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Matricially free random variables

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Abstract

We show that the framework developed by Voiculescu for free random variables can be extended to arrays of random variables whose multiplication imitates matricial multiplication. The associated notion of independence, called *matricial freeness*, can be viewed as a concept which not only leads to a natural generalization of freeness, but also underlies other fundamental types of noncommutative independence, such as monotone independence and boolean independence. At the same time, the sums of matricially free random variables, called *random pseudomatrices*, are closely related to random matrices. The main results presented in this paper concern the standard and tracial central limit theorems for random pseudomatrices and the corresponding limit distributions which can be viewed as matricial semicircle laws.

Keywords: Free probability; Matricial freeness; Strong matricial freeness; Matricially free random variable; Matricially free Fock space; Random matrix; Random pseudomatrix

1. Introduction

It has been shown by Voiculescu [25] that free random variables arise naturally as limits of random matrices. In particular, if we take symmetric matrices whose entries form a family of independent Gaussian random variables and we let the size of these matrices go to infinity, their joint distribution (with respect to normalized trace composed with classical expectation) converges to the joint distribution of freely independent random variables with the semicircle distribution obtained by Wigner [29] as the limit distribution of one Gaussian random matrix.

Therefore, we can study free random variables, using at least two different frameworks: operator algebras and random matrices. However, it is to some extent surprising that a random

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matrix framework, of quite different nature than that of operator algebras, exists for free random variables, and the connection between these two approaches does not seem to be very transparent. In this connection, our first motivation is to better understand the relation between the operatorial approach to free probability and random matrices.

The second motivation comes from the question whether different types of independence, like freeness of Voiculescu, monotone independence of Muraki [19] and boolean independence based on the regular free product of Bożejko [5], can be included in one natural framework. Note in this context that models with more than one state on a given algebra, like conditional freeness of Bożejko and Speicher [6] and freeness with infinitely many states of Cabanal-Duvillard and Ionesco [8,9] extend free probability and include, as shown by Franz [12], certain elements of monotone probability. For instance, this can be done for convolutions, but including monotone independence in the framework of conditional freeness can be done only under additional (rather restrictive) assumptions on the considered algebras. We show in this paper that one can remedy this situation by introducing a concept of 'independence' which reminds freeness, but at the same time has some 'matricial' features which places it somewhere between freeness and the model of random matrices.

A different reason to look for a new concept of independence arises from concrete examples of interpolations between free probability and monotone probability [17,18]. In particular, the continuous (p,q)-Brownian motions with Kesten distributions and related Poisson processes studied in [2,18] lead to the first example of a two-mode interacting Fock space introduced directly and not by means of orthogonal polynomials. This example has certain matricial features which also call for a new model of 'independence' that would be related to freeness.

The main result of our paper is the construction of a model, called *matricial freeness*, which is related to the concept of the free product of states introduced and studied by Ching [10] in the context of von Neumann algebras and by Avitzour [3] and Voiculescu [24] in the context of C^*-algebras. The underlying concept is that of the matricially free product of an array of Hilbert spaces with distinguished unit vectors

$$(\mathcal{H},\xi) = *_{i,j}^{M}(\mathcal{H}_{i,j},\xi_{i,j})$$

which reminds the free product of Hilbert spaces, to which we associate the matricially free product of states. Multiplication of 'matricially free random variables' under this product of states reminds the free product but it also satisfies the condition imitating matrix multiplication. Similarities between free probability and 'matricially free probability' hold also on other levels, some of which are studied in this paper.

Strictly speaking, however, one needs to take a 'restriction' of matricial freeness, in which not all products imitating matrix multiplication are included, called *strong matricial freeness*, to recover freeness and monotone independence without using the asymptotics. In particular, let $(X_{i,j})$ be a finite array of strongly matricially free random variables from some unital algebra $\mathcal{A}$ which includes the diagonal. Then

(A) the sums corresponding to the rows of a square array,

$$A_i := \sum_j X_{i,j}$$

are free with respect to φ,

(B) the sums corresponding to the rows of a lower-triangular array,

$$B_i := \sum_{j \leqslant i} X_{i,j}$$

are monotone independent with respect to φ,

(C) the diagonal variables $X_{j,j}$, $j \in I$, are boolean independent with respect to φ,

where φ is a distinguished state on $\mathcal{A}$ which lies everywhere on the diagonal of the associated array of states $(\varphi_{i,j})$. In the case of triangular arrays, we tacitly assume that the index sets involved are linearly ordered. Let us add that upper triangular arrays give anti-monotone random variables.

In this paper, of main interest to us are limit theorems, in which we consider a sequence $(X_{i,j}(n))_{1 \leqslant i,j \leqslant n}$ of matricially free arrays of variables taken from unital $*$-algebras $\mathcal{A}(n)$, respectively, equipped with distinguished states $\phi(n)$ and associated states $(\phi_j(n))_{1 \leqslant j \leqslant n}$ called 'conditions'. We study sums

$$S(n) = \sum_{i,j=1}^{n} X_{i,j}(n)$$

called *random pseudomatrices*, and their asymptotic distributions with respect to the states $\phi(n)$ and with respect to normalized traces

$$\psi(n) = \frac{1}{n} \sum_{j=1}^{n} \phi_j(n),$$

respectively. We assume that the distributions of the $X_{i,j}(n)$ in the states $\phi(n)$ and $\phi_j(n)$ depend on n in a suitable way and are block-identical.

It turns out that the central limit theorem for the distributions of random pseudomatrices in the states $\phi(n)$ may be viewed as an analog of the central limit theorem for free random variables (especially, if we take square arrays). In turn, the 'tracial' central limit theorem for random pseudomatrices in the states $\psi(n)$ is related to the limit theorem for random matrices (especially, if we take square arrays with block-symmetric variances). The limit distributions play then the role of multivariate generalizations of the semicircle distributions. Let us point out, however, that when we consider lower-triangular arrays, the central limit laws can be viewed as generalizations of the arcsine law.

More importantly, refinements of our limit theorems give a deeper analogy between random pseudomatrices and random matrices. By a refinement we understand the scheme in which pseudomatrices are replaced by smaller sums which play the role of blocks. Namely, in the case of block-symmetric variances

(D) the $\psi(n)$-distributions of symmetric blocks of random pseudomatrices agree asymptotically with those of random symmetric blocks, where random symmetric blocks are symmetric blocks of random matrices in the approach of Voiculescu [25] and Dykema [11]. It is worth mentioning that in this scheme the difference between blocks built from matricially free random variables and strongly matricially free random variables disappears as $n \to \infty$ (in both cases blocks are asymptotically matricially free).

This and other asymptotic properties of random pseudomatrices and their blocks are studied in a subsequent paper [16]. Among these properties are also asymptotic versions of (A)–(C) which refer to matricial freeness. Thus, the subarrays corresponding to row blocks of pseudomatrices are asymptotically free, asymptotically monotone independent, or asymptotically boolean independent, depending on whether the pseudomatrices are square, block lower-triangular, or block diagonal, respectively. Therefore, matricial freeness can be treated as a notion of independence which underlies the fundamental types of noncommutative independence as well as the asymptotic structure of random matrices.

In Section 2, we introduce the concepts of the 'matricially free product of states' and the 'matricially free Fock space'. We obtain from these structures their strong counterparts in Section 3. In Section 4, we introduce the notions of 'matricial freeness' and 'strong matricial freeness' and discuss the example of the discrete (strongly) matricially free Fock space. In Section 5, of combinatorial nature, we define and study certain real-valued functions on the set of non-crossing partitions, defined in terms of traces of certain matrices. In Section 6, we study the asymptotic behavior of random pseudomatrices and we prove standard and tracial central limit theorems. The limit distributions, which can be interpreted as matricial multivariate generalizations of semicircle laws, are studied in Section 7. Their decompositions in terms of s-free additive convolutions in the case of two-dimensional arrays are proved in Section 8. Two geometric realizations of the limit distributions, in terms of walks on weighted binary trees and in terms of weighted Catalan paths, are given in Section 9.

2. Matricially free products

In this section we introduce the notion of the matricially free product of states as well as the corresponding notions of the matricially free product of Hilbert spaces and the matricially free Fock space.

When speaking of arrays indexed by two indices, say i, j, we shall assume that $(i, j) \in J \subseteq I \times I$, where $\{(j, j)\colon\ j \in I\} = \Delta \subset J$ and I is an index set. This refers to the situation when we deal with *arrays which contain the diagonal*. In particular, this includes lower-triangular arrays of the form $J := \{(i, j)\colon\ i \geqslant j,\ i, j \in I\}$, in which case we shall tacitly assume that I is linearly ordered. Without loss of generality we can use the square array formulation most of the time and take their subarrays if needed. Of special interest will be the finite-dimensional case when $I = [n] := \{1, 2, \ldots, n\}$.

Definition 2.1. Let $\widehat{\mathcal{H}} := (\mathcal{H}_{i,j})$ be an array of complex Hilbert spaces. By the *matricially free Fock space over* $\widehat{\mathcal{H}}$ we understand the Hilbert space direct sum

$$\mathcal{M}(\widehat{\mathcal{H}}) = \mathbb{C}\Omega \oplus \bigoplus_{m=1}^{\infty} \bigoplus_{\substack{(i_1,i_2)\neq\cdots\neq(i_m,i_m)\\ n_1,\ldots,n_m\in\mathbb{N}}} \mathcal{H}_{i_1,i_2}^{\otimes n_1} \otimes \mathcal{H}_{i_2,i_3}^{\otimes n_2} \otimes \cdots \otimes \mathcal{H}_{i_m,i_m}^{\otimes n_m},$$

where Ω is a unit vector, with the canonical inner product.

Let us observe that the Hilbert space tensor powers which appear in the above direct sum have the following three properties:

(P1) the 'matricial property' – the second index of the preceding power agrees with the first index of the following power,
(P2) the 'freeness property' – the consecutive pairs of indices are different,
(P3) the 'diagonal subordination property' – the last pair is 'diagonal'.

The last property is needed to ensure existence of limit mixed moments in limit theorems with the square-root normalization. The term 'subordination' follows from the 'subordination' of the related additive convolution of an array of measures to the diagonal measures. Although we do not study this convolution in this paper, the basic idea can be understood on the example of the binary tree in Section 9, which can be viewed as a comb product of two graphs, of which the first one is labelled by a diagonal variance.

Of course, if the index set I consists of one element and thus $\widehat{\mathcal{H}}$ is just one Hilbert space $\mathcal{H}$, the corresponding matricially free Fock space reduces to the usual free Fock space $\mathcal{F}(\mathcal{H})$. In general, however, $\mathcal{M}(\widehat{\mathcal{H}})$ is a (usually, proper) subspace of the free Fock space $\mathcal{F}(\bigoplus_{i,j}\mathcal{H}_{i,j}) \cong *_{i,j}\mathcal{F}(\mathcal{H}_{i,j})$.

Related to the 'matricially free Fock space' is the 'matricially free product of Hilbert spaces'. The terminology parallels that introduced in free probability [24,28].

Definition 2.2. Let $(\mathcal{H}_{i,j}, \xi_{i,j})$ be an array of Hilbert spaces with distinguished unit vectors. By the *matricially free product of* $(\mathcal{H}_{i,j}, \xi_{i,j})$ we understand the pair $(\mathcal{H}, \xi)$, where

$$\mathcal{H} = \mathbb{C}\xi \oplus \bigoplus_{m=1}^{\infty} \bigoplus_{(i_1,i_2)\neq\cdots\neq(i_m,i_m)} \mathcal{H}^0_{i_1,i_2} \otimes \mathcal{H}^0_{i_2,i_3} \otimes \cdots \otimes \mathcal{H}^0_{i_m,i_m},$$

with $\mathcal{H}^0_{i,j} = \mathcal{H}_{i,j} \ominus \mathbb{C}\xi_{i,j}$ and ξ being a unit vector. We denote it $(\mathcal{H}, \xi) = *^M_{i,j}(\mathcal{H}_{i,j}, \xi_{i,j})$.

Proposition 2.1. *It holds that*

$$\big(\mathcal{M}(\widehat{\mathcal{H}}), \xi\big) \cong *^M_{i,j}\big(\mathcal{M}(\mathcal{H}_{i,j}), \xi_{i,j}\big).$$

Proof. We use the definition of the matricially free Fock space, the isomorphism $\mathcal{M}(\mathcal{H}_{i,j}) \cong \mathcal{F}(\mathcal{H}_{i,j})$ for any i, j and regroup terms. □

For any j, introduce *diagonal subspaces* of $\mathcal{H}$ of the form

$$\mathcal{H}(j,j) = \mathbb{C}\xi \oplus \bigoplus_{m=2}^{\infty} \bigoplus_{\substack{(j,i_2)\neq\cdots\neq(i_m,i_m)\\ i_2\neq j}} \mathcal{H}^0_{j,i_2} \otimes \mathcal{H}^0_{i_2,i_3} \otimes \cdots \otimes \mathcal{H}^0_{i_m,i_m},$$

and the associated *diagonal partial isometries* $V_{j,j} : \mathcal{H}_{j,j} \otimes \mathcal{H}(j,j) \to \mathcal{H}$:

$$\xi_{j,j} \otimes \xi \to \xi,$$
$$\mathcal{H}^0_{j,j} \otimes \xi \to \mathcal{H}^0_{j,j},$$
$$\xi_{j,j} \otimes \left(\mathcal{H}^0_{j,j_1} \otimes \cdots \otimes \mathcal{H}^0_{j_m,j_m}\right) \to \mathcal{H}^0_{j,j_1} \otimes \cdots \otimes \mathcal{H}^0_{j_m,j_m},$$
$$\mathcal{H}^0_{j,j} \otimes \left(\mathcal{H}^0_{j,j_1} \otimes \cdots \otimes \mathcal{H}^0_{j_m,j_m}\right) \to \mathcal{H}^0_{j,j} \otimes \mathcal{H}^0_{j,j_1} \otimes \cdots \otimes \mathcal{H}^0_{j_m,j_m},$$

where $m > 1$.

For any $i \neq j$ we introduce *off-diagonal subspaces* of $\mathcal{H}$ of the form

$$\mathcal{H}(i,j) = \bigoplus_{m=1}^{\infty} \bigoplus_{(j,i_2)\neq\cdots\neq(i_m,i_m)} \mathcal{H}^0_{j,i_2} \otimes \mathcal{H}^0_{i_2,i_3} \otimes \cdots \otimes \mathcal{H}^0_{i_m,i_m}$$

and the associated *off-diagonal partial isometries* $V_{i,j} : \mathcal{H}_{i,j} \otimes \mathcal{H}(i,j) \to \mathcal{H}$ for $i \neq j$:

$$\xi_{i,j} \otimes \left(\mathcal{H}^0_{j,j_1} \otimes \cdots \otimes \mathcal{H}^0_{j_m,j_m}\right) \to \mathcal{H}^0_{j,j_1} \otimes \cdots \otimes \mathcal{H}^0_{j_m,j_m},$$
$$\mathcal{H}^0_{i,j} \otimes \left(\mathcal{H}^0_{j,j_1} \otimes \cdots \otimes \mathcal{H}^0_{j_m,j_m}\right) \to \mathcal{H}^0_{i,j} \otimes \mathcal{H}^0_{j,j_1} \otimes \cdots \otimes \mathcal{H}^0_{j_m,j_m},$$

where $m \geqslant 1$.

Each $\mathcal{H}(i,j)$ is spanned by simple tensors which do not begin with vectors from $\mathcal{H}^0_{i,j}$ and for that reason it is suitable for the left free action of the operators creating such vectors. Thus, roughly speaking, both types of partial isometries jointly replace the unitary maps used in free probability. It is the diagonal subordination property which is responsible for distinguishing two types of isometries.

Consider an array of C^*-algebras $(\mathcal{A}_{i,j})$, each with a unit $1_{i,j}$ and a state $\varphi_{i,j}$, and let $(\mathcal{H}_{i,j}, \pi_{i,j}, \xi_{i,j})$ be the associated GNS triples, so that $\varphi_{i,j}(a) = \langle \pi_{i,j}(a)\xi_{i,j}, \xi_{i,j}\rangle$ for any $a \in \mathcal{A}_{i,j}$. For any i, j, let $\lambda_{i,j}$ be the $*$-representation of $\mathcal{A}_{i,j}$ given by

$$\lambda_{i,j}(a) = V_{i,j}\left(\pi_{i,j}(a) \otimes I_{\mathcal{H}(i,j)}\right)V^*_{i,j} \quad \text{for } a \in \mathcal{A}_{i,j},$$

where $I_{\mathcal{H}(i,j)}$ denotes the identity on $\mathcal{H}(i,j)$. Note that these representations are, in general, non-unital. In fact,

$$\lambda_{i,j}(1_{i,j}) = V_{i,j}V^*_{i,j} = r_{i,j} + s_{i,j},$$

where $r_{i,j}$ and $s_{i,j}$ are canonical projections in $B(\mathcal{H})$ given by

$$r_{i,j} = P_{\mathcal{H}(i,j)} \quad \text{and} \quad s_{i,j} = P_{\mathcal{K}(i,j)},$$

where $\mathcal{K}(i,j) = \mathcal{H}^0_{i,j} \otimes \mathcal{H}(i,j)$. For given i, j, the projections $r_{i,j}$ and $s_{i,j}$ are orthogonal and their sum is the canonical projection onto the subspace of $\mathcal{H}$ onto which $\lambda_{i,j}(\mathcal{A}_{i,j})$ acts non-trivially.

The $\lambda_{i,j}$'s remind the representations λ_i of free probability [24,28], but the corresponding operators $\lambda_{i,j}(a)$ have larger kernels. Using $\lambda_{i,j}$'s, we shall define product representations on

$$\mathcal{A} := \bigsqcup_{i,j} \mathcal{A}_{i,j},$$

the free product without identification of units, equipped with the unit $1_{\mathcal{A}}$, and products of states which are analogs of the free product representation and the free product of states, respectively.

Definition 2.3. The *matricially free product representation* $\pi_M = *_{i,j}^M \pi_{i,j}$ is the unital $*$-homomorphism $\lambda : \mathcal{A} \to B(\mathcal{H})$ given by the linear extension of

$$\lambda(1_{\mathcal{A}}) = \mathbf{1} \quad \text{and} \quad \lambda(a_1 a_2 \dots a_n) = \lambda_{i_1,j_1}(a_1)\lambda_{i_2,j_2}(a_2)\dots\lambda_{i_n,j_n}(a_n)$$

for any $a_k \in \mathcal{A}_{i_k,j_k}$, $k = 1, \dots, n$, with $(i_1, j_1) \neq (i_2, j_2) \neq \dots \neq (i_n, j_n)$. The associated state $\varphi = *_{i,j}^M \varphi_{i,j} : \mathcal{A} \to \mathbb{C}$ is given by

$$\varphi(a) = \langle \pi_M(a)\xi, \xi \rangle$$

and will be called the *matricially free product of* $(\varphi_{i,j})$.

Basic properties of the product state φ are collected in the proposition given below. Roughly speaking, they show that this state (on the free product of C^*-algebras without identification of units) has similar properties as the free product of states (on the free product of C^*-algebras with identification of units) except that the units of these algebras act as units only on 'matricial' tensor products and otherwise they act as null projections.

For that purpose, it will be useful to introduce sets of indices associated with 'matricial' tensor products:

$$\Lambda_n = \{((i_1, i_2), (i_2, i_3), \dots, (i_n, i_{n+1})) : (i_1, i_2) \neq (i_2, i_3) \neq \dots \neq (i_n, i_{n+1})\}$$

and their union $\Lambda = \bigcup_{n=1}^{\infty} \Lambda_n$. Finally, $\mathcal{I}$ stands for the unital subalgebra of $\mathcal{A}$ generated by the units $1_{i,j}$. In analogy to the notion of marginal laws in classical probability, by *marginal moments* we shall understand moments of the form $\varphi_{i,j}(a_1 \dots a_k)$, where $a_1 \dots a_k \in \mathcal{A}_{i,j}$ and i, j are arbitrary.

Proposition 2.2. *Let φ be the matricially free product of states $(\varphi_{i,j})$ and let $a_k \in \mathcal{A}_{i_k,j_k}$, where $k \in [n]$ and $(i_1, j_1) \neq \dots \neq (i_n, j_n)$.*

1. *If $a_k \in \operatorname{Ker} \varphi_{i_k,j_k}$ for $k \in [n]$, then $\varphi(a_1 a_2 \dots a_n) = 0$.*
2. *If $a_r = 1_{i_r,j_r}$ and $a_m \in \operatorname{Ker} \varphi_{i_m,j_m}$ for $r < m \leqslant n$, then*

$$\varphi(a_1 \dots a_n) = \begin{cases} \varphi(a_1 \dots a_{r-1} a_{r+1} \dots a_n) & \text{if } ((i_r, j_r), \dots, (i_n, j_n)) \in \Lambda, \\ 0 & \text{otherwise.} \end{cases}$$

3. *For any $a \in \mathcal{A}$, $u_1, u_2 \in \mathcal{I}$ and $i, j \in I$, it holds that*

$$\varphi(u_1 a u_2) = \varphi(u_1)\varphi(a)\varphi(u_2) \ \text{and} \ \varphi(1_{i,j}) = \delta_{i,j}.$$

4. *The restriction of φ to $\mathcal{A}_{j,j}$ is $\varphi_{j,j}$ for any $j \in I$.*
5. *The mixed moments $\varphi(a_1 a_2 \ldots a_n)$ are uniquely expressed in terms of marginal moments.*

Proof. If $a_k \in \operatorname{Ker}\varphi_{i_k,j_k}$ for $k \in [n]$, where $(i_1, j_1) \neq \cdots \neq (i_n, j_n)$, then it follows from the definition of the $\lambda_{i,j}$ that

1. $\pi_M(a_1 \ldots a_n)\xi = 0$ if $((i_1, j_1), \ldots, (i_n, j_n)) \notin \Lambda$ or $i_n \neq j_n$,
2. $\pi_M(a_1 \ldots a_n)\xi \in \mathcal{H}^0_{i_1,j_1} \otimes \cdots \otimes \mathcal{H}^0_{i_n,j_n}$ if $((i_1, j_1), \ldots, (i_n, j_n)) \in \Lambda$ and $i_n = j_n$.

In both cases we obtain a vector orthogonal to ξ on the RHS, which proves (1). Suppose now that the assumptions of (2) hold. If $((i_r, j_r), \ldots, (i_n, j_n)) \in \Lambda$, then $\lambda_{i_r,j_r}(1_{i_r,j_r})$ acts as a unit on $\mathcal{H}^0_{i_{r+1},j_{r+1}} \otimes \cdots \otimes \mathcal{H}^0_{i_n,j_n}$ by the definition of the representations $\lambda_{i,j}$. On the other hand, $\lambda_{i_r,j_r}(1_{i_r,j_r})$ kills any simple tensor beginning with $h \in \mathcal{H}^0_{i_{r+1},j_{r+1}}$ if $j_r \neq i_{r+1}$ or $((i_{r+1}, j_{r+1}), \ldots, (i_n, j_n)) \notin \Lambda$ since V_{i_r,j_r} does, which completes the proof of (2). In turn, (3) follows from the action of the $\lambda(1_{i,j})$ onto ξ. That φ agrees with $\varphi_{j,j}$ on $\mathcal{A}_{j,j}$ for any $j \in I$ follows from the action of the $\lambda_{j,j}(a)$, $a \in \mathcal{A}_{j,j}$, onto ξ, namely $\pi_M(a)\xi = (\pi_{j,j}(a)\xi)^0 + \varphi_{j,j}(a)\xi$, which gives (4). Finally, (5) is a consequence of (1)–(2). $\square$

In a similar way we can define states associated with other unit vectors from $\mathcal{H}$. We shall consider the simplest case of states associated with unit vectors $e_j \in \mathcal{H}^0_{j,j}$, $j \in I$, which are in the ranges of $\pi_{j,j}(\mathcal{A}_{j,j})$, respectively, namely $\varphi_j : \mathcal{A} \to \mathbb{C}$ defined by the formulas

$$\varphi_j(a) = \langle \pi_M(a)e_j, e_j \rangle,$$

called *conditions* associated with φ, which will be used for computing normalized traces. Most properties of the states φ_j are inherited from φ as the proposition given below demonstrates. However, $\varphi_j|\mathcal{I}$ is quite different than $\varphi|\mathcal{I}$ due to different normalization conditions.

Proposition 2.3. *Let φ_j, $j \in I$, be the conditions associated with φ and let $a_k \in \mathcal{A}_{i_k,j_k}$, where $k \in [n]$ and $(j, j) \neq (i_1, j_1) \neq \cdots \neq (i_n, j_n) \neq (j, j)$.*

1. *If $a_k \in \operatorname{Ker}\varphi_{i_k,j_k}$ for $k \in [n]$, then $\varphi_j(a_1 a_2 \ldots a_n) = 0$ for each j.*
2. *If $a_r = 1_{i_r,j_r}$ and $a_m \in \operatorname{Ker}\varphi_{i_m,j_m}$ for $r < m \leqslant n$, then*

$$\varphi_j(a_1 \ldots a_n) = \begin{cases} \varphi_j(a_1 \ldots a_{r-1} a_{r+1} \ldots a_n) & \text{if } ((i_r, j_r), \ldots, (i_n, j_n)) \in \Lambda, \\ 0 & \text{otherwise.} \end{cases}$$

3. *For any $a \in \mathcal{A}$, $u_1, u_2 \in \mathcal{I}$ and $i, j, k \in I$, it holds that*

$$\varphi_j(u_1 a u_2) = \varphi_j(u_1)\varphi_j(a)\varphi_j(u_2) \quad \text{and} \quad \varphi_j(1_{i,k}) = \delta_{j,k}.$$

4. *The restriction of φ_j to $\mathcal{A}_{i,j}$ is $\varphi_{i,j}$ for any $i \neq j$.*
5. *The mixed moments $\varphi_j(a_1 a_2 \ldots a_n)$ are uniquely expressed in terms of marginal moments.*

Proof. Properties (1) and (2) follow from (1) and (2) of Proposition 2.2. The normalization in (3) follows directly from the action of $\lambda_{i,k}(1_{i,k})$ onto $\mathcal{H}^0_{j,j}$. In this context, notice that the unit vectors e_j play the same role with respect to the action of the $\lambda_{i,j}(a)$ for any $i \neq j$ as $\xi_{i,j}$ plays with respect to the action of $\pi_{i,j}(a)$, where $a \in \mathcal{A}_{i,j}$, and thus $\varphi_j(a) = \langle \lambda_{i,j}(a)e_j, e_j\rangle = \langle \pi_{i,j}(a)\xi_{i,j}, \xi_{i,j}\rangle = \varphi_{i,j}(a)$, which gives (4) for $i \neq j$. Finally, there exists $b_j \in \mathcal{A}_{j,j} \cap \operatorname{Ker}\varphi$ such that

$$\varphi_j(w) = \varphi\big(b_j^* w b_j\big)$$

for any $w \in \bigsqcup_{i,j} \mathcal{A}_{i,j}$, which reduces computations of mixed moments in each state φ_j to computations of mixed moments in the state φ. This proves (5). $\square$

Remark 2.1. The states φ and (φ_j) share together the property of extending the array of states $(\varphi_{i,j})$. Thus, φ extends the diagonal states $\varphi_{j,j}$ for all j, but it does not extend the off-diagonal states $\varphi_{i,j}$ for $i \neq j$ since $\pi_M(a)\xi = 0$ for any $a \in \mathcal{A}_{i,j}$. In turn, φ_j extends the off-diagonal states $\varphi_{i,j}$, where $i \neq j$, but it does not extend $\varphi_{j,j}$. This is a natural consequence of differences in the definitions of the diagonal and off-diagonal partial isometries.

Finally, let us denote by $\lambda(\mathcal{I})$ the unital commutative $*$-subalgebra of $B(\mathcal{H})$ generated by the $\lambda(1_{i,j})$, where $i, j \in I$. By abuse of notation, $\lambda(1_{i,j})$ will also be denoted by $1_{i,j}$ (in general, these projections are not mutually orthogonal).

3. Strongly matricially free products

Of special importance is the subspace of the matricially free Fock space, called the 'strongly matricially free Fock space', in which the diagonal Hilbert spaces appear only at the end of tensor products. The main reason is that it is related to both free and monotone Fock spaces. We also study the associated product states which can be viewed as direct generalizations of both free and monotone products of states.

Definition 3.1. By the *strongly matricially free Fock space* over $\widehat{\mathcal{H}} := (\mathcal{H}_{i,j})$ we understand the subspace of $\mathcal{M}(\widehat{\mathcal{H}})$ of the form

$$\mathcal{R}(\widehat{\mathcal{H}}) = \mathbb{C}\Omega \oplus \bigoplus_{m=1}^{\infty} \bigoplus_{\substack{i_1 \neq \cdots \neq i_m \\ n_1,\ldots,n_m \in \mathbb{N}}} \mathcal{H}_{i_1,i_2}^{\otimes n_1} \otimes \mathcal{H}_{i_2,i_3}^{\otimes n_2} \otimes \cdots \otimes \mathcal{H}_{i_m,i_m}^{\otimes n_m},$$

with the canonical inner product.

A justification for the word 'strong' is that in this case the words $i_1 i_2 \ldots i_m$ which label the tensor products in the above definition satisfy $i_1 \neq i_2 \neq \cdots \neq i_m$.

Example 3.1. The simplest space of this type is associated with a two-dimensional square array $\widehat{\mathcal{H}}$. Then

$$\mathcal{R}(\widehat{\mathcal{H}}) = \bigoplus_{m=0}^{\infty} \mathcal{R}^{(m)}(\widehat{\mathcal{H}}),$$

where the first few summands are of the form

$$\mathcal{R}^{(0)}(\widehat{\mathcal{H}}) = \mathbb{C}\Omega,$$
$$\mathcal{R}^{(1)}(\widehat{\mathcal{H}}) = \mathcal{H}_{1,1} \oplus \mathcal{H}_{2,2},$$
$$\mathcal{R}^{(2)}(\widehat{\mathcal{H}}) = \mathcal{H}_{1,1}^{\otimes 2} \oplus \mathcal{H}_{2,2}^{\otimes 2} \oplus (\mathcal{H}_{1,2} \otimes \mathcal{H}_{2,2}) \oplus (\mathcal{H}_{2,1} \otimes \mathcal{H}_{1,1}),$$
$$\begin{aligned}\mathcal{R}^{(3)}(\widehat{\mathcal{H}}) = {} & \mathcal{H}_{1,1}^{\otimes 3} \oplus \mathcal{H}_{2,2}^{\otimes 3} \oplus \left(\mathcal{H}_{2,1} \otimes \mathcal{H}_{1,1}^{\otimes 2}\right) \oplus \left(\mathcal{H}_{1,2} \otimes \mathcal{H}_{2,2}^{\otimes 2}\right) \oplus \left(\mathcal{H}_{2,1}^{\otimes 2} \otimes \mathcal{H}_{1,1}\right) \\ & \oplus \left(\mathcal{H}_{1,2}^{\otimes 2} \otimes \mathcal{H}_{2,2}\right) \oplus (\mathcal{H}_{1,2} \otimes \mathcal{H}_{2,1} \otimes \mathcal{H}_{1,1}) \oplus (\mathcal{H}_{2,1} \otimes \mathcal{H}_{1,2} \otimes \mathcal{H}_{2,2}),\end{aligned}$$

etc. In contrast to $\mathcal{M}(\widehat{\mathcal{H}})$, we do not have tensor products like $\mathcal{H}_{2,2} \otimes \mathcal{H}_{2,1} \otimes \mathcal{H}_{1,1}$ and $\mathcal{H}_{1,1} \otimes \mathcal{H}_{1,2} \otimes \mathcal{H}_{2,2}$ in the summand of the third order.

Remark 3.1. For a given array of Hilbert spaces $\widehat{\mathcal{H}} = (\mathcal{H}_{i,j})$, we have inclusions

$$\mathcal{R}(\widehat{\mathcal{H}}) \subseteq \mathcal{M}(\widehat{\mathcal{H}}) \subseteq \mathcal{F}\left(\bigoplus_{i,j} \mathcal{H}_{i,j}\right)$$

which, in most cases, are proper. Moreover, if we have a square array and $\mathcal{H}_{i,j} \cong \mathcal{H}_i$ for any $i, j \in I$, where $(\mathcal{H}_i)_{i\in I}$ is a family of Hilbert spaces, then there is a natural isomorphism

$$\mathcal{R}(\widehat{\mathcal{H}}) \cong \mathcal{F}\left(\bigoplus_{i\in I} \mathcal{H}_i\right)$$

since $\mathcal{H}_{i_1,i_2}^{\otimes n_1} \otimes \mathcal{H}_{i_2,i_3}^{\otimes n_2} \otimes \cdots \otimes \mathcal{H}_{i_m,i_m}^{\otimes n_m} \cong \mathcal{H}_{i_1}^{\otimes n_1} \otimes \mathcal{H}_{i_2}^{\otimes n_2} \otimes \cdots \otimes \mathcal{H}_{i_m}^{\otimes n_m}$ for any $i_1, i_2, \ldots, i_m \in I$, $n_1, \ldots, n_m, m \in \mathbb{N}$. Similarly, if we have a lower-triangular array and $\mathcal{H}_{i,j} \cong \mathcal{H}_i$ for any $i \geqslant j$, then $\mathcal{R}(\widehat{\mathcal{H}})$ is isomorphic to the monotone Fock space.

Moreover, as expected, there is a product of Hilbert spaces related to the strongly matricially free Fock space, and an analog of Proposition 2.1 holds.

Definition 3.2. By the *strongly matricially free product of* $(\mathcal{H}_{i,j}, \xi_{i,j})$ we understand the pair $(\mathcal{G}, \xi)$, where $\mathcal{G}$ is the subspace of $\mathcal{H}$ of the form

$$\mathcal{G} = \mathbb{C}\xi \oplus \bigoplus_{m=1}^{\infty} \bigoplus_{i_1 \neq \cdots \neq i_m} \mathcal{H}_{i_1,i_2}^0 \otimes \mathcal{H}_{i_2,i_3}^0 \otimes \cdots \otimes \mathcal{H}_{i_m,i_m}^0.$$

We denote it $(\mathcal{G}, \xi) = *_{i,j}^S (\mathcal{H}_{i,j}, \xi_{i,j})$.

If we consider a family of unital C^*-algebras $(\mathcal{A}_i)_{i\in I}$, each equipped with a family of states $(\varphi_{i,j})_{j\in I}$, then we can look at this product space as follows. If $(\mathcal{H}_{i,j}, \pi_{i,j}, \xi_{i,j})$ is the GNS triple associated with the pair $(\mathcal{A}_i, \varphi_{i,j})$, then the Hilbert space $\mathcal{H}_{i,j}$ (as well as the corresponding state and representation) taken as the representation space for the algebra $\mathcal{A}_i$ at some given tensor site depends on the algebra $\mathcal{A}_j$ represented at the following tensor site on the space $\mathcal{H}_{j,k}$ for some k. It is worth noting that in this framework we can assume that for fixed $i \in I$ all vectors $\xi_{i,j}$,

$j \in I$, are identified since we can take the tensor product of Hilbert spaces $\bigotimes_j \mathcal{H}_{i,j}$ and set $\xi_i = \bigotimes_{j\in I} \xi_{i,j}$ for each $i \in I$. It is not hard to see that in this framework our model is related to freeness with infinitely many states [8,9].

The construction of the product state is similar to that of the matricially free product. The only difference in all definitions is that the sets Λ_n are replaced by

$$\Gamma_n = \big\{\big((i_1, i_2), (i_2, i_3), \ldots, (i_n, i_{n+1})\big)\colon i_1 \neq i_2 \neq \cdots \neq i_n\big\}$$

and their union Λ by $\Gamma = \bigcup_{n=1}^{\infty} \Gamma_n$. Note that the conditions which define Γ_n's are stronger than those which define Λ_n's and therefore all objects constructed in the strong case are obtained from the standard ones by a projection-type operation.

The partial isometries in the strong case, denoted by $W_{i,j}$'s, remind $V_{i,j}$'s except that they refer to $\mathcal{G}$ rather than $\mathcal{H}$. In particular, the diagonal partial isometries $W_{j,j} : \mathcal{H}_{j,j} \otimes \mathcal{G}(j, j) \to \mathcal{G}$ are given by

$$\xi_{j,j} \otimes \xi \to \xi \quad \text{and} \quad \mathcal{H}^0_{j,j} \otimes \xi \to \mathcal{H}^0_{j,j},$$

where $\mathcal{G}(j, j) = \mathbb{C}\xi$ for any j, whereas the off-diagonal partial isometries $W_{i,j}$ and the associated subspaces $\mathcal{G}(i, j)$ are similar to those in the standard case.

Definition 3.3. Let $\rho_{i,j}$ be the $*$-representation of $\mathcal{A}_{i,j}$ on $\mathcal{G}$ given by the formula

$$\rho_{i,j}(a) = W_{i,j}\big(\pi_{i,j}(a) \otimes I_{\mathcal{G}(i,j)}\big)W^*_{i,j} \quad \text{where } a \in \mathcal{A}_{i,j},$$

for any $(i, j) \in J$. The corresponding *strongly matricially free product representation* $\pi_S = *^S_{i,j}\pi_{i,j}$ and *strongly matricially free product of states* $*^S_{i,j}\varphi_{i,j}$ are defined in terms of the $\rho_{i,j}$ as in the matricially free case.

Using appropriate direct sums of these representations, we can reproduce products of C^*-probability spaces in free probability of Voiculescu and in monotone probability of Muraki. If $(\mathcal{H}_i, \xi_i)$ is a family of Hilbert spaces with distinguished unit vectors, then in the decomposition theorem given below, $*_{i\in I}(\mathcal{H}_i, \xi_i)$, $\rhd_{i\in I}(\mathcal{H}_i, \xi_i)$ and $\uplus_{i\in I}(\mathcal{H}_i, \xi_i)$, respectively, stand for free, monotone and boolean products of Hilbert spaces.

Theorem 3.1 *(Decomposition theorem). Let $(\mathcal{A}_{i,j}, \varphi_{i,j}) = (\mathcal{A}_i, \varphi_i)$ for any i, j and let $(\pi_i, \mathcal{H}_i, \xi_i)$ be the GNS triple associated with $(\mathcal{A}_i, \varphi_i)$ for any $i \in I$.*

1. *If $(\mathcal{A}_{i,j}, \varphi_{i,j})$ is a square array, then $(\mathcal{G}, \xi) \cong *_{i\in I}(\mathcal{H}_i, \xi_i)$ and each $\lambda_i = \bigoplus_{j\in I} \rho_{i,j}$ is the canonical $*$-representation of $\mathcal{A}_i$ on $*_{i\in I}(\mathcal{H}_i, \xi_i)$.*
2. *If $(\mathcal{A}_{i,j}, \varphi_{i,j})$ is a lower-triangular array, then $(\mathcal{G}, \xi) \cong \rhd_{i\in I}(\mathcal{H}_i, \xi_i)$ and each $\tau_i = \bigoplus_{i\geqslant j} \rho_{i,j}$ is the canonical $*$-representation of $\mathcal{A}_i$ on $\rhd_{i\in I}(\mathcal{H}_i, \xi_i)$.*
3. *If $(\mathcal{A}_{i,j}, \varphi_{i,j})$ is a diagonal array, then $(\mathcal{G}, \xi) \cong \uplus_{i\in I}(\mathcal{H}_i, \xi_i)$ and each $\rho_{i,i}$ is the canonical $*$-representation of $\mathcal{A}_i$ on $\uplus_{i\in I}(\mathcal{H}_i, \xi_i)$.*

Proof. First, let us remark that the representations λ_i and τ_i are well defined since the corresponding direct sums of operators $\bigoplus_j \rho_{i,j}(a)$ are convergent in the strong-operator topology on

$B(\mathcal{G})$ for any $a \in \mathcal{A}_i$ and $i \in I$. Now, since $\mathcal{H}_{i,j} \cong \mathcal{H}_i$ for any i, j, we have a natural isomorphism

$$\mathcal{H}^0_{i_1,i_2} \otimes \mathcal{H}^0_{i_2,i_3} \otimes \cdots \otimes \mathcal{H}^0_{i_m,i_m} \cong \mathcal{H}^0_{i_1} \otimes \mathcal{H}^0_{i_2} \otimes \cdots \otimes \mathcal{H}^0_{i_m}$$

for any $i_1 \neq i_2 \neq \cdots \neq i_m$ or any $i_1 > i_2 > \cdots > i_m$, in the case of square or lower-triangular arrays, respectively, which leads to the corresponding isomorphisms

$$(\mathcal{G}, \xi) \cong *_{i \in I}(\mathcal{H}_i, \xi_i) \quad \text{or} \quad (\mathcal{G}, \xi) \cong \rhd_{i \in I}(\mathcal{H}_i, \xi_i)$$

(recall our tacit assumption that I is linearly ordered when dealing with triangular arrays). Then partial isometries which lie in the same row of the array $(W_{i,j})$ are orthogonal in the sense that $W_{i,j}W_{i,k} = \delta_{j,k}W_{i,j}$ for any i, j, k (in the monotone case the array of partial isometries is lower-triangular and we have here $i > j > k$). Therefore, we have an orthogonal direct sum decomposition

$$\bigoplus_{j \in I} W_{i,j} = V_i, \quad i \in I,$$

of the unitaries V_i used in the definition of the free product representation [21,24], and an analogous decomposition in the monotone case. The direct sum decompositions of $\lambda_i(a)$ and $\tau_i(a)$ in terms of $\rho_{i,j}(a)$'s, where $a \in \mathcal{A}_i$, follow then immediately from Definition 3.3, which completes the proof of (1) and (2). In particular, it follows that each λ_i is unital, but τ_i are, in general, non-unital. Since the case of a diagonal array is rather elementary, the proof is completed. □

Consequently, if $\mathcal{A}$ is the C^*-algebra generated by the family $\{\lambda_i(\mathcal{A}_i)\}$ of subalgebras of $B(\mathcal{G})$ and $\varphi(.) = \langle .\xi, \xi\rangle$, then $(\mathcal{A}, \varphi)$ is the free product of C^*-probability spaces. Similar statements hold for the monotone and boolean products of C^*-probability spaces, except that the identity $I \in B(\mathcal{G})$ has to be added to the generators. In the natural way this leads to properties (A), (B) and (C) of the introduction, of which the first two can be viewed as decompositions of free and monotone independent random variables in terms of strongly matricially free ones.

The above theorem allows us to view the strongly matricially free product of states as a deformation of the free product of states, obtained by a natural direct sum decomposition of the free product of Hilbert spaces. The matricially free product of states is then obtained from its strong counterpart by an extension to a slightly larger Hilbert space which includes all products which arise naturally in matrix multiplication. It is clear that basic results on the strongly matricially free product of states are similar to those for the matricially free product and therefore we state them in an abbreviated form without a proof.

Proposition 3.1. *Let φ be the strongly matricially free product of states $(\varphi_{i,j})$ and let φ_j, $j \in I$, be the associated conditions. Then the statements of Propositions* 2.2–2.3 *remain true, with Λ replaced by Γ.*

As we have mentioned earlier, one of the advantages of using the strong structures is that they are straightforward generalizations of those in free probability (in the case of square arrays) and monotone probability (in the case of lower-triangular arrays). However, it is the matricial freeness which gives a connection with random matrices.

Remark 3.2. Slightly more general is the case when the diagonal and off-diagonal states differ, but the latter stay the same within each row, namely

$$(\mathcal{A}_{i,i}, \varphi_{i,i}) = (\mathcal{A}_i, \varphi_i) \quad \text{and} \quad (\mathcal{A}_{i,j}, \varphi_{i,j}) = (\mathcal{A}_i, \psi_i) \quad \text{for } i \neq j,$$

where each $\mathcal{A}_i$ is equipped with two states, φ_i and ψ_i, respectively. Similar reasoning to that in the free case leads then to the conditionally free product of states (for square arrays) and conditionally monotone products of states (for lower-triangular arrays).

4. Matricial freeness

Guided by the notion of the matricially free product of states, we shall introduce now the associated concept of independence called 'matricial freeness', and closely related to it, 'strong matricial freeness'. They involve arrays of noncommutative probability spaces and to some extent they remind models with many states [6,8,9], but they cannot be reduced in a natural way to any of these (freeness with infinitely many states has some non-empty intersection with 'strong matricial freeness' and conditional freeness is its special case). Moreover, we will study discrete (strong) matricially free Fock spaces.

Let $\mathcal{A}$ be a unital algebra with an array $(\mathcal{A}_{i,j})$ of subalgebras of $\mathcal{A}$. We will assume that each $\mathcal{A}_{i,j}$ has an internal unit $1_{i,j}$ which may be different from the unit of $\mathcal{A}$, and we assume that the unital subalgebra $\mathcal{I}$ generated by all internal units is commutative. Let φ be a distinguished state on $\mathcal{A}$ and let $\{\varphi_j\colon j \in I\}$ be a family of additional states on $\mathcal{A}$, where by a state we understand a normalized linear functional. If $\mathcal{A}$ is a unital $*$-algebra, then we assume that $\mathcal{A}_{i,j}$'s are $*$-subalgebras and all states are positive functionals. Further,

Definition 4.1. States φ_j, $j \in I$, will be called *conditions* associated with φ if for any $j \in I$ there exist $b_j, c_j \in \mathcal{A}_{j,j} \cap \mathrm{Ker}(\varphi)$ such that $\varphi_j(a) = \varphi(c_j a b_j)$ for any $a \in \mathcal{A}$. The array of states on $\mathcal{A}$ given by

$$\varphi_{j,j} = \varphi \quad \text{and} \quad \varphi_{i,j} = \varphi_j \quad \text{for any } i \neq j$$

will be said to be *defined* by φ and the associated conditions φ_j. In addition, if $\mathcal{A}$ is a $*$-algebra, we assume that $c_j = b_j^*$ for any j.

Throughout this paper we will assume the following normalization conditions:

$$\varphi(1_{i,j}) = \delta_{i,j} \quad \text{and} \quad \varphi_j(1_{i,k}) = \delta_{j,k}$$

for any i, j, k. Although they are natural in view of the Hilbert space formulation presented before and will be assumed in this paper, we prefer to exclude them from the definition given below. In particular, the normalization of each φ_j implies that if $a \in \mathcal{A}_{i,k}$ and $j \neq k$, then $\varphi_j(a) = 0$. Moreover, it is equivalent to scaling the variables c_j, b_j according to $\varphi(c_j b_j) = 1$.

Definition 4.2. Let $(\varphi_{i,j})$ be the array defined by φ and the associated conditions φ_j. We say that $(1_{i,j})$ is a *matricially free array of units* associated with $(\mathcal{A}_{i,j})$ and $(\varphi_{i,j})$ if

1. $\varphi(u_1 a u_2) = \varphi(u_1)\varphi(a)\varphi(u_2)$ for any $a \in \mathcal{A}$ and $u_1, u_2 \in \mathcal{I}$,
2. for $a_k \in \mathcal{A}_{i_k,j_k} \cap \operatorname{Ker}\varphi_{i_k,j_k}$, where $r < k \leqslant n$,

$$\varphi(a 1_{i_r,j_r} a_{r+1} \dots a_n) = \begin{cases} \varphi(a a_{r+1} \dots a_n) & \text{if } ((i_r, j_r), \dots, (i_n, j_n)) \in \Lambda, \\ 0 & \text{otherwise,} \end{cases}$$

   where $a \in \mathcal{A}$ is arbitrary and $(i_r, j_r) \neq \dots \neq (i_n, j_n)$.

The array $(1_{i,j})$ is called a *strongly matricially free array of units* if Λ is replaced by Γ.

The above definition enables us to define the concepts of *matricial freeness* and its strong version called *strong matricial freeness*. They both bear some resemblance to freeness, but the main difference is that the identified unit in the context of freeness is replaced by the (strongly) matricially free array of units. In fact, as we have already remarked, it is the strong matricial freeness which can be viewed as a direct generalization of freeness.

Definition 4.3. We say that $(\mathcal{A}_{i,j})$ is *matricially free* with respect to the array $(\varphi_{i,j})$ defined by φ and the associated conditions φ_j if

1. for any $a_k \in \operatorname{Ker}\varphi_{i_k,j_k} \cap \mathcal{A}_{i_k,j_k}$, where $k \in [n]$ and $(i_1, j_1) \neq \dots \neq (i_n, j_n)$,

$$\varphi(a_1 a_2 \dots a_n) = 0,$$

2. $(1_{i,j})$ is a matricially free array of units associated with $(\mathcal{A}_{i,j})$ and $(\varphi_{i,j})$.

In an analogous manner we define *strongly matricially free* arrays of subalgebras.

Definition 4.4. The array of variables $(a_{i,j})$ in a unital algebra $\mathcal{A}$ will be called (*strongly*) *matricially free* with respect to the array $(\varphi_{i,j})$ of states on $\mathcal{A}$ if the array $(\mathbb{C}[a_{i,j}, 1_{i,j}])$ is (strongly) matricially free with respect to $(\varphi_{i,j})$ for some array of elements $(1_{i,j})$ of $\mathcal{A}$ which is a (strongly) matricially free array of units. If $\mathcal{A}$ is a unital $*$-algebra, then, in addition, we require that the functionals $\varphi_{i,j}$ are positive, the $\mathcal{A}_{i,j}$ are $*$-subalgebras and the $1_{i,j}$ are projections. Then an array of variables $(a_{i,j})$ will be called $*$-(*strongly*) *matricially free* if the array of $*$-algebras $(\mathbb{C}\langle a_{i,j}, a_{i,j}^*, 1_{i,j}\rangle)$ is (strongly) matricially free.

Using the above definitions, we can uniquely express mixed moments under φ and thus under φ_j's of arbitrary matricially free random variables in terms of marginal moments under $\varphi_{i,j}$'s. To see how this computation works, it is convenient to assume that neighboring variables come from different algebras and use the recurrence given below.

Remark 4.1. Suppose that $(\mathcal{A}_{i,j})$ is matricially free with respect to the array $(\varphi_{i,j})$ defined by φ and the associated conditions φ_j. Writing $a_k = a_k^0 + \varphi_{i_k,j_k}(a_k)1_{i_k,j_k} \in \mathcal{A}_{i_k,j_k}$ for any $1 \leqslant k \leqslant n$, we obtain the recurrence

$$\begin{aligned}\varphi(a_1\ldots a_n) = &\sum_{1\leqslant k\leqslant n}\varphi_{i_k,j_k}(a_k)\varphi\big(a_1^0\ldots 1_{i_k,j_k}\ldots a_n^0\big)\\ &+\sum_{1\leqslant k<l\leqslant n}\varphi_{i_k,j_k}(a_k)\varphi_{i_l,j_l}(a_l)\varphi\big(a_1^0\ldots 1_{i_k,j_k}\ldots 1_{i_l,j_l}\ldots a_n^0\big)\\ &+\cdots\\ &+\varphi_{i_1,j_1}(a_1)\ldots\varphi_{i_n,j_n}(a_n)\varphi(1_{i_1,j_1}\ldots 1_{i_n,j_n}).\end{aligned}$$

If we assume that neighboring variables come from different algebras, we can reduce the moments under φ which appear on the RHS to sums of products of marginal moments (repeated application of Definitions 4.1 and 4.2 is needed). Let us remark that the recurrence generalizes that for free random variables [3] except that the units on the RHS are not equal to the unit of $\mathcal{A}$.

The above recurrence can also be used even if neighboring variables belong to the same algebra. In that case, with each product $a = a_1\ldots a_n$ we can associate a partition $\pi = \{\pi_1,\ldots,\pi_s\}$ of the set $\{1,\ldots,n\}$ in a natural way. Namely, numbers k, r will belong to the same block of π if and only if $(i_k, j_k) = (i_r, j_r)$. This allows us to derive basic properties of the mixed moments, collected in the lemma given below (compare with the free case [22]).

Moreover, observe that in the definition of (strong) matricial freeness, the mixed moments under φ and φ_j's are uniquely determined by marginal moments under $\varphi_k|\mathcal{A}_{i,k}$ for $i \neq k$ and those under $\varphi|\mathcal{A}_{k,k}$ for any k (cf. Proposition 2.3). Therefore, in order to determine φ uniquely on $\bigsqcup_{i,j}\mathcal{A}_{i,j}$, it suffices to specify linear functionals $\varphi_{i,j}$ on subalgebras $\mathcal{A}_{i,j}$ such that $\varphi_{i,j}(1_{i,j}) = 1$, where i, j are arbitrary, and set $\varphi(1) = 1$. Then φ_j's are determined uniquely on $\bigsqcup_{i,j}\mathcal{A}_{i,j}$, provided a pair $c_j, b_j \in \mathcal{A}_{j,j}\cap\operatorname{Ker}\varphi_{j,j}$, such that $\varphi_{j,j}(c_jb_j) = 1$, is chosen for any j.

We assume now that $(\mathcal{A}_{i,j})$ is an array of subalgebras of a unital algebra $\mathcal{A}$, equipped with an array of states $(\varphi_{i,j})$ defined by φ and associated conditions φ_j, with respect to which $(\mathcal{A}_{i,j})$ is matricially free. Without loss of generality, we assume in the lemma given below that neighbors come from different algebras.

Lemma 4.1. *Let $a = a_1\ldots a_n$, where $a_k \in \mathcal{A}_{i_k,j_k}$ for $1 \leqslant k \leqslant n$ and $(i_1, j_1) \neq \cdots \neq (i_n, j_n)$, and let $\pi = \{\pi_1,\ldots,\pi_s\}$ be the associated partition of the set $\{1,\ldots,n\}$.*

1. *If there exists k such that $(i_r, j_r) \neq (i_k, j_k)$ for $r \neq k$, and $a_k \in \operatorname{Ker}\varphi_{i_k,j_k}$, then $\varphi(a) = 0$.*
2. *If π is arbitrary, then $\varphi(a)$ is a sum of products of at least s marginal moments.*
3. *If π is crossing, then $\varphi(a)$ is a sum of products of more than s marginal moments.*
4. *If $i_k = j_k$ for any k, then $\varphi(a) = \varphi(a_1)\ldots\varphi(a_n)$.*
5. *If among $i_1, j_1,\ldots,i_m, j_m$ there are more than s different indices, then $\varphi(a) = 0$.*

Proof. The proof of (1) is similar to that of the recurrence of Remark 4.1. In fact, it is an easy consequence of the equation

$$\varphi(a) = \varphi_{i_k,j_k}(a_k)\varphi(a_1\ldots a_{k-1}1_{i_k,j_k}a_{k+1}\ldots a_n),$$

which, in turn, follows from

$$\varphi\big(a_1\ldots a_{k-1}a_k^0a_{k+1}\ldots a_n\big) = 0.$$

To justify the latter, we assume (without loss of generality) that neighboring variables come from different algebras. Then, decomposing $a_r = a_r^0 + \varphi_{i_r,j_r}(a_r)1_{i_r,j_r}$ for any $r \neq k$, we repeatedly apply Definitions 4.2–4.3 to the above moment until we are left with a sum of products of moments which do not contain units, have neighboring variables in the kernels of the corresponding states and come from different algebras. However, this means that each of these products vanishes since the variable a_k^0 does not participate in any reductions as it is the only one from $\mathcal{A}_{i_k,j_k}$ and that is why a_k^0 appears in one of the factors. This proves (1). Next, (2) follows from a repeated application of the recurrence of Remark 4.1 and Definitions 4.1–4.2. In order to prove (3), it suffices to consider the case when π has no singletons since if π is crossing, the partition obtained from π by removing singletons is still crossing (note that in the case when neighbors come from different algebras, each non-crossing partition has a singleton). Then consider a summand from the RHS of the same recurrence of the form

$$\varphi_{i_k,j_k}(a_k)\ldots\varphi_{i_r,j_r}(a_r)\varphi(a_1\ldots 1_{i_k,j_k}\ldots 1_{i_r,j_r}\ldots a_n).$$

There are two possibilities. If among $(i_k, j_k), \ldots, (i_r, j_r)$ there are no repetitions, among the remaining pairs there are still s different ones and further reduction of the above summand gives products of more than s marginal moments in view of (2). In turn, if among $(i_k, j_k), \ldots, (i_r, j_r)$ there is a repetition, then there must be at least $s-(r-1)$ different remaining pairs, and therefore, in view of (2), we obtain again a sum of products of more than s marginal moments, which completes the proof of (3). Property (4) follows from the proof of (1). In order to prove (5), it is enough to consider $a_k \in \{a_k^0, 1_{i_k,j_k}\}$ for any k. It is obvious that the number of different indices among $i_1, j_1, \ldots, i_n, j_n$ is smaller or equal to $2s$, where s is the number of blocks of π. However, in order to obtain a non-vanishing moment, the last variable in the product $a_1\ldots a_n$ which belongs to a given block π_r, say a_p, must have the second index equal to the first index of one of the variables which follow a_p and are associated with blocks different than π_r (this is not necessarily a_{p+1} if some variables which follow a_p are units). In particular, in order that

$$\varphi\big(a_1\ldots a_p a_{p+1}\ldots a_{q-1}a_q^0\ldots a_n^0\big) \neq 0,$$

where $a_{p+1}, \ldots, a_{q-1}$ are units, we must have $j_{q-1} = i_q$ and thus, by Definition 4.2, $j_k = i_q$ for $p < k < q-1$. In that case, the moment reduces to $\varphi(a_1\ldots a_p a_q^0\ldots a_n^0)$, which implies that we must have $j_p = i_q$. This shows how the indices associated with different blocks are tied together if they are separated by units. Therefore, the number of different indices among $i_1, j_1, \ldots, i_n, j_n$ is reduced to a number smaller or equal to s (strictly smaller if among them there are diagonal variables other than the last one), which completes the proof of (5). □

Let us note in this context that, in contrast to the free case, property (1) does not entail the factorization of pyramidally ordered products (see the computation of $\varphi(ab'a)$ in Example 4.1).

In general, strong matricial freeness generalizes freeness, monotone independence as well as boolean independence. This fact is rather natural in view of Theorem 3.1, but we shall state it below on the more abstract level of independence in the category of noncommutative probability spaces rather than on the level of products of states. For that purpose, assume that $(\mathcal{A}_{i,j})$ is an array of subalgebras of a unital algebra $\mathcal{A}$ which is strongly matricially free with respect to some array $(\varphi_{i,j})$ defined by a distinguished state φ and the associated conditions φ_j.

Proposition 4.1. *Let $\mathcal{A}_{i,j}$ be a replica of $\mathcal{A}_{i,i}$ with $a_{i,j}$ being a replica of $a_{i,i} \in \mathcal{A}_{i,i}$ such that $\varphi_{i,j}$-distribution of $a_{i,j}$ agrees with φ-distribution of $a_{i,i}$ for any i, j.*

1. *If $(\mathcal{A}_{i,j})$ is a finite square array, then the family $\{a_i := \sum_j a_{i,j},\ i \in I\}$ is free with respect to φ.*
2. *If $(\mathcal{A}_{i,j})$ is a finite lower-triangular array, then the family $\{a_i := \sum_j a_{i,j},\ i \in I\}$ is monotone independent with respect to φ.*
3. *If $(\mathcal{A}_{i,j})$ is a diagonal array, then the family $\{a_{i,i},\ i \in I\}$ is boolean independent with respect to φ.*

Proof. Let us denote $a_i^0 := \sum_j a_{i,j}^0$ and $1_i := \sum_j 1_{i,j}$ for any i, where the summation runs over those j's for which $(i, j) \in J$, with J denoting the index set of the array, and where $a_{i,j}^0 = a_{i,j} - \varphi(a_{i,j})1_{i,j}$ (in the case of lower-triangular arrays, the summation runs over $j \leqslant i$). Moreover,

$$\varphi_{i,j}(a_{i,j}) = \varphi(a_{i,i}) = \varphi(a_i) \quad \text{for any } (i, j) \in J$$

by the assumption on identical distributions of replicas and by the normalization $\varphi(1_{i,j}) = \delta_{i,j}$. In the case of a square array, we obtain the implication

$$\varphi\big(a_{i_1,j_1}^0 \dots a_{i_n,j_n}^0\big) = 0 \quad \Rightarrow \quad \varphi\big(a_{i_1}^0 \dots a_{i_n}^0\big) = 0$$

for $i_1 \neq \dots \neq i_n$. Moreover, using the assumption about identical distributions of replicas once again, we can write

$$a_i = a_i^0 + \varphi(a_i)1_i$$

for any i, and thus it is enough to observe that for any r it holds that

$$\varphi\big(a_{i_1,j_1} \dots 1_{i_r} a_{i_{r+1},j_{r+1}}^0 \dots a_{i_n,j_n}^0\big) = \varphi\big(a_{i_1,j_1} \dots a_{i_{r+1},j_{r+1}}^0 \dots a_{i_n,j_n}^0\big)$$

for any $r \leqslant n$, which can be used repeatedly to conclude that the 1_i's can be identified with the unit of $\mathcal{A}$ in all mixed moments under φ which involve the a_k's. This completes the proof for the free case. In the monotone case we have to slightly modify the approach since we have triangular arrays and units do not become identified. Note that monotone independence of the a_i's essentially follows from the following two equations:

$$\varphi(w_1 1_i 1_j 1_i w_2) = \varphi(w_1 1_i w_2) \quad \text{and} \quad \varphi\big(w_1 1_i a_j^0 1_i w_2\big) = 0$$

for any $i < j$, where w_1, w_2 are products of a_k's. These equations are proved by induction with repeated use of Definitions 4.2–4.3. In fact, it is enough to show that such equations hold if w_2 is an alternating product of $a_{k,l}^0$'s and $1_{k,l}$'s, and that can be reduced to the proof for mixed moments of the above type, in which w_2 is an alternating product of variables from the kernels, say $b_{k,l}^0 \in \mathcal{A}_{k,l} \cap \operatorname{Ker} \varphi_{i,j}$. To obtain the first equation, it then suffices to observe that in any such mixed moment $1_i 1_j$ can be identified with 1_i since the latter acts as a projection onto products w_2 which begin with some $b_{k,l}^0$ for $k \leqslant i$, and therefore 1_j maps $1_i w_2$ onto itself. The same kind of argument is used to prove the second equation, which completes the proof of (2). The proof of (3) is omitted since it is straightforward. $\square$

We assumed above that sums $\sum_j a_{i,j}$ are finite since in the considered category we do not have a topology. However, appropriately defined inductive limits of mixed moments of these sums considered above can be shown to exist for arbitrary arrays since under φ all mixed moments of these sums reduce to finite sums of mixed moments of the $a_{i,j}$'s. This implies that in the category of C^*-probability spaces these sums converge in the strong-operator topology on some Hilbert space (isomorphic to the associated free product of Hilbert spaces).

By a straightforward modification of the free case in Proposition 4.1 we obtain conditional freeness. We just need to take the array $(\varphi_{i,j})$ defined by φ and associated conditions φ_j which are assumed to agree with another state ψ on off-diagonal replicas (as in the free case, the conditions remain different as states on $\mathcal{A}$). Then the considered family of sums is conditionally free with respect to (φ, ψ). Some simple examples of mixed moments are given below, where we can also see that the factorization of mixed moments of (strongly) matricially free random variables depends not only on the associated partition, but also on the algebras to which variables belong. In particular, in contrast to the free case, a non-crossing partition associated with a given product may give products of more marginal moments than the number of blocks.

Example 4.1. Consider a two-dimensional array of strongly matricially free subalgebras of $\mathcal{A}$, with variables and states

$$\begin{pmatrix} a & a' \\ b' & b \end{pmatrix} \in \begin{pmatrix} \mathcal{A}_{1,1} & \mathcal{A}_{1,2} \\ \mathcal{A}_{2,1} & \mathcal{A}_{2,2} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \varphi & \varphi_2 \\ \varphi_1 & \varphi \end{pmatrix},$$

respectively. Using Remark 4.1 and Definitions 4.1–4.2, we obtain

$$\varphi(aba) = \varphi^2(a)\varphi(b), \qquad \varphi\big(ab'a\big) = \big(\varphi\big(a^2\big) - \varphi^2(a)\big)\varphi_1\big(b'\big).$$

Similar computations give higher-order mixed moments

$$\begin{aligned} \varphi\big(ab'ab\big) &= \varphi(b)\varphi_1\big(b'\big)\big(\varphi\big(a^2\big) - \varphi^2(a)\big), \\ \varphi(abab) &= \varphi^2(a)\varphi^2(b), \\ \varphi\big(aba'b\big) &= \varphi(a)\varphi_2\big(a'\big)\big(\varphi\big(b^2\big) - \varphi^2(b)\big). \end{aligned}$$

These are all non-vanishing moments that contribute to $\varphi(ABA)$ and $\varphi(ABAB)$, respectively, where $A = a + a'$ and $B = b + b'$. However, the latter agree with the mixed moments of A and B which are conditionally independent with respect to (φ, ψ), where ψ agrees with φ_1 on $\mathbb{C}[B]$ and with φ_2 on $\mathbb{C}[A]$. In turn, $\varphi(aBa)$ and $\varphi(aBaB)$, corresponding to the associated lower-triangular array, agree with the mixed moments of a, B which are monotone independent with respect to φ if φ agrees with φ_1 on $\mathbb{C}[B]$.

Definition 4.5. By a *discrete matricially free Fock space* we understand $\mathcal{M} = \mathcal{M}(\widehat{\mathcal{H}})$, where $\mathcal{H}_{i,j} = \mathbb{C}e_{i,j}$ for any $(i, j) \in \mathbb{N}$, and the array $(e_{i,j})$ forms an orthonormal basis of some Hilbert space. In an analogous manner we define the *discrete strongly matricially free Fock space* $\mathcal{R} = \mathcal{R}(\widehat{\mathcal{H}})$.

Both $\mathcal{M}$ and $\mathcal{R}$ are subspaces of the discrete free Fock space $\mathcal{F}(\bigoplus_{i,j} \mathbb{C}e_{i,j})$ and thus allow for the canonical action of free creation and annihilation operators. In order to specify the arrays of units, let us distinguish two types of their subspaces.

In the case of $\mathcal{M}$ these are

1. $\mathcal{M}(i,j)$, spanned by simple tensors which begin with $e_{j,k}$ for some k, where $(j,k) \neq (i,j)$, and, in addition, by Ω if $i = j$,
2. $\mathcal{K}(i,j)$, spanned by vectors which begin with $e_{i,j}$, where i, j are arbitrary,

and the direct sum $\mathcal{K}(i,j) \oplus \mathcal{M}(i,j)$ is the subspace of $\mathcal{M}$ onto which the $*$-algebra generated by $\ell(e_{i,j})$ restricted to $\mathcal{M}$ acts non-trivially, where $\ell(e_{i,j})$ denotes the canonical free creation operator associated with vector $e_{i,j}$.

The canonical projections onto such direct sums are natural candidates for the matricially free units $1_{i,j}$ and therefore we set

$$1_{i,j} := P_{\mathcal{K}(i,j)\oplus\mathcal{M}(i,j)} \quad \text{and} \quad \ell_{i,j} = \ell(e_{i,j})1_{i,j}$$

for any i, j. The adjoint of $\ell_{i,j}$ will be denoted by $\ell_{i,j}^*$. Note that the projection $1_{i,j}$ is an internal unit in the $*$-algebra $\mathcal{A}_{i,j} = \mathbb{C}\langle \ell_{i,j}, \ell_{i,j}^*\rangle$ and $\ell_{i,j}^*\ell_{i,j} = 1_{i,j}$ for any i, j. However, in the algebra $\mathbb{C}\langle \ell_{i,j}, \ell_{i,j}^*\colon i,j \in \mathbb{N}\rangle$ there are more relations as the proposition given below demonstrates.

Proposition 4.2. *In the algebra $\mathbb{C}\langle \ell_{i,j}, \ell_{i,j}^*\colon i,j \in \mathbb{N}\rangle$ the following relations hold*:

1. $\ell_{i,j}^*\ell_{i,j} = 1_{i,j}$ *for any* i, j,
2. $\ell_{i,j}^*\ell_{k,l} = 0$ *whenever* $(i,j) \neq (k,l)$,
3. $\ell_{i,j}\ell_{k,l} = 0$ *and* $1_{i,j}\ell_{k,l} = 0$ *whenever* $(i,j) \neq (k,l)$ *and* $j \neq k$,
4. $1_{i,j}\ell_{j,k} = \ell_{j,k}$ *for any* i, j, k.

Proof. We omit the elementary proof. □

In the case of $\mathcal{R}$, we proceed in a completely analogous fashion and distinguish the following subspaces:

$$\mathcal{R}(i,j) = \mathcal{M}(i,j) \cap \mathcal{R} \quad \text{and} \quad \mathcal{L}(i,j) = \mathcal{K}(i,j) \cap \mathcal{R}$$

which lead to the definitions of the strongly matricially free array of units and of the creation operators, respectively,

$$1_{i,j} = P_{\mathcal{L}(i,j)\oplus\mathcal{R}(i,j)} \quad \text{and} \quad k_{i,j} = \ell(e_{i,j})1_{i,j},$$

where, slightly abusing notation, we use the same symbols for the units as before.

Finally, we specify the states: φ will denote the vacuum state associated with Ω, the φ_j will be the states associated with the $e_{j,j}$, $j \in \mathbb{N}$, and $(\varphi_{i,j})$ will be the array defined by φ and the φ_j. Here, the same notation is used for states on $B(\mathcal{M})$ and $B(\mathcal{R})$.

Proposition 4.3. *The array of $*$-subalgebras $\mathcal{A}_{i,j} = \mathbb{C}\langle \ell_{i,j}, \ell_{i,j}^*\rangle$ of $B(\mathcal{M})$, where $i,j \in \mathbb{N}$, is matricially free with respect to $(\varphi_{i,j})$. The array of $*$-subalgebras $\mathcal{B}_{i,j} = \mathbb{C}\langle k_{i,j}, k_{i,j}^*\rangle$ of $B(\mathcal{R})$, where $i,j \in \mathbb{N}$, is strongly matricially free with respect to $(\varphi_{i,j})$.*

Proof. The proof is similar to that of Voiculescu for the discrete free Fock space given in [27]. We shall look at the case of $(\mathcal{A}_{i,j})$ since the case of $(\mathcal{B}_{i,j})$ is analogous. Each algebra $\mathcal{A}_{i,j}$ is spanned by operators of the form

$$\ell_{i,j}^{q}\ell_{i,j}^{*p}, \quad \text{where } p+q>0,$$

and the projection $1_{i,j}$. However, the corresponding moments vanish:

$$\varphi_{i,j}\big(\ell_{i,j}^{q}\ell_{i,j}^{*p}\big)=0$$

for any i, j since $p+q>0$. Moreover, $\varphi_{i,j}(1_{i,j})=1$ for any i, j. Therefore, in order to show that condition (1) of Definition 4.2 holds, it is enough to show that

$$\varphi\big(\ell_{i_1,j_1}^{q_1}\ell_{i_1,j_1}^{*p_1}\ldots\ell_{i_n,j_n}^{q_n}\ell_{i_n,j_n}^{*p_n}\big)=0$$

whenever $(i_1, j_1)\neq\cdots\neq(i_n, j_n)$ and $p_1+q_1>0,\ldots,p_n+q_n>0$. The same argument as in [27] allows us to reduce the proof to the case when $q_1=\cdots=q_n=0$, which implies that $p_1>0,\ldots,p_n>0$. But then the moment clearly vanishes. This proves condition (1) of Definition 4.2. Condition (2) follows easily from the definition of the projections $1_{i,j}$ in view of the relations given in Proposition 4.1. This completes the proof. □

Example 4.2. For $i, j\in I$, let $G_{i,j}\cong F(1)$ be the free group on one generator $g_{i,j}$ with unit $\epsilon_{i,j}$. Consider the subspace l_M^2 of $l^2(*_{i,j}G_{i,j})$ spanned by vectors of the form $\delta(g)$, where g is either the unit e of the free product $*_{i,j}G_{i,j}$, or a product of the form $g_1g_2\ldots g_m$, where $g_k\in G_{i_k,i_{k+1}}^0:=G_{i_ki_{k+1}}\setminus\{\epsilon_{i_k,i_{k+1}}\}$ for each k, with $(i_1, i_2)\neq\cdots\neq(i_m, i_{m+1})$ and $i_{m+1}=i_m$. The space l_M^2, which is a simple example of the matricially free Fock space, can be viewed as the space of square integrable functions on the 'matricially free product of groups' (which is not a group). Let $1_{i,j}$ denote the projection from $l^2(*_{i,j}G_{i,j})$ onto the subspace of l_M^2 spanned by vectors $\delta(g)$, where g begins with an element from $G_{i,j}^0$ or $G_{j,k}^0$ for some k or, in the case of $i=j$, also $g=e$. We can now define

$$\widetilde{\lambda}_{i,j}(g)=\lambda_{i,j}(g)1_{i,j}$$

for $g\in G_{i,j}$, where by $\lambda_{i,j}$ we denote the left regular representation of $G_{i,j}$ on $l^2(*_{i,j}G_{i,j})$. Then the array $(\mathcal{A}_{i,j})$, where $\mathcal{A}_{i,j}$ is the $*$-subalgebra of $B(l^2(*_{i,j}G_{i,j}))$ generated by $\widetilde{\lambda}_{i,j}(g_{i,j})$ and $1_{i,j}$, with the standard involution, is matricially free with respect to the array $(\varphi_{i,j})$, where the diagonal states coincide with $\varphi(.)=\langle .\delta(e),\delta(e)\rangle$ and the associated conditions are $\varphi_j(.)=\langle .\delta(g_{j,j}),\delta(g_{j,j})\rangle$, where $j\in I$. In a similar way we proceed in the case of functions on the 'strongly matricially free product of groups'. For classical results on random walks on free groups, see [13].

Example 4.3. To be more concrete, we consider a two-dimensional square array of free groups on one generator. Then our Hilbert space is

$$l_M^2=\operatorname{span}\big\{\delta(e),\delta(g)\colon\ g\in H_1\cup H_2\big\},$$

where

$$H_1 := \{g_{1,1}^k, g_{1,2}^k g_{2,2}^l, g_{1,1}^k g_{1,2}^l g_{2,2}^m, g_{1,2}^k g_{2,1}^l g_{1,1}^m, \ldots:\ k,l,m \in \mathbb{Z}_0\}$$

and

$$H_2 := \{g_{2,2}^k, g_{2,1}^k g_{1,1}^l, g_{2,2}^k g_{2,1}^l g_{1,1}^m, g_{2,1}^k g_{1,2}^l g_{2,2}^m, \ldots:\ k,l,m \in \mathbb{Z}_0\},$$

respectively, with $\mathbb{Z}_0$ denoting the set of non-zero integers. The diagonal units $1_{1,1}$ and $1_{2,2}$ are projections onto subspaces spanned by $\delta(e)$ and $\delta(g)$, where $g \in H_1$ and $g \in H_2$, respectively. The off-diagonal units $1_{1,2}$ and $1_{2,1}$ are defined in a similar way, so that we have decompositions $1_{1,1} + 1_{1,2} = 1_{2,1} + 1_{2,2} = 1$ of the unit 1 on l_M^2.

Example 4.4. In the case of square integrable functions on the 'strongly matricially free product of groups', the Hilbert space is smaller, namely

$$l_S^2 = \operatorname{span}\{\delta(e), \delta(g):\ g \in K_1 \cup K_2\},$$

where

$$K_1 := \{g_{1,1}^k, g_{1,2}^k g_{2,2}^l, g_{1,2}^k g_{2,1}^l g_{1,1}^m, \ldots:\ k,l,m \in \mathbb{Z}_0\}$$

and

$$K_2 := \{g_{2,2}^k, g_{2,1}^k g_{1,1}^l, g_{2,1}^k g_{1,2}^l g_{2,2}^m, \ldots:\ k,l,m \in \mathbb{Z}_0\},$$

which reflects the fact that the diagonal generators act non-trivially only on $\delta(e)$. Internal units are reduced accordingly. Note that in this case $l_S^2 \cong l^2(F(2))$, which conforms with Theorem 3.1 and the resulting decomposition of the left regular representation.

Example 4.5. In the case of an n-dimensional square array of copies of $F(1)$, we form a tree (a subtree of the homogeneous tree $\mathbb{H}_{2n^2}$) which corresponds to the 'matricially free product of n^2 free groups'. Suppose the root e corresponds to the 'father'. We distinguish 'sons' and 'daughters' in each 'generation' which correspond to the left action of $g_{j,j}$ or $g_{j,j}^{-1}$, and $g_{i,j}$ or $g_{i,j}^{-1}$, respectively, where $i \neq j$. The rules of drawing the tree follow from matricial freeness and are the following: each 'son' has 1 'son' and $2n-2$ 'daughters', whereas each 'daughter' has 2 'sons' and $2n-1$ 'daughters'. Therefore, 'sons' and 'daughters' correspond to vertices of valencies $2n$ and $2n+2$, respectively. In Fig. 1 we draw such a tree for $n=2$ (black and empty circles are assigned to 'sons' and 'daughters', respectively). If we make an additional assumption, for instance that 'daughters' cannot have 'sons' (this fact corresponds to strong matricial freeness, where diagonal generators kill words beginning with the off-diagonal ones), we recover $\mathbb{H}_{2n}$ of free probability.

Example 4.6. Let $\mathcal{R}(\widehat{\mathcal{H}})$ be the discrete strongly matricially free Fock space and let $\mathcal{R}(\widehat{\mathcal{H}}) \cong \mathcal{F}(\bigoplus_j \mathbb{C}e_j)$ be the natural isomorphism of Remark 3.1, where $\{e_j:\ j \in \mathbb{N}\}$ is an orthonormal basis of some Hilbert space. If $(w_{i,j})$ is an infinite matrix with non-negative parameters p and q

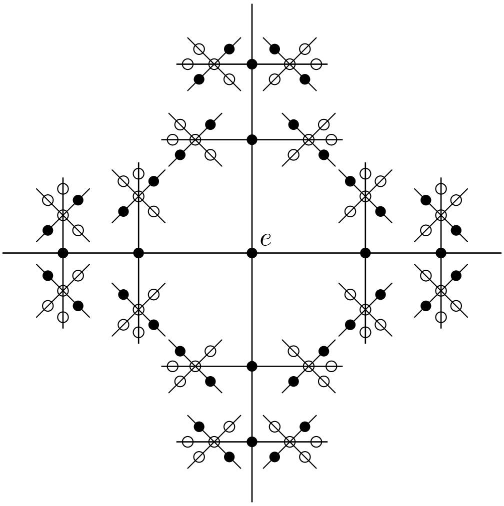


Fig. 1. Matricially free analog of $\mathbb{H}_4$.

above and below the main diagonal, respectively, and 1's on the diagonal, then it is easy to see that the (p,q)-creation operators studied in [18] can be identified (with the use of this isomorphism) with the strongly convergent sums

$$A_i = \sum_j w_{i,j} k_{i,j},$$

where $i \in \mathbb{N}$ and the (p,q)-annihilation operators are their adjoints. A similar approach can be applied to square arrays of arbitrary Hilbert spaces and $*$-representations, which leads to some notion of '(p,q)-independence'. Moreover, it can be carried out for more general matrices $(w_{i,j})$ within the framework of the strong matricial freeness, which generalizes notions of independence of this type.

5. Traces

In this section we introduce some real-valued functions on the set of non-crossing pair-partitions. These functions are obtained by computing traces of a square real-valued matrix. We will assume later that this matrix has non-negative entries which represent variances of probability measures on the real line $\mathcal{M}_{\mathbb{R}}$ and we will demonstrate that the functions introduced in this section describe the asymptotics of matricially free random variables in central limit theorems.

Let $\mathcal{NC}_m$ denote the set of non-crossing partitions of the set $[m]$, i.e. if $\pi = \{\pi_1, \pi_2, \ldots, \pi_k\} \in \mathcal{NC}_m$, then there are no numbers $i < p < j < q$ such that $i, j \in \pi_r$ and $p, q \in \pi_s$ for $r \neq s$. The block π_r is *inner* with respect to π_s if $p < i < q$ for any $i \in \pi_r$ and $p, q \in \pi_s$ (then π_s is *outer* with respect to π_r). It is clear that if π_r has outer blocks, then there exists a unique block among them, say π_s, which is *nearest* to π_r, i.e. if another block, say π_t, is outer with respect to π_r, then we must have $a < p < b$ for any $a, b \in \pi_t$ and $p \in \pi_s$. In that case we shall write $\pi_s = o(\pi_r)$ and call the pair $(\pi_r, o(\pi_r))$ the *nearest inner–outer pair* of blocks. If π_i does not have an outer block, it is called a *covering* block. It is convenient to extend each partition $\pi \in \mathcal{NC}_m^2$ to the partition $\widehat{\pi}$ obtained from π by adding one block, say $\pi_0 = \{0, m+1\}$, called the *imaginary block*.

If $\mathcal{B}(\pi)$ is the set of blocks of π, we shall denote by $F_r(\pi)$ the set of all mappings $f : \mathcal{B}(\pi) \to [r]$ called *colorings* of the blocks of π by the set $[r] := \{1, 2, \ldots, r\}$. Then the pair (π, f) plays the role of a colored partition.

Let $\mathcal{NCC}_m$ denote the set of *non-crossing covered partitions* of $[m]$, by which we understand the subset of $\mathcal{NC}_m$ consisting of those partitions in which 1 and m belong to the same block (if $m = 1$, we understand that the partition consists of one block). In terms of diagrams, all blocks of $\pi \in \mathcal{NCC}_m$, where $m > 1$, are covered by the block containing 1 and m. We denote by $\mathcal{NC}_m^2$ and $\mathcal{NCC}_m^2$ the sets of non-crossing pair-partitions of $[m]$ and non-crossing covered pair-partitions of $[m]$, respectively, and we set $\mathcal{NC}^2 = \bigcup_{m=1}^{\infty} \mathcal{NC}_m^2$ and $\mathcal{NCC}^2 = \bigcup_{m=1}^{\infty} \mathcal{NCC}_m^2$.

It is easy to see that each $\pi \in \mathcal{NC}_m$ can be decomposed as

$$\pi = \pi^{(1)} \cup \pi^{(2)} \cup \cdots \cup \pi^{(p)}, \tag{5.1}$$

where $\pi^{(1)}, \ldots, \pi^{(p)}$ are non-crossing covered partitions of subintervals $I_1, I_2, \ldots, I_p$ of $[m]$ whose union gives $[m]$. By a partition of a set I consisting of r elements we understand the corresponding partition of $[r]$.

On the other hand, each $\pi \in \mathcal{NCC}_m$ can be decomposed as

$$\pi = \pi^{(0)} \cup \pi^{(1)} \cup \cdots \cup \pi^{(r)}, \tag{5.2}$$

where $\pi^{(0)}$ is the block containing 1 and m and $\pi^{(1)}, \pi^{(2)}, \ldots, \pi^{(r)}$ are non-crossing covered partitions of subintervals $I_1, I_2, \ldots, I_r$ of $[m] \setminus \pi_0$.

Consider now a square real-valued matrix $V = (v_{i,j}) \in M_n(\mathbb{R})$, where $n \in \mathbb{N}$. The usual trace and the normalized trace will be denoted

$$\mathrm{Tr}(V) = \sum_{j=1}^{n} v_{j,j} \quad \text{and} \quad \mathrm{tr}(V) = \frac{1}{n} \sum_{j=1}^{n} v_{j,j},$$

respectively. For each n, we define the 'diagonalization mapping'

$$\tau : M_n(\mathbb{R}) \to D_n(\mathbb{R}), \quad \tau(V) = \mathrm{diag}\left(\sum_j v_{j,1}, \ldots, \sum_j v_{j,n} \right),$$

where $D_n(\mathbb{R})$ is the set of square diagonal real-valued matrices of dimension n (by abuse of notation, the same symbol τ is used for all n). In other words, τ computes the sum of all elements of V in each column separately and puts this value on the diagonal.

Using the above trace operations, we shall define two real-valued functions on the set $\mathcal{NC}^2$, denoted v and v_0, associated with given $V \in M_n(\mathbb{R})$. Although there is a close similarity between these functions when restricted to $\mathcal{NCC}^2$ (in particular, they are defined as traces of certain matrix-valued quasi-multiplicative functions), note that they are extended to $\mathcal{NC}^2$ in two different ways.

Definition 5.1. For a given matrix $V \in M_n(\mathbb{R})$, we define a mapping from $\mathcal{NC}^2$ to $D_n(\mathbb{R})$ by assigning to each $\pi \in \mathcal{NC}^2$ the matrix $V(\pi)$ by the following recursion:

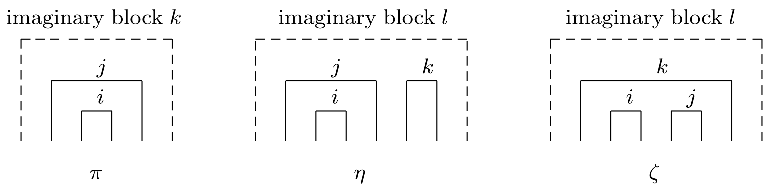


Fig. 2. Colored partitions with imaginary blocks.

1. if π consists of one block, we set $V(\pi) = \tau(V)$,
2. if $\pi \in \mathcal{NCC}^2$ consists of more than one block, then

$$V(\pi) = \tau\big(V\big(\pi^{(1)}\big) \dots V\big(\pi^{(r)}\big) V\big) \tag{5.3}$$

according to the decomposition (5.2),

3. if $\pi \in \mathcal{NC}^2$, then

$$V(\pi) = V\big(\pi^{(1)}\big) V\big(\pi^{(2)}\big) \dots V\big(\pi^{(p)}\big) \tag{5.4}$$

according to the decomposition (5.1).

Let $v : \mathcal{NC}^2 \to \mathbb{R}$ be the function defined by $v(\pi) = \operatorname{tr}(V(\pi))$.

Example 5.1. For some $V \in M_n(\mathbb{R})$, consider three partitions given in Fig. 2. Color each block by a number from the set $[n]$. Computation of the corresponding values of the function v gives

$$v(\pi) = \operatorname{tr}\big(\tau\big(\tau(V)V\big)\big) = \frac{1}{n} \sum_{i,j,k} v_{i,j} v_{j,k},$$

$$v(\eta) = \operatorname{tr}\big(\tau\big(\tau(V)V\big)\tau(V)\big) = \frac{1}{n} \sum_{i,j,k,l} v_{i,j} v_{j,l} v_{k,l},$$

$$v(\zeta) = \operatorname{tr}\big(\tau\big(\tau(V)\tau(V)V\big)\big) = \frac{1}{n} \sum_{i,j,k,l} v_{i,k} v_{j,k} v_{k,l}.$$

One can see that to each nearest inner–outer pair of blocks (π_r, π_s) we assign the matrix element $v_{p,q}$, where p and q are the colors of π_r and π_s respectively. Moreover, if a block does not have any outer blocks and is colored by q, then we assign to it the matrix element $v_{q,t}$, where t is assumed to be the same for all such blocks (one can imagine that we have an additional ‘imaginary block’ colored by t which covers all other blocks). At the end we sum over all colorings.

The function v_0 is defined on the set $\mathcal{NCC}^2$ in a very similar manner, except that we replace the right multiplier V in (5.3) by its main diagonal

$$V_0 := \operatorname{diag}(v_{1,1}, \dots, v_{n,n}),$$

which corresponds to changing only the contribution of the covering block. Then we extend v_0 to all of $\mathcal{NC}^2$ by multiplicativity of v_0.

Definition 5.2. For a given matrix $V_0 \in M_n(\mathbb{R})$, we define a mapping from $\mathcal{NCC}^2$ to $D_n(\mathbb{R})$ by assigning to each $\pi \in \mathcal{NCC}^2$ the matrix $V_0(\pi)$ by the following recursion:

1. if π consists of one block, we set $V_0(\pi) = V_0$,
2. if $\pi \in \mathcal{NCC}^2$ consists of more than one block, then

$$V_0(\pi) = \tau\big(V\big(\pi^{(1)}\big) \ldots V\big(\pi^{(r)}\big) V_0\big) \tag{5.5}$$

according to the decomposition (5.2).

Let $v_0 : \mathcal{NC}^2 \to \mathbb{R}$ be the function defined by $v_0(\pi) = \mathrm{Tr}(V_0(\pi))$ for $\pi \in \mathcal{NCC}^2$, extended to $\mathcal{NC}^2$ by multiplicativity $v_0(\pi) = v_0(\pi^{(1)}) \ldots v_0(\pi^{(p)})$ according to the decomposition (5.1).

Example 5.2. Let us compute the values of v_0 corresponding to the partitions in Fig. 2. We have

$$v_0(\pi) = \mathrm{Tr}\big(\tau(V)V_0\big) = \sum_{i,j} v_{i,j} v_{j,j},$$

$$v_0(\eta) = \mathrm{Tr}\big(\tau(V)V_0\big)\mathrm{Tr}(V_0) = \sum_{i,j,k} v_{i,j} v_{j,j} v_{k,k},$$

$$v_0(\zeta) = \mathrm{Tr}\big(\tau\big(\tau(V)\tau(V)V_0\big)\big) = \sum_{i,j,k} v_{i,k} v_{j,k} v_{k,k}.$$

Note that the main difference (apart from normalization) between $v_0(\pi)$ and $v(\pi)$ concerns the blocks which do not have outer blocks. Here, if such a block is colored by q, we assign to it the matrix element $v_{q,q}$ and we do not use ‘imaginary blocks’.

Below we shall prove a lemma which gives explicit combinatorial formulas for $v(\pi)$ and $v_0(\pi)$. For that purpose, to the blocks

$$\mathcal{B}(\pi, f) = \big\{(\pi_1, f), (\pi_2, f), \ldots, (\pi_k, f)\big\}$$

of each colored non-crossing partition (π, f), where $f \in F_r(\pi)$, we assign entries of a given real-valued matrix $V \in M_r(\mathbb{R})$ according to the definition given below. If the imaginary block is used, it is convenient to assume that it is also colored by a number from the set $[r]$.

Definition 5.3. Let (π, f) be a colored non-crossing partition with blocks as above, where $f \in F_r(\pi)$ and let $V \in M_r(\mathbb{R})$ be given. For any $0 \leqslant j \leqslant r$ we define

$$v_j(\pi, f) = v_j(\pi_1, f) v_j(\pi_2, f) \ldots v_j(\pi_k, f),$$

where the functions $v_j : \mathcal{B}(\pi, f) \to \mathbb{R}$ are given by the following rules:

1. $v_j(\pi_i, f) = v_{p,q}$ if $f(\pi_i) = p$ and $f(o(\pi_i)) = q$, where $0 \leqslant j \leqslant r$,
2. $v_j(\pi_i, f) = v_{p,j}$ if $f(\pi_i) = p$ and π_i does not have outer blocks, where $1 \leqslant j \leqslant r$,
3. $v_0(\pi_i, f) = v_{p,p}$ if $f(\pi_i) = p$ and π_i does not have outer blocks.

We are ready to prove purely combinatorial formulas for $v_j(\pi)$ for any $0 \leqslant j \leqslant n$ and any $\pi \in \mathcal{NC}^2$. In particular, if V is a square n-dimensional matrix with all entries equal to $1/n$ and f runs over $F_n(\pi)$, then $v_0(\pi) = 1$ and $v_j(\pi) = 1/n$ for any π. In that case we get $\sum_{\pi \in \mathcal{NC}^2_{2m}} v_0(\pi) = \sum_{j=1}^{n} \sum_{\pi \in \mathcal{NC}^2_{2m}} v_j(\pi) = c_m$, the m-th Catalan number. Limit theorems studied in Section 6 will give matricial deformations of Catalan numbers induced by the formulas given by the lemma proven below.

Lemma 5.1. *For any $V \in M_n(\mathbb{R})$, where $n \in \mathbb{N}$, and $\pi \in \mathcal{NC}^2$, it holds that*

$$v_0(\pi) = \sum_{f \in F_n(\pi)} v_0(\pi, f) \quad \textit{and} \quad v(\pi) = \frac{1}{n} \sum_{f \in F_n(\pi)} \sum_{j \in [n]} v_j(\pi, f),$$

where the summation over j corresponds to all colorings of the imaginary block.

Proof. We provide an induction proof for the function v (the proof for v_0 is similar). The main induction step will be carried out on the level of the (diagonal) matrices $V(\pi)$. We claim that its diagonal entries are of the form

$$\big(V(\pi)\big)_{q,q} = \sum_{f \in F_n(\pi)} v_q(\pi, f)$$

for any $\pi \in \mathcal{NCC}^2$ and $q \in [n]$. In view of (5.4), the required formula for $v(\pi)$ is then a straightforward consequence of the claim. Of course, if π consists of one block, then

$$\big(V(\pi)\big)_{q,q} = \sum_j v_{j,q}$$

and thus our assertion easily follows. Assume now that π has $k \geqslant 2$ blocks and suppose the assertion holds for non-crossing covered partitions which have less than k blocks. Since π has a decomposition of type (5.2), the assertion holds for $\pi^{(1)}, \pi^{(2)}, \ldots, \pi^{(r)}$ in this decomposition. We know that the matrix assigned to π has the form (5.3) and therefore the product of (diagonal) matrices corresponding to these subpartitions has (diagonal) matrix elements of the form

$$\big(V(\pi)\big)_{q,q} = \big(\tau(WV)\big)_{q,q} = \sum_j W_{j,j} v_{j,q}$$

where q is the color of the imaginary block of π and

$$W_{j,j} = \sum_{f_1 \in F_n(\pi^{(1)})} v_j\big(\pi^{(1)}, f_1\big) \ldots \sum_{f_r \in F_n(\pi^{(r)})} v_j\big(\pi^{(r)}, f_r\big),$$

by the inductive assumption. Now, since the blocks of $\pi^{(1)}, \ldots, \pi^{(r)}$ are colored independently, we have

$$v_j\big(\pi^{(1)}, f_1\big)\ldots v_j\big(\pi^{(r)}, f_r\big)v_{j,q} = v_q(\pi, f)$$

for a uniquely determined coloring f of the blocks of π in which j can be interpreted as the color of $\pi^{(0)}$ since $\pi^{(0)}$ covers $\pi^{(1)}, \ldots, \pi^{(r)}$ and q can be viewed as the color of the imaginary block of π. This and the above formula for $W_{j,j}$ gives the desired formula

$$V(\pi)_{q,q} = \sum_{f\in F_n(\pi)} v_q(\pi, f),$$

which proves our claim and thus completes the proof of the theorem. □

Under suitable assumptions on V, there is a simple connection between functions v and v_0 if $V_0 = aI_n/n$, where I_n is a unit matrix and a is a positive number. We shall express this relation in terms of the corresponding formal Laurent series, which turn out to be Cauchy transforms of (compactly supported) probability measures on the real line associated with appropriately constructed random variables.

Proposition 5.1. *Let $V \in M_n(\mathbb{R})$ be such that $V_0 = aI_n/n$, where $a > 0$. If $G(z) = \sum_{k=0}^{\infty} a_{2k}z^{-2k-2}$ and $G_0(z) = \sum_{k=0}^{\infty} b_{2k}z^{-2k-2}$ are formal Laurent series, where*

$$a_{2k} = \sum_{\pi\in\mathcal{NC}_{2k}^2} v(\pi) \quad \text{and} \quad b_{2k} = \sum_{\pi\in\mathcal{NC}_{2k}^2} v_0(\pi),$$

and both $v(\pi)$ and $v_0(\pi)$ are associated with V, then $G_0(z) = 1/(z - aG(z))$.

Proof. Observe that in the case when $v_{j,j} = a/n$ for all $j \in [n]$, we have

$$v(\pi) = \frac{v_0(\pi')}{a}$$

for any $\pi \in \mathcal{NC}_m^2$, where the partition $\pi' \in \mathcal{NCC}_{m+2}^2$ is obtained from π by adding to π the block that covers all blocks of π, say $\{0, m+1\}$. This leads to

$$\begin{aligned} A(z) := \sum_{m=0}^{\infty} a_{2m}z^{2m} &= 1 + \sum_{m=1}^{\infty}\sum_{\pi\in\mathcal{NC}_{2m}^2} v(\pi)z^{2m} \\ &= 1 + \frac{1}{a}\sum_{m=1}^{\infty}\sum_{\pi\in\mathcal{NCC}_{2m+2}^2} v_0(\pi)z^{2m} = \frac{C(z)-1}{az^2}, \end{aligned}$$

where $C(z)=\sum_{m=0}^{\infty}c_{2m}z^{2m}$ and $c_{2m}=\sum_{\pi\in\mathcal{NCC}_{2m}^{2}}v_0(\pi)$. Now, using the multiplicativity of v_0, we obtain

$$B(z):=\sum_{m=0}^{\infty}b_{2m}z^{2m}=1+\sum_{m=1}^{\infty}\bigl(C(z)-1\bigr)^m=\frac{1}{2-C(z)},$$

which leads to

$$A(z)=\frac{1}{az^2}\left(1-\frac{1}{B(z)}\right)\quad\text{and}\quad G_0(z)=\frac{1}{z-aG(z)},$$

where

$$G(z)=\frac{1}{z}A\left(\frac{1}{z}\right)\quad\text{and}\quad G_0(z)=\frac{1}{z}B\left(\frac{1}{z}\right),$$

which completes the proof. $\square$

6. Random pseudomatrices

In this section we will study the asymptotic behavior of random pseudomatrices for two sequences of states: the sequence of distinguished states, with respect to which our variables will be matricially free, and the sequence of traces which are normalized sums of conditions in the definition of matricial freeness. Under suitable assumptions, we will later obtain two types of limit theorems: the 'standard' central limit theorem as well as the 'tracial' central limit theorem for matricially free random variables, the latter being related to random matrix models.

Suppose we have a sequence $(\mathcal{A}(n))$ of unital $*$-algebras, each equipped with a distinguished state $\phi(n)$ and associated conditions $\{\phi_j(n)\colon\ 1\leqslant j\leqslant n\}$. For each n, let $(X_{i,j}(n))_{1\leqslant i,j\leqslant n}$ be an array of self-adjoint random variables in $\mathcal{A}(n)$. We are going to study the asymptotic behavior of random pseudomatrices

$$S(n)=\sum_{i,j=1}^{n}X_{i,j}(n),\tag{6.1}$$

where we assume that each array $(X_{i,j}(n))$ is matricially free with respect to the array $(\phi_{i,j}(n))$ defined by $\phi(n)$ and $\phi_j(n)$'s. In the first setting we shall compute the asymptotic moments of random pseudomatrices with respect to $\phi(n)$. This corresponds to the 'standard CLT' reminding the CLT for free (or, monotone) random variables [24].

In the second setting, we shall compute the asymptotic moments of random pseudomatrices with respect to the convex linear combinations of conditions

$$\psi(n):=\frac{1}{n}\sum_{j=1}^{n}\phi_j(n)\tag{6.2}$$

which play the role of traces in the context of random pseudomatrices, with $\phi_j(n)$ corresponding the classical expectation $\mathbb{E}$ composed with the state associated with vector e_i of the canonical orthonormal basis $(e_i)_{1\leqslant i\leqslant n}$ in $\mathbb{C}^n$. This will lead to the 'tracial CLT' for matricially free random variables reminding the limit theorem for random matrices [25]. Thanks to the properties of the matricially free product of states which essentially boil down to properties (P1) and (P3) of Section 2, the normalization of square root type works in both cases since the number of summands in $S(n)$ which give a non-zero contribution to the limits is in both cases of order n.

We partition the set $[n] := \{1, 2, \dots, n\}$ into disjoint non-empty intervals,

$$[n] = N_1 \cup N_2 \cup \dots \cup N_r,$$

where $r \in \mathbb{N}$, such that their relative sizes $n_j/n \to d_j$ as $n \to \infty$, where n_j is the cardinality of N_j for any $j \in [r]$. Then the numbers d_j form a diagonal matrix D of trace one called the *dimension matrix*.

Our results will involve the following assumptions:

(A1) $(X_{i,j}(n))$ is matricially free with respect to $(\phi_{i,j}(n))$ for any $n \in \mathbb{N}$,

(A2) the variables have zero expectations,

$$\phi_{i,j}(n)\big(X_{i,j}(n)\big) = 0$$

for all $i, j \in [n]$ and $n \in \mathbb{N}$,

(A3) their moments are uniformly bounded, i.e. $\forall m\ \exists M_m \geqslant 0$ such that

$$\big|\phi_{i,j}(n)\big(X_{i,j}^m(n)\big)\big| \leqslant \frac{M_m}{n^{m/2}}$$

for all $i, j \in [n]$ and $n \in \mathbb{N}$,

(A4) their variances are block-identical and are of order $1/n$, namely

$$\phi_{i,j}(n)\big(X_{i,j}^2(n)\big) = \frac{u_{p,q}}{n}$$

for any $i \in N_p$, $j \in N_q$, where each $u_{p,q}$ is a non-negative real number.

Of course, the uniform boundedness assumption is satisfied if we take variables of type $X_{i,j}(n) = X_{i,j}/\sqrt{n}$, where $(X_{i,j})$ is an infinite array of random variables whose distributions in the states $\phi_{i,j}(n)$, respectively, are identical and do not depend on n. Although it is convenient to think of the $X_{i,j}(n)$ as if they were of this form, we want to study similar variables, whose variances are of order $1/n$ and stay the same within blocks whose sizes become infinite as $n \to \infty$.

If the tuple $((i_1, j_1), \dots, (i_m, j_m)) \in (I \times I)^m$, where I is an index set, defines a partition $\pi = \{\pi_1, \dots, \pi_k\}$ of the set $[m]$, i.e. $(i_p, j_p) = (i_q, j_q)$ if and only if there exists r such that $p, q \in \pi_r$, we will write

$$P\big((i_1, j_1), \dots, (i_m, j_m)\big) = \pi.$$

Of course, if π is a non-crossing pair-partition, then each π_r is a two-element set. By a *k-block* we will understand a block consisting of k elements. In particular, a 1-block will be called a *singleton*. We will also adopt the convention that if m is odd, then $\mathcal{NC}_m^2 = \emptyset$ and the summation over $\pi \in \mathcal{NC}_m^2$ gives zero. This allows us to state results for moments of the $S(n)$ of all orders without distinguishing even and odd moments.

Lemma 6.1. *Let $(\mathcal{A}(n), \psi(n))$ be a sequence of noncommutative probability spaces, each with an array $(\phi_{i,j}(n))$ defined by $\phi(n)$ and $\phi_j(n)$'s, for which* (A1)–(A3) *hold, where $\psi(n)$ is given by* (6.2). *Then*

$$\psi(n)\big(S^m(n)\big) = \sum_{\pi \in \mathcal{NC}_m^2} v(\pi, n) + O\left(\frac{1}{\sqrt{n}}\right),$$

where $v(\pi, n) = \operatorname{tr}(V(\pi, n))$ is given by Definition 5.1 *and corresponds to the variance matrix $V(n) = (v_{i,j}(n))$, where $v_{i,j}(n) = \phi(n)(X_{i,j}^2(n))$.*

Proof. We have

$$\begin{aligned}\psi(n)\big(S^m(n)\big) &= \sum_{i_1, j_1, \ldots, i_m, j_m} \psi(n)\big(X_{i_1,j_1}(n) \ldots X_{i_m,j_m}(n)\big) \\ &= \sum_{\pi \in \mathcal{P}_m} \sum_{\substack{i_1, j_1, \ldots, i_m, j_m \\ P((i_1,j_1),\ldots,(i_m,j_m))=\pi}} \psi(n)\big(X_{i_1,j_1}(n) \ldots X_{i_m,j_m}(n)\big),\end{aligned}$$

where $\mathcal{P}_m$ denotes the set of all partitions of $[m]$. Arguments presented below allow us to conclude that for large n only non-crossing pair-partitions give relevant contributions.

1. If π has a singleton associated with some $(i_k, j_k) \neq (j, j)$, then the moment
$$\phi_j(n)\big(X_{i_1,j_1}(n) \ldots X_{i_m,j_m}(n)\big)$$
vanishes by Lemma 4.1 since in that case the variable X_{i_k,j_k} is the only variable from $\mathcal{A}_{i_k,j_k}$ in the corresponding moment under $\phi(n)$, by which we mean $\phi(n)(b_j^* a b_j)$, where $a = X_{i_1,j_1}(n) \ldots X_{i_m,j_m}(n)$ and $b_j \in \mathbb{C}[X_{j,j}(n), 1_{j,j}] \cap \operatorname{Ker} \phi(n)$. For that reason, in the remaining cases we can assume that there are no such singletons.
2. If π has exactly one singleton associated with some $(i_k, j_k) = (j, j)$, then the extended partition $\widehat{\pi}$ associated to $b_j^* a b_j$ does not have singletons and has at least one 3-block (this is the imaginary block colored with j). Therefore, $\widehat{\pi}$ has the same number of blocks as π and $s \leqslant (m+1)/2$. Since with each block we can associate at most one independent index to sum over (by Lemma 4.1), the sum of the above moments over $i_1, j_1, \ldots, i_m, j_m$ and j for the given π (with the $1/n$ normalization coming from $\psi(n)$) is $O(1/\sqrt{n})$ by (A3).
3. If π has no singletons and is not a pair-partition, then the number of blocks of $\widehat{\pi}$ is either s (if (j, j) is present among $(i_1, j_1), \ldots, (i_m, j_m)$) or $s + 1$ (in the opposite case). In both cases, we have that $s \leqslant (m+1)/2$ and the summation of the considered moments over $i_1, j_1, \ldots, i_m, j_m$ and j for the given π is $O(1/\sqrt{n})$.
4. If π is a crossing pair-partition, then the associated $\widehat{\pi}$ is a crossing partition for any j. It suffices to consider the case when $\widehat{\pi}$ is a pair-partition since otherwise it has a 4-block and a similar argument to that in (3) works. In turn, if $\widehat{\pi}$ is a pair-partition for some j, then by Lemma 4.1 and the mean zero assumption, the corresponding moment under $\phi_j(n)$ vanishes, which implies that the corresponding moment under $\psi(n)$ also vanishes.

The above arguments imply that only non-crossing pair-partitions contribute to the limit and we can write

$$\psi(n)\big(S^m(n)\big)=\sum_{\pi\in\mathcal{NC}_m^2}\ \sum_{\substack{i_1,j_1,\dots,i_m,j_m\\ P((i_1,j_1),\dots,(i_m,j_m))=\pi}}\psi(n)\big(X_{i_1,j_1}(n)\dots X_{i_m,j_m}(n)\big)+O\left(\frac{1}{\sqrt{n}}\right).$$

Now, suppose that m is even, $\pi\in\mathcal{NC}_m^2$ and the sequence of pairs $((i_1,j_1),\dots,(i_m,j_m))$ is compatible with the matricial multiplication, i.e. such that if (i_k,j_k) and (i_r,j_r) label an inner–outer pair of blocks, then $j_k=i_r$. If $\pi_j=\{r,r+1\}$ is the last block which has no inner blocks, then we can pull out the variance corresponding to that block, namely:

$$\begin{aligned}&\psi(n)\big(X_{i_1,j_1}(n)\dots X_{i_m,j_m}(n)\big)\\&\quad=v_{i_r,j_r}\psi(n)\big(X_{i_1,j_1}(n)\dots X_{i_{r-1},j_{r-1}}(n)X_{i_{r+2},j_{r+2}}(n)\dots X_{i_m,j_m}(n)\big),\end{aligned}$$

where we use the proof of Lemma 4.1. Continuing this procedure with other blocks which have no inner blocks, and summing over indices $i_1,j_1,\dots,i_m,j_m$, for which it holds that $P((i_1,j_1),\dots,(i_m,j_m))=\pi$, we arrive at

$$\frac{1}{n}\sum_{k_0,k_1\dots,k_m}v_{k_1,k_{o(1)}}(n)\dots v_{k_m,k_{o(m)}}(n)+O\left(\frac{1}{\sqrt{n}}\right),$$

where $o(r)=0$ if π_r has no outer blocks (k_0 labels the imaginary block) and $o(r)=j$ if the nearest outer block of π_r is labelled by k_j. Let us observe that we included in the above sum all possible labellings of the blocks of π. This is done for convenience since it enables us to express the final result in terms of $v(\pi)$. More explicitly, we allow $k_0,k_1,\dots,k_m$ to assume arbitrary values from the set $[m]$ (in particular, they can all be equal), which produces certain terms which cannot be obtained from the summation over all $((i_1,j_1),\dots,(i_m,j_m))$ which define π. For example, no nearest inner–outer pair of blocks can contribute $v_{j,j}v_{j,j}$, which appears in the above sum. However, all such terms are of order $1/\sqrt{n}$ due to insufficient number of different summation indices (there are fewer than $m/2$ independent indices) and therefore they can be included in the sum without changing the asymptotics. Using Lemma 5.1, we obtain our assertion. □

Lemma 6.2. *Let $(\mathcal{A}(n),\phi(n))$ be a sequence noncommutative probability spaces, each with an array $(\phi_{i,j}(n))$ of states defined by $\phi(n)$ and conditions $\phi_j(n)$, for which* (A1)–(A3) *hold. Then*

$$\phi(n)\big(S^m(n)\big)=\sum_{\pi\in\mathcal{NC}_m^2}v_0(\pi,n)+O\left(\frac{1}{\sqrt{n}}\right),$$

where $v_0(\pi,n)=\mathrm{Tr}(V_0(\pi,n))$ is given by Definition 5.2 *and corresponds to the variance matrix $V(n)=(v_{i,j}(n))$.*

Proof. The proof is similar to that of Lemma 6.1 (in fact, it is simpler since we only need to compute the moments under $\phi(n)$ and thus the complications involving moments under $\phi_j(n)$'s do not appear). □

In order to ensure existence of the limits of $v(\pi,n)$ and $v_0(\pi,n)$ as $n\to\infty$, we need to specify the sequence of matrices $(V(n))_{n\in\mathbb{N}}$ more closely. We shall assume that (A4) holds. Assuming that the variance matrices $V(n)$ are of this form, we can now state the standard and tracial central limit theorems, with limit distributions described in terms of traces of Section 5.

Theorem 6.1 *(Tracial central limit theorem). Under the assumptions of Lemma* 6.1*, if $V(n)$ is of the block form* (A4) *for each $n\in\mathbb{N}$, then*

$$\lim_{n\to\infty}\psi(n)\big(S^m(n)\big)=\sum_{\pi\in\mathcal{NC}_m^2}b(\pi), \tag{6.3}$$

for any $m\in\mathbb{N}$, where $b(\pi)=\mathrm{Tr}(B(\pi)D)$ and $B(\pi)$ is the diagonal matrix of Definition 5.1 *corresponding to π and the matrix $B=DU$.*

Proof. Clearly, if m is odd, we get zeros on both sides of the above formula (we use our convention that in this case $\mathcal{NC}_m^2=\emptyset$). The proof for $m=2k$, where $k\in\mathbb{N}$, is based on Lemmas 5.1 and 6.1. If, in the combinatorial expression for $v(\pi,n)$, we substitute for the matrix elements of $V(n)$ the assumed block form, then, using the partition of the set of colors $[n]=N_1\cup N_2\cup\cdots\cup N_r$, we can perform summations over the colorings which belong to each interval N_j separately. Thus, the contributions of various $[n]$-colorings of π to the limit laws reduce to those corresponding to $[r]$-colorings and are described in terms of numbers $u_j(\pi_i,f)$, where $i\in[k]$, $j\in[r]$ and $f\in F_r(\pi)$ (the number j is the color of the imaginary block). We have

$$\begin{aligned}\lim_{n\to\infty}v(\pi,n)&=\lim_{n\to\infty}\left(\frac{1}{n^{k+1}}\sum_{j\in[r]}n_j\sum_{f\in F_r(\pi)}n_{f(1)}u_j(\pi_1,f)\dots n_{f(k)}u_j(\pi_k,f)\right)\\&=\sum_{j\in[r]}d_j\sum_{f\in F_r(\pi)}d_{f(1)}u_j(\pi_1,f)\dots d_{f(k)}u_j(\pi_k,f)=\mathrm{Tr}\big(B(\pi)D\big),\end{aligned}$$

where $\pi\to B(\pi)$ is the matrix-valued function which corresponds to the matrix $B=DU$ in accordance with Definition 5.1. In terms of matrix multiplication, the expression on the right-hand side is obtained from that of Lemma 5.1 corresponding to matrix U by multiplying U from the left by the dimension matrix D and multiplying the whole product of matrices from the right by D. This proves our assertion. □

Theorem 6.2 *(Central limit theorem). Under the assumptions of Lemma* 6.2*, if $V(n)$ is of the block form* (A4) *for each $n\in\mathbb{N}$, then*

$$\lim_{n\to\infty}\phi(n)\big(S^m(n)\big)=\sum_{\pi\in\mathcal{NC}_m^2}b_0(\pi), \tag{6.4}$$

for any $m\in\mathbb{N}$, where $\pi\to b_0(\pi)$ is the real-valued function of Definition 5.2 *corresponding to the matrix $B=DU$.*

Proof. The proof is similar to that of Theorem 6.1 and is based on Lemmas 5.1 and 6.2. The only difference is that we do not use imaginary blocks to describe the colorings of all blocks of π. Thus, the contributions of various $[n]$-colorings of π to the limit laws reduce to those corresponding to $[r]$-colorings and are described in terms of numbers $u_0(\pi_i, f)$, where $i \in [k]$ and $f \in F_r(\pi)$. Namely, we have

$$\lim_{n\to\infty} v_0(\pi,n) = \lim_{n\to\infty}\left(\frac{1}{n^k}\sum_{f\in F_r(\pi)} n_{f(1)}u_0(\pi_1,f)\dots n_{f(k)}u_0(\pi_k,f)\right)$$
$$= \sum_{f\in F_r(\pi)} d_{f(1)}u_0(\pi_1,f)\dots d_{f(k)}u_0(\pi_k,f) = \mathrm{Tr}\big(B_0(\pi)\big)$$

as $n \to \infty$, where $\pi \to B_0(\pi)$ is the function defined by Definition 5.2, which proves our assertion. □

7. Matricial semicircle distributions

The results of Section 6 lead to combinatorial formulas for the asymptotic moments in the corresponding central limit theorems. In this section we are going to express the limits in terms of their Cauchy transforms represented in the form of continued fractions. They play the role of the (standard and tracial) 'matricial semicircle distributions'.

For that purpose let us recall definitions of certain convolutions of distributions, or more generally, of probability measures. If F_μ is the reciprocal Cauchy transform of some probability measure $\mu \in \mathcal{M}_{\mathbb{R}}$, then the K-transform of μ is given by $K_\mu(z) = z - F_\mu(z)$. The boolean additive convolution $\mu \uplus \nu$ can be defined by the equation

$$K_{\mu\uplus\nu}(z) = K_\mu(z) + K_\nu(z),$$

where $\mu, \nu \in \mathcal{M}_{\mathbb{R}}$ and $z \in \mathbb{C}^+$, respectively. In fact, this equation shows that the K-transform is the boolean analog of the logarithm of the Fourier transform [23].

We will also need another convolution, which reminds the monotone convolution [20], called the orthogonal additive convolution and defined by the equation

$$K_{\mu\vdash\nu}(z) = K_\mu\big(F_\nu(z)\big),$$

where $\mu, \nu \in \mathcal{M}_{\mathbb{R}}$ and $z \in \mathbb{C}^+$. It was introduced in [14], where we showed that the above formula defines a unique probability measure on the real line. Moreover, if μ and ν are compactly supported, both $\mu \vdash \nu$ and $\mu \uplus \nu$ are compactly supported.

Using these convolutions, we will now define certain important continued fractions (or, 'continued multifractions') which converge uniformly on the compact subsets of $\mathbb{C}^+$ to the K-transforms of some probability measures $\mu_{i,j} \in \mathcal{M}_{\mathbb{R}}$.

Lemma 7.1. *For given $B \in M_r(\mathbb{R})$ with non-negative entries, continued fractions of the form*

$$K_{i,j}(z) = \cfrac{b_{i,j}}{z - \sum_k \cfrac{b_{k,i}}{z - \sum_p \frac{b_{p,k}}{z - \cdots}}},$$

where $i, j \in [r]$, converge uniformly on the compact subsets of $\mathbb{C}^+$ to the K-transforms of some $\mu_{i,j} \in \mathcal{M}_{\mathbb{R}}$ with compact supports.

Proof. Let us define a sequence of functions which approximate the $K_{i,j}$. Namely, set $K_{i,j}^{(0)}(z) = b_{i,j}/z$ for any $i, j \in [r]$, which are the K-transforms of probability measures on $\mathbb{R}$ for any i, j (Bernoulli measures if $b_{i,j} > 0$ and δ_0 if $b_{i,j} = 0$). In order to use an inductive argument, let us establish the recurrence

$$K_{i,j}^{(m)}(z) = \frac{b_{i,j}}{z - \sum_k K_{k,i}^{(m-1)}(z)}$$

for $m \geqslant 1$. If the $K_{k,i}^{(m-1)}$ are the K-transforms of some $\mu_{k,i}^{(m-1)} \in \mathcal{M}_{\mathbb{R}}$ for any i and k, respectively, then the sums $\sum_k K_{k,i}^{(m-1)}$ are the K-transforms of their boolean convolution

$$\mu_i^{(m-1)} := \mu_{1,i}^{(m-1)} \uplus \mu_{2,i}^{(m-1)} \uplus \cdots \uplus \mu_{r,i}^{(m-1)} \in \mathcal{M}_{\mathbb{R}},$$

and next, each $K_{i,j}^{(m)}$ is the K-transform of the orthogonal convolution

$$\mu_{i,j}^{(m)} := \kappa_{i,j} \vdash \mu_i^{(m-1)} \in \mathcal{M}_{\mathbb{R}},$$

where the $\kappa_{i,j}$'s are the Bernoulli measures with K-transforms $K_{i,j}(z) = b_{i,j}/z$, respectively. It is easy to see that all these measures are compactly supported. Moreover, the properties of the orthogonal additive convolution (Corollary 5.3 in [14]) say that the moments of $\mu_{i,j}^{(n)}$ of orders $\leqslant 2m$ agree with the corresponding moments of $\mu_{i,j}^{(m)}$ for any $n > m$ and any given i, j. More precisely, in the formal Laurent series expansion

$$K_{i,j}^{(m)}(z) = \sum_{n=0}^{\infty} c_{-2n-1} z^{-2n-1},$$

the coefficients $c_{-1}, \ldots, c_{-2m-1}$ are uniquely determined by the constants which appear in the continued fraction of the corresponding Cauchy transform $G_{i,j}^{(m)}$ at depths $\leqslant 2m$, and these determine the moments of $\mu_{i,j}^{(m)}$ of orders $\leqslant 2m$. The recurrence for the K-transform given above is such that the corresponding Cauchy transforms $G_{i,j}^{(m)}$ and $G_{i,j}^{(m-1)}$ agree down to depth $m-1$ for any given i, j, which proves our assertion. Therefore, we have weak convergence

$$\mathrm{w} - \lim_{m\to\infty} \mu_{i,j}^{(m)} = \mu_{i,j}$$

to some $\mu_{i,j} \in \mathcal{M}_{\mathbb{R}}$ for any i, j. These measures are also compactly supported since $\sup_{i,j} b_{i,j}$ is finite. In turn, this implies that the corresponding Cauchy transforms (and thus K-transforms) converge uniformly to the Cauchy transform (K-transforms) of $\mu_{i,j}$ on compact subsets of $\mathbb{C}^+$. This completes the proof. $\square$

We are ready to state a theorem, which is another version of the tracial central limit theorem for matricially free random variables.

Theorem 7.1. *Under the assumptions of Theorem* 6.1*, the* $\psi(n)$*-distributions of* S_n *converge weakly to the distribution given by the convex linear combination*

$$\mu = \sum_{j=1}^{r} d_j \mu_j,$$

where $\mu_j = \mu_{1,j} \uplus \mu_{2,j} \uplus \cdots \uplus \mu_{r,j}$ *for each* $j = 1, \ldots, r$ *and* $\mu_{i,j}$ *is the distribution defined by* $K_{i,j}$ *for any* i, j.

Proof. By Theorem 6.1, we have combinatorial formulas for the moments M_m of the limit law in the tracial central limit theorem. The associated distribution extends to a unique compactly supported probability measure μ on the real line since its moments are bounded by the moments of the Wigner semicircle distribution σ_a with variance $a = \sup_{i,j} b_{i,j}$. Using the multiplicative formula (5.4) for $B(\pi)$, we can formally write the Cauchy transform of μ in the form

$$\begin{aligned} G_\mu(z) &= \sum_{k=0}^{\infty} M_{2k} z^{-2k-1} \\ &= \frac{1}{z} + \sum_{k=1}^{\infty} \Bigl(\sum_{\pi \in \mathcal{NC}_{2k}^2} \operatorname{Tr}\bigl(B(\pi) D\bigr) \Bigr) z^{-2k-1} \\ &= \operatorname{Tr}\bigl(\bigl(z - K(z)\bigr)^{-1} D \bigr), \end{aligned}$$

which can be called the 'trace formula' for G_μ, where

$$K(z) = \sum_{k=1}^{\infty} \sum_{\pi \in \mathcal{NCC}_{2k}^2} B(\pi) z^{-2k+1} \tag{7.1}$$

is a diagonal-matrix-valued formal power series. Moreover, we will show below that each function K_j on its diagonal is, in fact, the K-transform of some $\mu_j \in \mathcal{M}_{\mathbb{R}}$. Then, the formal power series given by the trace formula is the Cauchy transform of μ as a convex linear combination of Cauchy transforms of probability measures. In fact, using the definition of $B(\pi)$ and (5.3), we obtain the equation

$$K(z) = \tau\bigl(\bigl(z - K(z)\bigr)^{-1} B \bigr), \tag{7.2}$$

where $K(z) = \operatorname{diag}(K_1(z), \ldots, K_r(z))$. By analogy with the scalar-valued case, we can find its solution in the form of a continued fraction. Namely, observe that each $K_j(z)$ has the form of a formal Laurent series

$$K_j(z) = \sum_{n=0}^{\infty} c_{-2n-1} z^{-2n-1}$$

for some $c_{-1}, c_{-3}, \ldots$, and therefore the above vector equation can be solved by successive approximations. Namely, we set μ_j to be the (compactly supported) probability measure associated with the K-transform $K_j(z) = \sum_i K_{i,j}(z)$ for each $j \in [r]$, where the $K_{i,j}$ are given by Lemma 7.1 for $B = DU$. These K-transforms solve (7.2). This, together with the trace formula for G_μ, gives

$$G_\mu = \sum_{j=1}^{r} d_j G_{\mu_j},$$

where $G_{\mu_j}(z) = 1/(z - K_j(z))$ is the Cauchy transform of μ_j for $j = 1, \ldots, r$. That completes the proof. □

Remark 7.1. The Cauchy transform of each μ_j can be written as a continued fraction of the form

$$G_{\mu_j}(z) = \cfrac{1}{z - \sum_i \cfrac{b_{i,j}}{z - \sum_k \cfrac{b_{k,i}}{z - \sum_p \cfrac{b_{p,k}}{z - \cdots}}}}$$

which converges on the compact subsets of $\mathbb{C}^+$.

Next, we state a theorem, which is another version of the standard central limit theorem for matricially free random variables.

Theorem 7.2. *Under the assumptions of Theorem 6.2, the $\phi(n)$-distributions of S_n converge weakly to the distribution*

$$\mu_0 = \mu_{1,1} \uplus \mu_{2,2} \uplus \cdots \uplus \mu_{r,r},$$

where $\mu_{j,j}$ is the distribution defined by $K_{j,j}$ for each j.

Proof. By Theorem 6.2, we have combinatorial expressions for the limit moments M_m. The proof is similar to that of Theorem 7.1 and is based on the trace formula for the K-transform of μ_0

$$K_{\mu_0}(z) = \mathrm{Tr}\Big(\big(z - K(z)\big)^{-1} B_0\Big)$$

derived from the definition of the function b_0, which leads to the equation for the Cauchy transform

$$G_{\mu_0}(z) = \frac{1}{z - \sum_j K_{j,j}(z)},$$

which completes the proof. □

Remark 7.2. The Cauchy transform G_{μ_0} can be written as a continued fraction of the form

$$G_{\mu_0}(z) = \cfrac{1}{z - \sum_j \cfrac{b_{j,j}}{z - \sum_i \cfrac{b_{i,j}}{z - \sum_k \cfrac{b_{k,i}}{z - \cdots}}}}$$

which gives a matricial extension of the continued fraction of the Wigner semicircle distribution.

8. Decompositions in terms of subordinations

In the one-dimensional case, the limit distributions μ_0 and μ are related by Proposition 5.1. In particular, if each variance matrix $V(n)$ has identical entries equal to one, both central limit theorems (standard and tracial) give the Wigner semicircle distribution with variance 1 (of course, the standard case also follows from free probability, whereas the tracial case is related to random matrices).

In this section we will analyze in more detail the limit distributions for the two-dimensional case, namely when each variance matrix $V(n)$ consists of four blocks. They will be expressed in terms of two-dimensional arrays of distributions. Finding simple analytic formulas for the corresponding four-parameter Cauchy transforms and densities does not seem possible in the general case. However, we shall derive decomposition formulas for those measures in terms of s-free additive convolutions [14], which gives some insight into their structure (see also [21] for recent results on the multivariate case). The s-free additive convolution refers to the subordination property for free additive convolution, discovered by Voiculescu [26] and generalized by Biane [4]. As shown in [14] and [15], there is a notion of independence, called *freeness with subordination*, or simply *s-freeness*, associated with the s-free additive convolution and its multiplicative counterpart.

Recall that the s-free additive convolution of $\mu, \nu \in \mathcal{M}_{\mathbb{R}}$ is the unique probability measure $\mu \boxdot \nu \in \mathcal{M}_{\mathbb{R}}$ defined by the subordination equation

$$\nu \boxplus \mu = \nu \rhd (\mu \boxdot \nu),$$

where $\rhd$ denotes the monotone additive convolution [20]. Equivalently, the above subordination property can be written in terms of Cauchy transforms or their reciprocals.

Using s-free additive convolutions and the boolean convolution, we obtain a decomposition of the free additive convolution of the form

$$\mu \boxplus \nu = (\mu \boxdot \nu) \uplus (\nu \boxdot \mu),$$

which allows us to interpret both s-free additive convolutions appearing here as (in general, non-symmetric) halves of $\mu \boxplus \nu$. We find it interesting that the limit distributions in the two-dimensional case will turn out to be deformations of the free additive convolution of semicircle laws implemented by this decomposition. In other words, the subordination property and the associated convolutions give a natural framework for studying matricial generalizations of the semicircle law.

For simplicity, it will be convenient to use the indices-free notation for the two-dimensional matrix of K-transforms:

$$\begin{pmatrix} a(z) & b(z) \\ c(z) & d(z) \end{pmatrix} = \begin{pmatrix} K_{1,1}(z) & K_{1,2}(z) \\ K_{2,1}(z) & K_{2,2}(z) \end{pmatrix}$$

and

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \sqrt{b_{1,1}} & \sqrt{b_{1,2}} \\ \sqrt{b_{2,1}} & \sqrt{b_{2,2}} \end{pmatrix} = \sqrt{B}$$

where the square root is interpreted entry-wise.

Moreover, we will distinguish two laws by special notations: we denote by σ_α the Wigner semicircle distribution with the Cauchy transform

$$G_{\sigma_\alpha}(z) = \frac{z - \sqrt{z^2 - 4\alpha^2}}{2\alpha^2},$$

where the branch of $\sqrt{z^2-4\alpha^2}$ is chosen so that $\sqrt{z^2-4\alpha^2} > 0$ if $z \in \mathbb{R}$ and $z \in (2\alpha, \infty)$, and by κ_γ we denote the Bernoulli law with the Cauchy transform

$$G_{\kappa_\gamma}(z) = \frac{1}{z - \gamma^2/z},$$

i.e. $\sigma_\gamma = 1/2(\delta_{-\gamma} + \delta_\gamma)$.

Finally, we will also use the *boolean compressions* of $\mu \in \mathcal{M}_{\mathbb{R}}$, where $t \geqslant 0$, defined by multiplying its K-transform by t, namely we define $T_t\mu$ to be the (unique) probability measure on $\mathbb{R}$, for which

$$K_{T_t\mu} = tK_\mu.$$

These transformations were introduced and studied in [7] and called 't-transformations' of μ. We allow $t = 0$, in which case $T_0\mu = \delta_0$. In particular, we shall use two-parameter boolean compressions of semicircle distributions, $\sigma_{\alpha,\beta} = T_t\sigma_\alpha$ for $t = (\beta/\alpha)^2$, with

$$G_{\sigma_{\alpha,\beta}}(z) = \frac{(2\alpha^2 - \beta^2)z - \beta^2\sqrt{z^2 - 4\alpha^2}}{(2\alpha^2 - 2\beta^2)z^2 + 2\beta^4}$$

being their Cauchy transforms, where the branch of the square root is the same as in the case of G_{σ_α}.

Theorem 8.1. *If $\alpha, \beta, \gamma, \delta \neq 0$, then the diagonal measures $\mu_{j,j}$ defined by $K_{j,j}$, where $1 \leqslant j \leqslant 2$, have the form*

$$\mu_{1,1} = T_{1/t}(\sigma_{\alpha,\beta} \boxplus \sigma_{\delta,\gamma}),$$
$$\mu_{2,2} = T_{1/s}(\sigma_{\delta,\gamma} \boxplus \sigma_{\alpha,\beta}),$$

with the off-diagonal measures given by $\mu_{1,2} = T_t\mu_{1,1}$ and $\mu_{2,1} = T_s\mu_{2,2}$, where $t = (\beta/\alpha)^2$ and $s = (\gamma/\delta)^2$,

Proof. It is easy to see that the following algebraic relations hold:

$$a(z)=\frac{\alpha^2}{z-a(z)-c(z)}\quad\text{and}\quad d(z)=\frac{\delta^2}{z-b(z)-d(z)},$$
$$b(z)=\frac{\beta^2}{z-a(z)-c(z)}\quad\text{and}\quad c(z)=\frac{\gamma^2}{z-b(z)-d(z)}.$$

Thus, $b(z)=ta(z)$ and $c(z)=sd(z)$, which gives $\mu_{1,2}=T_t\mu_{1,1}$ and $\mu_{2,1}=T_s\mu_{2,2}$. In turn, from the equation for $a(z)$, we get

$$a(z)=\frac{z-c(z)-\sqrt{(z-c(z))^2-4\alpha^2}}{2}=K_{\sigma_\alpha}\bigl(z-c(z)\bigr)$$

and thus $\mu_{1,1}=\sigma_\alpha\vdash\mu_{2,1}$. In a similar manner we obtain $\mu_{2,2}=\sigma_\delta\vdash\mu_{1,2}$. Therefore, we arrive at the equations

$$\mu_{1,1}=\sigma_\alpha\vdash(T_s\sigma_\delta\vdash T_t\mu_{1,1}),$$
$$\mu_{2,2}=\sigma_\delta\vdash(T_t\sigma_\alpha\vdash T_s\mu_{2,2})$$

since $T_t(\mu\vdash\nu)=(T_t\mu)\vdash\nu$. In order to express the $\mu_{j,j}$ in terms of s-free additive convolutions, we need to use the properties of the orthogonal convolution. We have shown in [14] that the moment of order k of $\mu\vdash\nu$ depends on the moments of orders $\leqslant k$ of μ and the moments of orders $\leqslant k-2$ of ν. This leads to the conclusion that for any compactly supported $\mu,\nu\in\mathcal{M}_{\mathbb{R}}$ and the associated sequence of measures $(\mu\vdash_m\nu)$, defined recursively by

$$\mu\vdash_m\nu=\mu\vdash(\nu\vdash_{m-1}\mu)\quad\text{with}\quad\mu\vdash_1\nu=\mu\vdash\nu,$$

we have weak convergence $\mathrm{w}-\lim_{m\to\infty}\mu\vdash_m\nu=\mu\boxplus\nu$. If we take $\mu=T_t\sigma_\alpha$ and $\nu=T_s\sigma_\delta$ (these measures are compactly supported), we get the desired formulas. □

Corollary 8.1. *The measures μ_0, μ_1, μ_2 can be decomposed as*

$$\mu_0=T_{1/t}(\sigma_{\alpha,\beta}\boxplus\sigma_{\delta,\gamma})\uplus T_{1/s}(\sigma_{\delta,\gamma}\boxplus\sigma_{\alpha,\beta}),$$
$$\mu_1=T_{1/t}(\sigma_{\alpha,\beta}\boxplus\sigma_{\delta,\gamma})\uplus(\sigma_{\delta,\gamma}\boxplus\sigma_{\alpha,\beta}),$$
$$\mu_2=T_{1/s}(\sigma_{\delta,\gamma}\boxplus\sigma_{\alpha,\beta})\uplus(\sigma_{\alpha,\beta}\boxplus\sigma_{\delta,\gamma}),$$

where the assumptions and notations are the same as in Theorem 8.1.

Proof. These decompositions follow immediately from Theorems 7.1, 7.2 and 8.1. □

Remark 8.1. The formulas for the diagonal measures $\mu_{j,j}$ in the proof of Theorem 8.1 remind those for two-periodic continued fractions if we take $t=s=1$. The latter are of the same form, except that the semicircle distributions are replaced by much simpler Bernoulli laws. Nevertheless, if $t=s=1$, the formulas for the μ_j take a simple form

$$\mu_j=\sigma_\alpha\boxplus\sigma_\delta,$$

where $j = 0, 1, 2$. By Theorem 7.1 and Corollary 8.1, the same formula holds for μ. Therefore, all measures μ_0, μ_1, μ_2, μ can be viewed as deformations of the free additive convolution of two semicircle distributions, implemented by means of boolean compressions.

Let us consider now the situation in which some of the numbers α, β, γ, δ vanish. Suitable formulas can be derived algebraically, as we did in the proof of Theorem 8.1. However, one can also obtain the same results by taking weak limits in the formulas for the measures $\mu_{i,j}$, using the fact that all measures involved have compact supports. For that purpose, let us state a few useful facts about weak limits which will be of interest to us. Then, we consider eight cases, to which the remaining cases are similar (for instance, $\alpha = \beta = 0$ is similar to $\gamma = \delta = 0$).

Proposition 8.1. *Let $t = \beta^2/\alpha^2$, where $\alpha, \beta > 0$, and let $\mu \in \mathcal{M}_{\mathbb{R}}$ be compactly supported.*

1. *If $\alpha \to 0^+$, then*
   (a) $\mathrm{w} - \lim \sigma_\alpha = \delta_0$,
   (b) $\mathrm{w} - \lim(\sigma_{\alpha,\beta}) = \kappa_\beta$,
   (c) $\mathrm{w} - \lim(T_{1/t}(\sigma_{\alpha,\beta} \boxplus \mu)) = \delta_0$.
2. *If $\beta \to 0^+$, then*
   (a) $\mathrm{w} - \lim \sigma_{\alpha,\beta} = \delta_0$,
   (b) $\mathrm{w} - \lim(T_t\mu) = \delta_0$,
   (c) $\mathrm{w} - \lim(T_{1/t}(\sigma_{\alpha,\beta} \boxplus \mu)) = \sigma_\alpha \vdash \mu$.

Proof. If $\alpha \to 0^+$, then $K_{\sigma_\alpha}(z) \to 0$, which proves 1(a). Here, as well as in the remaining cases, convergence is uniform on compact subsets of $\mathbb{C}^+$. Moreover, $K_{\sigma_{\alpha,\beta}} = \beta^2/(z - \alpha^2 K_{\sigma_\alpha}) \to \beta^2/z = K_{\kappa_\beta}$, which gives 1(b). In turn, if $\beta \to 0^+$, then $K_{\sigma_{\alpha,\beta}}(z) = \beta^2/(z - K_{\sigma_\alpha}(z)) \to 0$, thus also $\mathrm{w} - \lim \sigma_{\alpha,\beta} = \delta_0$, which proves 2(a). If, in addition, $t \to 0^+$, then $T_t\mu \to \delta_0$ for any $\mu \in \mathcal{M}_{\mathbb{R}}$, which proves 2(b). Finally,

$$T_{1/t}(\sigma_{\alpha,\beta} \boxplus \mu) = T_{1/t}\big(T_t\sigma_\alpha \vdash (\mu \boxplus \sigma_{\alpha,\beta})\big) = \sigma_\alpha \vdash (\mu \boxplus \sigma_{\alpha,\beta})$$

and thus the right-hand side tends weakly to δ_0 as $\alpha \to 0^+$, which gives 1(c), and tends weakly to $\sigma_\alpha \vdash (\mu \boxplus \delta_0) = \sigma_\alpha \vdash \mu$ as $\beta \to 0^+$, which gives 2(c). This holds for any $\mu \in \mathcal{M}_{\mathbb{R}}$, and we also use the right unit property of δ with respect to the s-free additive convolution, namely $\mu \boxplus \delta_0 = \mu$. □

Corollary 8.2. *If some of the entries of the matrix A vanish, we can distinguish eight different cases, for which the distributions $\mu_{i,j}$ are given by Table* 1.

Proof. If $a_{i,j} = 0$, then $\mu_{i,j} = \delta_0$, which easily follows from the algebraic equations for the corresponding K-transforms. An alternative proof can be given by taking weak limits of the formulas of Theorem 8.1, as we proceed with the remaining measures. Thus, if $\delta \to 0^+$, then

$$\mu_{1,1} = \mathrm{w} - \lim T_{1/t}(\sigma_{\alpha,\beta} \boxplus \sigma_{\delta,\gamma}) = T_{1/t}(\sigma_{\alpha,\beta} \boxplus \kappa_\gamma),$$

$$\mu_{1,2} = \mathrm{w} - \lim(\sigma_{\alpha,\beta} \boxplus \sigma_{\delta,\gamma}) = \sigma_{\alpha,\beta} \boxplus \kappa_\gamma,$$

$$\mu_{2,1} = \mathrm{w} - \lim(\sigma_{\delta,\gamma} \boxplus \sigma_{\alpha,\beta}) = \kappa_\gamma \boxplus \sigma_{\alpha,\beta},$$

Table 1
Distributions $\mu_{i,j}$ in the case A has zero entries.

$a_{1,1}$	$a_{1,2}$	$a_{2,1}$	$a_{2,2}$	$\mu_{1,1}$	$\mu_{1,2}$	$\mu_{2,1}$	$\mu_{2,2}$
α	β	γ	0	$T_{1/t}(\sigma_{\alpha,\beta} \boxplus \kappa_\gamma)$	$\sigma_{\alpha,\beta} \boxplus \kappa_\gamma$	$\kappa_\gamma \boxplus \sigma_{\alpha,\beta}$	δ_0
0	β	γ	0	δ_0	$\kappa_\beta \boxplus \kappa_\gamma$	$\kappa_\gamma \boxplus \kappa_\beta$	δ_0
α	0	γ	δ	$\sigma_\alpha \vdash \sigma_{\delta,\gamma}$	δ_0	$\sigma_{\delta,\gamma}$	σ_δ
0	β	0	δ	δ_0	κ_β	δ_0	$\sigma_\delta \vdash \kappa_\beta$
0	0	γ	δ	δ_0	δ_0	$\sigma_{\delta,\gamma}$	σ_δ
α	0	0	δ	σ_α	δ_0	δ_0	σ_δ
α	0	0	0	σ_α	δ_0	δ_0	δ_0
0	β	0	0	δ_0	κ_β	δ_0	δ_0

by 1(b) of Proposition 8.1, which proves the first case in Table 1. The remaining cases are proved in a similar manner. □

In the case of arbitrary matrix A, finding the four-parameter densities of μ_0 and μ is unwieldy. Below we shall just consider two special cases, in which we can find nice formulas for these measures for matrices A of arbitrary dimension. These two cases are of special interest since they are associated with (asymptotic) freeness and (asymptotic) monotone independence.

Proposition 8.2. *If A is a square r-dimensional matrix with identical positive entries α_j in the j-th row, then*

$$\mu_j = \sigma_{\alpha_1} \boxplus \sigma_{\alpha_2} \boxplus \cdots \boxplus \sigma_{\alpha_r}$$

for each $j \in [r]$, and the measures μ_j coincide with μ and μ_0.

Proof. Since the columns of A are identical, the functions $K_{i,j}$ are the same for all j's. Denote them $L_i = K_{i,j}$, where $i, j \in [r]$. Moreover,

$$L_i(z) = \frac{b_i}{z - \sum_{j=1}^r L_j(z)} \quad \text{and} \quad \sum_{i=1}^r L_i(z) = \frac{\sum_{i=1}^r b_i}{z - \sum_{j=1}^r L_j(z)}$$

for any $i \in [r]$. Therefore, $\sum_{i=1}^r L_i(z)$ is the K-transform of the measure

$$\sigma_{\alpha_1} \boxplus \sigma_{\alpha_2} \boxplus \cdots \boxplus \sigma_{\alpha_r}$$

and since $K_{\mu_j}(z) = \sum_{i=1}^r K_{i,j}(z) = \sum_{i=1}^r L_i(z)$, the proof for μ_j is completed. It is then easy to see that we get the same result for μ and μ_0. □

Proposition 8.3. *If A is a lower-triangular r-dimensional matrix with identical positive entries α_j in the j-th row below and on the main diagonal, then*

$$\mu_j = \sigma_{\alpha_j} \triangleright \sigma_{\alpha_{j+1}} \triangleright \cdots \triangleright \sigma_{\alpha_r}$$

for each $j \in [r]$. Moreover, $\mu_0 = \mu_1$ and μ is the convex linear combination of the measures μ_j as in Theorem 7.1.

Proof. As in the proof of Proposition 8.2, note that the measures $\mu_{i,j}$ do not depend on j and thus we can set $L_i = K_{i,j}$ for any $i \geqslant j$. If $i = r$, we have

$$L_r(z) = \frac{b_r}{z - L_r(z)},$$

using the continued fraction for the $K_{r,j}$ of Lemma 7.1. Therefore, $\mu_{r,j} = \sigma_{\alpha_r}$ for any $j \leqslant r$. Next, we have

$$L_k(z) = \frac{b_k}{z - \sum_{i=k}^{r} L_i(z)},$$

which leads to

$$L_k(z) = K_{\sigma_{\alpha_k}}\left(z - \sum_{i=k+1}^{r} L_i(z)\right),$$

giving the orthogonal decomposition of $\mu_{k,j}$,

$$\mu_{k,j} = \sigma_{\alpha_k} \vdash (\mu_{k+1,j} \uplus \cdots \uplus \mu_{r,j})$$

for any $j \leqslant k < r$. Now, we claim that

$$\mu_{i,j} \uplus \cdots \uplus \mu_{r,j} = \sigma_{\alpha_i} \rhd \sigma_{\alpha_{i+1}} \rhd \cdots \rhd \sigma_{\alpha_r}$$

for any $1 \leqslant j \leqslant i \leqslant r$. Clearly, it holds for $i = r$ and any $j \leqslant r$ since we have already shown that $\mu_{r,j} = \sigma_{\alpha_r}$ for any $j \leqslant r$. Suppose now that this formula holds for $i > k$ and any $j \leqslant i$. We will show that it holds for $i = k$ and any $j \leqslant k$. Using the orthogonal decomposition of $\mu_{k,j}$ given above and the inductive assumption, we obtain

$$\mu_{k,j} = \sigma_{\alpha_k} \vdash (\sigma_{\alpha_{k+1}} \rhd \cdots \rhd \sigma_{\alpha_r}).$$

However, for any $\mu, \nu \in \mathcal{M}_{\mathbb{R}}$, we have a simple relation

$$(\mu \vdash \nu) \uplus \nu = \mu \rhd \nu,$$

which gives

$$\mu_{k,j} \uplus \cdots \uplus \mu_{r,j} = \sigma_{\alpha_k} \rhd \sigma_{\alpha_{k+1}} \rhd \cdots \rhd \sigma_{\alpha_r}$$

and the desired expression for μ_j. In a similar manner we obtain μ_0 and μ. □

Example 8.1. If $\alpha = \delta \neq 0$ and $\beta = \gamma = 0$, then we can use Table 1 to obtain $\mu_0 = \sigma_\alpha \uplus \sigma_\alpha$, which is the arcsine law with Cauchy transform $G_{\mu_0}(z) = 1/\sqrt{z^2 - \alpha^2}$ whereas μ is the Wigner semicircle distribution σ_α. In turn, if $\alpha = \delta = 0$ and $\beta = \gamma \neq 0$, then $\mu_0 = \delta_0$, whereas $\mu = \sigma_\beta$.

9. Weighted binary trees and Catalan paths

In this section we show how to express the limit distributions in terms of walks on weighted binary trees, or equivalently, in terms of weighted Catalan paths. The binary tree serves here as an example of the strongly matricially free Fock space.

The usual framework which gives a description of distributions in terms of walks on graphs is the following. Let $W(n)$ denote the set of root-to-root walks of length n on a rooted graph $(\mathcal{G}, e)$ and let μ be the spectral distribution of $(\mathcal{G}, e)$, i.e. the distribution given by the moments of the adjacency matrix $A(\mathcal{G})$ in the state φ associated with the vector δ_e on the space of square integrable functions on the set $V(\mathcal{G})$ of the vertices of $\mathcal{G}$. Then the n-th moment of $A(\mathcal{G})$ in the state φ is equal to the cardinality of the set $W(n)$. In particular, it is well known that the moments of the Wigner semicircle distribution of variance 1 can be expressed in terms of walks on the half-line $(\mathbb{T}_1, e)$ with the first vertex denoted by e and chosen as the root.

For many distributions we have to use a more general framework, in which the moments of these distributions are expressed in terms of root-to-root (random or, more generally, weighted) walks on some rooted graph, except that to each walk w on this graph we have to assign a real-valued weight $\xi(w)$. Then we can write

$$M_\mu(n) = \sum_{w \in W(n)} \xi(w)$$

for any $n \geqslant 1$, where μ is the considered distribution. In particular, we obtain the moments of σ_α for any $\alpha > 0$ by putting $\xi(w) = \alpha^n$, where $n = |w|$ is the length of w.

In the cases which are of interest to us, the weight function ξ is first defined on the set of edges $E(\mathcal{G})$ of $\mathcal{G}$ and then is extended to $W = \bigcup_{n \geqslant 1} W(n)$ by multiplicativity. Namely, if we are given a mapping $\xi : E(\mathcal{G}) \to \mathbb{R}$, we set

$$\xi(w) = \xi(E_1)\xi(E_2)\dots\xi(E_n),$$

where $w = (E_1, E_2, \dots, E_n)$ and $E_1, E_2, \dots, E_n$ are the edges of w (we choose to describe walks on graphs as sequences of edges). Such extension, by abuse of notation denoted also by ξ, will be called *multiplicative*.

For instance, it is easy to see that the moments of $\mu = \sigma_\alpha \boxplus \sigma_\delta$ can be expressed in this form. It is enough to take the free product of two half-lines, which is the binary tree $(\mathbb{T}_2, e)$ with root e. Let us color this graph in the natural way, namely each edge which belongs to a copy of the first half-line is colored by 1 and each edge which belongs to a copy of the second half-line is colored by 2. Then the above formula holds for the moments of μ if we take $\xi(E) = \alpha$ whenever E is colored by 1 and $\xi(E) = \delta$ whenever E is colored by 2.

We will demonstrate below that we can express our distributions μ_0, μ_1, μ_2 in a similar form (in particular, we can use the binary tree), except that the weight function ξ will depend on all four parameters which appear in the matrix A. Before formulating the theorem, let us introduce special weight functions related to matricial freeness.

Definition 9.1. Let $\mathbb{T}_r$ be an r-ary rooted tree with root e and let $A \in M_r(\mathbb{R})$. The weight function $\xi : E(\mathbb{T}_r) \to \mathbb{R}$ which assigns the entries of A to the edges of $\mathbb{T}_r$ is called *matricial* if, for any pair of edges $E_1, E_2 \in E(\mathbb{T}_r)$, incident on the same vertex and such that E_1 is the 'father' of E_2,

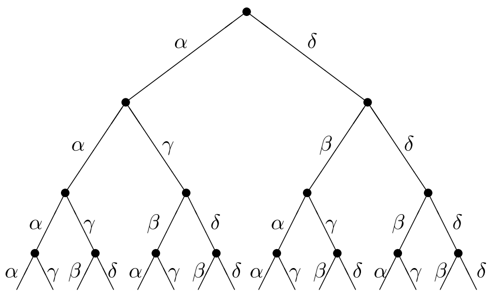


Fig. 3. Binary tree with a matricial weight function.

the following implication holds:

$$\xi(E_1) = a_{i,j} \quad \text{for some } i, j \quad \Rightarrow \quad \xi(E_2) = a_{k,i} \quad \text{for some } k.$$

The unique multiplicative extension of this weight function to the set of all walks on $\mathbb{T}_r$ will also be called *matricial*.

We specialize to $p = 2$ and the binary tree. Note that any matricial weight function ξ on the binary tree is uniquely determined (up to equivalence) by the set of those entries of the matrix A which are assigned to the set $\{E_1, E_2\}$ of two edges incident on the root of the tree, called the *initial weights*. In order to establish a connection with our limit distributions, we will assume, as in Section 8, that A is the 'square root' of $B = DU$ of Section 6. Then, in particular, the binary tree with the matricial weight function associated with A and the initial weights $\{\alpha, \delta\}$, shown in Fig. 3, describes μ_0 as we show below.

Finally, recall after [1,14] that if $(\mathcal{G}_1, e_1)$ and $(\mathcal{G}_2, e_2)$ are two locally finite simple graphs and μ_1 and μ_2 are the associated spectral distributions, then the s-free product of $\mathcal{G}_1$ and $\mathcal{G}_2$ (in that order) can be interpreted as this half of the free product $\mathcal{G}_1 * \mathcal{G}_2$ which 'begins' (starting from the root) with a copy $\mathcal{G}_1$. Moreover, the associated spectral distribution is given by $\mu_1 \boxplus \mu_2$. Let us add that a similar result holds in the multiplicative case: both the s-free multiplicative convolution and the associated s-free loop product of graphs were introduced in [15].

Below we shall use the s-free product of half-lines, which are (left and right) halves of the binary tree.

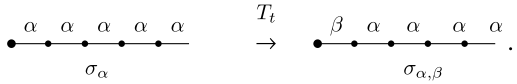
Theorem 9.1. *Let ξ_0, ξ_1, ξ_2 be the multiplicative matricial weight functions on the set of walks on $\mathbb{T}_2$ associated with matrix A and the initial weights $\{\alpha, \delta\}$, $\{\beta, \delta\}$ and $\{\alpha, \gamma\}$, respectively. Then*

$$M_{\mu_j}(n) = \sum_{w \in W(n)} \xi_j(w) \tag{9.1}$$

for $j = 0, 1, 2$ and any $n \in \mathbb{N}$, where $W(n)$ denotes the set of root-to-root walks on $\mathbb{T}_2$ of length n.

Proof. We need to translate the result of Corollary 8.1 to the language of graphs. It is well known that the moments of σ_α can be interpreted in terms of walks on the half-line with the weight α

assigned to each edge. The boolean compression T_t of σ_α changes only the weight assigned to the edge incident on the root, namely it multiplies it by $\sqrt{t} = \beta/\alpha$, which can be illustrated as



Now, the s-free additive convolutions of compressed semicircle distributions which appear in the decompositions of Corollary 8.1, namely

$$\sigma_{\alpha,\beta} \boxplus \sigma_{\delta,\gamma} \quad \text{and} \quad \sigma_{\delta,\gamma} \boxplus \sigma_{\alpha,\beta}$$

are the spectral distributions of the s-free products of these half-lines (taken in two different orders). These turn out to be the spectral distributions of the two halves of the binary tree $\mathbb{T}_2$ with the weight function defined by the initial set $\{\beta, \gamma\}$. In order to obtain μ_0, we still need to apply $T_{1/t}$ and $T_{1/s}$, respectively, to the left and right halves of the tree, which amounts to changing the initial weights from β and γ, respectively, to α and δ. As a result, we obtain the weight function associated with A and the initial set $\{\alpha, \delta\}$. This proves the statement concerning μ_0 (the corresponding weight function on the binary tree is shown in Fig. 3). The cases of μ_1 and μ_2 are very similar (at the end, a boolean compression is applied to only one half of the binary tree). $\square$

Another geometric interpretation of Corollary 8.1 can be given in terms of Catalan paths. In order to define a Catalan path, we begin with a function $f : [2n] \to [n]$, such that $f(0) = f(2n)$ and $|f(i) - f(i-1)| = 1$ for any $1 \leqslant i \leqslant 2n$, and then we define a *Catalan path* as its unique extension $f : [0, 2n] \to [0, n]$ (by abuse of notation, denoted by the same symbol) obtained by connecting each $(i-1, f(i-1))$ with $(i, f(i))$ with a segment, where $1 \leqslant i \leqslant 2n$. Clearly, each Catalan path consists of segments of two types: 'rises' $R_1, R_2, \ldots, R_n$, and 'falls' $F_1, F_2, \ldots, F_n$. Moreover, to each 'rise' R_j there corresponds the closest 'fall' $F_{\tau(j)}$ lying to the right of R_j and on the same (vertical) level.

There is a natural mapping from the set $W(2n)$ of walks of length $2n$ on the binary tree and the set $C(n)$ of Catalan paths of length $2n$. In order to rephrase Theorem 8.1 in terms of Catalan paths, we need to take the sets of *weighted Catalan paths*, by which we understand pairs (f, ξ), where f is a Catalan path and ξ is a real-valued weight function defined on the set of segments of f. The multiplicative formula

$$\xi(f) = \xi(R_1) \ldots \xi(R_n)\xi(F_1) \ldots \xi(F_n)$$

assigns the corresponding weight to f. Weighted Catalan paths of special type defined below allow us to rephrase Theorem 8.1.

Definition 9.2. A weighted Catalan path (f, ξ) of length $2n$ is called *matricially weighted* if ξ assigns entries of $A \in M_p(\mathbb{R})$ to the segments of f in such a way that the following implications hold:

$$\xi(R_1) = a_{i,j} \quad \text{for some } i, j \quad \Rightarrow \quad \xi(R_2) = a_{k,i} \quad \text{for some } k,$$
$$\xi(F_1) = a_{i,j} \quad \text{for some } i, j \quad \Rightarrow \quad \xi(F_2) = a_{j,k} \quad \text{for some } k,$$

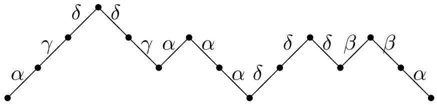


Fig. 4. A weighted Catalan path associated with A.

for any two consecutive 'rises' R_1, R_2 and two consecutive 'falls' F_1, F_2, and the same weights are assigned to R_i and $F_{\tau(i)}$ for each $i \in [n]$.

We will consider below the set of matricially weighted Catalan paths of length $2n$ associated with the matrix A. In order to rephrase Theorem 9.1, using weighted Catalan paths, we need to restrict the set of weights which can be assigned to the first segment of each path (by analogy with trees, we call them *initial weights*). An example of a weighted Catalan path contributing to μ_0 is given in Fig. 4.

Corollary 9.1. *Let $C_0(n)$, $C_1(n)$ and $C_2(n)$ be the sets of matricially weighted Catalan paths associated with A, with the initial weights $\{\alpha, \delta\}$, $\{\beta, \delta\}$ and $\{\gamma, \alpha\}$, respectively. Then*

$$M_{\mu_j}(2n) = \sum_{(f,\xi)\in C_j(n)} \xi(f)$$

for $j = 0, 1, 2$ and any $n \in \mathbb{N}$.

Proof. This is an easy consequence of Theorem 9.1 since there is a natural bijection between each $C_j(n)$ and the pair $(W(2n), \xi_j)$ for any n. □

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