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Stochastic Processes and their Applications 116 (2006) 917–938

stochastic processes and their applications

www.elsevier.com/locate/spa

A class of remarkable submartingales

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Received 23 August 2005; received in revised form 12 December 2005; accepted 12 December 2005
Available online 4 January 2006

Abstract

In this paper, we consider the special class of positive local submartingales (X_t) of the form $X_t = N_t + A_t$, where the measure (dA_t) is carried by the set $\{t : X_t = 0\}$. We show that many examples of stochastic processes studied in the literature are in this class and propose a unified approach based on martingale techniques for studying them. In particular, we establish some martingale characterizations for these processes and compute explicitly some distributions involving the pair (X_t, A_t). We also associate with X a solution to the Skorokhod's stopping problem for probability measures on the positive half-line.

MSC: 05C38; 15A15; 05A15; 15A18

Keywords: Submartingales; Azéma's submartingale; Bessel processes; Inequalities; General theory of stochastic processes; Skorokhod's stopping problem

1. Introduction

The study of deterministic functions or stochastic processes of the form

$$Y_t = Z_t + K_t, \tag{1.1}$$

where $Y \geq 0$ is continuous, and K is increasing and continuous with (dK_t) carried by the set $\{t : Y_t = 0\}$, has received much attention in probability theory: Eq. (1.1) is referred to as Skorokhod's reflection equation. It plays a key role in martingale theory: the family

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0304-4149/$ - see front matter
doi:10.1016/j.spa.2005.12.003

of Azéma–Yor martingales, the resolution of Skorokhod's embedding problem, the study of Brownian local times (see [16] Chapter VI for more references and examples; see also [19,11, 1,2,12]). It also plays an important role in the study of some diffusion processes (see [18,16,1]) and in the study of zeros of continuous martingales [4]. Well known examples of such stochastic processes are $(|M_t|)$, the absolute value of some local martingale M, or $(S_t - M_t)$, where S is the supremum process of M. Usually, in the literature, one studies the case of the standard Brownian Motion and its local time, then translates it into equivalent results for the Brownian Motion and its supremum process using Lévy's equivalence theorem, and then extends it to a wide class of continuous local martingales, using the Dubins–Schwarz theorem.

The aim of this paper is to provide a general framework and methods, based on martingale techniques, for dealing with a large class of stochastic processes, which can be discontinuous, and which contain all the previously mentioned processes. In particular, we shall extend some well known results in the Brownian setting (and whose proofs are based on excursion theory) to a wide class of stochastic processes which are not Markov and which can even be discontinuous.

More precisely, we shall consider the following class of local submartingales, whose definition goes back to Yor [21]:

Definition 1.1. Let (X_t) be a positive local submartingale, which decomposes as

$$X_t = N_t + A_t.$$

We say that (X_t) is of class (Σ) if:

(1) (N_t) is a càdlàg local martingale, with $N_0 = 0$;
(2) (A_t) is a continuous increasing process, with $A_0 = 0$;
(3) the measure $(\mathrm{d}A_t)$ is carried by the set $\{t : X_t = 0\}$.

If additionally, (X_t) is of class (D), we shall say that (X_t) is of class (ΣD).

In Section 2, we prove a martingale characterization for the processes of class (Σ) and then give some examples. In particular, we obtain a family of martingales reminiscent of the family of Azéma–Yor martingales.

In Section 3, we prove the following estimate for a large class of processes of class (Σ), generalizing a well known result for the pair (B_t, ℓ_t), where (B_t) is the standard one-dimensional Brownian Motion and (ℓ_t) its local time at 0:

$$\mathbb{P}\left(\exists t \geq 0, X_t > \varphi\left(A_t\right)\right) = 1 - \exp\left(-\int_0^\infty \frac{\mathrm{d}x}{\varphi\left(x\right)}\right), \tag{1.2}$$

where φ is a positive Borel function. We then use this estimate to obtain the law of the maximum for some processes involving the pair $(X_t, \varphi(A_t))$. We also use the domination relations of Lenglart [10] to obtain some probabilistic inequalities involving the pair $(X_t, \varphi(A_t))$.

In Section 4, we compute the distribution of A_∞ when (X_t) is of class (ΣD), and then deduce the law of A_T, where T is a stopping time chosen in a family of stopping times reminiscent of the stopping times used by Azéma and Yor for the resolution of the Skorokhod stopping problem. Among other applications of these results, we recover a result of Lehoczky [9] about the law of the maximum for some stopped diffusions. We also give the law of the increasing process of a conveniently stopped process which solves a stochastic differential equation of Skorokhod type with a reflecting condition at 0 (such processes are defined in [18] and play a key role in extensions of Pitman's theorem).

Finally in Section 5, inspired by a recent paper of Obłój and Yor [15], and using the estimates (1.2), we propose two different solutions to the Skorokhod embedding problem for a non-atomic probability measure on $\mathbb{R}_+$.

2. A first characterization and some examples

Let (X_t) be of class (Σ). We have the following martingale characterization for the processes of class (Σ):

Theorem 2.1. *The following are equivalent:*

(1) *The local submartingale (X_t) is of class (Σ).*
(2) *There exists an increasing, adapted and continuous process (C_t) such that for every locally bounded Borel function f, and $F(x) \equiv \int_0^x f(z)\,\mathrm{d}z$, the process*

$$F(C_t) - f(C_t)X_t$$

is a local martingale. Moreover, in this case, (C_t) is equal to (A_t), the increasing process of X.

Proof. (1) $\Longrightarrow$ (2). First, let us assume that f is $\mathcal{C}^1$ and let us take $C_t \equiv A_t$. An integration by parts and the fact that $\int_0^t X_{s-}\mathrm{d}A_s = \int_0^t X_s \mathrm{d}A_s$ since A is continuous yield

$$\begin{aligned} f(A_t)X_t &= \int_0^t f(A_u)\,\mathrm{d}X_u + \int_0^t f'(A_u)X_u \mathrm{d}A_u \\ &= \int_0^t f(A_u)\,\mathrm{d}N_u + \int_0^t f(A_u)\,\mathrm{d}A_u + \int_0^t f'(A_u)X_u \mathrm{d}A_u. \end{aligned}$$

Since $(\mathrm{d}A_t)$ is carried by the set $\{t : X_t = 0\}$, we have $\int_0^t f'(A_u)X_u \mathrm{d}A_u = 0$. As $\int_0^t f(A_u)\,\mathrm{d}A_u = F(A_t)$, we have thus obtained that

$$F(A_t) - f(A_t)X_t = -\int_0^t f(A_u)\,\mathrm{d}N_u, \tag{2.1}$$

and consequently $(F(A_t) - f(A_t)X_t)$ is a local martingale. The general case when f is only assumed to be locally bounded follows from a monotone class argument and the integral representation (2.1) is still valid.

(2) $\Longrightarrow$ (1). First take $F(a) = a$; we then obtain that $C_t - X_t$ is a local martingale. Hence the increasing process of X in its Doob–Meyer decomposition is C, and $C = A$. Next, we take $F(a) = a^2$ and we get

$$A_t^2 - 2A_t X_t$$

is a local martingale. But

$$A_t^2 - 2A_t X_t = 2\int_0^t A_s\,(\mathrm{d}A_s - \mathrm{d}X_s) - 2\int_0^t X_s \mathrm{d}A_s.$$

Hence, we must have

$$\int_0^t X_s \mathrm{d}A_s = 0.$$

Thus $\mathrm{d}A_s$ is carried by the set of zeros of X. $\square$

Remark 2.2. We have proved that for f a locally bounded Borel function, we have

$$f(A_t)X_t - F(A_t) = \int_0^t f(A_u)\,\mathrm{d}N_u.$$

When f is nonnegative, this means that

$$f(A_t)X_t = \int_0^t f(A_u)\,\mathrm{d}N_u + F(A_t),$$

is again of class (Σ), and its increasing process is $F(A_t)$.

Remark 2.3. When X is continuous, we can apply Skorokhod's reflection lemma to obtain

$$A_t = \sup_{s\leq t}(-N_s).$$

Then an application of the balayage formula (see [16], Chapter VI, p. 262) yields

$$F(C_t) - f(C_t)X_t$$

is a local martingale. In the continuous case, we can also note that X may be represented as $(S_t - M_t)$, where M is a continuous local martingale and S its supremum process.

One often needs to know when $(F(A_t) - f(A_t)X_t)$ is a true martingale. We shall deal with a special case which will be of interest to us later.

Corollary 2.4. *Let X be of class (ΣD). If f is a Borel bounded function with compact support, then $(F(A_t) - f(A_t)X_t)$ is a uniformly integrable martingale.*

Proof. There exist two constants $C > 0, K > 0$ such that $\forall x \geq 0, |f(x)| \leq C$, and $\forall x \geq K, f(x) = 0$. Consequently, we have

$$|F(A_t) - f(A_t)X_t| \leq CK + CX_t;$$

now, since $(CK + CX_t)$ is of class (D), we deduce that $(F(A_t) - f(A_t)X_t)$ is a local martingale of class (D) and hence it is a uniformly integrable martingale. □

Now, we shall give some examples of processes in the class (Σ).

(1) Let (M_t) be a continuous local martingale with respect to some filtration $(\mathcal{F}_t)$, starting from 0; then,

$$|M_t| = \int_0^t sgn(M_u)\,\mathrm{d}M_u + L_t(M),$$

is of class (Σ).

(2) Similarly, for any $\alpha > 0, \beta > 0$, the process

$$\alpha M_t^+ + \beta M_t^-$$

is of class (Σ).

(3) Let (M_t) be a local martingale (starting from 0) with only negative jumps and let $S_t \equiv \sup_{u\leq t} M_u$; then

$$X_t \equiv S_t - M_t$$

is of class (Σ). In this case, X has only positive jumps.

(4) Let (R_t) be a Bessel process, starting from 0, of dimension δ, with $\delta \in (0, 2)$. We borrow our results about these processes from [13] where all the proofs can be found. We shall introduce the parameter $\mu \in (0, 1)$, defined by $\delta = 2(1 - \mu)$. Then,

$$X_t \equiv R_t^{2\mu}$$

is of class (Σ) and can be decomposed as

$$R_t = N_t + L_t(R),$$

where $(L_t(R))$ is one normalization for the local time at 0 of R.

(5) Again, let (R_t) be a Bessel process (starting from 0) of dimension $2(1-\mu)$, with $\mu \in (0, 1)$. Define

$$g_\mu(t) \equiv \sup\{u \leq t : R_u = 0\}.$$

In the filtration $\mathcal{G}_t \equiv \mathcal{F}_{g_\mu(t)}$ of the zeros of the Bessel process R, the stochastic process

$$X_t \equiv \left(t - g_\mu(t)\right)^\mu,$$

is a submartingale of class (Σ), whose increasing process in its Doob–Meyer decomposition is given by

$$A_t \equiv \frac{1}{2^\mu \Gamma(1+\mu)} L_t(R),$$

where as usual Γ stands for Euler's gamma function. Recall that $\mu \equiv \frac{1}{2}$ corresponds to the absolute value of the standard Brownian Motion; thus for $\mu \equiv \frac{1}{2}$ the above result leads to nothing but the celebrated second Azéma's martingale ($X_t - A_t$; see [3,22]). In this example, X has only negative jumps.

3. Some estimates and distributions for the pair (X_t, A_t)

In the next sections, we shall introduce the family of stopping times T_φ, associated with a nonnegative Borel function φ, and defined by

$$T_\varphi \equiv \inf\{t \geq 0 : \varphi(A_t) X_t \geq 1\}.$$

These stopping times (associated with a suitable φ) play an important role in the resolution by Azéma and Yor [2] and Obłój and Yor [15] of the Skorokhod embedding problem for the Brownian Motion and the age of Brownian excursions. Some special cases of this family are also studied in [6] and [9]. One natural and important question is whether the stopping time T_φ is almost surely finite or not. The next theorem, reminiscent of some studies by Knight [7,8], answers this question, and generalizes a result of Yor [23] for the standard Brownian Motion. In particular, martingale techniques will allow us to get rid of the Markov property (which was one of the aims of Paul André Meyer when he developed systematically the general theory of stochastic processes). Before stating and proving our main theorem, we shall need an elementary, yet powerful lemma, which we have called Doob's maximal identity in [12]. For sake of completeness, we give again a short proof for it.

Lemma 3.1 (*Doob's Maximal Identity*). *Let (M_t) be a positive local martingale which satisfies*

$$M_0 = x, x > 0; \quad \lim_{t\to\infty} M_t = 0.$$

If we note

$$S_t \equiv \sup_{u \le t} M_u,$$

and if S *is continuous, then for any* $a > 0$*, we have*

(1)

$$\mathbf{P}\left(S_\infty > a\right) = \left(\frac{x}{a}\right) \wedge 1. \tag{3.1}$$

Hence, $\frac{x}{S_\infty}$ *is a uniform random variable on* $(0, 1)$.

(2) *For any stopping time* T,

$$\mathbf{P}\left(S^T > a \mid \mathcal{F}_T\right) = \left(\frac{M_T}{a}\right) \wedge 1, \tag{3.2}$$

where

$$S^T = \sup_{u \ge T} M_u.$$

Hence, $\frac{M_T}{S^T}$ *is also a uniform random variable on* $(0, 1)$*, independent of* $\mathcal{F}_T$.

Proof. Formula (3.2) is a consequence of (3.1) when applied to the martingale $(M_{T+u})_{u\ge0}$ and the filtration $(\mathcal{F}_{T+u})_{u\ge0}$. Formula (3.1) itself is obvious when $a \le x$, and for $a > x$, it is obtained by applying Doob's optional stopping theorem to the local martingale $\left(M_{t\wedge T_a}\right)$, where $T_a = \inf\{u \ge 0 : M_u \ge a\}$. □

Now, we state the main result of this section:

Theorem 3.2. *Let* X *be a local submartingale of the class* (Σ)*, with only negative jumps, such that* $A_\infty = \infty$*. Define* (τ_u) *the right continuous inverse of* A*:*

$$\tau_u \equiv \inf\{t : A_t > u\}.$$

Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *be a Borel function. Then, we have the following estimates:*

$$\mathbb{P}\left(\exists t \ge 0, X_t > \varphi(A_t)\right) = 1 - \exp\left(-\int_0^\infty \frac{\mathrm{d}x}{\varphi(x)}\right), \tag{3.3}$$

and

$$\mathbb{P}\left(\exists t \le \tau_u, X_t > \varphi(A_t)\right) = 1 - \exp\left(-\int_0^u \frac{\mathrm{d}x}{\varphi(x)}\right). \tag{3.4}$$

Proof. The proof is based on Theorem 2.1 and Lemma 3.1. We shall first prove Eq. (3.3), and for this, we first note that we can always assume that $\frac{1}{\varphi}$ is bounded and integrable. Indeed, let us consider the event

$$\Delta_\varphi \equiv \{\exists t \ge 0, X_t > \varphi(A_t)\}.$$

Now, if $(\varphi_n)_{n\ge1}$ is a decreasing sequence of functions with limit φ, then the events $\left(\Delta_{\varphi_n}\right)$ are increasing, and $\bigcup_n \Delta_{\varphi_n} = \Delta_\varphi$. Hence, by approximating φ from above, we can always assume that $\frac{1}{\varphi}$ is bounded and integrable.

Now, let

$$F(x) \equiv 1 - \exp\left(-\int_x^\infty \frac{\mathrm{d}z}{\varphi(z)}\right);$$

its Lebesgue derivative f is given by

$$f(x) = \frac{-1}{\varphi(x)} \exp\left(-\int_x^\infty \frac{\mathrm{d}z}{\varphi(z)}\right) = \frac{-1}{\varphi(x)}(1 - F(x)).$$

Now, from Theorem 2.1, $(M_t \equiv F(A_t) - f(A_t) X_t)$, which is also equal to $F(A_t) + X_t \frac{1}{\varphi(A_t)}(1 - F(A_t))$, is a positive local martingale (whose supremum is continuous since (M_t) has only negative jumps), with $M_0 = 1 - \exp\left(-\int_0^\infty \frac{\mathrm{d}x}{\varphi(x)}\right)$. Moreover, as (M_t) is a positive local martingale, it converges almost surely as $t \to \infty$. Let us now consider M_{τ_u}:

$$M_{\tau_u} = F(u) - f(u) X_{\tau_u}.$$

But since $(\mathrm{d}A_t)$ is carried by the zeros of X and since τ_u corresponds to an increase time of A, we have $X_{\tau_u} = 0$. Consequently,

$$\lim_{u \to \infty} M_{\tau_u} = \lim_{u \to \infty} F(u) = 0,$$

and hence

$$\lim_{u \to \infty} M_u = 0.$$

Now let us note that if for a given $t_0 < \infty$, we have $X_{t_0} > \varphi\left(A_{t_0}\right)$, then we must have

$$M_{t_0} > F\left(A_{t_0}\right) - f\left(A_{t_0}\right)\varphi\left(A_{t_0}\right) = 1,$$

and hence we easily deduce that

$$\begin{aligned}\mathbb{P}(\exists t \geq 0, X_t > \varphi(A_t)) &= \mathbb{P}\left(\sup_{t \geq 0} M_t > 1\right) \\ &= \mathbb{P}\left(\sup_{t \geq 0} \frac{M_t}{M_0} > \frac{1}{M_0}\right) \\ &= M_0,\end{aligned}$$

where the last equality is obtained by an application of Doob's maximal identity (Lemma 3.1).

To obtain the second identity of the theorem, it suffices to replace φ by the function φ_u defined as

$$\varphi_u(x) = \begin{cases} \varphi(x) & \text{if } x < u \\ \infty & \text{otherwise.} \end{cases} \qquad \square$$

Now, as an application of Theorem 3.2, we have the following corollaries:

Corollary 3.3. *Let X be a local submartingale of the class (Σ), with only negative jumps, such that $\lim_{t \to \infty} A_t = \infty$. If $\int_0^\infty \frac{\mathrm{d}x}{\varphi(x)} = \infty$, then the stopping time T_φ is finite almost surely, i.e. $T_\varphi < \infty$. Furthermore, if $T < \infty$ and if φ is locally bounded, then*

$$X_T = \frac{1}{\varphi(A_T)}. \tag{3.5}$$

Proof. At this stage, only formula (3.5) is not trivial. We note, from Remark 2.3 that

$$\varphi(A_t) X_t = \int_0^t \varphi(A_u) \, \mathrm{d}N_u + \Phi(A_t),$$

where $\Phi(x) = \int_0^x \mathrm{d}z \varphi(z)$. Hence, $(\varphi(A_t) X_t)$ has also only negative jumps and consequently $\varphi(A_T) X_T = 1$. □

Remark 3.4. It is remarkable that no more regularity than local boundedness is required for φ to have $X_T = \frac{1}{\varphi(A_T)}$. Usually, in the literature, one requires left or right continuity for φ.

Remark 3.5. Sometimes, in the literature (see Section 5), one is interested in the stopping time:

$$T \equiv \inf \{t \geq 0 : X_t \geq \varphi(A_t)\}.$$

In this case, if $\int_0^\infty \mathrm{d}x \varphi(x) = \infty$, then $T < \infty$. In the case where $T < \infty$ and $1/\varphi$ is locally bounded, we have

$$X_T = \varphi(A_T).$$

Corollary 3.6 (*Knight [8]*). *Let (B_t) denote a standard Brownian Motion, and S its supremum process. Then, for φ a nonnegative Borel function, we have*

$$\mathbb{P}(\forall t \geq 0, S_t - B_t \leq \varphi(S_t)) = \exp\left(-\int_0^\infty \frac{\mathrm{d}x}{\varphi(x)}\right).$$

Furthermore, if we let T_x denote the stopping time:

$$T_x = \inf\{t \geq 0 : S_t > x\} = \inf\{t \geq 0 : B_t > x\},$$

then for any nonnegative Borel function φ, we have

$$\mathbb{P}(\forall t \leq T_x, S_t - B_t \leq \varphi(S_t)) = \exp\left(-\int_0^x \frac{\mathrm{d}x}{\varphi(x)}\right).$$

Proof. It is a consequence of Theorem 3.2, with $X_t = S_t - B_t$, and $A_t = S_t$. □

Remark 3.7. The same result holds for any continuous local martingale such that $S_\infty = \infty$.

Corollary 3.8 (*[13]*). *Let R be a Bessel process of dimension $2(1-\mu)$, with $\mu \in (0,1)$, and L its local time at 0 (as defined in Section 2). Define τ, the right continuous inverse of L:*

$$\tau_u = \inf\{t \geq 0; L_t > u\}.$$

Then, for any positive Borel function φ, we have

$$\mathbb{P}(\exists t \leq \tau_u, R_t > \varphi(L_t)) = 1 - \exp\left(-\int_0^u \frac{\mathrm{d}x}{\varphi^{2\mu}(x)}\right),$$

and

$$\mathbb{P}(\exists t \geq 0, R_t > \varphi(L_t)) = 1 - \exp\left(-\int_0^\infty \frac{\mathrm{d}x}{\varphi^{2\mu}(x)}\right).$$

When $\mu = \frac{1}{2}$, we recover the well known estimates for the standard Brownian Motion and its local time (ℓ_t) (see [23]).

Proof. This follows from the fact that $X_t = R_t^{2\mu}$ is a submartingale satisfying the condition of Theorem 3.2 (see Section 2). $\square$

Corollary 3.9. *Let R be a Bessel process of dimension $2(1-\mu)$, with $\mu \in (0,1)$; define*

$$g_\mu(t) \equiv \sup\{u \le t : R_u = 0\}.$$

In the filtration $\mathcal{G}_t \equiv \mathcal{F}_{g_\mu(t)}$,

$$X_t \equiv \left(t - g_\mu(t)\right)^\mu,$$

is a submartingale of class (Σ) whose increasing process is

$$\frac{1}{2^\mu \Gamma(1+\mu)} L_t(R).$$

Consequently, we have

$$\mathbb{P}\left(\exists t \ge 0, g_\mu(t) < t - \varphi(L_t)\right) = 1 - \exp\left(-\frac{1}{2^\mu \Gamma(1+\mu)} \int_0^\infty \frac{\mathrm{d}x}{\varphi^{2\mu}(x)}\right),$$

and

$$\mathbb{P}\left(\exists t \le \tau_u, g_\mu(t) < t - \varphi(L_t)\right) = 1 - \exp\left(-\frac{1}{2^\mu \Gamma(1+\mu)} \int_0^{2^\mu \Gamma(1+\mu)u} \frac{\mathrm{d}x}{\varphi^{2\mu}(x)}\right).$$

Now, we shall use Theorem 3.2 to obtain the distribution of the supremum of some random variables involving the pair (X_t, A_t). The results we shall prove have been obtained by Yor [23] in the Brownian setting.

Corollary 3.10. *Consider for $q > p > 0$, the random variable*

$$S_{p,q} \equiv \sup_{t\ge 0}\left(X_t^p - A_t^q\right).$$

Then, under the assumptions of Theorem 3.2 for X,

$$S_{p,q} \overset{\text{(law)}}{=} \left(\mathbf{e}_{p,q}\right)^{\frac{pq}{p-q}},$$

where $\mathbf{e}_{p,q}$ is an exponential random variable with parameter $c_{p,q} = \frac{1}{q}\int_0^\infty \frac{\mathrm{d}z}{z^{1-\frac{1}{q}}(1+z)^{\frac{1}{p}}}$,

Proof. Let $a > 0$;

$$\mathbb{P}\left(S_{p,q} > a\right) = \mathbb{P}\left(\exists t \ge 0, X_t > \left(a + L_t^q\right)^{\frac{1}{p}}\right) = 1 - \exp\left(-\int_0^\infty \frac{\mathrm{d}x}{(a+x^q)}^{\frac{1}{p}}\right),$$

and the result follows from a straightforward change of variables in the last integral. $\square$

Corollary 3.11. *Let φ be a nonnegative integrable function and define the random variable S_φ as*

$$S_\varphi = \sup_{t\ge 0}\left(X_t - \varphi(A_t)\right).$$

Then, for all $a \geq 0$, under the assumptions of Theorem 3.2 for X, we have

$$\mathbb{P}\left(S_{\varphi} > a\right) = 1 - \exp\left(-\int_0^{\infty} \frac{\mathrm{d}x}{a + \varphi(x)}\right).$$

Proof. It suffices to remark that

$$\mathbb{P}\left(S_{\varphi} > a\right) = \mathbb{P}\left(\exists t \geq 0, X_t > a + \varphi(A_t)\right),$$

and then apply Theorem 3.2. □

To conclude this section, we shall give some inequalities for the pair (X_t, A_t). Inequalities for submartingales have been studied in depth by Yor in [21]: here we first simply apply the domination inequalities of Lenglart [10] to derive a simple inequality between $\left(X_t^* \equiv \sup_{s \leq t} X_s\right)$ and (A_t) and then we mention a result of Yor [21] for H^p norms.

Definition 3.12 (*Domination Relation*). A positive adapted right continuous process X ($X_0 = 0$) is dominated by an increasing process A ($A_0 = 0$) if

$$\mathbb{E}[X_T] \leq \mathbb{E}[A_T]$$

for any bounded stopping time T.

Lemma 3.13 (*[10], p. 173*). *If X is dominated by A and A is continuous, then for any $k \in (0, 1)$,*

$$\mathbb{E}\left[\left(X_{\infty}^*\right)^k\right] \leq \frac{2-k}{1-k}\mathbb{E}\left[A_{\infty}^k\right].$$

Now, applying this to positive submartingales yields:

Proposition 3.14. *Let X be a positive local submartingale which decomposes as $X_t = N_t + A_t$. Then, we have for any $k \in (0, 1)$*

$$\mathbb{E}\left[\left(X_{\infty}^*\right)^k\right] \leq \frac{2-k}{1-k}\mathbb{E}\left[A_{\infty}^k\right].$$

If furthermore the process $\left(X_t^*\right)$ is continuous (this is the case if X has only negative jumps), then we have for any $k \in (0, 1)$

$$\mathbb{E}\left[\left(X_{\infty}^*\right)^k\right] \leq \frac{2-k}{1-k}\mathbb{E}\left[A_{\infty}^k\right] \leq \left(\frac{2-k}{1-k}\right)^2 \mathbb{E}\left[\left(X_{\infty}^*\right)^k\right].$$

Proof. The proof is a consequence of

$$\mathbb{E}[X_T] \leq \mathbb{E}[A_T] \leq \mathbb{E}\left[X_T^*\right]$$

for any bounded stopping time, combined with an application of Lenglart's domination results. □

Remark 3.15. The previous proposition applies in particular to the pair $(|B_t|, \ell_t)$, where B is the standard Brownian Motion and (ℓ_t) its local time at 0. More generally, it holds for the pair $\left(R_t^{2\mu}, L_t(R)\right)$ where R is a Bessel process of dimension $2(1-\mu)$. It also applies to the age process and its local time at zero: $\left(\left(t-g_{\mu}(t)\right)^{\mu}, \frac{1}{2^{\mu}\Gamma(1+\mu)}L_t(R)\right)$. More precisely, for any stopping time T, and any $k \in (0,1)$, we have

$$\mathbb{E}\left[\left(\sup_{u\leq T}\left(t-g_{\mu}(t)\right)\right)^{k\mu}\right] \leq \frac{2-k}{(1-k)\,2^{k\mu}\Gamma(1+\mu)^{k}}\mathbb{E}\left[L_T^k\right]$$

$$\leq \left(\frac{2-k}{1-k}\right)^2 \mathbb{E}\left[\left(\sup_{u\leq T}\left(t-g_{\mu}(t)\right)\right)^{k\mu}\right].$$

There are also other types of inequalities for local times of Azéma martingales ($\mu=\frac{1}{2}$) obtained by Chao and Chou [5].

Now, we shall give some inequalities, obtained by Yor [21], for local submartingales of the class (Σ). In [21], the inequalities are given for continuous local submartingales; the proof in fact applies to local submartingales of the class (Σ) with only negative jumps if one uses the version of Skorokhod's reflection lemma for functions with jumps, as explained in [11] or [12].

First, for $X=N+A$ of the class (Σ), and $k\in(0,\infty)$, we define

$$\|X\|_{H^k} = \left\|[N,N]_{\infty}^{1/2}\right\|_{L^k} + \|A_{\infty}\|_{L^k}.$$

For $k\geq 1$, it defines a norm. Now, we state a slight generalization of Yor's inequalities (in this version negative jumps are allowed):

Proposition 3.16 (*Yor [21]*). *Let X be a local submartingale of class (Σ), which has only negative jumps. Then the following hold:*

(1) *for all $k\in[1,\infty)$, $\|A_{\infty}\|_{L^k} \leq \left\|N_{\infty}^{*}\right\|_{L^k}$, and the quantities $\|X\|_{H^k}$, $\left\|X_{\infty}^{*}\right\|_{L^k}$ and $\|N\|_{H^k}$ are equivalent.*
(2) *if N is continuous, then the above statement holds for all $k\in(0,\infty)$, and if $k\in(0,1)$, then $\|A_{\infty}\|_{L^k}$ is equivalent to the three former quantities.*

4. The distribution of A_{∞}

In this section, we shall compute the law of A_{∞}, for a large class of processes in (ΣD), and then we shall see that we are able to recover some well known results (for example, formulas for stopped diffusions [9]).

Theorem 4.1. *Let X be a process of class (ΣD), and define*

$$\lambda(x) \equiv \mathbb{E}\left[X_{\infty}|A_{\infty}=x\right].$$

Assume that $\lambda(A_{\infty})\neq 0$. Then, if we note

$$b \equiv \inf\{u : \mathbb{P}(A_{\infty}\geq u)=0\},$$

we have

$$\mathbb{P}(A_{\infty}>x) = \exp\left(-\int_0^x \frac{dz}{\lambda(z)}\right); \quad x<b. \tag{4.1}$$

Remark 4.2. The techniques we shall use for the proof of Theorem 4.1 are very close to those already used by Azéma and Yor [2] and more recently by Vallois [20] in the study of the law of the maximum of a continuous and uniformly integrable martingale. Our results here allow for some discontinuities (look at the examples in Section 2).

Proof. Let f be a bounded Borel function with compact support; from Corollary 2.4,

$$\mathbb{E}\left[F\left(A_{\infty}\right)\right]=\mathbb{E}\left[X_{\infty} f\left(A_{\infty}\right)\right],$$

where $F(x)=\int_0^x \mathrm{d}z f(z)$. Now, conditioning the right hand side with respect to A_{∞} yields

$$\mathbb{E}\left[F\left(A_{\infty}\right)\right]=\mathbb{E}\left[\lambda\left(A_{\infty}\right) f\left(A_{\infty}\right)\right]. \tag{4.2}$$

Now, since $\lambda(A_{\infty})>0$, if $\nu(\mathrm{d}x)$ denotes the law of A_{∞}, and $\overline{\nu}(x)\equiv\nu([x,\infty))$, (4.2) implies

$$\int_0^{\infty} \mathrm{d}z f(z)\,\overline{\nu}(z)=\int_0^{\infty} \nu(\mathrm{d}z)\, f(z)\,\lambda(z),$$

and consequently,

$$\overline{\nu}(z)\,\mathrm{d}z=\lambda(z)\,\nu(\mathrm{d}z). \tag{4.3}$$

Recall that $b\equiv\inf\{u:\mathbb{P}(A_{\infty}\geq u)=0\}$; hence for $x<b$,

$$\int_0^x \frac{\mathrm{d}z}{\lambda(z)}=\int_0^x \frac{\nu(\mathrm{d}z)}{\overline{\nu}(z)} \leq \frac{1}{\overline{\nu}(z)}<\infty,$$

and integrating (4.3) between 0 and x, for $x<b$ yields

$$\overline{\nu}(x)=\exp\left(-\int_0^x \frac{\mathrm{d}z}{\lambda(z)}\right),$$

and the result of the theorem follows easily. □

Remark 4.3. We note that

$$b=\inf\left\{x:\int_0^x \frac{\mathrm{d}z}{\lambda(z)}=+\infty\right\},$$

with the usual convention that $\inf\emptyset=\infty$.

Remark 4.4. We took the hypothesis $\lambda(A_{\infty})\neq 0$ to avoid technicalities. The result stated in Theorem 4.1 is general enough for our purpose; the reader interested in the general case can refer to [20] where the special case $X_t=S_t-M_t$, with M continuous, is dealt with in depth.

Now, we give a series of interesting corollaries.

Corollary 4.5. *Let (M_t) be a uniformly integrable martingale, with only negative jumps, and such that $\mathbb{E}(S_{\infty})<\infty$, and $M_{\infty}<S_{\infty}$. Then,*

$$\mathbb{P}(S_{\infty}>x)=\exp\left(-\int_0^x \frac{\mathrm{d}z}{z-\alpha(z)}\right),$$

where

$$\alpha(x)=\mathbb{E}\left[M_{\infty}\mid S_{\infty}=x\right].$$

Remark 4.6. This result, when the martingale M is continuous, is a special case of a more general result by Vallois [20] where $S_\infty - \alpha(S_\infty)$ can vanish. The reader can also refer to [17] for a discussion on the law of the maximum of a martingale and its terminal value.

Corollary 4.7. *Let X be a process of class (ΣD), such that*

$$\lim_{t\to\infty} X_t = a > 0, \ a.s.$$

Then,

$$\mathbb{P}(A_\infty > x) = \exp\left(-\frac{x}{a}\right),$$

i.e. A_∞ is distributed as a random variable with exponential law of parameter $\frac{1}{a}$.

Proof. It is a consequence of Theorem 4.1 with $\lambda(x) \equiv a$. □

Let (M_t) be a continuous local martingale such that $\langle M\rangle_\infty = \infty$, a.s.; let $T_1 = \inf\{t \geq 0 : M_t = 1\}$. Then an application of Tanaka's formula and Corollary 4.7 shows that $\frac{1}{2}L_{T_1}(M)$ follows the standard exponential law ($L_{T_1}(M)$ denotes the local time at 0 of the local martingale M). The result also applies to Bessel processes of dimension $2(1-\mu)$, with $\mu \in (0,1)$: taking $T_1 = \inf\{t \geq 0 : R_t = 1\}$, we have that $L_{T_1}(R)$ follows the standard exponential law. The result also applies to $\big(t - g_\mu(t)\big)$.

Corollary 4.8. *Let (M_t) be a continuous martingale such that $\lim_{t\to\infty} M_t = M_\infty$ exists and $|M_\infty| > 0$. Then, if (L_t) denotes its local time at 0, we have*

$$\mathbb{P}(L_\infty > x) = \exp\left(-\int_0^x \frac{\mathrm{d}z}{\mathbb{E}[|M_\infty| \mid L_\infty = z]}\right).$$

Corollary 4.9. *Let X be of the class (Σ) with only negative jumps and $A_\infty = \infty$ and let φ be a nonnegative locally bounded Borel function such that $\int_0^\infty \mathrm{d}x\varphi(x) = \infty$. Define the stopping time T as*

$$T \equiv \inf\{t : \varphi(A_t) X_t = 1\}.$$

Then $T < \infty$, a.s. and

$$\mathbb{P}(A_T > x) = \exp\left(-\int_0^x \mathrm{d}z\varphi(z)\right).$$

Proof. The fact that $T < \infty$, a.s. is a consequence of Theorem 4.1 and the rest follows from Theorem 4.1 with $\lambda(x) = \frac{1}{\varphi(x)}$. □

The following variant of Corollary 4.9 is sometimes useful:

Corollary 4.10. *Let X be of the class (Σ) with only negative jumps and $A_\infty = \infty$. Let ψ be a nonnegative Borel function such that $\frac{1}{\psi}$ is locally bounded and $\int_0^\infty \frac{\mathrm{d}x}{\psi(x)} = \infty$. Define the stopping time T as*

$$T \equiv \inf\{t : X_t \geq \psi(A_t)\}.$$

Then $T < \infty$, a.s. and

$$\mathbb{P}(A_T > x) = \exp\left(-\int_0^x \frac{\mathrm{d}z}{\psi(z)}\right).$$

Now, we shall apply the previous results to compute the maximum of a stopped continuous diffusion process; in particular, we are able to recover a formula discovered first by Lehoczky [9]. More precisely, let (Y_t) be a continuous diffusion process, with $Y_0 = 0$. Let us assume further that Y is recurrent; then, from the general theory of diffusion processes, there exists a unique continuous and strictly increasing function s, with $s(0) = 0$, $\lim_{x\to+\infty} s(x) = +\infty$, $\lim_{x\to-\infty} s(x) = -\infty$, such that $s(Y_t)$ is a continuous local martingale. Let θ be a Borel function with $\theta(x) > 0$, $\forall x > 0$. Define the stopping time

$$T \equiv \inf\left\{t : \overline{Y}_t - Y_t \geq \theta\left(\overline{Y}_t\right)\right\},$$

where $\overline{Y}_t = \sup_{u\leq t} Y_u$.

Proposition 4.11. *Let us assume that $T < \infty$, a.s. Then, the law of $\overline{Y}_T$ is given by*

$$\mathbb{P}\left(\overline{Y}_T > x\right) = \exp\left(-\int_0^x \frac{\mathrm{d}s(z)}{s(z) - s(z - \theta(z))}\right). \tag{4.4}$$

In particular, when $\theta(x) \equiv a$, with $a > 0$, we have

$$\mathbb{P}\left(\overline{Y}_T > x\right) = \exp\left(-\int_0^x \frac{\mathrm{d}s(z)}{s(z) - s(z-a)}\right).$$

Proof. First, we note that

$$\overline{Y}_T - Y_T = \theta\left(\overline{Y}_T\right).$$

Indeed, considering the process $K_t = \frac{\overline{Y}_t - Y_t}{\theta(\overline{Y}_t)}$, we have $T = \inf\{t : K_t = 1\}$, and each time $\theta\left(\overline{Y}_t\right)$ jumps corresponds to an increase time for $\overline{Y}_t$, and since at such a time $\overline{Y}_t - Y_t = 0$, K is in fact continuous and $K_T = 1$.

Now, let us define X of the class (Σ) by

$$X_t = s\left(\overline{Y}_t\right) - s(Y_t).$$

From the remark above, we have

$$X_T = s\left(\overline{Y}_T\right) - s\left(\overline{Y}_T - \theta\left(\overline{Y}_T\right)\right).$$

Now, with the notations of Theorem 4.1, we have

$$\lambda(x) = \mathbb{E}\left[X_T | \overline{Y}_T = x\right] = x - s\left(s^{-1}(x) - \theta\left(s^{-1}(x)\right)\right),$$

and consequently, since $\lambda(x) > 0$, from Theorem 4.1, we have

$$\mathbb{P}\left(s\left(\overline{Y}_T\right) > x\right) = \exp\left(-\int_0^x \frac{\mathrm{d}z}{z - s\left(s^{-1}(z) - \theta\left(s^{-1}(z)\right)\right)}\right),$$

and thus

$$\mathbb{P}\left(\overline{Y}_T > x\right) = \exp\left(-\int_0^{s(x)} \frac{\mathrm{d}z}{z - s\left(s^{-1}(z) - \theta\left(s^{-1}(z)\right)\right)}\right).$$

Now, making the change of variable $z = s(u)$ gives the desired result. □

Now, if Y is of the form

$$dY_t = b(Y_t)\,dt + \sigma(Y_t)\,dB_t,$$

where B is a standard Brownian Motion, and the coefficients b and σ are chosen such that uniqueness, existence and recurrence hold, then

$$s(x) = \int_0^x dy \exp(-\beta(y)),$$

with

$$\beta(x) = 2\int_0^x dy \frac{b(y)}{\sigma^2(y)}.$$

Under these assumptions, (4.4) takes the following form:

Corollary 4.12 (*Lehoczky [9]*). *With the assumptions and notation of Proposition 4.11, and the notation above, we have*

$$\mathbb{P}\left(\overline{Y}_T > x\right) = \exp\left(-\int_0^x \frac{\exp(-\beta(z))\,dz}{\int_{z-\theta(z)}^z \exp(-\beta(u))\,du}\right). \tag{4.5}$$

In the special case when $\theta(x) \equiv a$, with $a > 0$, we have

$$\mathbb{P}\left(\overline{Y}_T > x\right) = \exp\left(-\int_0^x \frac{\beta(z)\,dz}{\int_{z-a}^z \beta(u)\,du}\right). \tag{4.6}$$

To conclude this section, we mention a class of stochastic processes, which are solutions of a stochastic differential equation of Skorokhod type, with reflecting boundary condition at 0, and which look very similar to the local submartingales of class (Σ). These processes play an important role in the extension of Pitman's theorem to one-dimensional diffusion processes in Saisho and Tanemura's work [18]; they also appeared earlier in the works of Chaleyat-Maurel and El Karoui (see their paper in [1]).

More precisely, let $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ be a filtered probability space and (B_t) an $(\mathcal{F}_t)$ Brownian Motion. Let $\sigma, b : \mathbb{R} \to \mathbb{R}$ be Lipschitz continuous functions, and assume that $\sigma(x) > 0, \forall x \in \mathbb{R}$. Consider the stochastic differential equation of Skorokhod type:

$$Y_t = \int_0^t \sigma(Y_u + L_u)\,dB_u + \int_0^t b(Y_u + L_u)\,du + L_t, \tag{4.7}$$

where Y and L should be found under the conditions:

- Y is $(\mathcal{F}_t)$, continuous and $Y_t \geq 0$;
- L is continuous, nondecreasing, $L_0 = 0$ and $L_t = \int_0^t \mathbf{1}_{\{0\}}(Y_u)\,dL_u$.

It is proved in [18] that this equation has a unique solution. Although Y is not a process of the class (Σ), we shall apply the previous methods to compute the law of L conveniently stopped.

We let, as before, s denote the function

$$s(x) = \int_0^x dy \exp(-\beta(y)),$$

with

$$\beta(x) = 2\int_0^x \mathrm{d}y \frac{b(y)}{\sigma^2(y)}.$$

If (Y_t, L_t) is the solution of (4.7), define

$$X_t = s(Y_t + L_t) - s(L_t),$$
$$N_t = \int_0^t \sigma s'(Y_u + L_u)\,\mathrm{d}B_u,$$
$$A_t = s(L_t).$$

We shall need the following lemma:

Lemma 4.13. *The following equality holds:*

$$X_t = N_t + L_t,$$

and X is of the class (Σ).

Proof. An application of Itô's formula yields

$$s(Y_t + L_t) = \int_0^t \sigma s'(Y_u + L_u)\,\mathrm{d}B_u + 2\int_0^t s'(Y_u + L_u)\,\mathrm{d}L_u. \tag{4.8}$$

Now, since $(\mathrm{d}L_t)$ is carried by the set of zeros of Y, we have

$$\int_0^t s'(Y_u + L_u)\,\mathrm{d}L_u = \int_0^t s'(L_u)\,\mathrm{d}L_u = s(L_t) = A_t,$$

and consequently, from (4.8), we have

$$s(Y_t + L_t) - s(L_t) = \int_0^t \sigma s'(Y_u + L_u)\,\mathrm{d}B_u + s(L_t),$$

that is:

$$X_t = N_t + L_t.$$

It follows easily from the fact that s is continuous and strictly increasing that A is increasing and that $(\mathrm{d}A_t)$ is carried by the set of zeros of X. Hence X is of the class (Σ). □

Now, we can state an analogue of Lehoczky's result for the pair (Y_t, L_t):

Proposition 4.14. *Let T be the stopping time*

$$T \equiv \inf\{t : Y_t \geq \theta(L_t)\},$$

where $\theta : \mathbb{R}_+ \to \mathbb{R}_+$ is a Borel function such that $\theta(x) > 0, \forall x \geq 0$. Then, if $T < \infty$, a.s.,

$$\mathbb{P}(L_T > x) = \exp\left(-\int_0^x \frac{\exp(-\beta(z))\,\mathrm{d}z}{\int_z^{z+\theta(z)} \exp(-\beta(u))\,\mathrm{d}u}\right).$$

In the special case $\theta(x) \equiv a$, for some $a > 0$, we have

$$\mathbb{P}(L_T > x) = \exp\left(-\int_0^x \frac{\exp(-\beta(z))\,\mathrm{d}z}{\int_z^{z+a} \exp(-\beta(u))\,\mathrm{d}u}\right).$$

Proof. The proof follows exactly the same lines as the proof of Proposition 4.11, so we just give the main steps. Here again, we have

$$Y_T = \theta(L_T),$$

and consequently,

$$X_T = s(\theta(L_T) + L_T) - s(L_T),$$

and hence

$$\lambda(x) = s\left(s^{-1}(x) + \theta\left(s^{-1}(x)\right)\right) - x.$$

The end of the proof is now exactly the same as that of Proposition 4.11. □

Remark 4.15. If the ratio $\frac{b(x)}{\sigma^2(x)} \equiv \gamma$ is constant, then L_T, with $T \equiv \inf\{t : Y_t \geq a\}$, is exponentially distributed:

$$\mathbb{P}(L_T > x) = \exp\left(-\frac{2\gamma x}{1 - \exp(-2\gamma a)}\right).$$

5. The Skorokhod embedding problem for non-atomic probability measures on $\mathbb{R}_+$

The previous results can now be used to solve the Skorokhod stopping problem for non-atomic probability measures on $\mathbb{R}_+$. The literature on the Skorokhod stopping problem is vast (see [14]) and the aim here is not to give very specialized results on this topic, but rather to illustrate a general methodology which allows one to deal with a wide variety of stochastic processes, which may even be discontinuous. The reader interested in solving the problem for measures which have atoms can refer to the recent paper of Obłój and Yor [15], which inspired the ideas in the following, and which explains how to deal with such probability measures. Quite remarkably, the stopping times they propose for the reflected Brownian Motion and the age process of Brownian excursions (hence solving explicitly for the first time the Skorokhod embedding problem for a discontinuous process) can be used to solve the Skorokhod stopping problem for local submartingales of the class (Σ), with only negative jumps and with $A_\infty = \infty$. In particular, we will be able to solve the Skorokhod stopping problem for any Bessel process of dimension $2(1 - \mu)$, with $\mu \in (0, 1)$, and its corresponding age process. In [15], the authors mention that they have presented detailed arguments using excursion theory, but parallel arguments, using martingale theory, are possible as well. As far as we are concerned here, we shall detail arguments based on martingale theory, to show that the stopping times they propose have a universal feature (independent of any Markov or scaling property).

More precisely, let ϑ be a probability measure on $\mathbb{R}_+$, which has no atoms, and let X be a local submartingale of the class (Σ), with only negative jumps and such that $\lim_{t\to\infty} A_t = \infty$. Our aim is to find a stopping time T_ϑ, such that the law of X_{T_ϑ} is ϑ, and which coincides with the stopping time proposed by Obłój and Yor [15] when $X_t = B_t$ or $X_t = \sqrt{t - g_{1/2}(t)}$.

In the following, we write $\overline{\vartheta}(x) = \vartheta([x, \infty))$ for the tail of ϑ, and $a_\vartheta = \sup\left\{x \geq 0 : \overline{\vartheta}(x) = 1\right\}$, and $b_\vartheta = \inf\left\{x \geq 0 : \overline{\vartheta}(x) = 0\right\}$, $-\infty \leq a_\vartheta \leq b_\vartheta \leq \infty$, respectively, for the lower and upper bounds of the support of ϑ. Now, following [15], we introduce the dual Hardy–Littlewood function $\psi_\vartheta : [0, \infty) \to [0, \infty)$ through

$$\psi_\vartheta(x) = \int_{[0,x]} \frac{z}{\overline{\vartheta}(z)} \mathrm{d}\vartheta(z), \quad a_\vartheta \leq x < b_\vartheta,$$

and $\psi_\vartheta(x) = 0$ for $0 \leq x < a_\vartheta$, and $\psi_\vartheta(x) = \infty$ for $x \geq b_\vartheta$. The function ψ_ϑ is continuous and increasing, and we can define its right continuous inverse

$$\varphi_\vartheta(z) = \inf\{x \geq 0 : \psi_\vartheta(x) > z\},$$

which is strictly increasing. Now, we can state the main result of this subsection:

Theorem 5.1. *Let X be a local submartingale of the class (Σ), with only negative jumps and such that $A_\infty = \infty$. The stopping time*

$$T_\vartheta = \inf\{x \geq 0 : X_t \geq \varphi_\vartheta(A_t)\} \tag{5.1}$$

is a.s. finite and solves the Skorokhod embedding problem for X, i.e. the law of X_{T_ϑ} is ϑ.

Proof. Parts of the arguments that follow are tailored on those of Obłój and Yor [15], but at some stage, they use time change techniques, which would not deal with the case of negative jumps.

First, we note that (see [15])

$$\int_0^x \frac{dz}{\varphi_\vartheta(z)} < \infty, \qquad \text{for } 0 \leq x < b_\vartheta$$
$$\int_0^\infty \frac{dz}{\varphi_\vartheta(z)} = \infty,$$

and consequently, from Theorem 3.2 and Remark 3.5, $T_\vartheta < \infty$, a.s. and $X_{T_\vartheta} = \varphi_\vartheta\left(A_{T_\vartheta}\right)$.

Now, let $h : \mathbb{R}_+ \to \mathbb{R}_+$ be a strictly decreasing function, locally bounded, and such that $\int_0^\infty dz h(z) = \infty$. From Theorem 2.1, $h(A_t) X_t$ is again of the class (Σ), and its increasing process is $\int_0^{A_t} dz h(z) \equiv H(A_t)$. Now, define the stopping time:

$$R_h = \inf\{t : h(A_t) X_t = 1\},$$

which is finite almost surely from Theorem 3.2. Since $\lim_{t\to\infty} H(A_t) = \infty$, from Corollary 4.7, $H\left(A_{R_h}\right)$ is distributed as a random variable $\mathbf{e}$ which follows the standard exponential law, and hence

$$A_{R_h} \overset{\text{(law)}}{=} H^{-1}(\mathbf{e}),$$

and consequently

$$X_{R_h} \overset{\text{(law)}}{=} \frac{1}{h\left(H^{-1}(\mathbf{e})\right)}. \tag{5.2}$$

Now, we investigate the converse problem; that is, given a probability measure ϑ, we want to find h such that $X_{R_h} \overset{\text{(law)}}{=} \vartheta$. From (5.2), we deduce (recall that ϑ has no atoms)

$$\begin{aligned}
\overline{\vartheta}(x) &= \mathbb{P}\left(\frac{1}{h\left(H^{-1}(\mathbf{e})\right)} > x\right) = \mathbb{P}\left(h\left(H^{-1}(\mathbf{e})\right) < \frac{1}{x}\right) \\
&= \mathbb{P}\left(H^{-1}(\mathbf{e}) > h^{-1}\left(\frac{1}{x}\right)\right) \\
&= \mathbb{P}\left(\mathbf{e} > H\left(h^{-1}\left(\frac{1}{x}\right)\right)\right) = \exp\left(-H\left(h^{-1}\left(\frac{1}{x}\right)\right)\right).
\end{aligned} \tag{5.3}$$

Now, differentiating the last equality yields

$$-\mathrm{d}\overline{\vartheta}(x) = \overline{\vartheta}(x)\left[h\left(h^{-1}\left(\frac{1}{x}\right)\right)\right]\mathrm{d}\left(h^{-1}\left(\frac{1}{x}\right)\right) = \frac{\overline{\vartheta}(x)}{x}\mathrm{d}\left(h^{-1}\left(\frac{1}{x}\right)\right),$$

and hence

$$\mathrm{d}\left(h^{-1}\left(\frac{1}{x}\right)\right) = x\frac{\mathrm{d}\vartheta(x)}{\overline{\vartheta}(x)}. \tag{5.4}$$

Consequently, we have

$$h^{-1}\left(\frac{1}{x}\right) = \int_0^x \frac{z}{\overline{\vartheta}(z)}\mathrm{d}\vartheta(z) = \psi_\vartheta(x),$$

and

$$R_h = \inf\{t : X_t \geq \varphi_\vartheta(A_t)\} = T_\varphi,$$

and the proof of the theorem follows easily. □

Remark 5.2. One can easily check that

$$X_{t\wedge T\vartheta} \text{ is uniformly integrable } \Leftrightarrow \int_0^\infty x\mathrm{d}\vartheta(x) < \infty.$$

Theorem 5.1 shows that the stopping times proposed by Obłój and Yor for the pair $(|B_t|, \ell_t)$ and $\left(\sqrt{\frac{\pi}{2}\left(t - g_{1/2}(t)\right)}, \ell_t\right)$ have a universal aspect in the sense that they apply to a very wide class of processes. For example, Theorem 5.1 provides us with a solution for the Skorokhod stopping problem for Bessel processes of dimension $2(1-\mu)$, with $\mu \in (0,1)$, whilst these processes are not semimartingales for $\mu > 1/2$. With Theorem 5.1, we can also solve the Skorokhod stopping problem for the age processes associated with those Bessel processes. In fact, the Skorokhod embedding problem is solved for powers of these processes, but one can then easily deduce the stopping time for the process itself. More precisely:

Corollary 5.3. *Let (R_t) be a Bessel process, starting from 0, of dimension $2(1-\mu)$, with $\mu \in (0,1)$, and (L_t) its local time. Let ϑ be a non-atomic probability measure on $\mathbb{R}_+$; then the stopping time*

$$T_\vartheta = \inf\left\{x \geq 0 : R_t \geq \varphi_\vartheta^{1/2\mu}(L_t)\right\}$$

is a.s. finished and solves the Skorokhod embedding problem for $R_t^{2\mu}$. In particular, when $\mu = 1/2$, we obtain the stopping time proposed by Obłój and Yor for $|B_t|$.

Corollary 5.4. *With the notation and assumptions of the previous corollary, let*

$$g_\mu(t) \equiv \sup\{u \leq t : R_u = 0\}.$$

Recall that in the filtration $\left(\mathcal{G}_t \equiv \mathcal{F}_{g_\mu(t)}\right)$ of the zeros of the Bessel process R, $\left(t - g_\mu(t)\right)^\mu$, is a submartingale of class (Σ) whose increasing process in its Doob–Meyer decomposition is given *by* $A_t \equiv \frac{1}{2^{\mu}\Gamma(1+\mu)} L_t$*. Consequently, the stopping time*

$$T_{\vartheta} = \inf\left\{x \geq 0 : t - g_{\mu}(t) \geq \varphi_{\vartheta}^{1/\mu}\left(\frac{1}{2^{\mu}\Gamma(1+\mu)} L_t\right)\right\}$$

is a.s. finished and solves the Skorokhod embedding problem for $\left(t - g_{\mu}(t)\right)^{\mu}$*. In particular, when* $\mu = 1/2$*, we obtain the stopping time proposed by Obłój and Yor for* $\sqrt{t - g_{1/2}(t)}$*, the age process of a standard Brownian Motion.*

Remark 5.5. Our methodology does not allow us to solve the Skorokhod stopping problem for local submartingales of the class (Σ) which have positive jumps. It would thus be interesting to discover a method which would lead us to a solution for these processes.

One can also use the ideas contained in the proof of Theorem 5.1 to give another family of stopping times which solve the Skorokhod stopping problem. The stopping time we shall propose was obtained first by Obłój and Yor for the absolute value of the standard Brownian Motion. Here again, we shall prove that this stopping time solves the Skorokhod embedding problem for a much wider class of stochastic processes. Indeed, if in the proof of Theorem 5.1 we had supposed h strictly increasing instead of decreasing, then (5.3) would have equalled $\vartheta(x) \equiv \vartheta([0, x])$ instead of $\overline{\vartheta}(x)$ and (5.4) would read

$$\mathrm{d}\left(h^{-1}\left(\frac{1}{x}\right)\right) = -x\frac{\mathrm{d}\vartheta(x)}{\vartheta(x)}.$$

Therefore, we have the following proposition:

Proposition 5.6. *Let X be a local submartingale of the class (Σ), with only negative jumps and with $A_{\infty} = \infty$. Let ϑ be a probability measure on $\mathbb{R}_+$, without atoms, and define*

$$\overline{\psi}_{\vartheta}(x) = \int_{(x,\infty)} \frac{x\mathrm{d}\vartheta(x)}{\vartheta(x)},$$

and

$$\overline{\varphi}_{\vartheta}(x) = \inf\left\{z \geq 0 : \overline{\psi}_{\vartheta}(z) > x\right\}, \quad x < b_{\vartheta},$$

and $\overline{\varphi}_{\vartheta}(x) = 0, \forall x \geq b_{\vartheta}$*. Then the stopping time*

$$\overline{T}_{\vartheta} = \inf\left\{t \geq 0 : X_t \geq \overline{\varphi}_{\vartheta}(A_t)\right\}$$

embeds ϑ*, i.e.*

$$X_{\overline{T}_{\vartheta}} \overset{(\text{law})}{=} \vartheta.$$

Remark 5.7. Here again, the stopping time $\overline{T}_{\vartheta}$ embeds ϑ for the Bessel processes of dimension $2(1-\mu)$, $\mu \in (0, 1)$, and the corresponding age processes. The fact that $\overline{T}_{\vartheta}$ works for the Brownian age process was not noticed by Obłój and Yor.

Let us conclude this subsection with two simple examples.

Example 5.8 (*Exponential Law*). Let $\vartheta(\mathrm{d}x) = \rho \exp(-\rho x)\mathbf{1}_{\mathbb{R}_+}(x)\,\mathrm{d}x$, $\rho > 0$. We then have

$$\varphi_{\vartheta}(x) = \sqrt{\frac{2}{\rho}x}.$$

The stopping time takes the following form:

$$T_{\vartheta} = \inf \left\{ t : X_t \geq \sqrt{\frac{2}{\rho} A_t} \right\}.$$

Example 5.9 (*Uniform Law*). Let $\vartheta\,(\mathrm{d}x) = \frac{1}{b} \mathbf{1}_{[0,b]}\,(x)\,\mathrm{d}x$. Some simple calculations give

$$\overline{\psi}_{\vartheta}\,(x) = (b - x)\,,$$

and therefore

$$\overline{T_{\vartheta}} = \inf \{ t : X_t + A_t \geq b \}\,.$$

The reader can easily check that the corresponding formula for T_{ϑ} is not so simple.

Acknowledgement

I am very grateful to my supervisor Marc Yor for many helpful discussions and for correcting earlier versions of this paper.

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