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INVARIANT KNOTS OF FREE INVOLUTIONS OF S^4

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We show that knots invariant under a free involution of S^4 are unique up to equivariant concordance. We also give an easy construction of the unique non-smooth homotopy $\mathbb{R}P^4$.

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free involutions equivariant concordance
fake RP^4.

Introduction

The purpose of this paper is to investigate free involutions on S^4 and 2-spheres invariant under these actions. One main result is roughly that such knots are unique up to equivariant concordance. The other is a construction of the (unique) non-smoothable free involution on S^4.

Analogous questions have been extensively studied in higher dimensions using the methods of surgery theory [1, 8, 13]. Our method is to employ the viewpoint and some of the techniques of surgery theory, substituting four-dimensional ideas where the high-dimensional ones do not apply directly.

1. Uniqueness up to concordance

Most of the results are equally valid in the topological and smooth (= PL) categories. For simplicity we will generally stick to the smooth case and refer to the topological case only when there is a distinction to be made. Suppose $t:\Sigma^4\to\Sigma^4$ is a free involution on a homotopy sphere, and suppose K_0 and K_1 are 2-spheres invariant under t. It is well-known that K_0 and K_1 are concordant, i.e., that there is an imbedding of $S^2\times I$ in $\Sigma\times I$ giving $K_0\cup K_1$ on the boundary.

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Definition. K_0 and K_1 are *equivariantly concordant* if there is a concordance in $\Sigma \times I$ invariant under $t \times \mathrm{id}_I$ whose boundary is $K_0 \cup K_1$.

There is an equivalent notion which is easier to work with. Note that $\Sigma / t \simeq \mathbb{R}P^4$, and that $K_i / t \cong \mathbb{R}P^2$.

Definition (cf. [1]). A surface homeomorphic to $\mathbb{R}P^2$ in a homotopy $\mathbb{R}P^4$ carrying the correct homology class is called a *characteristic* $\mathbb{R}P^2$. A concordance of characteristic $\mathbb{R}P^2$'s is an imbedding of $\mathbb{R}P^2 \times I$ in $P^4 \times I$.

In the topological category we require that a characteristic $\mathbb{R}P^2$ or a concordance be locally flat. This is the same as requiring them to have normal bundles by [9] or [10] for surfaces in 4-manifolds and [7] in high dimensions. It is not hard to show that invariant knots in Σ are equivariantly concordant if and only if the corresponding characteristic $\mathbb{R}P^2$'s are concordant.

Our main theorem is the analogue in dimension four of a result of Cappell and Shaneson [1].

Theorem. *Let F_0 and F_1 be characteristic $\mathbb{R}P^2$'s of the homotopy projective space P^4. Then F_0 and F_1 are concordant.*

Equivalently, invariant knots of the same involution are equivariantly concordant. The outline of the proof follows [1]: One first shows that there is a cobordism V^5 containing a concordance $C \cong \mathbb{R}P^2 \times I$, with $\partial(V, C) = (P, F_0) \cup (P, F_1)$. Then one can do surgery on V (rel ∂V) and missing C to get an s-cobordism W containing C. At this point one would like to appeal to the s-cobordism theorem to make W a product. Of course the smooth s-cobordism theorem is not known in general, so we prove a special case of it.

Theorem 2. *Let W^5 be a smooth s-cobordism with both ends diffeomorphic to P, a given homotopy $\mathbb{R}P^4$. Then $W \cong P \times I$.*

The proof is an extension of the technique [11, 13] that gives the analogous result for S^4 or $S^3 \times S^1$. Theorems 1 and 2 both depend on being able to construct non-trivial self-homotopy equivalences of $\mathbb{R}P^2 \times I$ and $\mathbb{R}P^4 \times I$.

Lemma 3. *For $k = 2$ or 4, there is a self-homotopy equivalence G of $\mathbb{R}P^k \times I$ with $G | \mathbb{R}P^k \times \partial I = \mathrm{id}$ and with (the unique) nontrivial normal invariant in $[\mathbb{R}P^k \times I/\partial; G/PL] = [\mathbb{R}P^k \times I/\partial; G/TOP] = Z_2$.*

Proof of Lemma 3. For $k = 2$, define G by

$$\mathbb{R}P^2 \times I \to \mathbb{R}P^2 \times I \vee S^3 \xrightarrow{\mathrm{id} \vee g} \mathbb{R}P^2 \times I,$$

where the first map collapses the boundary of a 3-cell and $g: S^3 \to \mathbb{R}P^2 \times 1/2$ is the Hopf map $\gamma: S^3 \to S^2$ followed by the covering projection $\pi: S^2 \to \mathbb{R}P^2$. The normal invariant $\eta(G) \in [\mathbb{R}P^2 \times I/\partial; G/PL] = [\mathbb{R}P^2 \times I/\partial; G/TOP] = H^1(\mathbb{R}P^2; Z_2) = Z_2$ is detected by the Arf invariant of the induced normal map $G^{-1}(\mathbb{R}P^1 \times I) \to \mathbb{R}P^2 \times I$. G is already transverse to $\mathbb{R}P^1 \times I$, and $G^{-1}(\mathbb{R}P^1 \times I) = \mathbb{R}P^1 \times I \cup g^{-1}(\mathbb{R}P^1 \times \frac{1}{2})$ and the Arf invariant comes from $g^{-1}(\mathbb{R}P^1 \times \frac{1}{2})$. Now $g^{-1}(\mathbb{R}P^1 \times \frac{1}{2}) = \gamma^{-1}\pi^{-1}(\mathbb{R}P^1 \times \frac{1}{2})$ is a torus in S^3, where one factor, say $S^1 \times \text{pt.} = \gamma^{-1}$ (pt. on equator of S^2) is a fiber in the Hopf fibration, and the other factor pt. $\times S^1$ maps onto the equator of S^2 and hence double covers $\mathbb{R}P^1 \times \frac{1}{2}$.

To calculate the Arf invariant of this torus one must see the framing of the stable normal bundle that the torus acquires via the bundle map which covers $g|T^2$. It is easy to see the framing of the normal bundle on the Hopf fiber $S^1 \times$ pt.—it is exactly the framing given by the normal vectors perpendicular to T^2 in S^3, suitably stabilized. (See Fig. 1.) This framing is the non-trivial one (even stably) because it links $S^1 \times$ pt. an odd number of times.

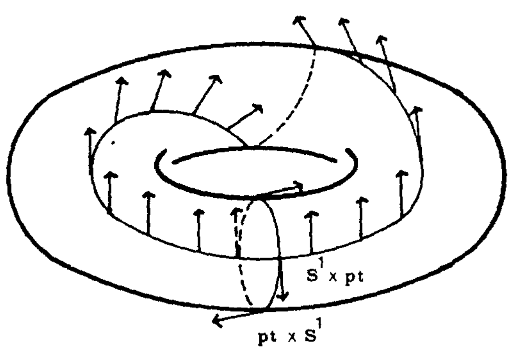

Fig. 1.

It is harder to see this normal framing on the other circle pt. $\times S^1$ because it only exists stably. So look instead at the map of stable tangent bundles; it covers the double covering map $S^1 \to S^1$ and hence must be the non-trivial (S^1-invariant) framing on pt. $\times S^1$. (See [2] for a similar argument.) Hence the normal framing is also non-trivial and the Arf invariant is non-zero.

For $k = 4$, the argument is much the same. G is defined by

$$\mathbb{R}P^4 \times I \to \mathbb{R}P^4 \times I \vee S^5 \xrightarrow{\text{id} \vee g} \mathbb{R}P^4 \times I$$

where $g: S^5 \to \mathbb{R}P^4 \times \frac{1}{2}$ is $\pi \circ (\Sigma^2\gamma)$ and $\Sigma^2\gamma$ generates $\pi_5(S^4)$. The normal invariant of G is detected by the Arf invariant of $G^{-1}(\mathbb{R}P^1 \times I)$ which again comes from a torus, now in S^5. The framing of this torus is essentially the suspension of the framing of the analogous torus in S^3, hence its Arf invariant is non-zero.

Note that in both cases $G|\partial(\mathbb{R}P^k \times I)$ is the identity, by construction.

Proof of Theorem 2 (We follow the argument in [11] and [13], Chapter 16). Let W^5 be an s-cobordism from P to itself. The idea is to show that W is s-cobordant (rel ∂W) to $P \times I$ using the surgery exact sequence and then to apply the 6-dimensional (relative) s-cobordism theorem. $Wh(Z_2) = 0$ so we never have to worry about torsion.

An s-cobordism (W, P, P) determines a homotopy equivalence $F:(W, P, P) \to (P \times I, P, P)$. Since $\mathbb{R}P^4$ has no self-homotopy equivalence except the identity the same is true for P, so we can assume that $F|\partial W = \text{id}$. Consider F as in $\mathcal{S}(P \times I, \partial)$ and look at the surgery sequence

$$\begin{array}{ccccccc} L_6(Z_2^-) & \xrightarrow{\partial} & \mathcal{S}(P \times I, \partial) & \xrightarrow{\eta} & [P \times I/\partial;\, G/PL] & \to & L_5(Z_2^-) = 0 \\ \| & & & & \| & & \\ Z_2 & & & & Z_2. & & \end{array}$$

Now $L_6(Z_2^-) = \text{im}(L_6(e))$ [13] and hence operates trivially on $\mathcal{S}(P \times I, \partial)$, so that $\eta : \mathcal{S}(P \times I, \partial) \xrightarrow{\cong} Z_2$. If F has trivial normal invariant, then it is s-cobordant to the identity and hence W is s-cobordant to $P \times I$, and so diffeomorphic to $P \times I$ by the 6-dimensional s-cobordism theorem. Suppose F has non-trivial normal invariant. By the lemma, we can vary F by a homotopy equivalence as follows:

$$W \xrightarrow{F} P \times I \xrightarrow{p} \mathbb{R}P^4 \times I \xrightarrow{G} \mathbb{R}P^4 \times I \xrightarrow{q} P \times I.$$

Here p is a homotopy equivalence and q its inverse, and G is the map from Lemma 3. We can fix up the composition to be the identity on ∂W without affecting the normal invariant and obtain a new homotopy equivalence $F^1 : (W, \partial W) \to (P \times I, \partial)$ with trivial normal invariant. In any event, W is s-cobordant to and hence diffeomorphic to $P \times I$.

Remark. The theorem and its proof work for a topological s-cobordism.

Proof of Theorem 1. We are given K_0 and K_1 which are characteristic $\mathbb{R}P^2$'s in P^4. Choose homotopy equivalences $f_i : P \to \mathbb{R}P^4$ such that $f_i \pitchfork \mathbb{R}P^2$ and $f_i^{-1}(\mathbb{R}P^2) = K_i$. Extend the f_i to a homotopy equivalence $F:(P \times I, \partial) \to (\mathbb{R}P^4 \times I, \partial)$ transverse to $\mathbb{R}P^2 \times I$ and let $U^3 = F^{-1}(\mathbb{R}P^2 \times I)$. By the lemma, $F|U$ is normally cobordant to a homotopy equivalence $\mathbb{R}P^2 \times I \to \mathbb{R}P^2 \times I$. Using the normal cobordism extension principle, we obtain a normal map $G:(V^5, \partial) \to (\mathbb{R}P^4 \times I, \partial)$ such that $\partial V = P \cup P$, $G|\partial V = F|\partial(P \times I)$ and $G^{-1}(\mathbb{R}P^2 \times I) \cong \mathbb{R}P^2 \times I$. Now the argument follows [1] exactly: Try to do surgery on V^5 away from $\mathbb{R}P^2 \times I$ and ∂V. There is a single obstruction in a group $\Gamma_5(Z[Z] \to Z[Z_2])$ which fortunately vanishes [1]. Hence we can surger V to obtain an s-cobordism containing $\mathbb{R}P^2 \times I$ with $\partial(\mathbb{R}P^2 \times I) = K_0 \cup K_1$. By Theorem 2, this is a product and we are done.

Theorem 1 can be applied to many examples of characteristic $\mathbb{R}P^2$'s in both the smooth and the topological cases. The Cappell–Shaneson fake $\mathbb{R}P^4$'s ([2]) all have characteristic $\mathbb{R}P^2$'s by construction but since it is not known if all these $\mathbb{R}P^4$'s are diffeomorphic, one cannot conclude that the knots are smoothly concordant.

The analysis in [13], (Chapter 14) shows that a homotopy RP^4 is determined up to s-cobordism by its normal invariant in $[RP^4, G/PL]$ or $[RP^4, G/TOP]$ and that there is a single non-trivial normal cobordism class in each group with vanishing surgery obstruction. So the Cappell–Shaneson RP^4's are all smoothly s-cobordant. Further, the map $\pi_4(G/PL) \to \pi_4(G/TOP)$ is $Z \xrightarrow{\times 2} Z$, so that any smooth homotopy RP^4 is topologically s-cobordant to RP^4. In Section 2, we construct a non-smooth homotopy RP^4, completing the topological classification. The recent work of Freedman [5] includes the topological s-cobordism theorem for finite fundamental groups, so that all smooth RP^4's are homeomorphic to RP^4. Using Theorem 1, we see that there is a topological concordance in a topological s-cobordism from knots in the Cappell–Shaneson RP^4's to the standard RP^2 in RP^4. The discussion above yields the following corollary to Theorem 1.

Corollary 4. *The Cappell–Shaneson $\mathbb{R}P^2$'s in $\mathbb{R}P^4$ are topologically concordant to the standard $\mathbb{R}P^2$ in $\mathbb{R}P^4$.*

Interesting smooth characteristic $\mathbb{R}P^2$'s in the standard $\mathbb{R}P^4$ are a little harder to come by. Motivated by Theorem 1, we give a construction for many such $\mathbb{R}P^2$'s. The construction starts with a knot $K \subset S^3$ which is $(-)$ amphicheiral. That is to say there is an orientation reversing diffeomorphism ϕ from S^3 to itself taking K to K and reversing the orientation of K. We can assume that ϕ is the antipodal map on some small unknotted ball pair in (S^3, K) so that ϕ gives an orientation reversing diffeomorphism of a pair (B^3, β) whose restriction to $\partial(B^3, \beta)$ is the antipodal map. Ignoring β, the map ϕ is isotopic to the antipodal map, so that $S^1 \times_\phi B^3 \cup D^2 \tilde{\times} P^2$ is diffeomorphic to $\mathbb{R}P^4$. Here $D^2 \tilde{\times} \mathbb{R}P^2$ is the normal disc bundle of the usual $\mathbb{R}P^2$ in $\mathbb{R}P^4$. Since ϕ preserves β, the Möbius band $S^1 \times_\phi \beta$ can be capped off with a D^2-fiber in $D^2 \tilde{\times} \mathbb{R}P^2$ to form a characteristic $\mathbb{R}P^2$ in $\mathbb{R}P^4$. More generally, the arc β can be twisted k times as B^3 goes around $S^1 \times_\phi B^3$, and then flipped over by ϕ.

Definition. This $\mathbb{R}P^2$ is the (k, ϕ)-twist spin of K (or the k-flip–twist spin of K) and is denoted $\phi_k(K)$.

The properties of flip–twist spinning are similar to those of ordinary twist spinning, and are summarised in the following theorem.

Theorem 5. *The (k, ϕ)-twist spin of K has the following properties:*

1) *If $k \neq 0$, then $\phi_k(K)$ is fibered, i.e., $\mathbb{R}P^4$-Neighborhood $(\phi_k(K))$ is a bundle over S^1 whose boundary is the usual fibering by S^2's of $\partial(D^2 \tilde{\times} \mathbb{R}P^2)$ over S^1.*

2) *The fiber is the punctured $2k$-fold cover of S^3 branched along K and the monodromy is a lift of ϕ to the branched cover whose square is the canonical $2k$-fold covering translation.*

3) *The double cover of $(\mathbb{R}P^4, \phi_k(K))$ is the $2k$-twist-spin of K.*

Note that by 3), the ± 1 flip–twist–spin is not unknotted. These facts can be proved following the proofs of Zeeman [14] for the analogous statements about twist-spinning. The third follows from the first two but can be seen directly by noting that in the double cover the arc β twists $2k$-times as it goes around $S^1 \times_{\phi^2} B^3 = S^1 \times B^3$.

Example. The simplest knot to use for K is the figure eight knot, drawn below in B^3. The symmetry ϕ can be taken to be the antipodal map on B^3 in this picture. The knot $\phi_1(K)$ can be given an alternative description: The figure eight knot is the rational knot $\frac{2}{5}$, and so has double branched cover lens space $L(5,2)$. Since $2^2 \equiv -1 \pmod 5$, there is an orientation reversing diffeomorphism t on $L(5,2)$ such that $t_* =$ multiplication by 2 on $H_1(L)$. One can verify that $S^1 \times_t L_0 \cup D^2 \tilde{\times} \mathbb{R}P^2$ is a homotopy $\mathbb{R}P^4$ with a characteristic $\mathbb{R}P^2$ in it. The diffeomorphism t is exactly the lift of the exhibited symmetry of the figure eight knot, so that this is indeed the knot $\phi_1(K)$ in the real $\mathbb{R}P^4$.

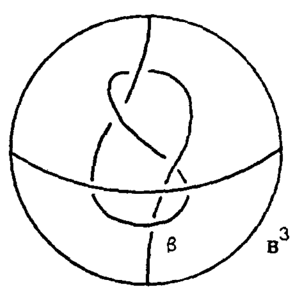

Fig. 2.

It would be interesting to see an explicit concordance of these flip–twist–spun knots to the standard $\mathbb{R}P^2$ in order to avoid the machinery that goes into Theorem 1. Similarly, it would be nice to find a proof of Theorem 1 based on ambient surgery analogous to the proof that knots in S^4 are slice.

2. A non-smooth real projective space

The recent work of Freedman [5] on non-simply connected topological 4-manifolds shows that surgery theory works for manifolds with finite π_1. A calculation with the surgery exact sequence ([13], Chapter 14) shows the existence of a unique non-smooth homotopy $\mathbb{R}P^4$. In this section we give a direct construction of this manifold analogous to the construction of smooth $\mathbb{R}P^4$'s by Fintushel and Stern [3].

The construction starts with a homology 3-sphere Σ^3 with the following properties:

1) $\mu(\Sigma) = 1$.
2) Σ admits a free orientation preserving involution t.
3) $\Sigma = \partial\Delta^4$ for Δ a contractible topological 4-manifold.

The Brieskorn sphere $\Sigma(5, 7, 11)$ will do for Σ. The involution t is contained in the circle action on $\Sigma(5, 7, 11)$ and is free because 5, 7 and 11 are all odd, and one easily calculates $\mu(\Sigma(5, 7, 11)) = 1$ via link calculus. The last condition comes for free as Freedman [4] has shown that every homology 3-sphere bounds a contractible manifold.

Following [3], write $S^4 = \Delta \cup_t \Delta$; there is then a free involution on S^4 which swaps the copies of Δ, so that the quotient P ($\cong \Delta/t$) is a homotopy $\mathbb{R}P^4$.

Intuitively, the μ-invariant of Σ is what causes P to be non-smooth, but showing this requires a little work because of the absence of a Rochlin-type theorem for non-orientable four-manifolds. The point is to identify the Kirby–Siebenmann obstruction to smoothing P with the μ-invariant of Σ even though P is non-orientable. Indeed our calculation in effect gives some version of a non-orientable Rochlin's theorem involving the μ-invariant of the double cover of a 3-manifold M^3 dual to w_1, the Arf invariant of a surface dual to w_2, and the signature of the 4-manifold cut open along M.

Proposition 6. *Let M^4 be a topological 4-manifold with the following structure: $M^4 = E^4 \cup \Delta^4$ where E^4 is smooth, Δ is contractible, and $\Sigma^3 = \partial E = \partial\Delta$ is a homology sphere with $\mu(\Sigma) = 1$. Then M is not smoothable.*

Proof. Let y be a point in Δ, so that $\Delta - y$ has a smooth structure inducing the unique structure Σ_{std} on its boundary. (Since Δ is contractible, one does not need to know the homotopy groups of TOP_4/O_4.) This puts a smooth structure on $(M-y)\times\mathbb{R}$, containing $\Sigma_{\text{std}}\times\mathbb{R}$ as a smooth submanifold. Now smooth structures on $(M-y)\times\mathbb{R}$ are classified [6] by $H^3((M-y)\times\mathbb{R})$ which may well be non-trivial. (Z_2-coefficients are understood here and for the rest of the proof.) If all the smooth structures contained $\Sigma_{\text{std}}\times\mathbb{R}$ as a smooth submanifold, we would be done, for then a smoothing of M induces one on $M\times\mathbb{R}$ which make $\Sigma_{\text{std}}\times\mathbb{R}$ the boundary of a smooth contractible 5-manifold homeomorphic to $\Delta\times\mathbb{R}$. Since $\mu(\Sigma) = 1$, this is impossible [12].

The proposition is thus proved by noticing that the smooth structures on $(M-y)\times\mathbb{R}$ all arise from structures on $(E\times\mathbb{R}, \Sigma_{\text{std}}\times\mathbb{R})$. This is because the latter are given by $H^3(E\times\mathbb{R}, \Sigma\times\mathbb{R}) \cong H^3((M-y)\times\mathbb{R}, (\Delta-y)\times\mathbb{R})$ and $r^*: H^3((M-y)\times\mathbb{R}, (\Delta-y)\times\mathbb{R}) \to H^3((M-y)\times\mathbb{R})$ is onto. The naturality of the Kirby–Siebenmann obstruction implies that that one can change smooth structure on $(M-y)\times\mathbb{R}$ by changing it on $E\times\mathbb{R}$ keeping $\Sigma_{\text{std}}\times\mathbb{R}$ a smooth submanifold.

Remark. It is an open question whether the Fintushel–Stern fake $\mathbb{R}P^4$'s admit characteristic $\mathbb{R}P^2$'s, so that it seems natural to wonder whether the non-smooth $\mathbb{R}P^4$ has a topological characteristic $\mathbb{R}P^2$. We cannot decide this at present, but merely remark that the splitting obstruction ([1], Section 8) to finding an $\mathbb{R}P^2$ vanishes. This can be used to show that $P^4 \times S^1$ admits an $\mathbb{R}P^2 \times S^1$, so by unwrapping the S^1, there is a characteristic $\mathbb{R}P^2 \times \mathbb{R}$ in $P^4 \times \mathbb{R}$.

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References

[1] S. Cappell and J. Shaneson, The codimension two placement problem and homology equivalent manifolds, Ann. Math. 99 (1974), 277–348.
[2] S. Cappell and J. Shaneson, Some new four-manifolds, Ann. Math. 104 (1976), 61–72.
[3] R. Fintushel and R. Stern, An exotic free involution on S^4, Ann. Math. 113 (1981), 357–365.
[4] M. Freedman, The topology of four-dimensional manifolds, J. Diff. Geo. 17 (1982), 357–453.
[5] M. Freedman, Lecture, Institute for Advanced Study, January 1983.
[6] R. Kirby and L. Siebenmann, Foundational essays on topological manifolds, smoothings, and triangulations, Annals of Mathematics Study 88 (Princeton University Press, Princeton, N.J., 1977).
[7] R. Kirby and L. Siebenmann, Normal bundles for codimension 2 locally flat imbeddings, in: L.C. Glaser, ed., Geometric Topology, Lecture Notes in Mathematics 438 (Springer, New York, 1975), 310–324.
[8] S. Lopez de Medrano, Involutions on Manifolds (Springer-Verlag, New York, 1971).
[9] F. Quinn, personal communication.
[10] D. Ruberman, Locally flat surfaces in four-manifolds have normal bundles, preprint (1983).
[11] J. Shaneson, Non-simply-connected surgery and some results in low dimensional topology, Comm. Math. Helv., 45 (1970), 333–352.
[12] L. Siebenmann, Disruption of low-dimensional handlebody theory by Rochlin's theorem, in: J.C. Cantrell and C.H. Edwards, eds., Topology of Manifolds (Markham, Chicago, 1970), 57–76.
[13] C.T.C. Wall, Surgery on Compact Manifolds (Academic Press, New York, 1971).
[14] E. Zeeman, Twisting spin knots, Trans. A.M.S. 115 (1965), 471–495.