding. The space $\Sigma(\overline{\Omega})$ is algebraically and metrically isomorphic to the dual $C(\overline{\Omega})$, the duality between both spaces given by $\langle \mu, y\rangle = \int_{\overline{\Omega}} y(x)\mu(dx)$. The dual of $C_0(\overline{\Omega})$ is the subspace $\Sigma_0(\overline{\Omega}) \subseteq \Sigma(\overline{\Omega})$ defined by $\mu(\Gamma) = 0$.

Define

$$A'_\Sigma = A_c^*. \tag{3.9}$$

It is clear from the definition of A_c^* and the weak definition of A'_1 that

$$A'_1 \subseteq A'_\Sigma. \tag{3.10}$$

It follows from (3.7) that

$$(A_c^{-1})^* = (A_c^*)^{-1} = A_\Sigma'^{-1} \tag{3.11}$$

and from (3.10) and (3.8) that

$$A_1'^{-1} \subseteq A_\Sigma'^{-1}, \qquad A_1'^{-1} = A_\Sigma'^{-1}|_{L^1(\Omega)}. \tag{3.12}$$

Finally, A'_Σ is not densely defined: in fact $L^1(\Omega)$ is not dense in $\Sigma(\overline{\Omega})$ or in $\Sigma_0(\overline{\Omega})$ and

$$D(A'_\Sigma) \subseteq L^1(\Omega). \tag{3.13}$$

To prove (3.13), note that a standard mollification argument shows that given $\mu \in \Sigma(\overline{\Omega})$ there exists a sequence $\{f_n\} \subseteq L^1(\Omega)$ with $\|f_n\|_{L^1(\Omega)}$ bounded and such that $\langle y, f_n\rangle \to \langle y, \mu\rangle$ as $n \to \infty$ for all $y \in C(\overline{\Omega})$. By compactness of $A_1'^{-1}$, $\{A_1'^{-1} f_n\} \subseteq L^1(\Omega)$ has a subsequence (equally named) such that $A_1'^{-1} f_n \to g$ in $L^1(\Omega)$. If $y \in C(\overline{\Omega})$, $\langle y, g\rangle = \lim\langle y, A_1'^{-1} f_n\rangle = \lim\langle y, A_\Sigma'^{-1} f_n\rangle = \lim\langle A_c y, f_n\rangle = \langle A_c y, \mu\rangle$ so that $A_\Sigma^{-1}\mu = g$.

4. DUALS AND ADJOINTS, II

Similar considerations apply to the operator A'_1. Define

$$A_\infty = A_1'^*. \tag{4.1}$$

Then the operator A_∞ is invertible with

$$A_\infty^{-1} = (A_1'^{-1})^*. \tag{4.2}$$

There are two definitions of A_∞, companions of the two definitions of A_c.

(a) *Strong definition of* A_∞. $y \in D(A_\infty)$ if and only if

$$y \in \bigcap_{p=1}^{\infty} W_\beta^{p,2}(\Omega) \qquad \text{and} \qquad A_\infty y = Ay \in L^\infty(\Omega).$$

(b) *Weak definition of* A_∞. $y \in D(A_\infty)$ if and only if there exists $z\,(= A_\infty y)$ in $L^\infty(\Omega)$ such that

$$\int_\Omega y(x)(A_1' v)(x)\,dx = \int_\Omega z(x)v(x)\,dx$$

for every $v \in D(A_1')$.

Plainly, (b) is just the definition of adjoint. Equivalence with the strong definition follows from the strong definition of A_p for $p > 1$ and the fact that $z(\cdot) \in L^p(\Omega)$ for all $p > 1$. Finally, it is clear that A_∞ extends A_c,

$$A_c \subseteq A_\infty, \qquad A_c^{-1} = A_\infty^{-1}|_{C(\overline{\Omega})} \tag{4.3}$$

the equality for the inverses following from (3.8). As an operator in $L^\infty(\Omega)$, A_∞ is not densely defined, since by Sobolev's imbedding theorem elements in the domain are (more than) continuous in $\overline{\Omega}$: this justifies the inclusion

$$D(A_\infty) \subseteq C(\overline{\Omega}), \qquad \overline{D(A_\infty)} = C(\overline{\Omega}). \tag{4.4}$$

The equality follows from the fact that $D(A_\infty) \supseteq D(A_c)$, the latter operator densely defined in $C(\overline{\Omega})$. The space $C(\overline{\Omega})$ in (4.3) and (4.4) is replaced by $C_0(\overline{\Omega})$ for the Dirichlet boundary condition.

Since $D(A_\infty)$ is not dense in $L^\infty(\Omega)$, the operator A_∞^* in $L^\infty(\Omega)^*$ is multivalued. In view of (4.4), the nullspace of $(A_\infty^*)^{-1}$ coincides with the subspace $\mathcal{N}_c(\Omega) \subseteq L^\infty(\Omega)^*$ consisting of all $\nu \in L^\infty(\Omega)^*$ such that $\langle \nu, y\rangle = 0$ for all $y \in C(\overline{\Omega})$. Due to (3.6) any two values of $A_\infty^*\eta$ differ by an element of $\mathcal{N}_c(\Omega)$. Equations (3.3) and (3.4) are

$$(A_\infty^{-1})^* A_\infty^* \subseteq I, \qquad A_\infty^*(A_\infty^{-1})^* = I + \mathcal{N}_c(\Omega). \tag{4.5}$$

We call $\Sigma_w(\Omega)$ the space of all finitely additive measures η defined in the field of all Lebesgue measurable sets of Ω, of bounded variation (this means $\|\eta\| = \sup \Sigma|\eta(e_j)| < \infty$, supremum over all finite partitions of $\overline{\Omega}$ in measurable sets) and such that

$$\eta(e) = 0 \qquad \text{if } |e| = 0, \tag{4.6}$$

where $|e|$ indicates the Lebesgue measure of e. The space $\Sigma_w(\Omega)$ is equipped with the total variation norm $\|\eta\|_{\Sigma_w(\Omega)} = \|\eta\|$.

THEOREM 4.1. *The space* $\Sigma_w(\Omega)$ *is algebraically and metrically isomorphic to the dual* $L^\infty(\Omega)^*$. *The duality between* $L^\infty(\Omega)$ *and* $\Sigma_w(\Omega)$ *is given by* $\langle \eta, y\rangle = \int_\Omega y(x)\eta(dx)$.

Proof. This is from [4, p. 296, Theorem 16]; more information on integration with respect to finitely additive measures can be found in [4, Chap. 3]. Let $\eta \in \Sigma_w(\Omega)$, $y(\cdot) \in L^\infty(\Omega)$. Select a sequence of *simple* functions $u_n(x) = \Sigma_{\text{finite}} c_{nk}\chi_{nk}(x)$ (the $\chi_{nk}(\cdot)$ characteristic functions of pairwise disjoint measurable sets e_{nk}) such that

$$\|u(\cdot) - u_n(\cdot)\|_{L^\infty(\Omega)} \to 0 \qquad (n \to \infty), \tag{4.7}$$

and define

$$\Phi(u) = \lim_{n\to\infty} \sum c_{nk}\eta(e_{nk}) = \int_\Omega u(x)\eta(dx). \tag{4.8}$$

Then Φ is a linear functional in $L^\infty(\Omega)$ (the integral (4.8) does not depend on the sequence $\{u_n(\cdot)\}$) and $\|\Phi\|_{L^\infty(\Omega)^*} \le \|\eta\|_{\Sigma_w(\Omega)}$; moreover, if $\{e_j\}$ is a finite partition of Ω in measurable sets and $u(x) = \Sigma\, \text{sign}\,\eta(e_k)\chi_k(x)$ then $\Phi(u) = \Sigma|\eta(e_j)|$, so that $\|\Phi\|_{L^\infty(\Omega)^*} = \|\eta\|_{\Sigma_w(\Omega)}$.

Conversely, let Φ be a bounded linear functional in $L^\infty(\Omega)$. Define $\eta(e) = \Phi(\chi_e(\cdot))$, $\chi_e(\cdot)$ the characteristic function of e. Then η is a finitely additive measure defined on Lebesgue measurable sets and satisfying (4.7). Accordingly, if $u_n(\cdot)$ is the sequence in (4.7) then (4.8) holds. This ends the proof.

Remark 4.2. The meaning of condition (4.6) may not be immediately apparent. For instance, let $m = 1$, $\Omega = (0, 1)$, and consider the measure

$$\eta(e) = \text{LIM}_{n\to\infty} \frac{n}{2} \int_{e\cap(\bar{x}-1/n,\, \bar{x}+1/n)} dx,$$

where $\bar{x} \in (0, 1)$ is fixed and $\text{LIM}_{n\to\infty}$ is a Banach limit (for these limits, see [4, Exercise 22, p. 73]). Clearly, the measure η satisfies condition (4.6) and is merely finitely additive; in fact, $\eta((\bar{x}, 1)) = 1/2$, whereas $\eta((\bar{x} + (1-\bar{x})/(n+1),\ \bar{x} + (1-\bar{x})/n)) = 0$ for all $n = 1, 2, \ldots$. Also, if $u(\cdot)$ is continuous in $\overline{\Omega}$ we have

$$\int_{\overline{\Omega}} u(x)\eta(dx) = u(\bar{x});$$

hence η is an extension of the Dirac delta $\delta_{\bar{x}}$, a measure that misses (4.6) completely. This "anomaly" is typical: every $\mu \in \Sigma(\overline{\Omega})$ (whether or not it satisfies (4.6)) as a functional in $C(\overline{\Omega})$ can be extended by the Hahn–Banach theorem to $L^\infty(\Omega)$ (that is, to an element of $\eta \in L^\infty(\Omega)^* = \Sigma_w(\Omega)$) with the same norm: $\|\mu\|_{\Sigma(\overline{\Omega})} = \|\eta\|_{\Sigma_w(\overline{\Omega})}$. Of course, "$\mu$ can be extended to η" does not mean that $\eta(e) = \mu(e)$ on Borel sets e, as the example above shows. The result below goes the other way.

THEOREM 4.3. *Let $\eta \in L^\infty(\Omega)^*$. Then there exists $\mu \in \Sigma(\overline{\Omega})$ such that*

$$\int_{\overline{\Omega}} y(x)\eta(dx) = \int_{\overline{\Omega}} y(x)\mu(dx) \qquad (y \in C(\overline{\Omega})). \tag{4.9}$$

The measure η is defined on Lebesgue measurable sets, in particular in the field generated by the open sets of Ω. Hence, we can apply [4, Theorem 2, p. 262] to produce a regular, finitely additive measure λ such that $\int_{\overline{\Omega}} y(x)\lambda(dx) = \int_{\overline{\Omega}} y(x)\eta(dx)$ for $y \in C(\overline{\Omega})$. We then apply Alexandroff's theorem [4, Theorem 13, p. 138] and deduce that there exists $\mu \in \Sigma(\overline{\Omega})$ with $\int_{\overline{\Omega}} y(x)\mu(dx) = \int_{\overline{\Omega}} y(x)\lambda(dx)$ for $y \in C(\overline{\Omega})$. This ends the proof.

Remark 4.4. The measure μ in Theorem 4.3 may vanish even if $\eta \neq 0$. For instance since $C(\overline{\Omega})$ is a proper closed subspace of $L^\infty(\Omega)$ we may use the Hahn–Banach theorem to construct a nonzero bounded functional in $L^\infty(\Omega)$ that vanishes on $C(\overline{\Omega})$. If η is the measure representing this functional, (4.9) can only be true with $\mu = 0$.

COROLLARY 4.5. (a) $D(A_\infty^*) = (A_\infty^{-1})^* L^\infty(\Omega)^* \subseteq L^1(\Omega)$. (b) *If $\eta \in D(A_\infty^*)$ with $A_\infty^* \eta \in L^1(\Omega)$ then $\eta \in D(A_1')$ and $A_\infty^*\eta = A_1'\eta$.*

Proof. (a) Let $\eta \in D(A_\infty^*)$, $A_\infty^*\eta = \nu \in L^\infty(\Omega)^*$. Then, due to the inclusion in (4.5), $\eta = (A_\infty^{-1})^*\nu$ is uniquely defined from ν. The equation giving η is $\langle y, \nu\rangle = \langle A_\infty y, \eta\rangle$ $(y \in D(A_\infty))$, or

$$\int_\Omega A_\infty y(x)\eta(dx) = \int_\Omega y(x)\nu(dx) \qquad (y(\cdot) \in D(A_\infty)). \tag{4.10}$$

Now (see (4.4)) $D(A_\infty) \subseteq C(\overline{\Omega})$, thus by Theorem 4.3 there exists $\mu \in \Sigma(\overline{\Omega})$ such that $\int_\Omega y(x)\mu(dx) = \int_\Omega y(x)\nu(dx)$ for $y(\cdot) \in D(A_\infty)$. Restricting $y(\cdot)$ to $D(A_c)$, (4.10) becomes

$$\int_\Omega A_c y(x)\eta(dx) = \int_\Omega y(x)\mu(dx) \qquad (y(\cdot) \in D(A_c)), \tag{4.11}$$

and it follows from the definition of A_Σ' and the fact that A_Σ' is invertible that (4.11), as an equation with η as unknown, has a (unique) solution $\lambda \in D(A_\Sigma') \subset \Sigma(\overline{\Omega})$ whose action on the continuous function $A_c y(x)$ is the same as that of η. Moreover, as $D(A_\Sigma') \subset L^1(\Omega)$ (see (3.13)), $\lambda(x) = z(x)dx$ with $z(\cdot) \in L^1(\Omega)$. Hence

$$\int_\Omega A_c y(x) z(x)\, dx = \int_\Omega y(x)\mu(dx) \qquad (y(\cdot) \in D(A_c)). \tag{4.12}$$

To pass from (4.12) to

$$\int_\Omega A_\infty y(x) z(x)\, dx = \int_\Omega y(x)\mu(dx) \qquad (y(\cdot) \in D(A_\infty)), \tag{4.13}$$

we only have to note that $A_c D(A_c) = C(\overline{\Omega})(A_c D(A_c) = C_0(\overline{\Omega})$ for the Dirichlet boundary condition) and $A_\infty D(A_\infty) = L^\infty(\Omega)$, so that, given $y(\cdot) \in D(A_\infty)$ we can select a sequence $\{y_n(\cdot)\} \subseteq D(A_c)$ such that $A_c y_n(\cdot) \to A_\infty y(\cdot)$ a.e. with$A_c y_n(x)$ uniformly bounded. Since A_c^{-1} is compact (a subsequence of) the sequence $\{y_n(\cdot)\}$ is convergent in $C(\overline{\Omega})$. Hence, we obtain (4.13) taking limits in (4.12) with $y = y_n$ (on the left, we use the dominated convergence theorem). We return then to (4.10) and use uniqueness of η to deduce that $\eta(dx) = z(x)dx$ as claimed. To show (b), it is enough to observe that if $\mu(dx) = f(x)dx$ with $f(\cdot) \in L^1(\Omega)$ then (4.13) is the weak definition of $A_1' z = f$ (see Section 3). In fact (4.13) is a little more than required as $C_\beta^{(2)}(\overline{\Omega}) \subseteq D(A_\infty)$.

5. DUAL SEMIGROUPS

The semigroup $S_1'(t)$ generated by A_1' in $L^1(\Omega)$ is analytic in the space $(L^1(\Omega),\ L^1(\Omega))$ for $t > 0$; hence the adjoint semigroup $S_\infty(t) = S_1'(t)^*$ is analytic in $(L^\infty(\Omega), L^\infty(\Omega))$ for $t > 0$. Noting that $S_1'(t)A_1'$ (domain $=$ $D(A_1')$) is bounded, taking adjoints and applying (3.2) we deduce that $A_\infty S_\infty(t)$ is everywhere defined and bounded, so that the inclusion

$$S_\infty(t)L^\infty(\Omega) \subseteq D(A_\infty) \subseteq C(\overline{\Omega}), \qquad S_\infty(t)|_{C(\overline{\Omega})} = S_c(t) \quad (t > 0) \tag{5.1}$$

follows (with $C_0(\overline{\Omega})$ for the Dirichlet boundary condition). To show the second relation, we use

$$(\lambda I - A_c)^{-1} = (\lambda I - A_\infty)^{-1}|_{C(\overline{\Omega})} \tag{5.2}$$

proved like the restriction in (4.3). This implies that if $y(\cdot) \in C(\overline{\Omega})$,

$$S_c(t)y = \int_\Gamma (\lambda I - A_c)^{-1} y\, d\lambda = \int_\Gamma (\lambda I - A_\infty)^{-1} y\, d\lambda = S_\infty(t)y, \tag{5.3}$$

where Γ is the union of two lines $|\arg \lambda| = \phi > \pi/2$. The first equality in (5.3) follows from the theory of analytic semigroups, the second from (5.2), and the third from taking the adjoint of

$$S_1'(t) = \int_\Gamma (\lambda I - A_1')^{-1} d\lambda$$

and using $(\lambda I - A_\infty)^{-1} = ((\lambda I - A_1')^{-1})^*$, a companion of (4.2).

The semigroup $S_\infty(t)^*$ in $L^\infty(\Omega)^*$ is analytic in $(L^\infty(\Omega)^*, L^\infty(\Omega)^*)$ for $t > 0$. Taking the adjoint of the densely defined, bounded operator $A_1' S_1'(t)$ and using (3.1), we obtain $S_\infty(t)A_\infty = S_1'(t)^* A_1'^* \subseteq (A_1' S_1'(t))^*$; hence $S_\infty(t)A_\infty$ is bounded. Since $S_\infty(t)$ is bounded, we can apply (3.2) and obtain $A_\infty^* S_\infty(t)^* = (S_\infty(t)A_\infty)^*$. It follows that $A_\infty^* S_\infty(t)^*$ is everywhere defined, thus $S_\infty(t)^* L^\infty(\Omega)^* \subseteq D(A_\infty^*) = (A_\infty^{-1})^* L^\infty(\Omega)^*$. By Corollary 4.5 the inclusion

$$S_\infty(t)^* L^\infty(\Omega)^* \subseteq L^1(\Omega), \qquad S_\infty(t)^*|_{L^1(\Omega)} = S_1'(t) \tag{5.4}$$

holds. The restriction relation results from $S_\infty(t)^* = S_1'(t)^{**}$.

It follows from (5.4) and from Corollary 4.5 that if $h > 0$ then we have $S_\infty(t)^* = S_1'(t-h)S_\infty(h)^*$ so that $S_\infty(t)^* L^\infty(\Omega)^* \subseteq D(A_1')$ and

$$A_1' S_\infty(t)^* = A_1' S_1'(t-h) S_\infty(h)^* \tag{5.5}$$

is analytic in $(L^1(\Omega), L^1(\Omega))$ for $t > h$, thus for $t > 0$.

We also use the semigroup $S_\Sigma'(t) = S_c(t)^*$ in $\Sigma(\overline{\Omega})$; again, by analyticity of $S_c(t)$ in $(C(\overline{\Omega}), C(\overline{\Omega}))$, $S_\Sigma'(t)$ is analytic in $(\Sigma(\overline{\Omega}), \Sigma(\overline{\Omega}))$ in $t > 0$. Although not strongly continuous at $t = 0$, $S_\Sigma'(t)$ is $C(\overline{\Omega})$-weakly continuous; if $\mu \in \Sigma(\overline{\Omega})$ and $y(\cdot) \in C(\overline{\Omega})$ then $\langle y, S_\Sigma'(t)\mu\rangle = \langle S_c(t)y, \mu\rangle \to \langle y, \mu\rangle$ as $t \to 0$. For the Dirichlet boundary condition, $C(\overline{\Omega})$ is replaced by $C_0(\overline{\Omega})$ and $\Sigma(\overline{\Omega})$ is replaced by $\Sigma_0(\overline{\Omega})$.

LEMMA 5.1. *Let $\nu \in \mathcal{N}_c(\Omega)$. Then $S_\infty(t)^*\nu = 0$ for $t > 0$. Conversely, assume that $S_\infty(t)^*\nu = 0$ for a single $t > 0$. Then $\nu \in \mathcal{N}_c(\Omega)$.*

Proof. Assume $\nu \in \mathcal{N}_c(\Omega)$. By (5.1), $S_\infty(t)L^\infty(\Omega) \subseteq C(\overline{\Omega})$, thus $\langle S_\infty(t)^*\nu, y\rangle = \langle \nu, S_\infty(t)y\rangle = 0\,(y \in L^\infty(\Omega))$. Conversely, assume that $S_\infty(t)^*\nu = 0$ for some $t > 0$; to show that $\nu \in \mathcal{N}_c(\Omega)$ it suffices to show that $S_\infty(t)L^\infty(\Omega)$ is dense in $C(\overline{\Omega})$, and, in view of (5.1), it is enough to show that $S_c(t)C(\overline{\Omega})$ is dense in $C(\overline{\Omega})$. Assume this is false. Then there exists $\mu \in \Sigma(\overline{\Omega})$ such that $\langle S_\Sigma'(t)\mu, y\rangle = \langle \mu, S_c(t)y\rangle = 0$ for all $y \in C(\overline{\Omega})$. This means $S_\Sigma'(t)\mu = 0$, thus $S_\Sigma'(s)\mu = 0$ for $s \geq t$ by the semigroup equation; by analyticity, this equation can be extended to all $s > 0$. But

$$\mu = C(\overline{\Omega})\text{-weak} \lim_{h \to 0+} S_\Sigma'(h)\mu$$

($S_\Sigma'(t)$ is $C(\overline{\Omega})$-weakly continuous at $t = 0$) so that $\mu = 0$. This ends the proof.

For the Dirichlet boundary condition we show in the same way that $S_c(t)C_0(\overline{\Omega})$ is dense in $C_0(\overline{\Omega})$.

6. SEPARATION, I

The control space for (2.1) is $L^\infty((0, T) \times \Omega)$. To fit it to the model (2.3) we use weak-L^∞ spaces. If X is a Banach space, $L_w^\infty(0, T; X^*)$ consists of all X^*-valued, X-weakly measurable functions $u(\cdot)$ such that $|\langle y, u(\cdot)\rangle| \leq C$ a.e. for all $y \in X$ ("a.e." depending on y; the least C that does the job serves as the norm of $u(\cdot)$). With this definition, $L^\infty((0,T)\times\Omega) = L^\infty_w(0,T;L^\infty(\Omega))$. Solutions of (2.3) with $\zeta \in L^\infty(\Omega)$ are defined by the variation-of-constants formula

$$y(t) = S_\infty(t)\zeta + \int_0^t S_\infty(t-\sigma)u(\sigma)\,d\sigma. \tag{6.1}$$

In view of (5.1), the integrand $S_\infty(t-\sigma)u(\sigma)$ takes values in $C(\overline{\Omega})$ for $\sigma < t$; moreover, although $u(\cdot) \in L^\infty_w(0,T;L^\infty(\Omega))$ is merely $L^1(\Omega)$-weakly measurable, the function $\sigma \to S_\infty(t-\sigma)u(\sigma)$ is strongly measurable (see [9, Chap. 7] for details). It is easily shown that $y(\cdot)$ is continuous in the norm of $C(\overline{\Omega})$ in $t > 0$ (in $t \geq 0$ if $\zeta \in C(\overline{\Omega})$).

The subspace of $C(\overline{\Omega})$ of all elements of the form

$$y = \int_0^t S_\infty(t-\sigma)u(\sigma)\,d\sigma \qquad (u(\cdot) \in L^\infty_w(0,T;L^\infty(\Omega)))$$

is denoted $R^\infty(t)$ and given the norm

$$\|y\|_{R^\infty(t)} = \inf\left\{\|u\|_{L^\infty_w(0,T;L^\infty(\Omega))};\ \int_0^t S_\infty(t-\sigma)u(\sigma)\,d\sigma = y\right\}.$$

The unit ball of this space is named $B^\infty(t)$. It can be easily shown [4, 11] that $R^\infty(t)$ is independent of $t > 0$ and that all the norms $\|\cdot\|_{R^\infty(t)}$ are equivalent for $t > 0$.

We equip $D(A_\infty) \subseteq C(\overline{\Omega})$ with its graph norm $\|y\|_{D(A_\infty)} = \|A_\infty y\|_{L^\infty(\Omega)}$.

LEMMA 6.1. *For every $t > 0$ we have $D(A_\infty) \hookrightarrow R^\infty(t)$.*

Proof. The imbedding will follow if we prove the formula

$$y = \int_0^t S_\infty(t-\sigma)\frac{1}{t}(y - \sigma A_\infty y)\,d\sigma \tag{6.2}$$

for all $y \in D(A_\infty)$. We begin by noting that the equality is true (integration by parts) when $y \in D(A_c)$ (see [6, (3.5)]). Using this particular case and (5.1), we obtain

$$S_\infty(h)y = \int_0^t S_c(t-\sigma)\frac{1}{t}(S_\infty(h)y - \sigma A_c S_\infty(h)y)\,d\sigma. \tag{6.3}$$

We apply the element of $L^\infty(\Omega)$ on both sides of (6.3) to $z \in L^1(\Omega)$, obtaining

$$\begin{aligned}\langle S_1'(h)z, y\rangle &= \langle z, S_\infty(h)y\rangle \\ &= \frac{1}{t}\int_0^t \langle S_1'(t-\sigma)z, S_\infty(h)y - \sigma A_c S_\infty(h)y\rangle d\sigma \\ &= \frac{1}{t}\int_0^t \langle S_1'(t+h-\sigma)z, y - \sigma A_\infty y\rangle d\sigma,\end{aligned}$$

and take limits as $h \to 0$; the result is

$$\langle z, y\rangle = \frac{1}{t}\int_0^t \langle S_1'(t-\sigma)z, y - \sigma A_\infty y\rangle\, d\sigma,$$

which, since $z \in L^1(\Omega)$ is arbitrary, is equivalent to (6.2). This ends the proof.

The equality

$$S_\infty(t)\int_0^T S_\infty(T-\sigma)u(\sigma)\,d\sigma = \int_0^T S_\infty(T-\sigma)S_\infty(t)u(\sigma)\,d\sigma$$

shows that $S_\infty(t) \in (R^\infty(T), R^\infty(T))$. We have

$$(S_\infty(t+h) - S_\infty(t))\int_0^T S_\infty(T-\sigma)u(\sigma)\,d\sigma$$
$$= \int_0^T S_\infty(T-\sigma)(S_\infty(t+h) - S_\infty(t))u(\sigma)\,d\sigma,$$

so that $S_\infty(t)$ is continuous in the norm of $(R^\infty(T), R^\infty(T))$ for $t > 0$, although not necessarily strongly continuous at the origin (see [11]).

Assume there is an admissible control driving an initial condition ζ to a target $\bar{y}$ in time $\bar{t}$. This means

$$S_\infty(\bar{t})\zeta + \int_0^{\bar{t}} S_\infty(\bar{t}-\sigma)\bar{u}(\sigma)\,d\sigma = \bar{y},$$

so that $\bar{y} - S_\infty(\bar{t})\zeta \in B^\infty(\bar{t})$.

LEMMA 6.2. *Let $\bar{u}(\cdot)$ be time optimal in the interval $0 \le t \le \bar{t}$. Then $\bar{y} - S_\infty(\bar{t})\zeta$ is a boundary point of $B^\infty(\bar{t})$.*

Proof. If this is false, on account of $R^\infty(T)$-continuity of $S_\infty(t)\zeta$ there exist $s < \bar{t}$ and $r > 1$ such that $r^{-1}(\bar{y} - S_\infty(s))\zeta \in B^\infty(T)$, which translates into

$$\bar{y} = S_\infty(s)\zeta + \int_0^{\bar{t}} S_\infty(\bar{t}-\sigma)u(\sigma)\,d\sigma \tag{6.4}$$

with $\|u\|_{L^\infty_w(0,\bar{t};L^\infty(\Omega))} = \|u\|_{L^\infty((0,\bar{t})\times\Omega)} \le r$. We use now the equality

$$\int_0^{\bar{t}} S_\infty(\bar{t}-\sigma)u(\sigma)d\sigma = \int_0^s S_\infty(s-\sigma)\left(\frac{S_\infty(\sigma)}{s}\int_0^{\bar{t}-s} S_\infty(\bar{t}-s-\tau)u(\tau)d\tau\right)d\sigma$$
$$+\int_0^s S_\infty(s-\sigma)u(\sigma+\bar{t}-s)d\sigma$$
$$= \int_0^s S_\infty(s-\sigma)v(\sigma)d\sigma, \tag{6.5}$$

where

$$\|v(\sigma)\| \le \frac{M^2(\bar{t} - s)}{s} r + r, \tag{6.6}$$

M a bound for $\|S_\infty(\sigma)\|$ in $0 \le \sigma \le \bar{t}$. We may now take s so close to $\bar{t}$ that the right side of (6.6) is ≤ 1. Then we have

$$\bar{y} = S_\infty(s)\zeta + \int_0^s S_\infty(s - \sigma)v(\sigma)\, d\sigma$$

with $\|v\|_{L_w^\infty(0,\, s; L^\infty(\Omega))} \le 1$, which contradicts time optimality. This ends the proof.

The "natural" venue for the separation argument is of course the space $R^\infty(t)$. However, there is evidence in [11, 12] that the dual of $R^\infty(\bar{t})$ may be too large for a good version of the maximum principle. Accordingly, separation will be performed in the space $D(A_\infty)$ *under the condition that* $\bar{y} - S_\infty(\bar{t})\zeta \in D(A_\infty)$. (Note that, by (5.1), $S_\infty(\bar{t})L^\infty(\Omega) \subseteq D(A_\infty)$, hence this condition reduces to (1.15).) The point $\bar{y} - S_\infty(\bar{t})\zeta$ will be separated from $B^\infty(\bar{t}) \cap D(A_\infty)$ by means of a functional $z \in D(A_\infty)^*$; that separation is possible is guaranteed by Lemma 6.2, which says $\bar{y} - S_\infty(\bar{t})\zeta$ is a boundary point of $B^\infty(\bar{t})$ (thus of $B^\infty(\bar{t}) \cap D(A_\infty)$) and Lemma 6.1, which states that $B^\infty(\bar{t}) \cap D(A_\infty)$ contains interior points in $D(A_\infty)$.

7. SEPARATION, II

Below, $[\eta]$ is an equivalence class in the space $L^\infty(\Omega)^*/\mathcal{N}_c(\Omega)$ equipped with the usual quotient norm $\|[\eta]\|_{L^\infty(\Omega)^*/\mathcal{N}_c(\Omega)} = \inf_{\eta \in [\eta]} \|\eta\|_{L^\infty(\Omega)^*}$.

LEMMA 7.1. *Every bounded linear functional Φ in $D(A_\infty)$ is given by*

$$\Phi(y) = \langle [\eta], A_\infty y \rangle = \langle \eta, A_\infty y \rangle \tag{7.1}$$

for some $[\eta] \in L^\infty(\Omega)^*/\mathcal{N}_c(\Omega)$. Moreover,

$$\|\Phi\|_{D(A_\infty)^*} = \|[\eta]\|_{L^\infty(\Omega)^*/\mathcal{N}_c(\Omega)}. \tag{7.2}$$

Proof. Let $\eta \in L^\infty(\Omega)^*$. Then (7.1) defines a bounded linear functional in $D(A_\infty)$; since $\mathcal{N}_c(\Omega)$ is the nullspace of $(A_\infty^{-1})^*$, (7.1) only depends on the equivalence class of $[\eta]$, and (7.2) follows. Conversely, let Φ be a bounded linear functional in $D(A_\infty)$. Then $\Phi(A_\infty^{-1}y)$ is a bounded linear functional in $L^\infty(\Omega)$, so that $\Phi(A_\infty^{-1}y) = \langle \eta, y \rangle$, which is equivalent to (7.1).

THEOREM 7.2 (The Maximum Principle). *Let $\bar{u}(\cdot)$ be time optimal in the interval $0 \le t \le \bar{t}$ for a target satisfying* (1.15). *Then there exists $\eta \in L^\infty(\Omega)^*$ such that*

$$\int_0^{\bar{t}} \|A_1' S_\infty(\bar{t}-t)^* \eta\|_{L^1(\Omega)} < \infty, \tag{7.3}$$

and[3]

$$\langle A_1' S_\infty(\bar{t}-t)^* \eta, \bar{u}(t)\rangle = \max_{\|u\|_{L^\infty(\Omega)} \le 1} \langle A_1' S_\infty(\bar{t}-t)^* \eta, u\rangle, \tag{7.4}$$

where the angled brackets indicate the duality of $L^1(\Omega)$ and $L^\infty(\Omega)$. Finally,

$$A_1' S_\infty(\bar{t}-t)^* \eta \ne 0 \qquad (0 \le t < \bar{t}). \tag{7.5}$$

Proof. We apply the separation argument and obtain $\eta \in L^\infty(\Omega)^*$, $\eta \ne 0$, such that

$$\left\langle \eta, A_\infty \int_0^{\bar{t}} S_\infty(\bar{t}-\sigma) u(\sigma)\, d\sigma \right\rangle \le \left\langle \eta, A_\infty \int_0^{\bar{t}} S_\infty(\bar{t}-\sigma) \bar{u}(\sigma)\, d\sigma \right\rangle = 1 \tag{7.6}$$

for all admissible controls $u(\cdot)$ such that

$$\int_0^{\bar{t}} S_\infty(\bar{t}-\sigma) u(\sigma)\, d\sigma \in D(A_\infty) \tag{7.7}$$

(the angled brackets in (7.6) indicate the duality of $L^\infty(\Omega)$ and $L^\infty(\Omega)^*$). Among controls satisfying (7.7) are those $u(\cdot)$ with $u(\sigma) = 0$ for $\bar{t} - \delta \le t \le \bar{t}$, and for them we may introduce A_∞ inside the integrals on the left of (7.6). By (5.4) we have

$$\begin{aligned} \langle \eta, A_\infty S_\infty(t) y\rangle &= \langle \eta, S_\infty(t/2) A_\infty S_\infty(t/2) y\rangle \\ &= \langle S_\infty(t/2)^* \eta, A_\infty S_\infty(t/2) y\rangle = \langle A_1' S_\infty(t/2)^* \eta, S_\infty(t/2) y\rangle \\ &= \langle S_\infty(t/2)^* A_1' S_\infty(t/2)^* \eta, y\rangle = \langle A_1' S_\infty(t)^* \eta, y\rangle \end{aligned}$$

for $t > 0$. Using this in the separation inequality (7.6) we obtain

$$\int_0^{\bar{t}-\delta} \langle A_1' S_\infty(\bar{t}-t)^* \eta, u(\sigma)\rangle\, d\sigma \le C$$

independently of δ, which shows (7.3). This proved, we may stick in (7.6) admissible controls $u(\cdot)$ such that $u(t) = \bar{u}(t)$ $(\bar{t} - \delta \le t \le \bar{t})$, cross out the integrals in the interval $(\bar{t}-\delta, \bar{t})$, and obtain

$$\int_0^{\bar{t}-\delta} \langle A_1' S_\infty(\bar{t}-\sigma)^* \eta, u(\sigma)\rangle d\sigma \le \int_0^{\bar{t}-\delta} \langle A_1' S_\infty(\bar{t}-\sigma)^* \eta, \bar{u}(\sigma)\rangle d\sigma.$$

[3]Condition (7.3) is independent of the maximum principle (7.4) but seems to be necessary for most, if not all applications of the maximum principle. Also, it is a sufficient condition for time optimality in certain cases. See [11, 12].

This gives (7.4) in every interval $0 \le t \le \bar{t} - \delta$, thus in $0 \le t < \bar{t}$.

To prove (7.5), assume that $A_1' S_\infty(t)^* \eta = 0$ for a single $t > 0$. Since A_1' is 1–1, $S_\infty(t)^* \eta = 0$; by the semigroup equation we have $S_\infty(\sigma)^* \eta = 0$ for $\sigma \ge t$ and then $S_\infty(\sigma)^* \eta = 0$ for all $\sigma > 0$ by analyticity. Now, we have

$$\int_0^{\bar{t}} S_\infty(\bar{t} - \sigma)\bar{u}(\sigma) d\sigma = \lim_{\delta \to 0} \int_0^{\bar{t}-\delta} S_\infty(\bar{t} - \sigma)\bar{u}(\sigma)\, d\sigma$$

in the norm of $L^\infty(\Omega)$, so that

$$\begin{aligned}\left\langle \eta, \int_0^{\bar{t}} S_\infty(\bar{t} - \sigma)\bar{u}(\sigma)\, d\sigma \right\rangle &= \lim_{\delta \to 0} \left\langle \eta, \int_0^{\bar{t}-\delta} S_\infty(\bar{t} - \sigma)\bar{u}(\sigma)\, d\sigma \right\rangle \\ &= \lim_{\delta \to 0} \int_0^{\bar{t}-\delta} \langle \eta, S_\infty(\bar{t} - \sigma)\bar{u}(\sigma) \rangle d\sigma \\ &= \lim_{\delta \to 0} \int_0^{\bar{t}-\delta} \langle S_\infty(\bar{t} - \sigma)^* \eta, \bar{u}(\sigma) \rangle d\sigma = 0,\end{aligned}$$

which contradicts the equality in (7.6). This ends the proof of Theorem 7.2.

One objection to this maximum principle is that the costate $z(t) = A_1' S_\infty(\bar{t} - t)^* \eta$ does not appear as a solution of the reverse adjoint equation $z'(t) = -A_1' z(t), z(\bar{t}) = z$. A fix is to "commute A_1' and $S_\infty(t)^*$": formally, $z(t) = A_1' S_\infty(\bar{t} - t)^* \eta = S_\infty(\bar{t} - t)^* A_\infty^* \eta$. To give sense to this we need to construct a space $\mathscr{Z} \supseteq L^\infty(\Omega)^*$ such that $A_\infty^* L^\infty(\Omega)^* = \mathscr{Z}$. ($A_\infty^*$ suitably extended to $L^\infty(\Omega)^*$). We do this below.

8. DUALS AND ADJOINTS, III

Let X be a Banach space, $B : X \to X$ linear, bounded, and one-to-one. We assume $B^2 X$ dense in BX in the norm of X; BX need not be dense in X.

THEOREM 8.1. *There exists a Banach space* $\mathscr{Z}, X \hookrightarrow \mathscr{Z}$ *and an operator* $\mathscr{B} : \mathscr{Z} \to X$ *such that* (a) $\mathscr{B}$ *is an isometry onto*, (b) $\mathscr{B}$ *is an extension of* B.

We define the space X_{-1} as the completion of X in the norm $\|y\|_{X_{-1}} = \|By\|_X$. Clearly, every $y \in X$ belongs to X_{-1} with

$$\|y\|_{X_{-1}} = \|By\|_X \le \|B\|_{(X,X)} \|y\|_X, \tag{8.1}$$

so that $X \hookrightarrow X_{-1}$. If $y \in X_{-1}$ then there exists a sequence $\{y_n\} \subseteq X$ with $y_n \to y$ in X_{-1}; this means that $\{By_n\}$ is Cauchy in X, thus $By_n \to z \in X$. If we define $B_1 y = z$, it is plain that the definition doesn't depend on $\{y_n\}$ and that $B_1 : X_{-1} \to X$ is linear and an extension of B. Finally, $\|z\|_X =$

$\lim \|By_n\|_X = \lim \|y_n\|_{X_{-1}} = \|y\|_{X_{-1}}$, so that $B_1 : X_{-1} \to X$ is an isometry, with

$$B_1 X_{-1} = \overline{BX} \qquad \text{(closure in } X\text{)}. \tag{8.2}$$

By virtue of (8.1) and the isometric character of $B_1 : X_{-1} \to X$ we have

$$\|B_1 y\|_{X_{-1}} \leq \|B\|_{(X,X)} \|B_1 y\|_X = \|B\|_{(X,X)} \|y\|_{X_{-1}}, \tag{8.3}$$

so that $B_1 : X_{-1} \to X_{-1}$ is a bounded operator with

$$\|B_1\|_{(X_{-1},X_{-1})} \leq \|B\|_{(X,X)}. \tag{8.4}$$

We apply the argument above to this operator and obtain a second space $X_{-2} = (X_{-1})_{-1}$ and an isometric extension $B_2 : X_{-2} \to X_{-1}$ of B_1. Every $y \in X_{-1}$ belongs to X_{-2} with

$$\|y\|_{X_{-2}} = \|B_2 y\|_{X_{-1}} \leq \|B_1\|_{(X_{-1},X_{-1})} \|y\|_{X_{-1}} \leq \|B\|_{(X,X)} \|y\|_{X_{-1}} \tag{8.5}$$

(the last inequality coming from (8.4)), thus $X_{-1} \hookrightarrow X_{-2}$ and

$$B_2 X_{-2} = \overline{B_1 X_{-1}}^1 \qquad \text{(closure in } X_{-1}\text{)}. \tag{8.6}$$

Now, (8.2) and the fact that the norm of X dominates the norm of X_{-1} imply

$$\overline{B_1 X_{-1}}^1 = \overline{\overline{BX}}^1 \subseteq \overline{\overline{BX}^1}^1 = \overline{BX}^1. \tag{8.7}$$

On the other hand, $B_1 X_{-1} \supseteq BX$, so that taking closure in X_{-1} the inclusion opposite to (8.7) results. We have then proved

$$\overline{B_1 X_{-1}}^1 = \overline{BX}^1, \tag{8.8}$$

and the equality in

$$B_2 X_{-2} = \overline{BX}^1 \supseteq X \tag{8.9}$$

follows. To show the inclusion, recall that $B^2 X$ is dense in BX in the norm of X. Hence, if $y \in X$ we have a sequence $\{y_n\} \subseteq X$ such that $\|By_n - y\|_{X_{-1}} = \|B^2 y_n - By\|_X \to 0$.

Since $B^2 : X_{-2} \to X_{-1}$ is an isometry, if $z \in X_{-2}$ is not zero, then $B_2 z$ is not zero in X_1; hence if $B_2 z \in X$ then $B_2 z \neq 0$ in X. If we define

$$\mathscr{Z} = \{\zeta \in X_{-2}; B_2 \zeta \in X\}, \tag{8.10}$$

it follows that the operator $B_2 : \mathscr{Z} \to X$ is 1–1 and (from (8.9)) onto; we rename it $\mathscr{B}$ and renorm $\mathscr{Z}$ with $\|\zeta\|_{\mathscr{Z}} = \|\mathscr{B}\zeta\|_X$. Note that

$$\|\zeta\|_{X_{-2}} = \|B_2 \zeta\|_{X_{-1}} \leq \|B\|_{(X,X)} \|B_2 \zeta\|_X = \|B\|_{(X,X)} \|\zeta\|_{\mathscr{Z}}, \tag{8.11}$$

so that $\mathcal{Z} \hookrightarrow X_{-2}$. To show that $\mathcal{Z}$ is a Banach space, let $\{\zeta_n\}$ be a Cauchy sequence in $\mathcal{Z}$. Then, after (8.11), $\{\zeta_n\}$ is Cauchy in X_{-2}; thus there exists $\zeta \in X_{-2}$ with $\|\zeta_n - \zeta\|_{X_{-2}} \to 0$. On the other hand, $\{B_2\zeta_n\}$ is Cauchy in X, thus there exists $y \in X$ with $\|B_2\zeta_n - y\|_X \to 0$. It then follows that $\|B_2\zeta_n - y\|_{X_{-1}} \to 0$, thus $y = B_2\zeta$. We then conclude that $\zeta \in \mathcal{Z}$ and $\|\zeta_n - \zeta\|_{\mathcal{Z}} = \|B_2\zeta_n - B_2\zeta\| \to 0$. This ends the proof.

We apply Theorem 8.1 to

$$X = L^\infty(\Omega)^*/\mathcal{N}_c(\Omega), \qquad B = (A_\infty^{-1})^*. \tag{8.12}$$

Since $\mathcal{N}_c(\Omega) \subseteq L^\infty(\Omega)^*$ is the nullspace of $(A_\infty^{-1})^*$, $B = (A_\infty^{-1})^*$ is one-to-one in $X = L^\infty(\Omega)^*/\mathcal{N}_c(\Omega)$. The other property of B we need is

Lemma 8.2. *The space $(A_\infty^{-1})^*(A_\infty^{-1})^*(L^\infty(\Omega)^*/\mathcal{N}_c(\Omega))$ is dense in the space $(A_\infty^{-1})^*(L^\infty(\Omega)^*/\mathcal{N}_c(\Omega))$.*

Proof. It is enough to show that $(A_\infty^{-1})^*(A_\infty^{-1})^*L^\infty(\Omega)^*$ is dense in $(A_\infty^{-1})^*L^\infty(\Omega)^*$. We have shown in Corollary 4.5 that $(A_\infty^{-1})^*L^\infty(\Omega)^* \subseteq L^1(\Omega)$. On the other hand, $(A_\infty^{-1})^*L^\infty(\Omega)^* = D(A_\infty^*)$ contains $C^{(2)}(\overline{\Omega})$, so that

$$\overline{(A_\infty^{-1})^*L^\infty(\Omega)^*} = L^1(\Omega) \tag{8.13}$$

(closure in $L^\infty(\Omega)^*$). Now, in view of (4.2), $A_\infty^{-1} = (A_1'^{-1})^*$. Taking adjoints, we obtain $(A_\infty^{-1})^* = (A_1'^{-1})^{**}$, and it follows from adjoint theory that

$$(A_\infty^{-1})^*|_{L^1(\Omega)} = (A_1'^{-1})^{**}|_{L^1(\Omega)} = A_1'^{-1}. \tag{8.14}$$

In view of (8.13), it is enough to show that $(A_\infty^{-1})^*L^1(\Omega)$ is dense in $L^1(\Omega)$ or, by (8.14), that $A_1'^{-1}L^1(\Omega) = D(A_1')$ is dense in $L^1(\Omega)$, which is obvious from the definition of A_1'. This ends the proof.

Let $\mathcal{Z}, \mathcal{B}$ be the space and operator provided by Theorem 8.1 for the space and operator (8.12).

Theorem 8.3. *The space $D(A_\infty)^*$ is algebraically and metrically isomorphic to $\mathcal{Z}$, the duality given by*

$$\langle\langle \zeta, y\rangle\rangle = \langle \mathcal{B}\zeta, A_\infty y\rangle. \tag{8.15}$$

Proof. By Lemma 7.1, every bounded linear functional in $D(A_\infty)$ is given by $\Phi(y) = \langle [\eta], A_\infty y\rangle = \langle \eta, A_\infty y\rangle$; by Theorem 8.1, $[\eta] = \mathcal{B}\zeta$ ($\zeta \in \mathcal{Z}$) with equality of norms: we then have

$$\|\Phi\|_{D(A_\infty)^*} = \|[\eta]\|_{L^\infty(\Omega)^*/\mathcal{N}_c(\Omega)} = \|\zeta\|_{\mathcal{Z}}.$$

as claimed.

To extend A_∞^* from $D(A_\infty^*)$ to $L^\infty(\Omega)^*/\mathcal{N}_c(\Omega)$ we only have to note that if $\mathscr{B}$ is the operator constructed in Theorem 8.1, then $\mathscr{B}^{-1}$ is an extension of B^{-1}. The extension $(A_\infty^*)_e : (L^\infty(\Omega)^*/\mathcal{N}_c(\Omega)) \to \mathfrak{X}$ is

$$(A_\infty^*)_e = \mathscr{B}^{-1}. \tag{8.16}$$

We extend $S_\infty(t)^*$ to $\mathfrak{X}$ (under the name $(S_\infty(t)^*)_e$) setting

$$(S_\infty(t)^*)_e(A_\infty^*)_e[\eta] = A_1' S_\infty(t)^*[\eta]. \tag{8.17}$$

Obviously, this definition is consistent in that, by Lemma 5.1, it does not depend on the element η in the equivalence class $[\eta]$. To check that $(S_\infty(t)^*)_e$ is actually an extension of $S_\infty(t)^*$ we must prove (8.17), as a theorem, for $[\eta] \in D(A_\infty^*)$. To do this, let $y \in D(A_\infty)$. Then

$$\begin{aligned}\langle (S_\infty(t)^*)_e(A_\infty^*)_e[\eta], y\rangle &= \langle S_\infty(t)^* A_\infty^* \eta, y\rangle = \langle A_\infty^* \eta, S_\infty(t) y\rangle \\ &= \langle \eta, A_\infty S_\infty(t) y\rangle = \langle \eta, S_\infty(t) A_\infty y\rangle \\ &= \langle S_\infty(t)^* \eta, A_\infty y\rangle = \langle A_\infty^* S_\infty(t)^* \eta, y\rangle \\ &= \langle A_1' S_\infty(t)^* \eta, y\rangle,\end{aligned}$$

where in the last step we have used Corollary 4.5.

THEOREM 8.4 (The Maximum Principle). *Let $\bar{u}(\cdot)$ be a time optimal control in the interval $0 \le t \le \bar{t}$ for a target satisfying* (1.15). *Then there exists $\zeta \in \mathfrak{X}$ such that*

$$\int_0^{\bar{t}} \|S_\infty(\bar{t} - t)^* \zeta\|_{L^1(\Omega)} < \infty \tag{8.18}$$

and

$$\langle S_\infty(\bar{t} - t)^* \zeta, \bar{u}(t)\rangle = \max_{\|u\|_{L^\infty(\Omega)} \le 1} \langle S_\infty(\bar{t} - t)^* \zeta, u\rangle, \tag{8.19}$$

where the angled brackets indicate the duality of $L^1(\Omega)$ and $L^\infty(\Omega)$. Finally,

$$S_\infty(\bar{t} - t)^* \zeta \neq 0 \qquad (0 \le t < \bar{t}). \tag{8.20}$$

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