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Colouring Eulerian Triangulations

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We show that for every orientable surface Σ there is a number c so that every Eulerian triangulation of Σ with representativeness $\geqslant c$ is 4-colourable.

1. INTRODUCTION

Collins and Hutchinson [3] conjectured that every Eulerian triangulation of an orientable surface is 4-colourable if its representativeness is sufficiently high, and obtained some partial results for the torus. (The *representativeness* of a graph drawn in a surface is the minimum number of times a non-null-homotopic closed curve must hit the drawing.) It is easy to see that Eulerian triangulations of the torus need not be 3-colourable, because for instance their duals need not be bipartite, and so the number 4 is best possible in Collins and Hutchinson's conjecture. It follows from [10] that all these graphs can be 5-coloured.

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Our objective is to prove that conjecture; we shall show that the result holds for every orientable surface, but not for the projective plane. More precisely:

(1.1) *For every orientable surface Σ of genus $\geqslant 1$ there is a number $c(\Sigma)$ so that every Eulerian triangulation of Σ with representativeness $\geqslant c(\Sigma)$ is* 4-*colourable.*

(1.2) *For the projective plane Σ there is no $c(\Sigma)$ as in* (1.1).

We prove (1.1) in Section 4, after some preliminary lemmas in Sections 2 and 3; and prove (1.2) in Section 5.

Since for $i \geqslant 1$, K_{12i+3} can be embedded as an Eulerian triangulation in the orientable surface of genus $i(12i-1)$, the condition about representativeness cannot be omitted from (1.1). (On the other hand, we do not know whether $c(\Sigma)$ must depend on Σ—it seems possible that (1.1) is true with $c(\Sigma)=100$, for all Σ.) Also, examples of Ballantine [2] and of Fisk [4] show that (1.1) does not hold when a triangulation contains two odd-degree vertices.

Incidentally, an application of our main lemma (2.5)(i) gives an alternative proof of the main result of [6], that every quadrangulation of an orient able surface can be 3-coloured provided its representativeness is sufficiently high.

2. A HOMOTOPY LEMMA

Let us make some terms more precise. A *surface* means a compact, connected 2-manifold without boundary. We need to define homotopy for several different kinds of objects in a surface. First, a *closed curve* in a surface Σ means a continuous map $\phi\colon [0,1] \to \Sigma$ such that $\phi(0)=\phi(1)$, and its *basepoint* is $\phi(0)$. We speak of (fixed basepoint) homotopy of closed curves with a given basepoint in the usual way. The equivalence class of curves homotopic to a given curve ϕ is denoted by $\langle\phi\rangle$ and called the *homotopy type* of ϕ. The natural product on homotopy types (defined by concatenation) yields a group, the *fundamental group of Σ* (with the given basepoint, v say), which we denote by $\pi_1(\Sigma, v)$.

Second, we need *free homotopy* of closed curves; closed curves ϕ, ψ: $[0,1] \to \Sigma$ are *freely homotopic* if there is a continuous map $w\colon [0,1]\times[0,1] \to \Sigma$ such that

$$w(x,0)=\phi(x) \qquad (0 \leqslant x \leqslant 1)$$
$$w(x,1)=\psi(x) \qquad (0 \leqslant x \leqslant 1)$$
$$w(0,y)=w(1,y) \qquad (0 \leqslant y \leqslant 1).$$

In particular, ϕ and ψ need not have the same basepoint to be freely homotopic.

Third, an *O-arc* in Σ means a subset of Σ homeomorphic to a circle. A closed curve $\phi\colon [0, 1] \to \Sigma$ is said to *trace* an O-arc F if

(a) $\phi(x) \in F$ $(0 \leqslant x \leqslant 1)$

(b) for each $y \in F$ there is a unique $x \in [0, 1)$ with $\phi(x) = y$.

We say two O-arcs are *homotopic* if there are closed curves tracing them that are freely homotopic; and similarly an O-arc F is *homotopic* to a closed curve ψ if there is a closed curve ϕ tracing F freely homotopic to ψ.

Fourth and fifth, given a drawing G in Σ (defined below), if W is a closed walk in G then we may speak of a closed curve "tracing" W with the natural meaning, and this enables us to speak of homotopy of walks (free homotopy, or with fixed basepoint).

A *drawing* G in a surface Σ is a pair $(U(G), V(G))$, where $U(G) \subseteq \Sigma$ is closed, $V(G) \subseteq U(G)$, $|V(G)|$ is finite, $U(G) - V(G)$ has only finitely many connected components, and for every connected component e of $U(G) - V(G)$, its closure $\bar{e}$ contains precisely two elements $u, v \in V(G)$, and $\bar{e}$ is homeomorphic to $[0, 1]$. We regard drawings as graphs in the usual way. Thus we permit multiple edges, but not loops.

Let G be a drawing in a surface Σ, not the sphere. We say G has *representativeness* $\geqslant k$ if $|F \cap U(G)| \geqslant k$ for every non-null-homotopic O-arc F.

Let G be a drawing in Σ and $k \geqslant 0$ an integer. A closed curve ϕ is said to be *k-wide* in G if ϕ is not null-homotopic, and there are circuits $C_1, ..., C_k$ of G, pairwise vertex-disjoint and each homotopic to ϕ. (*Circuits* by definition have no "repeated" vertices or edges.) A homotopy type is *k-wide* if its members are k-wide. An O-arc is *k-wide* if some closed curve tracing it is *k-wide.*

The main result of this section is the following.

(2.1) *For every orientable surface Σ of genus $\geqslant 1$ and every integer $k \geqslant 0$ there exists c such that for every drawing G in Σ with representativeness $\geqslant c$, every $v \in \Sigma$, and every homomorphism $\lambda\colon \pi_1(\Sigma, v) \to S_3$ (the group of permutations of three objects) there exists $\delta \in \pi_1(\Sigma, v)$ such that $\lambda(\delta)$ is the identity of S_3 and δ is k-wide in G.*

First we need the following lemma.

(2.2) *Let S_3 be the group of permutations of a 3-element set, with identity* 1 *(say).*

(i) *If* $x, y \in S_3$ *belong to an abelian subgroup of* S_3 *then at least one of* x, y, xy, xy^{-1} *equals* 1.

(ii) *If* $x, y, z \in S_3$ *then at least one of* $x, y, z, xy, xy^{-1}, yz, yz^{-1}, zx, zx^{-1}, xyz, zyx, xyxz$ *equals* 1.

Proof. For (i) we may assume $1, x, y$ are all distinct. But they belong to an abelian subgroup of S_3, and all such subgroups have $\leqslant 3$ elements, and so $xy = 1$ as required.

For (ii), we may assume $1, x, y, z$ are all distinct. Hence each of x, y, z has order 2 or 3; say k of them have order 3. Then $0 \leqslant k \leqslant 2$ (since there are only two elements of order 3 in S_3), If $k = 0$ then $xyxz = 1$. If $k = 1$ then one of $xyz, zyx = 1$; and if $k = 2$ then one of $xy, yz, zx = 1$. Q.E.D.

We need the following theorem of [9].

(2.3) *For every surface* Σ *except the sphere, and every drawing* H *in* Σ, *there is a number* c *with the following property. For every drawing* G *in* Σ *with representativeness* $\geqslant c$ *there is a drawing* H' *in* Σ *so that*

(i) H' *can be obtained from a subdrawing of* G *by contracting edges*

(ii) *there is a homeomorphism of* Σ *to itself taking* H *to* H'.

From (2.3) we deduce

(2.4) *For every surface* Σ *except the sphere, and every choice of finitely many O-arcs* $F_1, ..., F_n \subseteq \Sigma$, *each non-null-homotopic and two-sided, and every integer* $k > 0$, *there exists* c *with the following property. For every drawing* G *in* Σ *with representativeness* $\geqslant c$, *there is a homeomorphism* θ *of* Σ *to itself such that* $\theta(F_i)$ *is k-wide in* G $(1 \leqslant i \leqslant n)$.

Proof. For $1 \leqslant i \leqslant n$, since F_i is simple and two-sided, there are k pairwise disjoint O-arcs in Σ each homotopic to F_i. Consequently there is a drawing H in Σ such that for $1 \leqslant i \leqslant n$, F_i is k-wide in H. Choose c as in (2.3) (with the given Σ and H). Now let G be a drawing in Σ with representativeness $\geqslant c$. By (2.3), there is a drawing H' in Σ as in (2.3)(i) and a homeomorphism θ of Σ to itself taking H to H'. It follows that for $1 \leqslant i \leqslant n$, $\theta(F_i)$ is k-wide in H' and hence in G, as required. Q.E.D.

We use (2.4) to show the following.

(2.5) *For every orientable surface* Σ *except the sphere, and every integer* $k \geqslant 1$, *there is a number* c *with the following property. For every drawing* G *in* Σ *with representativeness* $\geqslant c$ *and every* $v \in \Sigma$

(i) *there exist* $\alpha, \beta \in \pi_1(\Sigma, v)$ *such that* $\alpha, \beta, \alpha\beta, \alpha\beta^{-1}$ *are all k-wide*

(ii) *if* Σ *is not a torus, there exist* $\alpha, \beta, \gamma \in \pi_1(\Sigma, v)$ *such that*

$$\alpha, \beta, \gamma, \alpha\beta, \alpha\beta^{-1}, \beta\gamma, \beta\gamma^{-1}, \gamma\alpha, \gamma\alpha^{-1}, \alpha\beta\gamma, \gamma\beta\alpha, \alpha\beta\alpha\gamma$$

are all k-wide in G.

Proof. We assume first that Σ has genus $\geqslant 2$. Let H_1 be the graph with four vertices v_0, v_1, v_2, v_3 and six edges $e_1, f_2, e_3, f_1, e_2, f_3$ where for $1 \leqslant i \leqslant 3$, e_i and f_i both have ends v_0 and v_i. Take a drawing of H_1 in Σ so that $e_1 e_2 e_3 f_1 f_2 f_3$ occur in this cyclic order around v_0. (This is possible since Σ has genus $\geqslant 2$.) Let the closed walks v_0, e_i, v_i, f_i, v_0 have homotopy type α_i $(i = 1, 2, 3)$ (with basepoint v_0) and choose the drawing so that there is no non-trivial relation between α_1, α_2 and α_3.

In particular, none of

$$\alpha_1, \alpha_2, \alpha_3, \alpha_1\alpha_2, \alpha_2\alpha_3, \alpha_3\alpha_1, \alpha_1\alpha_2^{-1}, \alpha_2\alpha_3^{-1}, \alpha_3\alpha_1^{-1}, \alpha_1 a_2\alpha_3, \alpha_3\alpha_2\alpha_1, \alpha_1\alpha_2\alpha_1\alpha_3$$

is the identity. But for each of these twelve homotopy types, δ say, there is an O-arc F_δ so that F_δ is homotopic to a member of δ. Each F_δ is two-sided, since Σ is orientable, and each is non-null-homotopic by choice of $\alpha_1, \alpha_2, \alpha_3$. By (2.4) (with $n = 12$) there is an integer c as in (2.4). We claim c satisfies (2.5)(ii). For let G be a drawing in Σ with representativeness $\geqslant c$. By (2.4) there is a homeomorphism θ of Σ to itself, such that $\theta(\delta)$ is k-wide in G for each δ.

Now if (2.5) is true (for given G, Σ) for some choice of v, then it is true for all v. To see this, let v' be some other choice of v, let ϕ be a curve from v to v', and for each $\alpha \in \pi_1(\mathrm{S}, v)$ define $f(\alpha) \in \pi_1(\Sigma, v')$ by choosing $\psi \in \alpha$, letting ψ' be the concatenation of ϕ^{-1}, ψ and ϕ, and letting $f(\alpha)$ be the member of $\pi_1(\Sigma, v')$ containing ψ'. This is well-defined, and f is an isomorphism from $\pi_1(\Sigma, v)$ to $\pi_1(\Sigma, v')$; and if α is k-wide then so is $f(\alpha)$. Thus for instance if α, β, γ, satisfy (2.5)(ii) for v, then $f(\alpha)$, $f(\beta)$, $f(\gamma)$ satisfy (2.5)(ii) for v'. This proves our claim.

Consequently it suffices to show that (2.5) holds for one particular value of v, so let us assume that $v = \theta(v_0)$. Since θ is a homeomorphism, θ induces an isomorphism from $\pi_1(\Sigma, v_0)$ to $\pi_1(\Sigma, v)$.

In particular, let $\alpha_i' = \theta(\alpha_i)$ $(i = 1, 2, 3)$; then $\alpha_1'\alpha_2' = \theta(\alpha_1\alpha_2)$, and so on for the other eight members of $\pi_1(\Sigma, v_0)$ of interest to us. But $\theta(\delta)$ is k-wide in G, for each δ, and so if we set $\alpha = \alpha_1'$, $\beta = \alpha_2'$, $\gamma = \alpha_3'$ then (2.5)(ii) holds.

The proof of (2.5)(i) is similar but easier, and we omit it. Q.E.D.

Proof of (2.1). Let Σ, k be as in (2.1), and let c be as in (2.5). We claim c satisfies (2.1). For let G, v and λ be as in (2.1). Then by (2.5), (2.5)(i) and (2.5)(ii) hold.

Suppose first that Σ is not a torus, and let α, β, γ be as in (2.5)(ii). By (2.2)(ii), $\lambda(\delta)$ is the identity of S_3 for some δ among the twelve listed in (2.5)(ii). But δ is k-wide in G, and so satisfies (2.1).

Now suppose Σ is a torus, and let α, β be as in (2.5)(1). Then $\pi_1(\Sigma, v)$ is abelian, and so the range of λ is an abelian subgroup of S_3. By (2.2)(i), $\lambda(\delta)$ is the identity for some

$$\delta \in \{\alpha, \beta, \alpha\beta, \alpha\beta^{-1}\}.$$

But δ is k-wide in G, and so satisfies (2.1). Q.E.D.

3. ANGLE PERMUTATIONS

A drawing G in Σ is said to be *closed* 2-*cell* if every region is homeomorphic to an open disc and has boundary $U(C)$ for some circuit C of G. For such a region, r say, bounded by a circuit C, we say a closed walk

$$v_0, e_1, v_1, ..., e_k, v_k = v_0$$

is a *perimeter walk* of r if $e_1, ..., e_k$ are all distinct and $E(C) = \{e_1, ..., e_k\}$. In general, a region has several perimeter walks, depending on the choice of basepoint and orientation.

An *angle* is a pair (v, r) where $v \in V(G)$ and r is a region incident with v. For a vertex v, we define

$$\nabla(v) = \{(v, r)\colon r \text{ is incident with } v\},$$

the set of all "angles at v". Thus, in a closed 2-cell drawing, $|\nabla(v)|$ equals the degree of v.

A vertex is *cubic* if it has degree 3; in fact we shall only be concerned with $\nabla(v)$ when v is cubic.

Let G be a closed 2-cell drawing, and let $e \in E(G)$ with ends v_1, v_2, both cubic. Let r_1, r_2 be the two regions incident with e, and for $i = 1, 2$ let s_i be the third region incident with v_i. Thus

$$\nabla(v_i) = \{(v_i, r_1), (v_i, r_2), (v_i, s_i)\} \qquad (i = 1, 2).$$

We define $\pi_{v_1 e v_2}$ to be the bijection from $\nabla(v_1)$ to $\nabla(v_2)$ mapping (v_1, r_1), (v_1, r_2), (v_1, s_1) to (v_2, r_1), (v_2, r_2), (v_2, s_2) respectively.

If W is a walk $v_0, e_1, v_1, e_2, \ldots, e_n, v_n$ of G, such that $v_0, \ldots, v_n$ are all cubic (a so-called *cubic* walk), we define π_W to be the product of the $\pi_{v_{i-1} e_i v_i}$ for $1 \leqslant i \leqslant n$; thus, for $x \in \nabla(v_0)$,

$$\pi_W(x) = \pi_{v_{n-1} e_n v_n}(\cdots(\pi_{v_1 e_2 v_2}(\pi_{v_0 e_1 v_1}(x)))\cdots).$$

We observe that, obviously,

(3.1) (i) *If W_1 is a cubic walk from a to b, and W_2 is a cubic walk from b to c, and W_3 is their concatenation, then*

$$\pi_{W_3}(x) = \pi_{W_2}(\pi_{W_1}(x)) \qquad (x \in \nabla(a)).$$

(ii) *If W is a cubic walk u, e, v, e, u then π_W is the identity.*

A closed cubic walk W is *balanced* in G if π_W is the identity. Let W be

$$v_0, e_1, v_1, e_2, \ldots, e_n, v_n = v_0;$$

if W is balanced, then so is

$$v_i, e_{i+1}, v_{i+1}, \ldots, e_n, v_n, e_1, v_1, \ldots, e_i, v_i$$

for any i $(1 \leqslant i \leqslant n-1)$, and also the reverse of W is balanced. Thus, we may speak of a circuit C of G being balanced without ambiguity (meaning that some, and hence every, closed walk

$$v_0, e_1, v_1, \ldots, e_n, v_n$$

with $e_1, \ldots, e_n$ all distinct and $E(C) = \{e_1, \ldots, e_n\}$ is balanced).

We are basically concerned with cubic drawings in Σ, but for inductive purposes we need to permit a few, widely-separated non-cubic vertices. Let us say an *arrangement* in Σ is a pair (G, X) such that

(i) G is a closed 2-cell drawing in Σ

(ii) $X \subseteq V(G)$, and $G \backslash X$ is closed 2-cell ($G \backslash X$ denotes the drawing obtained from G by deleting the vertices in X and all incident edges)

(iii) no region of G is incident with more than one member of X

(iv) every vertex of G not in X is cubic.

An arrangement (G, X) is *even* if for every region of $G \backslash X$, the circuit bounding it is balanced (in G).

(3.2) *If (G, X) is an even arrangement in Σ, then every null-homotopic closed walk in $G \backslash X$ is balanced in G.*

Proof. This follows easily from (3.1)(i) and (3.1)(ii), since (G, X) is even. Q.E.D.

Let G be a drawing in a surface Σ. Let $T \subseteq \Sigma$ be homeomorphic to

$$\{(x, y) \in \mathbf{R}^2 : 1 \leqslant x^2 + y^2 \leqslant 2\}.$$

Then the boundary of T consists of two disjoint O-arcs A, B say. If in addition $k \geqslant 2$ is an integer and

(a) A, B are non-null-homotopic in Σ

(b) $A, B \subseteq U(G)$, and hence there are circuits C_1, C_k of G with $U(C_1) = A$ and $U(C_k) = B$

(c) there are circuits $C_2, ..., C_{k-1}$ of G with $U(C_1 \cup \cdots \cup C_k) \subseteq T$, so that $C_1, ..., C_k$ are pairwise disjoint and pairwise homotopic

then we call T a *k-wide handle of* G (in Σ), and we call C_1, C_k the *end-circuits* of T.

(3.3) If G is a drawing in Σ and $v \in \Sigma$, and $\delta \in \pi_1(\Sigma, v)$ is k-wide in G where $k \geqslant 2$, then there is a k-wide handle in G with end-circuits homotopic to δ.

(In case (3.3) presents any difficulty to the reader, let us mention an alternative approach—define $\delta \in \pi_1(\Sigma, v)$ to be "k-wide" only when there is a handle T as in (3.3); then the proofs of the previous section still work, and we bypass the need for (3.3).)

The main result of this section is the following:

(3.4) For any orientable surface Σ of genus $\geqslant 1$, and every pair of integers $k \geqslant 2$ and $n \geqslant 0$, there exists $c \geqslant 0$ with the following property. If (G, X) is an even arrangement in Σ with $|X| \leqslant n$ and G has representativeness $\geqslant c$, then there is a k-wide handle T in G with $T \cap X = \varnothing$ and with balanced end-circuits.

Proof. Let $k' = k(n+1)$, and choose c' so that (2.1) holds (with c, k replaced by c', k'). Let $c = n + c'$; we shall show that c satisfies (3.4). For let (G, X) be an even arrangement in Σ with $|X| \leqslant n$ such that G has representativeness $\geqslant c$. Then $G \backslash X$ has representativeness $\geqslant c - n = c'$.

Choose $v \in V(G) - X$. For each $\alpha \in \pi_1(\Sigma, v)$, define $\lambda(\alpha)$ as follows: choose a closed walk W in $G \backslash X$ with basepoint v and homotopy type α (this is possible since $G \backslash X$ is 2-cell) and let $\lambda(\alpha) = \pi_W$. (By (3.2), this does not depend on the choice of W.) From (3.1)(i), λ is a homomorphism from $\pi_1(\Sigma, v)$ into $S_3(v)$, the group of permutations of $\nabla(v)$. By (2.1) applied to $G \backslash X$, c' and k', there exists $\delta \in \pi_1(\Sigma, v)$ such that $\lambda(\delta)$ is the identity of $S_3(v)$ and δ is k'-wide in $G \backslash X$. By (3.3) applied to $G \backslash X$, there is a k'-wide handle T' of $G \backslash X$ in Σ, with end-circuits balanced in G. Let us choose k' circuits of G, $C_1, \ldots, C_{k'}$ say, pairwise disjoint and pairwise homotopic, with $U(C_1 \cup \cdots \cup C_{k'}) \subseteq T'$, where C_1 and $C_{k'}$ are the end-circuits of T'; and let us number $C_1, \ldots, C_{k'}$ in order on T'. For $1 \leqslant i < j \leqslant k'$, let $T_{i,j} \subseteq T'$ be the handle with end-circuits C_i and C_j.

Since $|X| \leqslant n$ and $k' = k(n+1)$, there exists i with $1 \leqslant i \leqslant k' - k$ such that $X \cap T_{i,i+k-1} = \varnothing$; let $T = T_{i,i+k-1}$. Then T is a k-wide handle of G, and $T \cap X = \varnothing$, and its end-circuits C_i, C_{i+k-1} are balanced since they have homotopy type δ. Q.E.D.

4. THE MAIN PROOF

Let (G, X) be an arrangement in Σ. A 4-*colouring* of (G, X) means a 4-colouring of the regions of G, so that

(i) as usual, any two regions that share an edge receive different colours

(ii) no region incident with a vertex in X receives colour 4

(iii) no region incident with a vertex in X shares an edge with any region that receives colour 4.

The main result of the paper is the following:

(4.1) For every orientable surface Σ except the sphere, and for every $n \geqslant 0$, there exists $c \geqslant 0$ such that every even arrangement (G, X) in Σ has a 4-colouring provided that $|X| \leqslant n$ and G has representativeness $\geqslant c$.

If T is an Eulerian triangulation in Σ, and T^* is its geometric dual in Σ, then $(T^*, \varnothing)$ is an even arrangement, and since T and T^* have the same representativeness, we see that (1.1) follows from (4.1) taking $n = 0$. We permit $n > 0$ in (4.1) for inductive purposes. To prove (4.1) we need the following lemma; with $X = \varnothing$ this result is due to Heawood [5].

(4.2) If (G, X) is an even arrangement in a sphere Σ then G is 3-region-colourable.

Proof. Choose $z \in V(G) - X$.

(1) *If* (v, r) *is an angle of* G *with* $v \notin X$, *and* W_1, W_2 *are walks of* $G \backslash X$ *from* v *to* z, *then*

$$\pi_{W_1}(v, r) = \pi_{W_2}(v, r).$$

Subproof. This follows from (3.2) since Σ is a sphere and (G, X) is even.

Let us define $f(v, r)$ to be the common value of $\pi_W(v, r)$ over all walks W of $G\backslash X$ from v to z.

(2) *If r is a region of G and $v_1, v_2 \in V(G)-X$ are both incident with r, then $f(v_1, r) = f(v_2, r)$.*

Subproof. Let C be the circuit of G bounding r. By condition (iii) in the definition of "arrangement", at most one vertex of C is in X, and consequently to prove (2) in general it suffices to prove it when some edge e of C has ends v_1, v_2. Let W_2 be a walk of $G\backslash X$ from v_2 to z, let W_0 be the walk v_1, e, v_2, and let W_1 be formed by concatenating W_0 and W_2. Then

$$f(v_1, r) = \pi_{W_1}(v_1, r) = \pi_{W_2}(\pi_{W_0}(v_1, r))$$

by (3.1). But $\pi_{W_0}(v_1, r) = (v_2, r)$ by definition of π_{W_0}, and so

$$f(v_1, r) = \pi_{W_2}(\pi_{W_0}(v_1, r)) = \pi_{W_2}(v_2, r) = f(v_2, r).$$

This proves (2).

For each region r of G, let us define $f(r)$ to be the common value of $f(v, r)$ over all vertices $v \in V(G)-X$ incident with r. (There is such a vertex since all circuits have length $\geqslant 2$, by definition of a drawing.)

(3) *For any edge e of G, let r_1, r_2 be the regions of G incident with e; then $f(r_1) \neq f(r_2)$.*

Subproof. Let v be an end of e not in X, and let W be a walk in $G\backslash X$ from v to z. Then

$$f(r_1) = f(v, r_1) = \pi_W(v, r_1) \neq \pi_W(v, r_2) = f(v, r_2) = f(r_2).$$

This proves (3).

Since $f(r) \in \nabla(z)$ for every region r of G, and $|\nabla(z)| = 3$, it follows from (3) that f is a 3-region-colouring of G. Q.E.D.

Proof of (4.1). We proceed by induction on the genus of Σ. For every orientable surface Σ' (not a sphere) of genus smaller than that of Σ, and every integer n', let $c(\Sigma', n')$ be such that (4.1) holds with Σ, n, c replaced by Σ', n', $c(\Sigma', n')$.

Let t be the maximum of $c(\Sigma', n+2)$ over all such Σ'. Let $k = 2t+4$, and choose c so that (3.4) holds (with Σ, k, n unchanged). We may assume (by increasing c) that $c \geqslant t$ and $c \geqslant 2$. We claim that c satisfies (4.1). For let

(G, X) be an even arrangement in Σ, such that $|X| \leqslant n$ and G has representativeness $\geqslant c$. We must show that (G, X) has a 4-colouring.

By (3.4) and the choice of c, there is a k-wide handle T in G with $T \cap X = \varnothing$ and with balanced end-circuits. Let $C_1, \ldots, C_k$ be circuits as in the definition of "k-wide handle". By choosing C_t as close to C_{t+1} as possible, we may assume that every region of G between C_t and C_{t+1} incident with a vertex of C_t is also incident with a vertex of C_{t+1} (let us call this the *bridge property*). Similarly, choose C_{k-t+1} as close to C_{k-t} as possible.

Let Σ' be obtained from Σ as follows; we delete from Σ the part strictly between $U(C_{t+1})$ and $U(C_{k-t})$, and paste new discs onto the O-arcs $U(C_{t+1}), U(C_{t+4})$ respectively. Then Σ' is a 2-manifold, but it might not be connected. If it is not connected then it has exactly two components, both with genus $\geqslant 1$ and strictly less than the genus of Σ, and the argument below can easily be adapted (working with these two components separately) to cover this case. However, we shall assume for simplicity that Σ' remains connected.

Let Δ_1 be the disc in Σ' bounded by $U(C_t)$ containing $U(C_{t+1})$, and let Δ_2 be the disc in Σ' bounded by $U(C_{t+5})$ containing $U(C_{t+4})$. Let G' be a drawing in Σ' obtained from G as follows. First we delete all vertices and edges of G strictly between $U(C_{t+1})$ and $U(C_{t+4})$, forming G_1 say, which we may regard as a drawing in Σ'. Now contract all edges of G_1 that have both ends strictly inside Δ_1, and similarly for Δ_2. The result is a drawing G' in Σ' with precisely one vertex (say x_i) in the interior of Δ_i $(i = 1, 2)$, because of the bridge property. There is a natural 1-1 correspondence between the regions of G' inside Δ_1 and the regions of G between $U(C_t)$ and $U(C_{t+1})$ incident with an edge of C_t.

(1) *G' is closed 2-cell in Σ', and if Σ' is not a sphere then G' has representativeness $\geqslant t$.*

Subproof. For the first, it suffices to check that $\bar{r}$ is bounded by a circuit of G' for every region r of G' incident with x_1. But all neighbours of x_1 belong to C_t, and there are at least two such neighbours since G is closed 2-cell, so G' is closed 2-cell. For its representativeness, let F be an O-arc with $|F \cap U(G')| < t$. If no point of F is in the interior of Δ_1 or Δ_2, then F is an O-arc in Σ with $|F \cap U(G)| < t \leqslant c$, and so F is null-homotopic in Σ and hence in Σ' as required. We may assume then that some point of F is in the interior of Δ_1, say. Let $\Delta_0 \subseteq \Sigma'$ be the closed disc bounded by $U(C_1)$ that includes Δ_1. Since $|F \cap U(G')| < t$, F does not meet all of $U(C_1), \ldots, U(C_t)$, and in particular $F \subseteq \Delta_0$, and consequently F is null-homotopic in Σ' as required. This proves (1).

Let $X' = X \cup \{x_1, x_2\}$; then (G', X') is an even arrangement in Σ', since C_t and C_{k-t+1} are balanced (in Σ and hence in Σ').

(2) *(G', X') has a 4-colouring.*

Subproof. If Σ' is a sphere this follows from (4.2). If Σ' has genus > 0 then $t \geqslant c(\Sigma', n+2)$ and the claim follows from (1) and the definition of $c(\Sigma', n+2)$. This proves (2).

Let κ_1 be a 4-colouring of (G', X'). For $i = 1, ..., 5$, let B_i be the part of Σ (non-strictly) between $U(C_{t-1+i})$ and $U(C_{t+i})$, and let $\mathscr{R}_i$ be the set of regions of G included in B_i. Let $\mathscr{S}_1$ be the set of regions of G incident with an edge of $U(C_t)$, and $\mathscr{S}_2$ the regions incident with an edge of $U(C_{t+5})$. Thus, $\mathscr{S}_1 \nsubseteq \mathscr{R}_1$ but $\mathscr{S}_1 \cap \mathscr{R}_1 \neq \emptyset$. From the definition of 4-colouring an arrangement, $\kappa_1(r) \in \{1, 2, 3\}$ for every $r \in \mathscr{S}_1 \cup \mathscr{S}_2$ (identifying the regions of G' incident with x_1 or x_2 with regions of G in the natural way.)

For any set $\mathscr{R}$ of the regions of G and any subset Y of $E(G)$, a *d-colouring of $\mathscr{R}$ relative to Y* means a map $\phi: \mathscr{R} \to \{1, ..., d\}$ such that $\phi(r_1) \neq \phi(r_2)$ for every edge $e \in Y$ such that r_1, r_2 are the regions on either side of e and $r_1, r_2 \in \mathscr{R}$. By adding to $B_1 \cup \cdots \cup B_5$ discs bounded by $U(C_t)$ and $U(C_{k-t+1})$, and drawing a new vertex in each disc adjacent to the vertices in the boundary of the disc which have degree 2 in $G \mid (B_1 \cup \cdots \cup B_5)$, and letting X'' be the set of the two new vertices, we obtain an even arrangement in a sphere, which consequently is 3-region-colourable by (4.2).

Let Y be the set of all edges of G with at least one end in $B_1 \cup \cdots \cup B_5$. It follows that there is a 3-colouring of $\mathscr{S}_1 \cup \mathscr{R}_1 \cup \mathscr{R}_2 \cup \cdots \cup \mathscr{R}_5 \cup \mathscr{S}_2$ relative to Y, say κ_2.

Let Z be the set of edges of G with an end in C_t. The restrictions of both κ_1 and κ_2 to $\mathscr{S}_1$ yield 3-colourings of $\mathscr{S}_1$ relative to Z. But $\mathscr{S}_1$ is uniquely 3-colourable relative to Z, and so the restrictions of κ_1 and κ_2 to $\mathscr{S}_1$ are equal (up to permuting colours), and we may therefore choose κ_2 so that $\kappa_1(r) = \kappa_2(r)$ $(r \in \mathscr{S}_1)$. By the same argument applied to $\mathscr{S}_2$, we may choose a permutation $\pi: \{1, 2, 3\} \to \{1, 2, 3\}$ so that $\kappa_1(r) = \pi(\kappa_2(r))$ $(r \in \mathscr{S}_2)$. There are, up to symmetry, three possibilities for π, namely

$$\text{(i)} \qquad \pi(i) = i \qquad (1 \leqslant i \leqslant 3)$$

$$\text{(ii)} \qquad \pi(1) = 2, \quad \pi(2) = 1, \quad \pi(3) = 3$$

$$\text{(iii)} \qquad \pi(1) = 3, \quad \pi(2) = 1, \quad \pi(3) = 2.$$

We shall show that the result holds in each case.

In case (i), define $\kappa(r) = \kappa_1(r)$ $(r \nsubseteq B_1 \cup \cdots \cup B_5)$ and $\kappa(r) = \kappa_2(r)$ $(r \subseteq B_1 \cup \cdots \cup B_5)$; then κ is a 4-colouring of (G, X) as required.

In case (ii), for each region r of G, define $\kappa(r)$ as follows. If $r \notin \mathscr{R}_1 \cup \mathscr{R}_2 \cup \cdots \cup \mathscr{R}_5$ let $\kappa(r) = \kappa_1(r)$. If $r \in \mathscr{R}_1$, let $\kappa(r) = \kappa_2(r)$. If $r \in \mathscr{R}_2$ let

$$\kappa(r) = \begin{cases} 4 & \text{if} \quad \kappa_2(r) = 1 \\ \kappa_2(r) & \text{otherwise.} \end{cases}$$

If $r \in \mathscr{R}_3$ let

$$\kappa(r) = \begin{cases} 4 & \text{if} \quad \kappa_2(r) = 1 \\ 1 & \text{if} \quad \kappa_2(r) = 2 \\ 3 & \text{if} \quad \kappa_2(r) = 3. \end{cases}$$

If $r \in \mathscr{R}_4 \cup \mathscr{R}_5$ let $\kappa(r) = \pi(\kappa_2(r))$. Then κ is a 4-colouring of (G, X), as required.

In case (iii), for each region r of G we define $\kappa(r)$ as follows. If $r \notin \mathscr{R}_1 \cup \cdots \cup \mathscr{R}_5$ let $\kappa(r) = \kappa_1(r)$. If $r \in \mathscr{R}_1$ let $\kappa(r) = \kappa_2(r)$. If $r \in \mathscr{R}_2$ let

$$\kappa(r) = \begin{cases} 4 & \text{if} \quad \kappa_2(r) = 1 \\ \kappa_2(r) & \text{otherwise.} \end{cases}$$

If $r \in \mathscr{R}_3$ let

$$\kappa(r) = \begin{cases} 4 & \text{if} \quad \kappa_2(r) = 1 \\ 1 & \text{if} \quad \kappa_2(r) = 2 \\ 3 & \text{if} \quad \kappa_2(r) = 3. \end{cases}$$

If $r \in \mathscr{R}_4$ let

$$\kappa(r) = \begin{cases} 4 & \text{if} \quad \kappa_2(r) = 1 \\ 1 & \text{if} \quad \kappa_2(r) = 2 \\ 2 & \text{if} \quad \kappa_2(r) = 3. \end{cases}$$

If $r \in \mathscr{R}_5$ let $\kappa(r) = \pi(\kappa_2(r))$. Then again κ is a 4-colouring of (G, X), as required. Q.E.D.

5. THE PROJECTIVE PLANE

Finally we show (1.2), that the analogue of (1.1) is false for the projective plane. The following result is implicit in Youngs [11], and we include a proof (essentially that of [11]) for completeness.

(5.1) *Let G be a drawing in the projective plane so that every region is bounded by a circuit of length* 4. *If G is not bipartite, then for every vertex-colouring (in any number of colours) there is a region r of G so that the four vertices incident with r receive four different colours.*

Proof. Let $\phi: V(G) \to \{1, ..., k\}$ be the vertex-colouring. Let us direct every edge of G with ends $\{u, v\}$ from u to v where $\phi(u) < \phi(v)$. Let C be an odd circuit of G (necessarily non-null-homotopic), and let C have length t say. Then (by cutting along $U(C)$) there is a drawing H in the plane, such that the infinite region of H is bounded by a circuit C_0 of length $2t$, and every finite region by a circuit of length 4, such that if we number the vertices and edges of C_0 as

$$v_0, e_1, v_1, ..., e_{2t}, v_{2t} = v_0$$

in order, then G is obtained by identifying v_i and v_{t+i} $(1 \leqslant i \leqslant t)$ and e_i with e_{t+i} $(1 \leqslant i \leqslant t)$. Let us direct the edges of H in the same way that their images in G are directed. Now for each region r of H, let $a(r)$ be the number of edges of the circuit $C(r)$ bounding r that are traversed in positive direction as $C(r)$ is traversed in clockwise direction; and $b(r) = |E(C(r))| - a(r)$. If r_0 is the infinite region of H, then (by counting the contribution of each edge to each region) we see that

$$a(r_0) - b(r_0) = \sum_{r \neq r_0} (a(r) - b(r)).$$

Now for $1 \leqslant i \leqslant t$, e_i contributes to $a(r_0)$ if and only if e_{t+i} does so; and so $a(r_0)$ is even, and since $a(r_0) + b(r_0)$ is not divisible by 4, it follows that $a(r_0) - b(r_0) \neq 0$. Hence there is a finite region r of H with $a(r) - b(r) \neq 0$, by the equation above. The corresponding region of G satisfies the theorem. Q.E.D.

Proof of (1.2). Take G as in (5.1), with high representativeness and not bipartite (it is easy to see this is possible). Now add a new vertex of degree 4 in each region, forming an Eulerian triangulation. By (5.1) this is not 4-colourable. Q.E.D.

Since this article was submitted for publication, the non-orientable case has been completely analyzed. It is now known precisely when a highly representative quadrangulation and when a highly representative Eulerian triangulation of a non-orientable surface has chromatic number 2, 3, 4, or 5. In particular, for every non-orientable surface, there is a highly representative 5-chromatic Eulerian triangulation. See [1, 7, 8].

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