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Sequences of labeled trees related to Gelfand–Tsetlin patterns ☆

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ABSTRACT

By rewriting the famous hook-content formula it easily follows that there are $\prod_{1 \leqslant i < j \leqslant n} \frac{k_j - k_i + j - i}{j - i}$ semistandard tableaux of shape $(k_n, k_{n-1}, \ldots, k_1)$ with entries in $\{1, 2, \ldots, n\}$ or, equivalently, Gelfand–Tsetlin patterns with bottom row $(k_1, \ldots, k_n)$. In this article we introduce certain sequences of labeled trees, the signed enumeration of which is also given by this formula. In these trees, vertices as well as edges are labeled, the crucial condition being that each edge label lies between the vertex labels of the two endpoints of the edge. This notion enables us to give combinatorial explanations of the shifted antisymmetry of the formula and its polynomiality. Furthermore, we propose to develop an analogous approach of combinatorial reasoning for monotone triangles and explain how this may lead to a combinatorial understanding of the alternating sign matrix theorem.

1. Introduction

One possible way to see that the expression

$$\prod_{1 \leqslant i < j \leqslant n} \frac{k_j - k_i + j - i}{j - i} \tag{1.1}$$

is an integer for any choice of $(k_1, \ldots, k_n) \in \mathbb{Z}^n$ is to find combinatorial objects that are enumerated by this quantity. If $k_1 \leqslant k_2 \leqslant \cdots \leqslant k_n$, this is, for instance, accomplished by *Gelfand–Tsetlin patterns*

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with prescribed bottom row $k_1, k_2, \dots, k_n$. A Gelfand–Tsetlin pattern (see [13, p. 313] or [9, (3)] for the first appearance) is a triangular array of integers with n rows of the following shape

$$\begin{array}{ccccccccccc}
 & & & & & a_{1,1} & & & & & \\
 & & & & a_{2,1} & & a_{2,2} & & & & \\
 & & & \dots & & \dots & & \dots & & & \\
 & & a_{n-2,1} & & \dots & & \dots & & a_{n-2,n-2} & & \\
 & a_{n-1,1} & & a_{n-1,2} & & \dots & & \dots & & a_{n-1,n-1} & \\
a_{n,1} & & a_{n,2} & & a_{n,3} & & \dots & & \dots & & a_{n,n}
\end{array}$$

that is monotone increasing along northeast diagonals and southeast diagonals, i.e. $a_{i,j} \leqslant a_{i-1,j}$ for $1 \leqslant j < i \leqslant n$ and $a_{i,j} \leqslant a_{i+1,j+1}$ for $1 \leqslant j \leqslant i < n$. It is conceivable to assume that $(k_1, \dots, k_n) \in \mathbb{Z}^n_{\geqslant 0}$, as Gelfand–Tsetlin patterns with bottom row $(k_1, \dots, k_n)$ are obviously in bijective correspondence with Gelfand–Tsetlin patterns with bottom row $(k_1 + t, \dots, k_n + t)$ for any integer $t \in \mathbb{Z}$. Under this assumption, they are equivalent to *semistandard tableaux* of shape $(k_n, k_{n-1}, \dots, k_1)$ with entries in $\{1, 2, \dots, n\}$, the latter being fillings of the Ferrers diagram associated with the integer partition $(k_n, k_{n-1}, \dots, k_1)$ that are weakly increasing along rows and strictly increasing along columns.[1] Next we give an example of a Gelfand–Tsetlin pattern and the corresponding semistandard tableaux.

$$\begin{array}{ccccccccccc}
 & & & & & 2 & & & & & \\
 & & & & 2 & & 2 & & & & \\
 & & & 1 & & 2 & & 4 & & & \\
 & & 1 & & 1 & & 3 & & 4 & & \\
 & 0 & & 1 & & 3 & & 3 & & 5 & \\
0 & & 0 & & 2 & & 3 & & 5 & & 6
\end{array}
\qquad\qquad
\begin{array}{cccccc}
1 & 1 & 3 & 3 & 5 & 6 \\
2 & 2 & 4 & 6 & 6 & \\
3 & 5 & 5 & & & \\
4 & 6 & & & &
\end{array}$$

In general, given a Gelfand–Tsetlin pattern $(a_{i,j})_{1 \leqslant j \leqslant i \leqslant n}$, the corresponding semistandard tableau is constructed by placing the integer i in the cells of the skew shape

$$(a_{i,i}, a_{i,i-1}, \dots, a_{i,1}) / (a_{i-1,i-1}, a_{i-1,i-2}, \dots, a_{i-1,1}).$$

Semistandard tableaux of fixed shape (and thus Gelfand–Tsetlin patterns) are known to be enumerated by the hook-content formula [13, Corollary 7.21.4], which is easily seen to be equivalent to (1.1), see also [13, Lemma 7.21.1]. A common way to prove this formula is to translate the problem into the enumeration of families of non-intersecting lattice paths with a certain set of fixed starting points and end points. To complement the treatment given in this article, we sketch this point of view in Appendix A. A direct proof of the fact that Gelfand–Tsetlin patterns with bottom row $k_1, k_2, \dots, k_n$ are enumerated by (1.1) can be found in [4, Section 5]. There we have actually proven a more general result, which we describe in the following paragraph.

The reader will have noticed that the combinatorial interpretations that we have given so far only provide an explanation for the integrality of (1.1) if the sequence $k_1, k_2, \dots, k_n$ is weakly increasing. This can be overcome[2] by extending the combinatorial interpretation of Gelfand–Tsetlin patterns with bottom row $(k_1, \dots, k_n)$ to all n-tuples of integers $(k_1, \dots, k_n)$ and working with a signed enumeration as follows: a *generalized Gelfand–Tsetlin pattern* is an array of integers $(a_{i,j})_{1 \leqslant j \leqslant i \leqslant n}$ such that the following condition is fulfilled: for any $a_{i,j}$ with $1 \leqslant j \leqslant i \leqslant n-1$ we have $a_{i+1,j} \leqslant a_{i,j} \leqslant a_{i+1,j+1}$ if

[1] Note that there is actually no dependency between the number of feasible values for the entries of the semistandard tableaux and the number of parts in the integer partition: semistandard tableaux of shape $(k_m, k_{m-1}, \dots, k_1)$ with entries in $\{1, 2, \dots, n\}$ are equivalent to semistandard tableaux of shape $(k_m, k_{m-1}, \dots, k_1, 0^{n-m})$ with entries in $\{1, 2, \dots, n\}$ if $n \geqslant m$ and there exists no semistandard tableau otherwise if $n < m$ and $k_1 \neq 0$.

[2] Of course, this also follows by choosing a permutation $\sigma \in \mathcal{S}_n$ with $k_{\sigma_1} + \sigma_1 \leqslant k_{\sigma_2} + \sigma_2 \leqslant \dots \leqslant k_{\sigma_n} + \sigma_n$ and then observing that $\prod_{1 \leqslant i < j \leqslant n} \frac{k_{\sigma_j} - k_{\sigma_i} + \sigma_j - \sigma_i}{j-i} = \operatorname{sgn} \sigma \prod_{1 \leqslant i < j \leqslant n} \frac{k_j - k_i + j - i}{j-i}$ is the number of Gelfand–Tsetlin patterns with bottom row $(k_{\sigma_1} + \sigma_1 - 1, k_{\sigma_2} + \sigma_2 - 2, \dots, k_{\sigma_n} + \sigma_n - n)$.

$a_{i+1,j} \leqslant a_{i+1,j+1}$ and $a_{i+1,j} > a_{i,j} > a_{i+1,j+1}$ if $a_{i+1,j} > a_{i+1,j+1}$. (In particular, there exists no generalized Gelfand–Tsetlin pattern with $a_{i+1,j} = a_{i+1,j+1} + 1$.) In the latter case we say that $a_{i,j}$ is an *inversion*. The *weight* (or *sign*) of a given Gelfand–Tsetlin pattern is $(-1)^{\#\text{ of inversions}}$. With this, (1.1) is the signed enumeration of all Gelfand–Tsetlin patterns with bottom row $k_1, k_2, \ldots, k_n$.

The main task of the present paper is to provide a whole family of sets of objects that come along with a rather canonical notion of a sign, the signed enumeration of each of these sets is given by (1.1). We call these objects *Gelfand–Tsetlin tree sequences* as Gelfand–Tsetlin patterns are one special member of this family. The definition of these objects is given in Section 2. This enables us to give a combinatorial proof of the fact that Gelfand–Tsetlin patterns are enumerated by (1.1). Interestingly, this combinatorial proof is not based on a bijection between Gelfand–Tsetlin patterns and a second type of objects which are more easily seen to be enumerated by (1.1). Rather than that we give combinatorial proofs of the facts that the replacement $(k_i, k_j) \to (k_j + j - i, k_i - i + j)$ in the enumeration formula for the number of Gelfand–Tsetlin patterns with prescribed bottom row only causes the inversion of the sign (Section 3) as well as that the enumeration formula must be a polynomial in $(k_1, \ldots, k_n)$ of degree no greater than $n-1$ in every k_i (Section 4). For each of these properties, this is accomplished by providing an appropriate member of the family for which the respective property is almost obvious. Then, it is not hard to see that these properties essentially determine the enumeration formula, which is the only algebraic part of the proof. Note that the first property can obviously only be understood combinatorially after having extended the combinatorial interpretation of Gelfand–Tsetlin patterns with bottom row $k_1, k_2, \ldots, k_n$ to arbitrary $(k_1, k_2, \ldots, k_n) \in \mathbb{Z}^n$ as the sequence $k_1, \ldots, k_{i-1}, k_j + j - i, k_{i+1}, \ldots, k_{j-1}, k_i + i - j, k_{j+1}, \ldots, k_n$ cannot be weakly increasing if $k_1, k_2, \ldots, k_n$ is weakly increasing. Also the inversion of the sign surely indicates that a signed enumeration must be involved.

However, the original motivation for this paper is the intention to translate some of the research we have done on *monotone triangles* into a more combinatorial setting. Monotone triangles are Gelfand–Tsetlin patterns with strictly increasing rows and their significance is due to the fact that they are in bijective correspondence with alternating sign matrices when prescribing $1, 2, \ldots, n$ as bottom row. It took a lot of effort to enumerate $n \times n$ alternating sign matrices and all proofs known so far cannot be considered as combinatorial proofs as they usually involve heavy algebraic manipulations, see [2]. Also the long-standing "Gog–Magog conjecture" [11], which is a generalization of the fact that $n \times n$ alternating sign matrices are in bijective correspondence with $2n \times 2n \times 2n$ totally symmetric self-complementary plane partitions is still unsolved, which is another indication for the fact that alternating sign matrices (as well as plane partitions) are combinatorial objects that are rather persistent against combinatorial reasonings. However, partial progress in this direction is accomplished in [1,3,14].

Our own proof of the alternating sign matrix theorem [6] makes us believe that it could be helpful to work with signed enumerations: let $\alpha(n; k_1, \ldots, k_n)$ denote the number of monotone triangles with bottom row $k_1, \ldots, k_n$. The key identity in this proof is the following:

$$\alpha(n; k_1, \ldots, k_n) = (-1)^{n-1} \alpha(n; k_2, \ldots, k_n, k_1 - n). \tag{1.2}$$

Obviously, this identity does not make any sense at first as $k_2, k_3, \ldots, k_n, k_1 - n$ is not strictly increasing if $k_1, k_2, \ldots, k_n$ is strictly increasing. However, it is not hard to see that, for fixed n, the quantity $\alpha(n; k_1, \ldots, k_n)$ can in fact be represented by a (unique) polynomial in $k_1, \ldots, k_n$ and so (1.2) can be understood as an identity for this polynomial. On the other hand, it is also possible to give $\alpha(n; k_1, \ldots, k_n)$ a combinatorial interpretation for all $(k_1, \ldots, k_n) \in \mathbb{Z}^n$ in terms of a signed enumeration. We have provided such an interpretation in [7] and provide two additional interpretations in the concluding section of this article. These extensions provide combinatorial interpretations of (1.2) and to give also a combinatorial proof of this identity could be an important step towards a combinatorial understanding of the alternating sign matrix theorem as we explain in Section 6. It is hoped that a combinatorial proof of this identity as well as of other interesting identities involving monotone triangles follows the same lines as the combinatorial reasonings we present in this article for Gelfand–Tsetlin patterns.

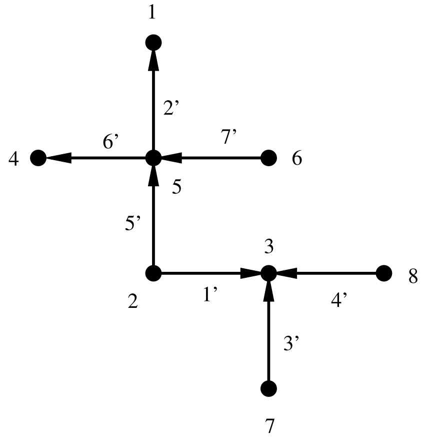

Fig. 1. An 8-tree.

2. Definition of Gelfand–Tsetlin tree sequences

In this paper, an *n-tree* is a directed tree with n vertices such that the vertices are identified with integers in $\{1, 2, \ldots, n\}$ and the edges are identified with primed integers in $\{1', 2', \ldots, (n-1)'\}$. In Fig. 1, we give an example of an 8-tree. We consider sequences of trees: a *tree sequence of order n* is a sequence of trees $\mathcal{T} = (T_1, T_2, \ldots, T_n)$ such that T_i is an i-tree for each i, see Fig. 4 for an example of order 5. Each member of the family, the signed enumeration of which is given by (1.1), will have a fixed underlying tree sequence of order n. The actual objects will be certain admissible labelings (vertices and edges are labeled with integers; the labels must not be confused with the fixed "names" of the vertices and edges, which are chosen once and for all from the sets $\{1, 2, \ldots, n\}$ and $\{1', 2', \ldots, (n-1)'\}$, respectively) of the underlying tree sequence. Gelfand–Tsetlin patterns will be one member of this family; in the underlying tree sequence $\mathcal{B} = (B_1, B_2, \ldots, B_n)$, the i-trees B_i are paths with the canonical labeling, i.e. $j' = (j, j+1) \in E(B_i)$ for $j = 1, 2, \ldots, i-1$. (See Fig. 3 (left) for the case $n = 6$.) In the following, the tree B_i will be referred to as the *basic i-tree*.

We work towards defining *admissible labelings* of tree sequences.

Definition 1. Let T be an n-tree and $\mathbf{k} = (k_1, \ldots, k_n) \in \mathbb{Z}^n$. A vector $\mathbf{l} = (l_1, \ldots, l_{n-1}) \in \mathbb{Z}^{n-1}$ is said to be admissible for the pair $(T, \mathbf{k})$ if for each edge $j' = (p, q)$ of T the following is fulfilled: if $k_p + p < k_q + q$ then $k_p + p \leqslant l_j + j < k_q + q$ and otherwise $k_q + q \leqslant l_j + j < k_p + p$. In the latter case we say that the edge j' is an inversion of the pair $(T, \mathbf{k})$.

Phrased differently, if we label vertex i with $k_i + i$ and edge j' with $l_j + j$ for all i and j then, for each edge, the edge label is greater than or equal to the minimum of the two vertex labels on the endpoints of the edge but smaller than the maximum. The edge is an inversion if it is directed from the maximum vertex label to the minimum vertex label. If, for an edge, the label of the tail coincides with the label of the head then there exists no vector $\mathbf{l}$ that is admissible for the pair $(T, \mathbf{k})$. In the following, we address the vectors $\mathbf{k} + (1, 2, \ldots, n)$ and $\mathbf{l} + (1, 2, \ldots, n-1)$ as the vertex labeling, respectively edge labeling of the tree and the vectors $\mathbf{k}$ and $\mathbf{l}$ as the shifted labelings.

For instance, consider the 8-tree T in Fig. 1 and the vector $\mathbf{k} = (4, 1, 7, 2, 4, 2, 6, 1) \in \mathbb{Z}^8$. Then the vector $\mathbf{l} = (6, 3, 9, 5, 1, 2, 1)$ is admissible for $(T, \mathbf{k})$, see Fig. 4. We suppress the "names" of the vertices and edges in order to avoid a confusion with the labelings. However, the labelings of vertices and edges are split into two summands in such a way that the "names" are just the second summands of the labelings. The inversions are $2', 3', 6'$. Also observe that there is no admissible shifted labeling $\mathbf{l}$ if $\mathbf{k} = (4, 1, 7, 2, 4, 2, 6, 2)$ as there is no l_4 with $2 + 8 \leqslant l_4 + 4 < 7 + 3$.

Now we are in the position to define *Gelfand–Tsetlin tree sequences*.

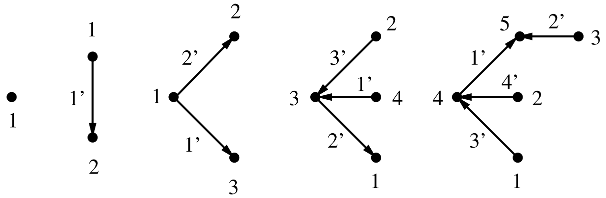

Fig. 2. A tree sequence of order 5.

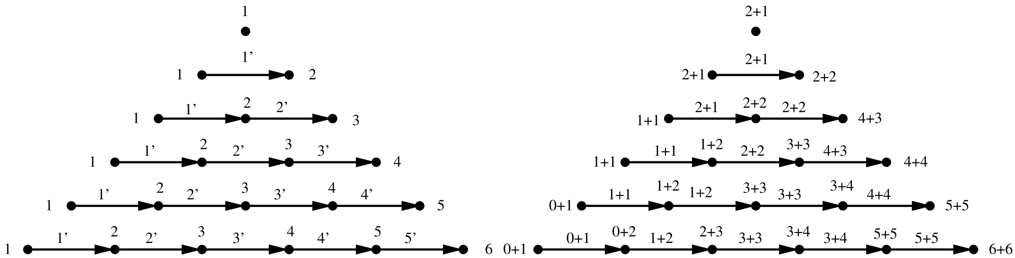

Fig. 3. Tree sequence for Gelfand–Tsetlin patterns of order 6 and an example of an admissible labeling.

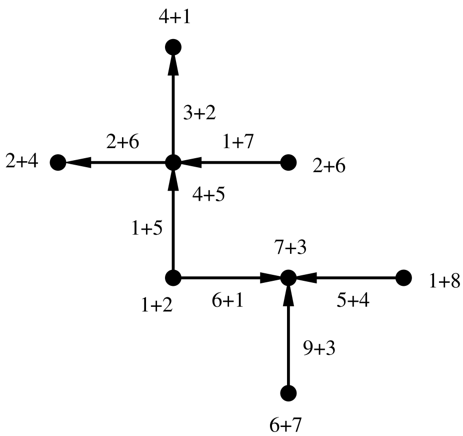

Fig. 4. An example of an admissible labeling.

Definition 2. A Gelfand–Tsetlin tree sequence associated with a tree sequence $\mathcal{T} = (T_1, \ldots, T_n)$ of order n and a shifted labeling $\mathbf{k} \in \mathbb{Z}^n$ of the vertices of T_n is a sequence $(\mathbf{l}_1, \mathbf{l}_2, \ldots, \mathbf{l}_n)$ of vectors $\mathbf{l}_i \in \mathbb{Z}^i$ with $\mathbf{l}_n = \mathbf{k}$ such that $\mathbf{l}_{i-1}$ is admissible for the pair $(T_i, \mathbf{l}_i)$ if $i = 2, 3, \ldots, n$. We let $\mathcal{L}_n(\mathcal{T}, \mathbf{k})$ denote the set of these Gelfand–Tsetlin tree sequences.

In Fig. 5, we give an example of a Gelfand–Tsetlin tree sequence associated with the tree sequence displayed in Fig. 2. Observe that $\mathbf{k} = (5, 6, 3, -3, 0)$ in this case. An edge label is displayed in italic type if the corresponding edge is an inversion. In Fig. 3 (right), we display the Gelfand–Tsetlin pattern from Section 1 as a Gelfand–Tsetlin tree sequence associated with $(B_1, B_2, B_3, B_4, B_5, B_6)$. Again the labelings are split into two summands in such a way that the first summands corresponds to the respective entry of the Gelfand–Tsetlin pattern in the introduction, while the second summands are just the names of the vertices and edges, respectively.

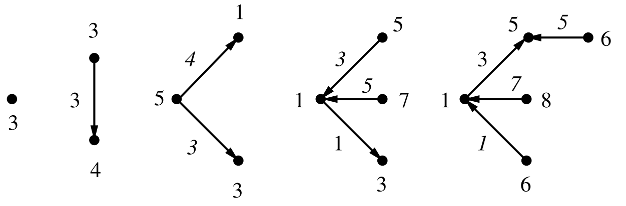

Fig. 5. A Gelfand–Tsetlin tree sequence.

We give a preliminary definition of the sign of a Gelfand–Tsetlin tree sequence: the inversions of a Gelfand–Tsetlin tree sequence are the inversions of the pairs $(T_i, \mathbf{l}_i)$ for $i = 2, 3, \ldots, n$ and the sign is defined as $(-1)^{\#\text{ of inversions}}$. The (preliminary) sign of the Gelfand–Tsetlin tree sequence given in Fig. 5 is -1 as there are 7 inversions. We will see that the signed enumeration of Gelfand–Tsetlin tree sequences associated with a fixed tree sequence $\mathcal{T} = (T_1, \ldots, T_n)$ of order n and a fixed shifted labeling $\mathbf{k} = (k_1, \ldots, k_n)$ of the vertices of T_n is, up to a sign, equal to (1.1). This sign only depends on the underlying unlabeled tree sequence $\mathcal{T}$ and will be defined next. After that we adjust the definition of the sign of a Gelfand–Tsetlin tree sequence by multiplying this global sign.

For this purpose, we define the *sign of an n-tree T*: fix a *root vertex r* of the tree. The *standard orientation* with respect to this root is the orientation in which each edge is oriented away from the root; these orientations transform the tree into an arborescence. An edge in T is said to be a *reversed* edge if its orientation does not coincide with the standard orientation. If, in our example in Fig. 1, we choose 2 to be the root then the reversed edges are $3'$, $4'$ and $7'$. Except for the root, each vertex is the head of a unique edge with respect to the standard orientation. We obtain a permutation π of $\{1, 2, \ldots, n\}$, if we order the head vertices of the edges in accordance with their edge names (i.e. for the edges $i' = (a, b)$ and $j' = (c, d)$ with $i < j$, the vertex b comes before vertex d in the permutation) and prepend the root r at the beginning of the permutation. In our running example, we obtain the permutation $\pi = 2\,3\,1\,7\,8\,5\,4\,6$. Then the sign of T is defined as follows

$$\operatorname{sgn} T = (-1)^{\#\text{ of reversed edges}} \operatorname{sgn} \pi. \tag{2.1}$$

The sign of the tree in Fig. 1 is 1 as there are 3 reversed edges and $\operatorname{sgn} \pi = -1$.

We need to show that the sign does not depend on the choice of the root: suppose s is a vertex adjacent to the root r. If we change from root r to root s, we have to interchange r and s in the permutation π, which reverses the sign of π. This is because the standard orientation with respect to the root s coincides with the standard orientation with respect to the root r except for the edge incident with r and s, where the orientation is reversed. For the same reason, shifting the root from r to s, either increases or decreases the number of reversed edges by 1. Consequently, the product in (2.1) remains unaffected.

The *sign of a tree sequence* $\mathcal{T} = (T_1, T_2, \ldots, T_n)$ is defined as the product of the signs of the i-trees in the sequence, i.e.

$$\operatorname{sgn} \mathcal{T} = \operatorname{sgn} T_1 \cdot \operatorname{sgn} T_2 \cdots \operatorname{sgn} T_n.$$

The sign of the tree sequence in Fig. 2 is 1 as $\operatorname{sgn} T_1 = 1$, $\operatorname{sgn} T_2 = 1$, $\operatorname{sgn} T_3 = -1$, $\operatorname{sgn} T_4 = 1$, and $\operatorname{sgn} T_5 = -1$. Concerning Gelfand–Tsetlin patterns we obviously have $\operatorname{sgn} B_i = 1$, which implies $\operatorname{sgn} \mathcal{B} = 1$.

Here is the final definition of the *sign of a Gelfand–Tsetlin tree sequence* $\mathbf{L} = (\mathbf{l}_1, \mathbf{l}_2, \ldots, \mathbf{l}_n) \in \mathcal{L}_n(\mathcal{T}, \mathbf{k})$:

$$\operatorname{sgn} \mathbf{L} = (-1)^{\#\text{ of inversions of } \mathbf{L}} \cdot \operatorname{sgn} \mathcal{T}.$$

The signed enumeration of elements in $\mathcal{L}_n(\mathcal{T}, \mathbf{k})$ is denote by $L_n(\mathcal{T}, \mathbf{k})$. The sign of the Gelfand–Tsetlin tree sequence given in Fig. 5 is -1 as there are 7 inversions and the sign of the underlying unlabeled tree sequence is 1. We are in the position to state an important result of this paper.

Theorem 1. *The signed enumeration of Gelfand–Tsetlin tree sequences associated with a fixed underlying unlabeled tree sequence $\mathcal{T} = (T_1, \dots, T_n)$ of order n and a shifted labeling $\mathbf{k} = (k_1, \dots, k_n)$ of the vertices of T_n is given by*

$$\prod_{1 \leqslant i < j \leqslant n} \frac{k_j - k_i + j - i}{j - i}.$$

Before we turn our attention to searching for properties of $L_n(\mathcal{T}, \mathbf{k})$ that determine this quantity uniquely, we want to mention an obvious generalization of Gelfand–Tsetlin tree sequences, which we do not consider in this article, but might be interesting to look at: the notion of admissibility makes perfect sense if the tree T is replaced by any other graph. Are there any nice assertions to be made on "Gelfand–Tsetlin graph sequences"?

3. Properties of $L_n(\mathcal{T}, \mathbf{k})$: independence and shift-antisymmetry

We say that a function $f(k_1, \dots, k_n)$ on $\mathbb{Z}^n$ is *shift-antisymmetric* iff

$$f(k_1, \dots, k_n) = -f(k_1, \dots, k_{i-1}, k_j + j - i, k_{i+1}, \dots, k_{j-1}, k_i + i - j, k_{j+1}, \dots, k_n)$$

for all i, j with $1 \leqslant i < j \leqslant n$ and all $(k_1, \dots, k_n) \in \mathbb{Z}^n$. In this section we prove by induction with respect to n that the signed enumeration $L_n(\mathcal{T}, \mathbf{k})$ has the following two properties.

- *Independence*: $L_n(\mathcal{T}, \mathbf{k})$ does not depend on the tree sequence $\mathcal{T}$.
- *Shift-antisymmetry*: $L_n(\mathcal{T}, \mathbf{k})$ is shift-antisymmetric in $\mathbf{k} = (k_1, \dots, k_n)$. In fact, we prove the following stronger result: fix i, j with $1 \leqslant i < j \leqslant n$. We construct a tree sequence of order n, denoted by $\mathcal{S}_n^{i,j}$, and an associated sign reversing involution on the set of Gelfand–Tsetlin tree sequences of the tree sequence $\mathcal{S}_n^{i,j}$ such that the shifted vertex labeling $\mathbf{k} \in \mathbb{Z}^n$ of the largest tree is transformed into

$$E_{k_j}^{j-i} E_{k_i}^{i-j} S_{k_i,k_j} \mathbf{k} = (k_1, \dots, k_{i-1}, k_j + j - i, k_{i+1}, \dots, k_{j-1}, k_i + i - j, k_{j+1}, \dots, k_n),$$

  where $S_{x,y} f(x, y) = f(y, x)$ and $E_x p(x) = p(x + 1)$.

The proofs are combinatorial in the following sense: suppose we are given two sets A and B and a signed enumeration $|.|_-$ on each of the sets such that $|A|_- = |B|_-$. Then we find decompositions of A and B into two sets A_1, A_2 and B_1, B_2, respectively, such that there is a sign preserving bijection between A_1 and B_1 and $|A_2|_- = |B_2|_- = 0$, where the latter identities are proven by giving sign reversing involutions on A_2 and B_2, respectively. However, if we have $|A|_- = -|B|_-$ then the bijection between A_1 and B_1 is sign reversing.

Observe that there is nothing to prove for $n = 1$. We deal with the independence first. The strategy is as follows: we define two operations on m-trees that allow us to transform any m-tree into any other m-tree (Lemma 1), and then show that $L_n(\mathcal{T}, \mathbf{k})$ is invariant under the application of the two operations to the trees in the tree sequence $\mathcal{T}$ (Lemma 2). The two operations are as follows:

- *Reversing*: the first operations simply allows us to *reverse the orientation of any edge* in an m-tree.
- *Sliding*: for this operation we assume $m \geqslant 3$. It is illustrated in Fig. 6 and defined as follows: suppose that i' and j' are two edges in the m-tree T_m that have a vertex q in common. Let T'_m be the tree we obtain from T_m by replacing vertex q in i' with the vertex of j' which is different from q. Then we say that T'_m is obtained from T_m by *sliding edge i' along edge j'*.

Lemma 1. *Every m-tree can be obtained from every other m-tree by means of the two operations "reversing" and "sliding". Concerning the sign of the m-tree, reversing the orientation of an edge changes the sign, while sliding an edge along another edge leaves the sign invariant.*

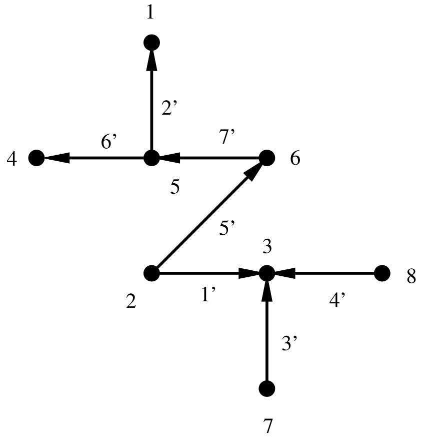

Fig. 6. The 8-tree is obtained from the 8-tree in Fig. 1 by sliding edge $5'$ along edge $7'$.

Proof. We start by showing the assertions on the sign. It is obvious for reversing the orientation of an edge. Suppose we let edge i' slide along edge j' and denote by q the vertex that have i' and j' in common. In the following argument, we let p be the vertex of i' in T_m that is different from q and r be the vertex of j' that is different from q. Choose q to be the root of the tree. The head of the old edge i' (i.e. in T_n) as well as of the new edge i' (i.e. in T'_n) is p with respect to the standard orientation. Moreover, the edge i' is reversed in T_n if and only if it is reversed in T'_n. There is no change for the remaining edges, since the standard orientation does not change there. Hence, neither the permutation π nor the set of reversed edges is changed.

As both operations are in fact involutions, in order to prove the first assertion it suffices to show that every m-tree can be transformed into the basic m-tree B_m. First of all, it is obvious that sliding and reversing can be used to transform a given m-tree into a directed path. Hence, it suffices to show that it is possible to interchange vertices as well as edges. In both cases, it suffices to consider adjacent vertices, respectively edges. Concerning edges, suppose x' and y' are adjacent edges. By possibly reversing the orientation of one edge, we may assume without loss of generality that $x' = (a,b)$ and $y' = (b,c)$. Then the following sequence of operations interchanges the edges:

$$\begin{aligned} &x' = (a,b),\ y' = (b,c) \to x' = (a,c),\ y' = (b,c) \to x' = (a,c),\ y' = (b,a) \\ &\quad \to x' = (b,c),\ y' = (b,a) \to x' = (b,c),\ y' = (a,b). \end{aligned}$$

(Note that all operations except for the last are slides, which implies that interchanging edges reverses the sign of the m-tree.) Concerning swapping vertices, assume that we want to interchange vertex a and b and that $x' = (a,b)$ is an edge. We reverse the orientation of x' and slide all edges incident with a but different from x' along x' to b as well as all edges incident with b but different from x' along x' to a. (Again we see that swapping vertices reverses the sign.) $\square$

Lemma 2. *The independence and shift-antisymmetry for order $n-1$ implies the independence for order n.*

Proof. For a tree sequence $\mathcal{T} = (T_1, T_2, \ldots, T_n)$ of order n we have

$$L_n(\mathcal{T}, \mathbf{k}) = \operatorname{sgn} T_n \cdot (-1)^{\#\text{ of inversions of }(T_n,\mathbf{k})} \sum_{\mathbf{l} \in \mathbb{Z}^{n-1} \text{ is admissible for } (T_n,\mathbf{k})} L_{n-1}(\mathcal{T}_{<n}, \mathbf{l}),$$

where $\mathcal{T}_{<n} = (T_1, T_2, \ldots, T_{n-1})$. The independence for $n-1$ implies that $L_n(\mathcal{T}, \mathbf{k})$ is invariant under the replacement of $\mathcal{T}_{<n}$ by any other tree sequence of order $n-1$. We have to show that it is also

invariant under the replacement of T_n by any other n-tree. By Lemma 1, it suffices to show this invariance for the two tree operations.

Let T_n be an n-tree and T'_n be an n-tree which is obtained from T_n by reversing the orientation of a single edge. Then $\operatorname{sgn} T_n = -\operatorname{sgn} T'_n$, the number of inversions of $(T_n, \mathbf{k})$ differs from the number of inversions of $(T'_n, \mathbf{k})$ by 1 and $\mathbf{l} \in \mathbb{Z}^{n-1}$ is admissible for $(T_n, \mathbf{k})$ if and only if $\mathbf{l}$ is admissible for $(T'_n, \mathbf{k})$. This implies that $L_n(\mathcal{T}, \mathbf{k})$ is invariant under the replacement of T_n by T'_n.

In order to show that $L_n(\mathcal{T}, \mathbf{k})$ is invariant under the replacement of T_n by T'_n, we first note that $\operatorname{sgn} T_n = \operatorname{sgn} T'_n$ by Lemma 1. We have to distinguish between the six possibilities for the relative positions of $k_p + p, k_q + q, k_r + r$. As we have a symmetry between vertex q and vertex r we may assume without loss of generality that $k_q + q \leqslant k_r + r$. We let $\mathcal{T}'$ denote the tree sequence that we obtain from $\mathcal{T}$ by replacing T_n by T'_n.

Case 1. $k_p + p \leqslant k_q + q \leqslant k_r + r$: we decompose $\mathcal{L}_n(\mathcal{T}', \mathbf{k})$ into two sets as follows. Let $\mathbf{l} \in \mathbb{Z}^{n-1}$ be an admissible shifted edge labeling of T'_n. The first set contains the Gelfand–Tsetlin tree sequences where the label of edge i' fulfills $k_p + p \leqslant l_i + i < k_q + q$, whereas for the second set we have $k_q + q \leqslant l_i + i \leqslant k_r + r$. The signed enumeration of the first set is obviously equal to $\mathcal{L}_n(\mathcal{T}, \mathbf{k})$, since the edge i' is an inversion of T_n if and only if it is an inversion of T'_n. We have to show that the signed enumeration of the second set reduces to zero: we replace $\mathcal{T}_{<n}$ by $\mathcal{S}^{i,j}_{n-1}$. As $k_q + q \leqslant l_i + i < k_r + r$ and $k_q + q \leqslant l_j + j < k_r + r$, the sign reversing involution on the set of all Gelfand–Tsetlin tree sequence associated with $\mathcal{S}^{i,j}_{n-1}$ induces a sign reversing involution on the second subset of $\mathcal{L}_n(\mathcal{T}', \mathbf{k})$.

Case 2. $k_q + q \leqslant k_p + p \leqslant k_r + r$: if $\mathbf{l} \in \mathbb{Z}^{n-1}$ is an admissible shifted edge labeling of T_n for an element of $\mathcal{L}_n(\mathcal{T}, \mathbf{k})$ then we have $k_q + q \leqslant l_i + i < k_p + p$; in $\mathcal{L}_n(\mathcal{T}', \mathbf{k})$ we have $k_p + p \leqslant l_i + i < k_r + r$. The edge i' is an inversion for the pair $(T_n, \mathbf{k})$ if and only if it is no inversion for the pair $(T'_n, \mathbf{k})$. We decompose both sets into two sets according to the edge label of j': in the first set we have $k_q + q \leqslant l_j + j < k_p + p$ and in the second set we have $k_p + p \leqslant l_j + j < k_r + r$. If we replace $\mathcal{T}_{<n}$ by $\mathcal{S}^{i,j}_{n-1}$, we see that in case of $\mathcal{L}_n(\mathcal{T}, \mathbf{k})$ the signed enumeration of the first set is zero, while for $\mathcal{L}_n(\mathcal{T}', \mathbf{k})$ the signed enumeration of the second set is zero. For the two other sets, the replacement of $(l_i, l_j) \to (l_j + j - i, l_i + i - j)$ of the shifted edge labels of the largest tree and performing the sign reversing involution on $\mathcal{S}^{i,j}_{n-1}$ is a sign preserving involution.

Case 3. $k_q + q \leqslant k_r + r \leqslant k_p + p$: for the edge label of i' in T_n we have $k_q + q \leqslant l_i + i < k_p + p$. We decompose $\mathcal{L}_n(\mathcal{T}, \mathbf{k})$ into two sets, where we have $k_q + q \leqslant l_i + i < k_r + r$ and $k_r + r \leqslant l_i + i < k_p + p$, respectively. As $k_q + q \leqslant l_j + j < k_r + r$, the signed enumeration of the first set is zero, while the signed enumeration of the second set coincides with the signed enumeration of the elements in $\mathcal{L}_n(\mathcal{T}', \mathbf{k})$. $\square$

Now we turn to the shift-antisymmetry.

Lemma 3. *The independence for order n implies the shift-antisymmetry for order n.*

Proof. Fix i, j with $1 \leqslant i < j \leqslant n$. We define a tree sequence $\mathcal{S}^{i,j}_n = (T_1, \ldots, T_n)$ of order n: let S_m be the directed tree with m vertices sketched in Fig. 7 and, for $3 \leqslant m \leqslant n$, let this be the underlying tree for T_m. (Note that there is no choice for the underlying tree if $m = 1, 2$.) There are no restrictions on the names of the vertices and edges except that the two sinks in T_n are i and j, the two sinks in T_{n-1} are the unprimed versions of the edges incident with i and j in T_{n-1}, the two sinks in T_{n-2} are the unprimed versions of the edges incident with the two sinks in T_{n-1}, etc. Let $\mathbf{k} = (k_1, \ldots, k_n) \in \mathbb{Z}^n$ and $\mathbf{k}' = E^{j-i}_{k_j} E^{i-j}_{k_i} S_{k_i,k_j} \mathbf{k}$. Then the following is a sign reversing involution between the Gelfand–Tsetlin tree sequence associated with $\mathcal{S}^{i,j}_n$ and fixed shifted vertex labeling $\mathbf{k}$ of T_n and those where the shifted vertex labeling of T_n is given by $\mathbf{k}'$: for $m \geqslant 3$, we interchange in T_m the labels of the two sink vertices as well as the labels of the two edges incident with the sinks; in T_2 we interchange the two vertex labels. This either produces or resolves an inversion in T_2 and concludes the proof of Lemma 3.

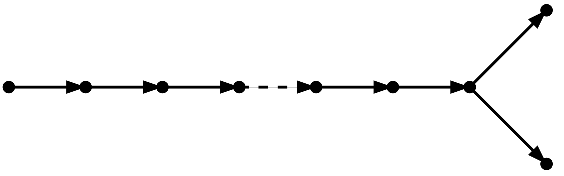

Fig. 7. The tree S_m.

Alternatively, we can also argue as follows: let T'_n be the tree which we obtain from T_n by interchanging vertex i and vertex j (the underlying tree remains unaffected) and $\mathcal{T}' = (T_1, \dots, T_{n-1}, T'_n)$. As $\operatorname{sgn} T_n = -\operatorname{sgn} T'_n$, we obviously have

$$L_n(\mathcal{T}, \mathbf{k}) = -E_{k_j}^{j-i} E_{k_i}^{i-j} S_{k_i,k_j} L_n\big(\mathcal{T}', \mathbf{k}\big).$$

The assertion follows from Lemma 2 since $L_n(\mathcal{T}', \mathbf{k}) = L_n(\mathcal{T}, \mathbf{k})$. $\square$

In Appendix B, a direct combinatorial proof of the shift-antisymmetry of the enumeration formula for Gelfand–Tsetlin patterns is sketched, which does not make use of the notion of Gelfand–Tsetlin tree sequences.

4. Taking differences – $L_n(\mathcal{T}, \mathbf{k})$ is a polynomial

The quantity $L_n(\mathcal{T}, \mathbf{k})$ is not characterized by the properties we have derived so far. Next, we show that $L_n(\mathcal{T}, \mathbf{k})$ is a polynomial of degree no greater than $n-1$ in every k_i, which is the last ingredient to finally see that it is equal to (1.1).

In order to show that $p(x)$ is a polynomial in x of degree no greater than $n-1$, it suffices to prove that $\Delta_x^n p(x) = 0$ where $\Delta_x := E_x - \mathrm{id}$ is the difference operator. Thus it suffices to show the following.

Lemma 4. *For $i \in \{1, 2, \dots, n\}$ we have $\Delta_{k_i}^n L_n(\mathcal{T}, \mathbf{k}) = 0$.*

Proof. We define a convenient tree sequence $\mathcal{R}_{n,i} = (R_1, \dots, R_n)$ (see Fig. 8) and find a combinatorial interpretation for $\Delta_{k_i}^j L_n(\mathcal{R}_{n,i}, \mathbf{k})$ if $j \in \{0, 1, \dots, n-1\}$: in R_n, we require $i =: i_n$ to be a leaf, in R_{n-1} we require the unprimed version i_{n-1} of the edge incident with i_n in R_n to be a leaf, in R_{n-2} we require the unprimed version i_{n-2} of the edge incident with i_{n-1} in R_{n-1} to be a leaf etc. As for the orientations of the edges $i'_1, i'_2, \dots, i'_{n-1}$, we choose the vertices $i_2, i_3, \dots, i_n$ to be sinks. By $l_{i_1} + i_1, l_{i_2} + i_2, \dots, l_{i_{n-1}} + i_{n-1}$, we denote the respective edge labels (which are of course also vertex labels in the next level).

We define $\Delta_{k_i}^j \mathcal{L}_n(\mathcal{R}_{n,i}, \mathbf{k})$: it is the set of labeled tree sequences on the unlabeled tree sequence $\mathcal{R}_{n,i}$ such that the conditions on the edge labels are as for Gelfand–Tsetlin tree sequence in $\mathcal{L}_n(\mathcal{R}_{n,i}, \mathbf{k})$, except for the edges $i'_{n-j}, i'_{n-j+1}, \dots, i'_{n-1}$ in $R_{n-j+1}, R_{n-j+2}, \dots, R_n$, respectively, where we require $l_{i_{n-j}} + i_{n-j} = l_{i_{n-j+1}} + i_{n-j+1} = \cdots = l_{i_{n-1}} + i_{n-1} = k_i + i$. As for the sign, we compute it as usual only we ignore the contributions of the edges $i'_{n-j} \in E(R_{n-j+1}), i'_{n-j+1} \in E(R_{n-j+2}), \dots, i'_{n-1} \in E(R_n)$ if they are inversions.

Then, by induction with respect to j, the signed enumeration of these labeled tree sequences on $\mathcal{R}_{n,i}$ is equal to $\Delta_{k_i}^j L_n(\mathcal{R}_{n,i}, \mathbf{k})$: for $j = 0$ this is obvious. It suffices to show that

$$\Delta_{k_i} \big|\Delta_{k_i}^j \mathcal{L}_n(\mathcal{R}_{n,i}, \mathbf{k})\big|_- = \big|\Delta_{k_i}^{j+1} \mathcal{L}_n(\mathcal{R}_{n,i}, \mathbf{k})\big|_-.$$

Consider an element from $E_{k_i} \Delta_{k_i}^j \mathcal{L}_n(\mathcal{R}_{n,i}, \mathbf{k})$ such that the vertex label of i_{n-j} in R_{n-j} (which is $l_{i_{n-j}} + i_{n-j} = k_i + i + 1$) is greater than the vertex label of the other endpoint of the edge i'_{n-j}.

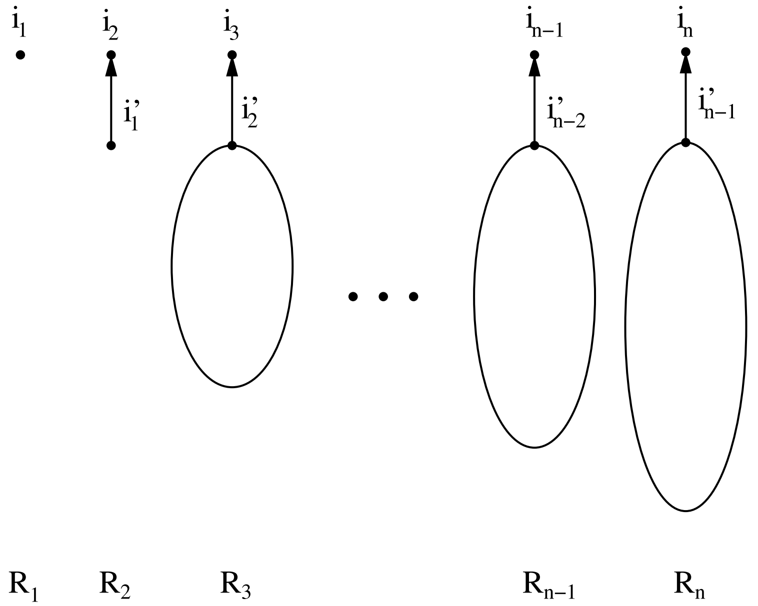

Fig. 8. Tree sequence in the proof of Lemma 4.

We decrease the labels $l_{i_{n-j}}+i_{n-j}, l_{i_{n-j+1}}+i_{n-j+1}, \ldots, l_{i_{n-1}}+i_{n-1}, k_i+i+1$ (which are all equal) by 1. If $l_{i_{n-j-1}}+i_{n-j-1}<k_i+i$ we obtain an element of $\Delta^j_{k_i}\mathcal{L}_n(\mathcal{R}_{n,i},\mathbf{k})$ and these elements cancel in the difference $\Delta_{k_i}|\Delta^j_{k_i}\mathcal{L}_n(\mathcal{R}_{n,i},\mathbf{k})|_-$. Otherwise, if, $l_{i_{n-j-1}}+i_{n-j-1}=k_i+i$ in R_{n-j}, we obtain an element of $\Delta^{j+1}_{k_i}\mathcal{L}_n(\mathcal{R}_{n,i},\mathbf{k})$. To be more precise, we obtain exactly the elements of $\Delta^{j+1}_{k_i}L_n(\mathcal{R}_{n,i},\mathbf{k})$ such that the edge i'_{n-j-1} is no inversion in R_{n-j}. On the other hand, if the edge i'_{n-j-1} is an inversion for an element of $\Delta^j_{k_i}\mathcal{L}_n(\mathcal{R}_{n,i},\mathbf{k})$, then, by increasing the labels $l_{i_{n-j}}+i_{n-j}, l_{i_{n-j+1}}+i_{n-j+1}, \ldots, l_{i_{n-1}}+i_{n-1}, k_i+i$ by 1, we obtain a corresponding element in $E_{k_i}\Delta^j_{k_i}\mathcal{L}_n(\mathcal{R}_{n,i},\mathbf{k})$, except for the case when $l_{i_{n-j-1}}+i_{n-j-1}=k_i+i$. This way, we obtain exactly the elements of $\Delta^{j+1}_{k_i}\mathcal{L}_n(\mathcal{R}_{n,i},\mathbf{k})$ such that the edge i'_{n-j-1} is an inversion in R_{n-j}. The sign that comes from the inversion i'_{n-j-1} in $R_{n,n-j}$ takes into account for the fact that we "subtract" the greater set from the smaller set in this case.

Now observe that in fact $\Delta^{n-1}_{k_i}\mathcal{L}_n(\mathcal{R}_{n,i},\mathbf{k})$ does not depend on k_i and, consequently, $\Delta^n_{k_i}L_n(\mathcal{R}_{n,i},\mathbf{k})$ must be zero. □

We are finally in the position of prove Theorem 1.

Proof of Theorem 1. By the shift-antisymmetry (Lemma 3), we conclude that the polynomial (Lemma 4) $L_n(\mathcal{T},\mathbf{k})$ vanishes if $k_i+i=k_j+j$ for distinct $i,j\in\{1,2,\ldots,n\}$. This implies that the expression in (1.1) has to be a factor of $L_n(\mathcal{T},\mathbf{k})$. Again by Lemma 4, we know that it is a polynomial of degree no greater than $n-1$ and since (1.1) is of degree $n-1$ in every k_i, this implies that

$$L_n(\mathcal{T},\mathbf{k})=C\cdot\prod_{1\leqslant i<j\leqslant n}\frac{k_j-k_i+j-i}{j-i},$$

where $C\in\mathbb{Q}$. As there is only one Gelfand–Tsetlin pattern with bottom row $(1,1,\ldots,1)\in\mathbb{Z}^n$, we can conclude that $C=1$. □

5. A combinatorial proof of $e_\rho(\Delta_{k_1},\ldots,\Delta_{k_n})L_n(\mathcal{T},\mathbf{k})=0$

The combinatorial interpretation of $\Delta_{k_i}^{j} L_n(\mathcal{R}_{n,i},\mathbf{k})$ was surely the main ingredient in the proof of Lemma 4. This section is devoted to use basically the same idea to give a combinatorial proof of the identity

$$e_\rho(\Delta_{k_1},\ldots,\Delta_{k_n})L_n(\mathcal{T},\mathbf{k})=0, \tag{5.1}$$

which holds for $\rho \geqslant 1$ and where

$$e_\rho(X_1,\ldots,X_n)=\sum_{1\leqslant i_1<i_2<\cdots<i_\rho\leqslant n} X_{i_1}X_{i_2}\cdots X_{i_\rho}$$

is the ρ-th elementary symmetric function. (An algebraic proof, which already uses the fact that $L_n(\mathcal{T},\mathbf{k})$ is equal to (1.1) as well as the presentation of (1.1) in terms of a determinant (see (A.1)), can be found in [6, Lemma 1].) This identity is of interest as it is the crucial fact in the proof of (1.2) given in [6].

Even though the ideas are straightforward, this combinatorial proof of (5.1) is a bit elaborate. (However, nothing else is to be expected when a statement is related to alternating sign matrix counting.) In fact, the benefit of this exercise is not primarily the proof of (5.1) but an improvement of the understanding of how to interpret the application of difference operators to enumerative quantities such as $L_n(\mathcal{T},\mathbf{k})$ combinatorially. To give a hint as to why such an understanding could be of interest, observe that the proof of (5.1) relies on a combinatorial interpretation of

$$\Delta_{k_{i_1}}\Delta_{k_{i_2}}\ldots\Delta_{k_{i_\rho}}L_n(\mathcal{T},\mathbf{k}) \tag{5.2}$$

for subsets $\{i_1,\ldots,i_\rho\}\subseteq[n]$. As the number of monotone triangles with bottom row $(k_1,\ldots,k_n)$ is given by

$$\alpha(n;k_1,\ldots,k_n)=\left(\prod_{1\leqslant p<q\leqslant n}(\mathrm{id}+\Delta_{k_p}\delta_{k_q})\right)L_n(\mathcal{T},\mathbf{k}), \tag{5.3}$$

where $\delta_x=\mathrm{id}-E_x^{-1}$ is a second type of difference operator (see Section 6), ideas along these lines might also lead to a combinatorial proof of this formula.

We need a more general notion of admissibility. The idea is simple and very roughly as follows: we require each vertex of a fixed vertex set R of the tree T to have an associated edge incident with it such that the edge label takes on the extreme label given by the vertex label.

Definition 3. Given an n-tree T, an n-tuple $\mathbf{k}=(k_1,\ldots,k_n)\in\mathbb{Z}^n$ and a subset $R\subseteq[n]=:\{1,2,\ldots,n\}$ of vertices of T, we define a vector $\mathbf{l}=(l_1,\ldots,l_{n-1})\in\mathbb{Z}^{n-1}$ to be *weakly R-admissible* for the pair $(T,\mathbf{k})$ as follows.

- For each vertex $r\in R$ of T, there exists a unique edge $i(r)'$ of T incident with r such that $k_r+r=l_{i(r)}+i(r)$.
- For the edges $j'=(p,q)$ that do not appear in the image $i(R)'$ we have $\min(k_p+p,k_q+q)\leqslant l_j+j<\max(k_p+p,k_q+q)$. (Note that for those edges we do not allow $l_j+j=k_p+p$ or $l_j+j=k_q+q$ if $p\in R$ or $q\in R$, respectively.)

The vector $\mathbf{l}$ is said to be *R-admissible* if the function $i:R\to[n-1]$ is injective. If the function is not injective then we choose for each pair of distinct vertices $r,s\in R$ that share an edge $i(r)'=i(s)'$ one endpoint to be the dominating endpoint.

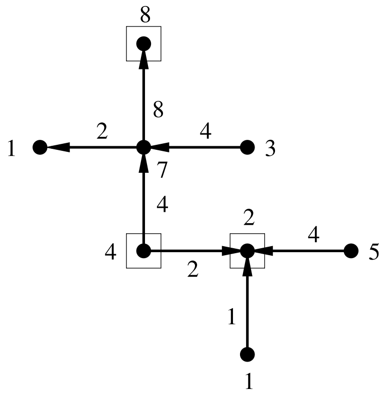

Fig. 9. An R-admissible labeling; the vertices of R are enclosed by squares.

An example is given in Fig. 9. For the extreme cases concerning R, we have the following: the weak $\emptyset$-admissibility coincides with the ordinary admissibility and there exists no $[n]$-admissible vector as there is no injective function $i : [n] \to [n-1]$. If $n = 1$ then there exists an R-admissible vector if and only if $R = \emptyset$, namely the empty set.

We introduce the *sign* which we associate with $(T, \mathbf{k})$, $i : R \to [n-1]$ and a choice of dominating vertices (if necessary). The following manner of speaking will turn out to be useful: if we refer to the minimum of an edge then we mean the minimum of the two labels of the endpoints of the edge or, by abuse of language, the respective vertex where this minimum is attained; similar for the maximum. If, for an edge j', the labels on the two endpoints coincide then the edge must be in the image $i(R)'$. In this case the maximum of an edge is determined as follows: if $i^{-1}(j)$ contains a unique vertex then we define this to be the "maximum" of the edge and if $i^{-1}(j)$ contains both endpoints then the dominating vertex is defined as the "maximum"; in both cases the other endpoint is defined as the minimum. As for the sign, we let each edge that is an inversion contribute a -1 (which is the case when it is directed from its maximum to its minimum) as well as each $r \in R$ that is the minimum of the edge $i(r)'$.

We define (n, m, R)-*Gelfand–Tsetlin tree sequence* as follows.

Definition 4. Let $m \leqslant n$ be positive integers, $\mathcal{T} = (T_1, \dots, T_n)$ be a tree sequence, $R \subseteq [m]$ be a set of vertices of T_m and $\mathbf{k} \in \mathbb{Z}^n$ be a shifted labeling of the tree T_n. An (n, m, R)-*Gelfand–Tsetlin tree sequence* associated with $\mathcal{T}$ and $\mathbf{k}$ is a sequence $\mathbf{L} = (\mathbf{l}_1, \mathbf{l}_2, \dots, \mathbf{l}_n)$ with $\mathbf{l}_i \in \mathbb{Z}^i$ and $\mathbf{l}_n = \mathbf{k}$ which has the following properties:

- The shifted labeling $\mathbf{l}_{i-1}$ is admissible for the pair $(T_i, \mathbf{l}_i)$ if $i \in \{2, 3, \dots, n\} \setminus \{m\}$.
- The shifted labeling $\mathbf{l}_{m-1}$ is weakly R-admissible for the pair $(T_m, \mathbf{l}_m)$.

If the function $i : R \to [m-1]$, which manifests the weak R-admissibility is not injective then the (n, m, R)-*Gelfand–Tsetlin tree sequence* comes along with a set of dominating vertices as described in Definition 3; all choices are possible. We let $\mathcal{L}_{n,m,R}(\mathcal{T}, \mathbf{k})$ denote the set of these sequences. For an integer $\rho \leqslant m$, we denote by $\mathcal{L}_{n,m,\rho}(\mathcal{T}, \mathbf{k})$ the union over all ρ-subsets R of $[m]$. Concerning the sign, we define

$$\operatorname{sgn} \mathbf{L} = (-1)^{\#\text{ of inversions of } \mathbf{L}} \cdot (-1)^{\#\text{ of vertices } r \in R \text{ s.t. } r \text{ is the minimum of } i(r)'} \cdot \operatorname{sgn} \mathcal{T}.$$

We let $L_{n,m,R}(\mathcal{T}, \mathbf{k})$, respectively $L_{n,m,\rho}(\mathcal{T}, \mathbf{k})$, denote the signed enumeration of these objects.

The following is obvious but crucial: the quantity $L_{n,m,R}(\mathcal{T},\mathbf{k})$ does not change if we pass in Definition 4 from weak R-admissibility to R-admissibility as changing the dominating vertex from one endpoint of a shared edge to the other is a sign-reversing involution.

We are in the position to give the combinatorial interpretation for the expression in (5.2). In order to state the result, we introduce a convenient notation: if $R=\{i_1,\dots,i_\rho\}\subseteq[n]$ then

$$\Delta_{\mathbf{k}_R} f(\mathbf{k}) := \Delta_{k_{i_1}}\cdots\Delta_{k_{i_\rho}} f(k_1,\dots,k_n).$$

(The analogous convention for $E_{\mathbf{k}_R} f(\mathbf{k})$ will be used below.)

Proposition 1. *Let $R\subseteq[n]$. Then $\Delta_{\mathbf{k}_R} L_n(\mathcal{T},\mathbf{k}) = L_{n,n,R}(\mathcal{T},\mathbf{k})$.*

This immediately implies the following combinatorial interpretation for the left-hand side of (5.1).

Corollary 1. *Let $\rho\in\{0,1,\dots,n\}$. Then $e_\rho(\Delta_{k_1},\dots,\Delta_{k_n})L_n(\mathcal{T},\mathbf{k}) = L_{n,n,\rho}(\mathcal{T},\mathbf{k})$.*

The following lemma is used in several places of our proofs in the remainder of this section.

Lemma 5. *Let $\mathcal{T}=(T_1,T_2,\dots,T_n)$ be a tree sequence and $m\leqslant n$. For an integer $t<m$, we fix a set P of pairs of edges of T_{t+1} and let $\mathcal{L}_{n,m,R,P}(\mathcal{T},\mathbf{k})$ denote the subset of labeled tree sequences in $\mathcal{L}_{n,m,R}(\mathcal{T},\mathbf{k})$ such that for each pair in P the edge labels of the respective edges of T_{t+1} are distinct. Then the signed enumeration of this subset is equal to the signed enumeration of the whole set.*

Proof. We consider the complement of $\mathcal{L}_{n,m,R,P}(\mathcal{T},\mathbf{k})$ and suppose that for $(i,j)\in P$ the edge labeling $\mathbf{l}_t+(1,2,\dots,t)$ of T_{t+1} is equal in the coordinates i and j. If there is more than one pair then we choose the pair which is minimal with respect to a fixed order on P. Then, we may replace the tree T_t in $\mathcal{T}$ by a tree where vertex i and j are adjacent. The assertion follows as such as tree does not possess an admissible edge labeling. □

Proof of Proposition 1. In this proof we use arguments that are similar to the ones used in Lemma 4.

We consider subsets of $\mathcal{L}_n(\mathcal{T},\mathbf{k})$ indexed by two disjoint subsets $P,Q\subseteq[n]$ of vertices of T_n: let $\mathcal{L}_n(\mathcal{T},\mathbf{k},P,Q)$ denote the set of Gelfand–Tsetlin tree sequences in $\mathcal{L}_n(\mathcal{T},\mathbf{k})$ such that for the edge labeling $\mathbf{l}\in\mathbb{Z}^{n-1}$ of the largest tree T_n in the tree sequence $\mathcal{T}$ the following is fulfilled:

- For each $p\in P$, there exists an edge $i(p)'$ of T_n incident with p such that k_p+p is the minimum of $i(p)'$ and $l_{i(p)}+i(p)=k_p+p$.
- For each $q\in Q$, there exists an edge $i(q)'$ in T_n incident with q such that k_q+q is the maximum of $i(q)'$ and $l_{i(q)}+i(q)=k_q+q-1$.

We denote the respective signed enumeration by $L_n(\mathcal{T},\mathbf{k},P,Q)$. Suppose $r\notin P,Q$. Then

$$\Delta_{k_r} L_n(\mathcal{T},\mathbf{k},P,Q) = E_{k_r} L_n\bigl(\mathcal{T},\mathbf{k},P,Q\cup\{r\}\bigr) - L_n\bigl(\mathcal{T},\mathbf{k},P\cup\{r\},Q\bigr). \tag{5.4}$$

In order to see this, consider an element of $E_{k_r}\mathcal{L}_n(\mathcal{T},\mathbf{k},P,Q)$ with the following property: for each edge i' of T_n that is incident with vertex r of T_n and such that the vertex label of the other endpoint of i' is smaller than k_r+r+1 we have that the respective edge label l_i+i is smaller than k_r+r. In this case, we may change the vertex label of r to k_r+r to obtain an element of $\mathcal{L}_n(\mathcal{T},\mathbf{k},P,Q)\setminus\mathcal{L}_n(\mathcal{T},\mathbf{k},P\cup\{r\},Q)$. Thus, these elements cancel in the difference on the left-hand side and we are left with the elements on the right-hand side.

This implies by induction with respect to the size of $R\subseteq[n]$ that

$$\Delta_{\mathbf{k}_R} L_n(\mathcal{T},\mathbf{k}) = \sum_{Q\subseteq R}(-1)^{|R|+|Q|} E_{\mathbf{k}_Q} L_n(\mathcal{T},\mathbf{k},R\setminus Q,Q), \tag{5.5}$$

if $R = \emptyset$ there is nothing to prove since $L_n(\mathcal{T}, \mathbf{k}, \emptyset, \emptyset) = L_n(\mathcal{T}, \mathbf{k})$; otherwise let $r \in R$ and we have, by the induction hypothesis and (5.4),

$$\Delta_{\mathbf{k}_R} L_n(\mathcal{T}, \mathbf{k}) = \Delta_{k_r} \Delta_{\mathbf{k}_{R \setminus \{r\}}} L_n(\mathcal{T}, \mathbf{k}) = \sum_{Q \subseteq R \setminus \{r\}} (-1)^{|R|+|Q|} E_{\mathbf{k}_Q} \Delta_{k_r} L_n(\mathcal{T}, \mathbf{k}, R \setminus Q, Q)$$

$$= \sum_{Q \subseteq R \setminus \{r\}} (-1)^{|R|+|Q|} E_{\mathbf{k}_Q} \left(E_{k_r} L_n\big(\mathcal{T}, \mathbf{k}, R \setminus Q, Q \cup \{r\}\big) - L_n\big(\mathcal{T}, \mathbf{k}, R \setminus Q \cup \{r\}, Q\big)\right).$$

The right-hand side of (5.5) is in fact equal to the signed enumeration of $\mathcal{L}_{n,n,R}(\mathcal{T}, \mathbf{k})$: in order to see this, we may assume by Lemma 5 that the edge labels of T_n are distinct, both in $\mathcal{L}_{n,n,R}(\mathcal{T}, \mathbf{k})$ and in $E_{\mathbf{k}_Q} \mathcal{L}_n(\mathcal{T}, \mathbf{k}, R \setminus Q, Q)$. This implies that for each tree sequence in $E_{\mathbf{k}_Q} \mathcal{L}_n(\mathcal{T}, \mathbf{k}, R \setminus Q, Q)$ and each $r \in R$, there is a unique edge $i(r)'$ of T_n with $l_{i(r)} + i(r) = k_r + r$. Now, we may convert elements of $E_{\mathbf{k}_Q} \mathcal{L}_n(\mathcal{T}, \mathbf{k}, R \setminus Q, Q)$ into elements of $\mathcal{L}_{n,n,R}(\mathcal{T}, \mathbf{k})$ by decreasing the labels of the vertices in Q by 1. We obtain elements, where for $r \in Q$, the vertex label $k_r + r$ is the maximum of $i(r)'$ and, for $r \in R \setminus Q$, the vertex label $k_r + r$ is the minimum of $i(r)'$ – attached with a sign according to the number cases where $k_r + r$ is the minimum of the edge $i(r)'$. The fact that the edge labels are distinct and since there always exists an edge label that is equal to $k_r + r$ implies that it is irrelevant that the intervals for the possible labels of the edges incident with r were slightly changed when passing from $E_{\mathbf{k}_Q} \mathcal{L}_n(\mathcal{T}, \mathbf{k}, R \setminus Q, Q)$ to $\mathcal{L}_{n,n,R}(\mathcal{T}, \mathbf{k})$.

However, by decreasing the vertex label of a vertex $q \in Q$ of an element in $E_{\mathbf{k}_Q} \mathcal{L}_n(\mathcal{T}, \mathbf{k}, R \setminus Q, Q)$ by 1 to $k_q + q$, this value may reach the vertex label $k_p + p$ of a vertex p that is adjacent to q; in this case we have to guarantee that $k_q + q$ can still be identified as the maximum of the edge j' connecting p and q. The assumption implies $i(q) = j$. If $p \notin R$ then, when considering the labeled tree sequence as an element of $\mathcal{L}_{n,n,R}(\mathcal{T}, \mathbf{k})$, the vertex q is the maximum of j' by definition. If, on the other hand, $p \in R$, then we also have $i(p) = j$ and we let q be the dominating vertex of the edge to remember that it used to be the maximum of the edge j'. Thus it is clear how to reverse the procedure. $\square$

In the definition of the R-admissibility, we have fixed a set R of vertices of T. However, we may as well fix the image $i(R) =: R'$ of the injective function $i : R \to [n-1]$, which corresponds to a set of edges of T.

Definition 5. Let T be an n-tree, $\mathbf{k} \in \mathbb{Z}^n$ and $R' \subseteq [n-1]$. A vector $\mathbf{l} \in \mathbb{Z}^{n-1}$ together with an injective function $t : R' \to [n]$ is said to be *R'-edge-admissible* for the pair $(T, \mathbf{k})$ if $\mathbf{l}$ is $t(R')$-admissible for the pair $(T, \mathbf{k})$, where $t^{-1} : t(R') \to [n-1]$ is the function that proves the $t(R')$-admissibility.

In analogy to Definition 4, it is also clear how to define Gelfand–Tsetlin tree sequences associated with a triple (n, m, R'), where $m \leqslant n$ are positive integers and $R' \subseteq [m-1]$ corresponds to a subset of *edges* of T_m. We denote this set by $\mathcal{L}^{R'}_{n,m}(\mathcal{T}, \mathbf{k})$ and by $L^{R'}_{n,m}(\mathcal{T}, \mathbf{k})$ its signed enumeration. Note that $\mathcal{L}_{n,m,\rho}(\mathcal{T}, \mathbf{k})$ is also the union of $\mathcal{L}^{R'}_{n,m}(\mathcal{T}, \mathbf{k})$, where R' runs over all ρ-subsets of $[m-1]$.

In the proof of the next proposition, it will be helpful to replace the R'-edge-admissibility in the definition of $L^{R'}_{n,m}(\mathcal{T}, \mathbf{k})$ by a more general notion, which we call *weak R'-edge-admissibility* and define as follows.

Definition 6. Let T be an n-tree, $\mathbf{k} \in \mathbb{Z}^n$ and $R' \subseteq [n-1]$. A vector $\mathbf{l} \in \mathbb{Z}^{n-1}$ is said to be *weakly R'-edge-admissible* for the pair $(T, \mathbf{k})$ if there exists a function $t : R' \to [n]$ such that the following conditions are fulfilled:

- For all $r \in R$, the edge r' of T is incident with the vertex $t(r)$ of T and $l_r + r = k_{t(r)} + t(r)$.
- For all $r \in [n-1] \setminus R'$, we have $\min(k_p + p, k_q + q) \leqslant l_r + r < \max(k_p + p, k_q + q)$, where $r' = (p, q)$ in T.

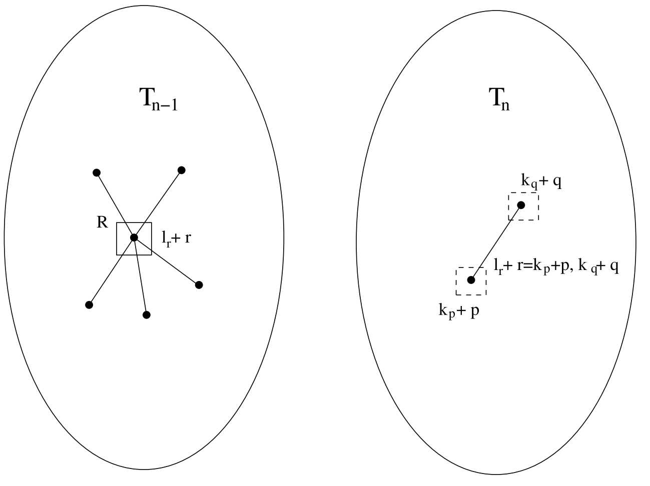

Fig. 10. Situation in the proof of Proposition 2.

The sign we associate is defined as follows: it is -1 raised to the number of inversions plus the number of edges r' of R' such that $t(r)$ is the minimum of the edge. (If the two vertex labels of an edge r' coincide then it must be an element of R' and we define $t(r)$ as the "maximum" of the edge.)

To obtain the ordinary edge-admissibility we have to require in addition that for all $r \in R'$ the following is fulfilled: suppose s' is an edge of T incident with vertex $t(r)$ such that $l_s + s = k_{t(r)} + t(r)$ then we have $r = s$. However, the violation of this condition would require two edges of T to have the same label, which can be avoided for an element of $L^{R'}_{n,m}(\mathcal{T}, \mathbf{k})$ by the argument given in Lemma 5.

The following proposition will finally imply (5.1).

Proposition 2. *Let $R \subseteq [m-1]$. Then $L^R_{n,m}(\mathcal{T}, \mathbf{k}) = L_{n,m-1,R}(\mathcal{T}, \mathbf{k})$.*

An immediate consequence is the following.

Corollary 2. *Let ρ be a non-negative integer. Then $L_{n,m,\rho}(\mathcal{T}, \mathbf{k}) = L_{n,m-1,\rho}(\mathcal{T}, \mathbf{k})$.*

The corollary implies $L_{n,m,\rho}(\mathcal{T}, \mathbf{k}) = 0$ if ρ is non-zero as $\mathcal{L}_{n,m,\rho}(\mathcal{T}, \mathbf{k}) = \emptyset$ if $\rho \geqslant m$, since there is no injective function from $[\rho]$ to $[m-1]$. By Corollary 1, (5.1) finally follows.

Proof of Proposition 2. We restrict our considerations to the case that $m = n$ as the general case is analogous. By Lemma 5, we assume that the edge labels of T_{n-1} are distinct, both in $\mathcal{L}^R_{n,n}(\mathcal{T}, \mathbf{k})$ and in $\mathcal{L}_{n,n-1,R}(\mathcal{T}, \mathbf{k})$.

We consider an element $\mathcal{S}$ of $\mathcal{L}^R_{n,n}(\mathcal{T}, \mathbf{k})$, denote by $\mathbf{l} \in \mathbb{Z}^{n-1}$ the respective shifted edge labeling of T_n and by $t: R \to [n]$ the function that proves the weak R-edge-admissibility of the vector $\mathbf{l}$ for the pair $(T_n, \mathbf{k})$. Suppose $r \in R$ and that p, q are the vertices of the edge r' in T_n then we have either $t(r) = p$ or $t(r) = q$. We denote the first subset of $\mathcal{L}^R_{n,n}(\mathcal{T}, \mathbf{k})$ by $M_{r,p}$ and the second subset by $M_{r,q}$. The situation is sketched in Fig. 10.

Assuming w.l.o.g. that $k_p + p \leqslant k_q + q$, we first observe that we can restrict our attention to the case that there is at least one edge incident with vertex r in T_{n-1}, the label of which lies in the interval $[k_p + p, k_q + q)$. This is because for the other elements, $l_r + r \to k_q + q$ and $t(r) \to q$ induces a sign reversing bijection from $M_{r,p}$ to $M_{r,q}$. In the following, we address these edges as the relevant edges of r.

In order to construct an element of $\mathcal{L}_{n,n-1,R}(\mathcal{T},\mathbf{k})$ we perform the following shifts to the labels $l_r + r$ for all $r \in R$: if $\mathcal{S}$ is an element of $M_{r,p}$, we shift $l_r + r$ to the minimum of incident edge labels in T_{n-1} no smaller than $k_p + p$ and let $j(r)'$ be the respective edge, while for elements of $M_{r,q}$ we shift $l_r + r$ to the maximum of incident edge labels in T_{n-1} smaller than $k_q + q$ and let $j(r)'$ be the respective edge. These edges $j(r)$ are unique as the edge labels are assumed to be distinct. The contribution of -1 to the sign of the elements in $M_{r,p}$ that comes from the fact that the edge label of r' in T_n is equal to the minimum of the edge translates in the new element into the contribution of -1 of the edge $j(r)'$ in T_{n-1} as its edge label is also equal to the minimum of the edge. If this procedure causes two distinct vertices $r, s \in R$ to share an edge $j(r)' = j(s)'$ then we let the dominating vertex be the maximum of the respective edge in the original element.

The precise description of the elements in $\mathcal{L}_{n,n-1,R}(\mathcal{T},\mathbf{k})$ that appear as a result of this procedure is the following. For each $r \in R$, one of the following two possibilities applies: suppose p, q are the endpoints of r' in T_n and w.l.o.g. $k_p + p \leqslant k_q + q$ then either

- the vertex r is the minimum of the edge $j(r)'$ and the edge label of $j(r)'$ is the minimum under all relevant edges of r, or
- the vertex r is the maximum of the edge $j(r)'$ and the edge label of $j(r)'$ is the maximum under all relevant edges of r.

For such an element it is also clear how to invert the procedure to reobtain an element of $\mathcal{L}^R_{n,n}(\mathcal{T},\mathbf{k})$.

Finally, we define a sign-reversing involution on the set of elements of $\mathcal{L}_{n,n-1,R}(\mathcal{T},\mathbf{k})$ that do not fulfill this requirement: suppose that $r \in R$ is minimal such that the requirement is not met and that r is the minimum of the edge $j(r)'$. Let i' be the relevant edge of r, the edge label of which is maximal with the property that it is smaller than $l_r + r$. We shift $l_r + r$ to this edge label and set $j(r) = i$. If necessary we choose the dominating vertices such that the set of inversions remains unaffected. Then, r is the maximum of the edge $j(r)'$. Likewise when r is the maximum of the edge. The fact that we only work with relevant edges guarantees that we are able to perform the shift accordingly for the edge label of r' in T_n. $\square$

To conclude this section, we demonstrate that also (5.1) implies that $L_n(\mathcal{T},\mathbf{k})$ is a polynomial in $k_1, \dots, k_n$ of degree no greater than $n-1$ in every k_i.

Lemma 6. *Suppose that $A(k_1, \dots, k_n)$ is a function with*

$$e_\rho(\Delta_{k_1}, \dots, \Delta_{k_n}) A(k_1, \dots, k_n) = 0$$

for all $\rho > 0$. Then $\Delta^n_{k_i} A(k_1, \dots, k_n) = 0$ for all $i \in \{1, 2, \dots, n\}$.

Proof. We define

$$A_{\rho,i}(k_1, \dots, k_n) = e_\rho(\Delta_{k_1}, \dots, \widehat{\Delta_{k_i}}, \dots, \Delta_{k_n}) A(k_1, \dots, k_n),$$

where $\widehat{\Delta_{k_i}}$ indicates that Δ_{k_i} does not appear in the argument. We use the identity

$$e_\rho(X_1, \dots, X_n) = e_\rho(X_1, \dots, \widehat{X_i}, \dots, X_n) + X_i e_{\rho-1}(X_1, \dots, \widehat{X_i}, \dots, X_n)$$

and the assumption to see that

$$A_{\rho,i}(k_1, \dots, k_n) = -\Delta_{k_i} A_{\rho-1,i}(k_1, \dots, k_n).$$

This implies

$$A_{\rho,i}(k_1, \dots, k_n) = (-1)^\rho \Delta^\rho_{k_i} A(k_1, \dots, k_n)$$

by induction with respect to ρ. As $A_{n,i}(k_1, \dots, k_n) = 0$, the assertion follows. $\square$

6. Monotone triangles

I would like to see an analogous "theory" for monotone triangles (Gelfand–Tsetlin patterns with strictly increasing rows), which seems conceivable as there are several properties of the unrestricted patterns for which we have a corresponding (though in some cases more complicated) property of monotone triangles. For instance, it is known [5] that the number $\alpha(n; k_1, \ldots, k_n)$ of monotone triangles with bottom row $k_1, k_2, \ldots, k_n$ is given by

$$\prod_{1 \leqslant p < q \leqslant n} \left(E_{k_p} + E_{k_q}^{-1} - E_{k_p} E_{k_q}^{-1}\right) \prod_{1 \leqslant i < j \leqslant n} \frac{k_j - k_i + j - i}{j - i}$$
$$= \prod_{1 \leqslant p < q \leqslant n} (\mathrm{id} + \Delta_{k_p} \delta_{k_q}) \prod_{1 \leqslant i < j \leqslant n} \frac{k_j - k_i + j - i}{j - i}, \tag{6.1}$$

where $\delta_x := \mathrm{id} - E_x^{-1}$. To start with, we give two different combinatorial extensions of $\alpha(n; k_1, \ldots, k_n)$ to all $(k_1, \ldots, k_n) \in \mathbb{Z}^n$ in this section, and then present certain other properties of $\alpha(n; k_1, \ldots, k_n)$, for which it would be nice to have combinatorial proofs of the type as we have presented them in this article for Gelfand–Tsetlin patterns. This is because these properties imply, on the one hand, (6.1) and, on the other hand, the refined alternating sign matrix theorem. The latter will be explained at the end of this section.

6.1. Two combinatorial extensions of $\alpha(n; k_1, \ldots, k_n)$ to all $(k_1, \ldots, k_n) \in \mathbb{Z}^n$

The quantity $\alpha(n; k_1, \ldots, k_n)$ obviously satisfies the following recursion for any sequence $(k_1, k_2, \ldots, k_n)$ of strictly increasing integers.

$$\alpha(n; k_1, \ldots, k_n) = \sum_{\substack{(l_1, \ldots, l_{n-1}) \in \mathbb{Z}^{n-1} \\ k_1 \leqslant l_1 \leqslant k_2 \leqslant l_2 \leqslant k_3 \leqslant \cdots \leqslant k_{n-1} \leqslant l_{n-1} \leqslant k_n, l_i \neq l_{i+1}}} \alpha(n-1; l_1, \ldots, l_{n-1}). \tag{6.2}$$

To obtain an extension of the combinatorial interpretation of $\alpha(n; k_1, \ldots, k_n)$ to all $(k_1, \ldots, k_n) \in \mathbb{Z}^n$, it is convenient to write this summation in terms of "simple" summations $\sum_{i=a}^{b} f(i)$, i.e. summations over intervals. This is because we can then use the extended definition of the summation, i.e. $\sum_{i=a}^{a-1} f(i) = 0$ and $\sum_{i=a}^{b} f(i) = -\sum_{i=b+1}^{a-1} f(i)$ if $b + 1 \leqslant a - 1$. Note that if $p(i)$ is a polynomial in i then there exists a polynomial $q(i)$ with $\Delta_i q(i) = p(i)$, which implies $\sum_{i=a}^{b} p(i) = q(b+1) - q(a)$ if $a \leqslant b$ and, consequently, that this sum is a polynomial in a and b. The extension of the simple summation we have just introduced was chosen such that the latter identity is true for all $a, b \in \mathbb{Z}$. After we have given at least one representation of the summation in (6.2) in terms of simple summations, this shows that $\alpha(n; k_1, \ldots, k_n)$ can be represented by a polynomial in $k_1, k_2, \ldots, k_n$ if $k_1 < k_2 < \cdots < k_n$. (This polynomial is in fact uniquely determined by its values on the set of n-tuples $(k_1, k_2, \ldots, k_n) \in \mathbb{Z}^n$ with $k_1 < k_2 < \cdots < k_n$.) The extended monotone triangles with prescribed bottom row $k_1, k_2, \ldots, k_n$ will be chosen such that these objects are enumerated by this polynomial for all $(k_1, \ldots, k_n) \in \mathbb{Z}^n$. In particular, it will certainly not be the naive extension, which sets $\alpha(n; k_1, \ldots, k_n) = 0$ if $k_1, k_2, \ldots, k_n$ is not strictly increasing.

6.1.1. First extension

If we assume that $k_1 < k_2 < \cdots < k_n$, then one possibility to write the summation in (6.2) in terms of simple summations is the following: we choose a subset $\{l_{i_1}, l_{i_2}, \ldots, l_{i_p}\} \subseteq \{l_1, \ldots, l_{n-1}\}$ for which we have $l_{i_j} = k_{i_j}$. For all other l_q we have $k_q < l_q \leqslant k_{q+1}$, except for the case that $q+1 = i_j$ for a j, where we have $k_q < l_q < k_{q+1}$. More formally,

$$\sum_{p \geqslant 0} \sum_{1 \leqslant i_1 < i_2 < \cdots < i_p \leqslant n-1} \sum_{l_1=k_1+1}^{k_2} \sum_{l_2=k_2+1}^{k_3} \cdots \sum_{l_{i_1-1}=k_{i_1-1}+1}^{k_{i_1}-1} \sum_{l_{i_1}=k_{i_1}}^{k_{i_1}} \cdots \sum_{l_{i_p-1}=k_{i_p-1}+1}^{k_{i_p}-1} \sum_{l_{i_p}=k_{i_p}}^{k_{i_p}} \cdots \sum_{l_{n-1}=k_{n-1}+1}^{k_n}$$

where in the exceptional case that $i_j = i_{j-1}+1$ the expression $\sum_{l_{i_j-1}=k_{i_j-1}+1}^{k_{i_j}-1} \sum_{l_{i_j}=k_{i_j}}^{k_{i_j}}$ is replaced by $\sum_{l_{i_{j-1}}=k_{i_{j-1}}}^{k_{i_{j-1}}} \sum_{l_{i_j}=k_{i_j}}^{k_{i_j}}$. This leads to the following extension: a *monotone triangle of order n and type* 1 is a triangular array $(a_{i,j})_{1 \leqslant j \leqslant i \leqslant n}$ of integers such that the following conditions are fulfilled.

- There is a subset of special entries $a_{i,j}$ with $i < n$ for which we require $a_{i,j} = a_{i+1,j}$. We mark these entries with a star on the left.
- If $a_{i,j}$ is not a special entry then we have to distinguish between the case that $a_{i,j}$ is the left neighbor of a special entry or not.
  - If $a_{i,j+1}$ is not special (which includes also the case that $a_{i,j+1}$ does not exist) then $a_{i+1,j} < a_{i,j} \leqslant a_{i+1,j+1}$ in case that $a_{i+1,j} < a_{i+1,j+1}$ and $a_{i+1,j+1} < a_{i,j} \leqslant a_{i+1,j}$ otherwise. (There exists no pattern with $a_{i+1,j} = a_{i+1,j+1}$.) In the latter case we have an inversion.
  - If $a_{i,j+1}$ is special then $a_{i+1,j} < a_{i,j} < a_{i+1,j+1}$ or $a_{i+1,j+1} \leqslant a_{i,j} \leqslant a_{i+1,j}$. (There exists no pattern with $a_{i+1,j+1} = a_{i+1,j}+1$.) In the latter case we have an inversion.

The sign of a monotone triangle is -1 to the number of inversions. Then $\alpha(n; k_1, \ldots, k_n)$ is the signed enumeration of monotone triangles with $a_{n,i} = k_i$. Here is an example of such an array.

$$\begin{array}{ccccccccc} & & & & 3 & & & & \\ & & & {}^*2 & & 6 & & & \\ & & 2 & & 4 & & {}^*6 & & \\ & 3 & & {}^*1 & & 6 & & 7 & \\ 3 & & 1 & & 7 & & 5 & & 8 \end{array}$$

Although we prefer the second extension, the first extension has the property that, in case $k_1 < k_2 < \cdots < k_n$, the removal of all stars leads to a monotone triangle in the original sense and no array is assigned a minus sign, i.e. we have a plain enumeration in this case. Note that we provide two variations of this extension in Appendix C.

6.1.2. Second extension

In order to explain the representation of (6.2) in terms of simple summations which is used for the second extension, it is convenient to use the operator $V_{x,y} := E_x^{-1} + E_y - E_x^{-1} E_y$. Then, for any function $a : \mathbb{Z}^{n-1} \to \mathbb{C}$,

$$\sum_{\substack{k_1 \leqslant l_1 \leqslant k_2 \leqslant l_2 \leqslant \cdots \leqslant k_{n-1} \leqslant l_{n-1} \leqslant k_n, \\ l_i \neq l_{i+1}}} a(l_1, \ldots, l_{n-1})$$
$$= V_{k_1,k_1'} V_{k_2,k_2'} \cdots V_{k_n,k_n'} \sum_{l_1=k_1'}^{k_2} \sum_{l_2=k_2'}^{k_3} \cdots \sum_{l_{n-1}=k_{n-1}'}^{k_n} a(l_1, \ldots, l_{n-1}) \Big|_{k_i'=k_i},$$

if $k_1 < k_2 < \cdots < k_n$ is strictly increasing. (Note that $V_{k_1,k_1'}$ as well as $V_{k_n,k_n'}$ can also be removed as the application of $V_{x,y}$ to a function which does not depend on x *and* y acts as the identity. In order to convince oneself that this is indeed a valid representation of the summation in (6.2), one can use induction with respect to n to transform it into the representation of the first extension.) This leads to the following extension, which we think is the nicest: a *monotone triangle of order* n *and type* 2 is an integer array $(a_{i,j})_{1 \leqslant j \leqslant i \leqslant n}$ together with a function f which assigns to each $a_{i,j}$ an element of $\{\leftarrow, \rightarrow, \leftrightarrow\}$ such that the following conditions are fulfilled for any element $a_{i,j}$ with $i < n$: we have to distinguish cases depending on the assignment of the arrows to the elements $a_{i+1,j}$ and $a_{i+1,j+1}$.

(1) $f(a_{i+1,j}) = \leftarrow$, $f(a_{i+1,j+1}) = \leftarrow, \leftrightarrow$: $a_{i+1,j} \leqslant a_{i,j} < a_{i+1,j+1}$ or $a_{i+1,j+1} \leqslant a_{i,j} < a_{i+1,j}$;
(2) $f(a_{i+1,j}) = \leftarrow$, $f(a_{i+1,j+1}) = \rightarrow$: $a_{i+1,j} \leqslant a_{i,j} \leqslant a_{i+1,j+1}$ or $a_{i+1,j+1} < a_{i,j} < a_{i+1,j}$;
(3) $f(a_{i+1,j}) = \leftrightarrow, \rightarrow$, $f(a_{i+1,j+1}) = \leftarrow, \leftrightarrow$: $a_{i+1,j} < a_{i,j} < a_{i+1,j+1}$ or $a_{i+1,j+1} \leqslant a_{i,j} \leqslant a_{i+1,j}$;
(4) $f(a_{i+1,j}) = \leftrightarrow, \rightarrow$, $f(a_{i+1,j+1}) = \rightarrow$: $a_{i+1,j} < a_{i,j} \leqslant a_{i+1,j+1}$ or $a_{i+1,j+1} < a_{i,j} \leqslant a_{i+1,j}$.

In Case 1 and Case 4, there exists no pattern if $a_{i+1,j} = a_{i+1,j+1}$, in Case 2, we have no pattern if $a_{i+1,j} = a_{i+1,j+1} + 1$ and, in Case 3, there is no pattern if $a_{i+1,j+1} = a_{i+1,j} + 1$. In each case, we say that $a_{i,j}$ is an inversion if the second possibility applies. We define the sign of a monotone triangle to be -1 to the number of inversions plus the number of elements that are assigned the element "$\leftrightarrow$". Then $\alpha(n; k_1, \ldots, k_n)$ is the signed enumeration of monotone triangles $(a_{i,j})_{1 \leqslant j \leqslant i \leqslant n}$ of order n with $a_{n,i} = k_i$. Here is an example.

$$
\begin{array}{ccccccccc}
 & & & & \overset{\leftrightarrow}{5} & & & & \\
 & & & \overset{\leftarrow}{5} & & \overset{\rightarrow}{6} & & & \\
 & & \overset{\leftarrow}{4} & & \overset{\leftarrow}{6} & & \overset{\leftrightarrow}{7} & & \\
 & \overset{\leftarrow}{2} & & \overset{\leftarrow}{6} & & \overset{\rightarrow}{7} & & \overset{\rightarrow}{5} & \\
\overset{\rightarrow}{3} & & \overset{\leftrightarrow}{1} & & \overset{\rightarrow}{8} & & \overset{\rightarrow}{5} & & \overset{\rightarrow}{4}
\end{array}
$$

In order to see that this extension comes from the presentation given above, note that, when expanding

$$V_{k_1,k_1'} V_{k_2,k_2'} \cdots V_{k_n,k_n'} = \left(E_{k_1}^{-1} + E_{k_1'} - E_{k_1}^{-1} E_{k_1'}\right)\left(E_{k_2}^{-1} + E_{k_2'} - E_{k_2}^{-1} E_{k_2'}\right) \cdots \left(E_{k_n}^{-1} + E_{k_n'} - E_{k_n}^{-1} E_{k_n'}\right),$$

the assignment of "$\leftarrow$" to the entry k_i in the bottom row corresponds to choosing $E_{k_i}^{-1}$ from the operator $V_{k_i,k_i'}$, while the assignment of "$\rightarrow$" to k_i corresponds to choosing $E_{k_i'}$ and the assignment of "$\leftrightarrow$" corresponds to choosing $E_{k_i}^{-1} E_{k_i'}$.

In both cases, the combinatorial extension of $\alpha(n; k_1, \ldots, k_n)$ is, generally speaking, a signed enumeration, which reduces to a plain enumeration in the first case if $k_1, k_2, \ldots, k_n$ is strictly increasing. This can be generalized as follows.

Proposition 3. *Suppose* $k_1, k_2, \ldots, k_n$ *is a weakly increasing sequence of integers then* $\alpha(n; k_1, \ldots, k_n)$ *is the number of Gelfand–Tsetlin patterns with prescribed bottom row* $k_1, \ldots, k_n$ *and where all other rows are strictly increasing.*

Proof. In order to see this, we use the first extension. Suppose $k_j = k_{j+1}$ and $(a_{i,j})_{1 \leqslant j \leqslant i \leqslant n}$ is a respective pattern. As $a_{n,j} = a_{n,j+1}$ it follows that $a_{n-1,j}$ equal to this quantity as well and at least one of $a_{n-1,j}$ and $a_{n-1,j+1}$ must be special. We can exclude the latter possibility by the following sign reversing involution on the extended monotone triangles where $a_{n-1,j+1}$ is special in such a situation: let j be maximal with this property. Then, changing the status of $a_{n-1,j}$ (from special to not special or vice versa) is a sign reversing involution. Thus we can assume that $a_{n-1,j+1}$ is not special (and, consequently, $a_{n-1,j}$ must be special) whenever we have $a_{n,j} = a_{n,j+1}$.

This can be used to show that $\alpha(n; k_1, \ldots, k_n) = 0$ if there are p, q with $1 \leqslant p < q \leqslant n-1$ such that $k_p = k_{p+1}$, $k_q = k_{q+1}$ and $k_j + 1 = k_{j+1}$ for $p < j < q$, which is one special case of the statement: as $a_{n-1,p+1}$ can be assumed not to be special (which already settles the case $q = p+1$) we can deduce that $a_{n-1,p+2}$ is not special (otherwise we would have no choice for $a_{n-1,p+1}$) and, by iterating this argument, we can see that $a_{n-1,j}$ is not special for $p+1 \leqslant j \leqslant q-1$. This implies that $a_{n-1,p+1} = a_{n,p+2}, a_{n-1,p+2} = a_{n,p+3}, \ldots, a_{n-1,q-1} = a_{n,q}$. On the other hand, the fact that $a_{n-1,q}$ is special implies $a_{n-1,q-1} = a_{n,q-1}$, which is a contradiction.

Thus we may assume that such p, q do not exist for our sequence $k_1, k_2, \ldots, k_n$. Consequently, if $k_j = k_{j+1}$ then $k_{j-1} < k_j$ and $k_{j+1} < k_{j+2}$. As $a_{n-1,j}$ is special and $a_{n-1,j+1}$ is not, we have $a_{n-1,j-1} < a_{n-1,j} < a_{n-1,j+1}$. $\square$

It should be remarked that the signed enumeration in both extensions is in general not a plain enumeration if $k_1, \ldots, k_n$ is weakly increasing but not strictly increasing. Also note that the proposition is equivalent to the fact that, for weakly increasing sequences $k_1, k_2, \ldots, k_n$, the application of the summation in (6.2) to $\alpha(n-1; l_1, \ldots, l_{n-1})$ is equivalent to the application of the representation of this summation in terms of simple summations to $\alpha(n-1; l_1, \ldots, l_{n-1})$. (If the sequence is not increasing then the summation in (6.2) is over the empty set and therefore zero.) As a next step, it would be interesting to figure out whether there is a notion analogous to that of Gelfand–Tsetlin tree sequences for monotone triangles. This could be helpful in understanding the properties of $\alpha(n; k_1, \ldots, k_n)$, which we list next.

6.1.3. Properties of $\alpha(n; k_1, \ldots, k_n)$

In previous papers we have shown that $\alpha(n; k_1, \ldots, k_n)$ has the following properties.

(1) For $n \geqslant 1$ and $i \in \{1, 2, \ldots, n-1\}$, we have

$$\left(\mathrm{id} + E_{k_{i+1}} E_{k_i}^{-1} S_{k_i,k_{i+1}}\right) V_{k_i,k_{i+1}} \alpha(n; k_1, \ldots, k_n) = 0.$$

(This is proved in [5].)

(2) For $n \geqslant 1$ and $i \in \{1, 2, \ldots, n\}$, we have $\deg_{k_i} \alpha(n; k_1, \ldots, k_n) \leqslant n-1$. (See [5].)

(3) For $n \geqslant 1$, we have $\alpha(n; k_1, \ldots, k_n) = (-1)^{n-1} \alpha(n; k_2, \ldots, k_n, k_1 - n)$. (A proof can be found in [6].)

(4) For $n \geqslant 1$ and $p \geqslant 1$, we have

$$e_p(\Delta_{k_1}, \ldots, \Delta_{k_n}) \alpha(n; k_1, \ldots, k_n) = 0.$$

(See Lemma 1 in [6].)

The first property is obviously the analog of the shift-antisymmetry of $L_n(\mathcal{T}, \mathbf{k})$ as the latter can obviously be formulated as follows.

$$\left(\mathrm{id} + E_{k_{i+1}} E_{k_i}^{-1} S_{k_i,k_{i+1}}\right) L_n(\mathcal{T}, \mathbf{k}) = 0.$$

It is interesting to note that a special case of this property for $\alpha(n; k_1, \ldots, k_n)$ follows from Proposition 3: if we specialize $k_{i+1} = k_i - 1$ then the first property simplifies to

$$\begin{aligned} &\alpha(n; k_1, \ldots, k_{i-1}, k_i - 1, k_i - 1, k_{i+2}, \ldots, k_n) + \alpha(n; k_1, \ldots, k_{i-1}, k_i, k_i, k_{i+2}, \ldots, k_n) \\ &\quad - \alpha(n; k_1, \ldots, k_{i-1}, k_i - 1, k_i, k_{i+2}, \ldots, k_n) = 0. \end{aligned}$$

However, for integers $k_1, k_2, \ldots, k_i, k_{i+2}, \ldots, k_n$ with $k_1 < k_2 < \cdots < k_{i-1} < k_i - 1$ and $k_i < k_{i+2} < \cdots < k_{n-1} < k_n$, Proposition 3 implies this identity: in a monotone triangle $(a_{i,j})_{1 \leqslant j \leqslant i \leqslant n}$ with bottom row $k_1, \ldots, k_{i-1}, k_i - 1, k_i, k_{i+2}, \ldots, k_n$ we have either $a_{n-1,i} = k_i - 1$, which corresponds to the case that we have $k_1, \ldots, k_{i-1}, k_i - 1, k_i - 1, k_{i+2}, \ldots, k_n$ as bottom row, or $a_{n-1,i} = k_i$, which corresponds to the case that $k_1, \ldots, k_{i-1}, k_i, k_i, k_{i+2}, \ldots, k_n$ is the bottom row. As a polynomial in $k_1, k_2, \ldots, k_i, k_{i+2}, \ldots, k_n$ is uniquely determined by its values on the set of these elements $(k_1, k_2, \ldots, k_i, k_{i+2}, \ldots, k_n) \in \mathbb{Z}^{n-1}$, the identity follows.

Concerning the second property, we have seen that it also holds for $\mathcal{L}_n(\mathcal{T}, \mathbf{k})$. Both properties together actually imply (6.2), see [5], and thus it would be interesting to give combinatorial proofs of these properties.

6.1.4. Property (3) implies the refined alternating sign matrix theorem

The third property is interesting as it holds also for Gelfand–Tsetlin patterns where it can easily be deduced from the shift-antisymmetry. However, it is a mystery that it also holds for monotone triangles, as we do not see how it can be deduced from the first property. Quite remarkably, it can be used to deduce the refined alternating sign matrix theorem as we explain next.

The number $\overline{A}_{n,i}$ of $n \times n$ alternating sign matrices, where the unique 1 in the first row is located in the i-th column is equal to the number of monotone triangles with bottom row $1, 2, \dots, n$ and i appearances of 1 in the first NE-diagonal, or, equivalently, the number of monotone triangles with bottom row $1, 2, \dots, n$ and i appearances of n in the last SE-diagonal. (This follows immediately from the standard bijection between alternating sign matrices and monotone triangles.) If we assume that $k_1 \leqslant k_2 < \cdots < k_n$, then the number of "partial" monotone triangles with n rows, where the entries $a_{n,1}, a_{n-1,1}, \dots, a_{n-i+1,1}$ are removed, no entry is smaller than k_1 and $a_{n,i} = k_i$ for $i = 2, 3, \dots, n$ is equal to

$$(-1)^{i-1} \Delta_{k_1}^{i-1} \alpha(n; k_1, \dots, k_n)\big|_{(k_1,\dots,k_n)=(1,1,2,\dots,n-1)}.$$

(A proof is given in [8].) In fact, it follows quite easily by induction with respect to i as

$$\begin{aligned}
-\Delta_{k_1} \Biggl(& \sum_{\substack{(l_1,\dots,l_{n-1}) \in \mathbb{Z}^{n-1} \\ k_1 \leqslant l_1 \leqslant k_2 \leqslant l_2 \leqslant k_3 \leqslant \cdots \leqslant k_{n-1} \leqslant l_{n-1} \leqslant k_n, l_i \neq l_{i+1}}} a(l_1, \dots, l_{n-1}) \Biggr) \\
&= \sum_{\substack{(l_2,\dots,l_{n-1}) \in \mathbb{Z}^{n-2} \\ k_2 \leqslant l_2 \leqslant k_3 \leqslant \cdots \leqslant k_{n-1} \leqslant l_{n-1} \leqslant k_n, l_i \neq l_{i+1}}} a(k_1, l_2, \dots, l_{n-1}),
\end{aligned}$$

for any function $a : \mathbb{Z}^{n-1} \to \mathbb{C}$. This implies the first identity in

$$\begin{aligned}
\overline{A}_{n,i} &= (-1)^{i-1} \Delta_{k_1}^{i-1} \alpha(n; k_1, \dots, k_n)\big|_{(k_1,\dots,k_n)=(1,1,2,\dots,n-1)} \\
&= \delta_{k_n}^{i-1} \alpha(n; k_1, \dots, k_n)\big|_{(k_1,\dots,k_n)=(1,2,\dots,n-1,n-1)}.
\end{aligned}$$

The proof of the fact that the first expression is also equal to the last expression is similar. Therefore, by property (3),

$$\begin{aligned}
\overline{A}_{n,i} &= (-1)^{i+n} \Delta_{k_1}^{i-1} \alpha(n; k_2, \dots, k_n, k_1 - n)\big|_{(k_1,\dots,k_n)=(1,1,2,\dots,n-1)} \\
&= (-1)^{i+n} \delta_{k_1}^{i-1} E_{k_1}^{-2n+1+i} \alpha(n; k_2, \dots, k_n, k_1)\big|_{(k_2,\dots,k_n,k_1)=(1,2,\dots,n-1,n-1)}.
\end{aligned}$$

We use $E_x^{-m} = (\mathrm{id} - \delta_x)^m = \sum_{j=0}^{m} \binom{m}{j} (-1)^j \delta_x^j$ to see that this is equal to

$$\begin{aligned}
&(-1)^{i+n} \delta_{k_1}^{i-1} \sum_{j=0}^{2n-1-i} \binom{2n-1-i}{j} (-1)^j \delta_{k_1}^j \alpha(n; k_2, \dots, k_n, k_1)\Big|_{(k_2,\dots,k_n,k_1)=(1,2,\dots,n-1,n-1)} \\
&= \sum_{j=0}^{2n-1-i} \binom{2n-1-i}{j} (-1)^{i+j+n} \overline{A}_{n,i+j}.
\end{aligned}$$

This shows that the refined alternating sign matrix numbers $\overline{A}_{n,i}$ are a solution of the following system of linear equations:

$$\overline{A}_{n,i} = \sum_{k=1}^{n} \binom{2n-1-i}{k-i} (-1)^{k+n} \overline{A}_{n,k}, \quad 1 \leqslant i \leqslant n.$$

This identity already appeared in [6]; the derivation given here is simpler. In [6], it was also shown that this system of linear equations together with the obvious symmetry $\overline{A}_{n,i} = \overline{A}_{n,n+1-i}$ determines the numbers $\overline{A}_{n,i}$ inductively with respect to n.

It is worth mentioning that a similar reasoning can be applied to the doubly refined enumeration $\overline{\underline{A}}_{n,i,j}$ of $n \times n$ alternating sign matrices with respect to the position i of the 1 in first row and the position j of the 1 in the last row. This number is equal to the number of monotone triangles with bottom row $1, 2, \ldots, n$ and i appearances of 1 in the first NE-diagonal and j appearances of n in the last SE-diagonal, which implies (see [8]) that

$$\overline{\underline{A}}_{n,i,j} = (-1)^{i-1} \Delta_{k_1}^{i-1} \delta_{k_n}^{j-1} \alpha(n; k_1, \ldots, k_n)\big|_{(k_1,\ldots,k_n)=(2,2,\ldots,n-1,n-1)}.$$

Using the first and the third property of $\alpha(n; k_1, \ldots, k_n)$ displayed above we deduce the following identity:

$$\begin{aligned}
&\left(\mathrm{id} + E_{k_n}^{n-1} E_{k_1}^{-n+1} S_{k_1,k_n}\right) V_{k_n,k_1} \alpha(n; k_1, \ldots, k_n) \\
&\quad = (-1)^{n-1} \left(\mathrm{id} + E_{k_n}^{n-1} E_{k_1}^{-n+1} S_{k_1,k_n}\right) V_{k_n,k_1} \alpha(n; k_2, \ldots, k_n, k_1 - n) = 0.
\end{aligned}$$

We apply $(-1)^{i-1} \Delta_{k_1}^{i-1} \delta_{k_n}^{j-1}$ to the equivalent identity

$$\begin{aligned}
0 &= \alpha(n; k_1, \ldots, k_n) + \Delta_{k_1} \delta_{k_n} \alpha(n; k_1, \ldots, k_n) \\
&\quad + E_{k_1}^{-2n+4} E_{k_n}^{2n-4} \alpha(n; k_n - n + 3, k_2, \ldots, k_{n-1}, k_1 + n - 3) \\
&\quad + E_{k_1}^{-2n+4} E_{k_n}^{2n-4} \delta_{k_1} \Delta_{k_n} \alpha(n; k_n - n + 3, k_2, \ldots, k_{n-1}, k_1 + n - 3)
\end{aligned}$$

to see that

$$\begin{aligned}
0 &= (-1)^{i-1} \Delta_{k_1}^{i-1} \delta_{k_n}^{j-1} \alpha(n; k_1, \ldots, k_n) - (-1)^{i} \Delta_{k_1}^{i} \delta_{k_n}^{j} \alpha(n; k_1, \ldots, k_n) \\
&\quad + E_{k_1}^{-2n+3+i} E_{k_n}^{2n-3-j} (-1)^{i-1} \delta_{k_1}^{i-1} \Delta_{k_n}^{j-1} \alpha(n; k_n - n + 3, k_2, \ldots, k_{n-1}, k_1 + n - 3) \\
&\quad + E_{k_1}^{-2n+3+i} E_{k_n}^{2n-3-j} (-1)^{i-1} \delta_{k_1}^{i} \Delta_{k_n}^{j} \alpha(n; k_n - n + 3, k_2, \ldots, k_{n-1}, k_1 + n - 3).
\end{aligned}$$

Now we use the expansions

$$E_{k_1}^{-2n+3+i} = (\mathrm{id} - \delta_{k_1})^{2n-3-i} = \sum_{p=0}^{2n-3-i} \binom{2n-3-i}{p} (-1)^p \delta_{k_1}^p$$

and

$$E_{k_n}^{2n-3-j} = (\mathrm{id} + \Delta_{k_n})^{2n-3-j} = \sum_{q=0}^{2n-3-j} \binom{2n-3-j}{q} \Delta_{k_n}^q$$

to see that

$$\begin{aligned}
0 = {} & (-1)^{i-1}\Delta_{k_1}^{i-1}\delta_{k_n}^{j-1}\alpha(n;k_1,\dots,k_n) - (-1)^i\Delta_{k_1}^{i}\delta_{k_n}^{j}\alpha(n;k_1,\dots,k_n) \\
& + \sum_{p=0}^{2n-3-i}\sum_{q=0}^{2n-3-j}\binom{2n-3-i}{p}\binom{2n-3-j}{q} \\
& \times (-1)^{i-1+p}\delta_{k_1}^{p+i-1}\Delta_{k_n}^{q+j-1}\alpha(n;k_n-n+3,k_2,\dots,k_{n-1},k_1+n-3) \\
& + \sum_{p=0}^{2n-3-i}\sum_{q=0}^{2n-3-j}\binom{2n-3-i}{p}\binom{2n-3-j}{q} \\
& \times (-1)^{i-1+p}\delta_{k_1}^{i+p}\Delta_{k_n}^{j+p}\alpha(n;k_n-n+3,k_2,\dots,k_{n-1},k_1+n-3).
\end{aligned}$$

We evaluate at $(k_1,k_2,\dots,k_{n-1},k_n)=(2,2,3,\dots,n-2,n-1,n-1)$ to arrive at

$$\begin{aligned}
& \underline{\overline{A}}_{n,i+1,j+1} - \underline{\overline{A}}_{n,i,j} \\
& = \sum_{p=0}^{2n-3-i}\sum_{q=0}^{2n-3-j}\binom{2n-3-i}{p}\binom{2n-3-j}{q}(-1)^{i+j+p+q}(\underline{\overline{A}}_{n,q+j,p+i} - \underline{\overline{A}}_{n,q+j+1,p+i+1}).
\end{aligned}$$

(This identity is not included in [6] and seems to be new.) Computer experiments led us to the conjecture that this identity together with the obvious relations $\underline{\overline{A}}_{n,i,j} = \underline{\overline{A}}_{n,j,i}$ and $\underline{\overline{A}}_{n,i,j} = \underline{\overline{A}}_{n,n+1-i,n+1-j}$ determine the doubly refined enumeration numbers $\underline{\overline{A}}_{n,i,j}$ uniquely inductively with respect to n.

6.1.5. Properties (1) and (4) imply property (3).

The analog of the fourth property is true for Gelfand–Tsetlin tree sequences, see (5.1), for which we gave a combinatorial proof in Section 4. The significance of this property is that it can be used to deduce the third property from the first property. Since every symmetric polynomial in $X_1, X_2, \dots, X_n$ can be written as a polynomial in the elementary symmetric functions, this property implies that

$$p(E_{k_1},\dots,E_{k_n})\alpha(n;k_1,\dots,k_n) = p(1,1,\dots,1)\alpha(n;k_1,\dots,k_n)$$

for every symmetric polynomial $p(X_1,\dots,X_n)$ in $X_1,\dots,X_n$. This extends to symmetric polynomials in $X_1, X_1^{-1},\dots,X_n,X_n^{-1}$: let $p(X_1,\dots,X_n)$ be such a polynomial and $t \in \mathbb{Z}$ such that $p(X_1,\dots,X_n)X_1^t\cdots X_n^t =: q(X_1,\dots,X_n)$ is a symmetric polynomial in $X_1,\dots,X_n$ then

$$\begin{aligned}
p(E_{k_1},\dots,E_{k_n})\alpha(n;k_1,\dots,k_n) &= E_{k_1}^t\cdots E_{k_n}^t p(E_{k_1},\dots,E_{k_n})\alpha(n;k_1-t,\dots,k_n-t) \\
&= E_{k_1}^t\cdots E_{k_n}^t p(E_{k_1},\dots,E_{k_n})\alpha(n;k_1,\dots,k_n) \\
&= q(1,1,\dots,1)\alpha(n;k_1,\dots,k_n) \\
&= p(1,1,\dots,1)\alpha(n;k_1,\dots,k_n).
\end{aligned}$$

In particular, this shows that (5.1) is also true if all "Δ"s are replaced by "δ"s. Now we are ready to deduce property (3) from property (1) and property (4): note that the operator $V_{x,y}$ is invertible as an operator on polynomials in x and y: this follows as $V_{x,y} = \mathrm{id} + \delta_x\Delta_y$ and

$$V_{x,y}^{-1} = \sum_{i=0}^{\infty}(-1)^i\delta_x^i\Delta_y^i.$$

(The sum is finite when applied to polynomials.) Property (1) is obviously equivalent to

$$\alpha(n; k_1, \dots, k_{i-1}, k_{i+1}+1, k_i-1, k_{i+2}, \dots, k_n) = -V_{k_i,k_{i+1}} V^{-1}_{k_{i+1},k_i} \alpha(n; k_1, \dots, k_n).$$

This implies

$$\begin{aligned}(-1)^{n-1}\alpha(n; k_2, \dots, k_n, k_1-n) &= (-1)^{n-1}\alpha(n; k_2+1, \dots, k_n+1, k_1-n+1) \\ &= \prod_{i=2}^{n} V_{k_1,k_i} V^{-1}_{k_i,k_1} \alpha(n; k_1, \dots, k_n).\end{aligned}$$

Therefore, in order to show the third property, we have to prove that

$$\left(\prod_{i=2}^{n} V_{k_1,k_i} - \prod_{i=2}^{n} V_{k_i,k_1}\right)\alpha(n; k_1, \dots, k_n) = 0.$$

This follows from the fourth property as

$$\begin{aligned}\prod_{i=2}^{n} V_{k_1,k_i} - \prod_{i=2}^{n} V_{k_i,k_1} &= \prod_{i=2}^{n}(\mathrm{id} + \delta_{k_1}\Delta_{k_i}) - \prod_{i=2}^{n}(\mathrm{id} + \Delta_{k_1}\delta_{k_i}) \\ &= \sum_{r=0}^{n-1} \delta_{k_1}^{r} e_r(\Delta_{k_2}, \dots, \Delta_{k_n}) - \sum_{r=0}^{n-1} \Delta_{k_1}^{r} e_r(\delta_{k_2}, \dots, \delta_{k_n}) \\ &= \sum_{r=0}^{n-1}\big(\delta_{k_1}^{r}\big(e_r(\Delta_{k_1}, \dots, \Delta_{k_n}) - \Delta_{k_1} e_{r-1}(\Delta_{k_2}, \dots, \Delta_{k_n})\big) \\ &\quad - \Delta_{k_1}^{r}\big(e_r(\delta_{k_1}, \dots, \delta_{k_n}) - \delta_{k_1} e_{r-1}(\delta_{k_2}, \dots, \delta_{k_n})\big)\big) \\ &= \sum_{r=0}^{n-1}\big(\delta_{k_1}^{r} e_r(\Delta_{k_1}, \dots, \Delta_{k_n}) - \Delta_{k_1}^{r} e_r(\delta_{k_1}, \dots, \delta_{k_n})\big) \\ &\quad - \sum_{r=1}^{n-1}\big(\delta_{k_1}^{r}\Delta_{k_1} e_{r-1}(\Delta_{k_2}, \dots, \Delta_{k_n}) - \Delta_{k_1}^{r}\delta_{k_1} e_{r-1}(\delta_{k_2}, \dots, \delta_{k_n})\big) = \cdots \\ &= \sum_{s=1}^{n}\sum_{r=1}^{n-s}(-1)^s\big(\Delta_{k_1}^{r+s-1}\delta_{k_1}^{s-1} e_r(\delta_{k_1}, \dots, \delta_{k_n}) - \delta_{k_1}^{r+s-1}\Delta_{k_1}^{s-1} e_r(\Delta_{k_1}, \dots, \Delta_{k_n})\big).\end{aligned}$$

(This derivation is a simplified version of the proof of [6, Lemma 5].)

Appendix A. The non-intersecting lattice paths point of view

In Fig. 11, the family of non-intersecting lattice paths that corresponds to the Gelfand–Tsetlin pattern given in the introduction is displayed: in general, the lattice paths join the starting points $(0,0), (-1,1), \dots, (-n+1, n-1)$ to the end points $(1,k_1), (1,k_2+1), \dots, (1,k_n+n-1)$, where the lattice paths can take east and north steps of length 1 and end with a step to the east. As indicated in the drawing, the heights of the horizontal steps of the i-th path, counted from the bottom, can be obtained from the i-th southeast diagonal of the Gelfand–Tsetlin pattern, counted from the left, by adding i to the entries in the respective diagonal of the Gelfand–Tsetlin pattern. By a well-known

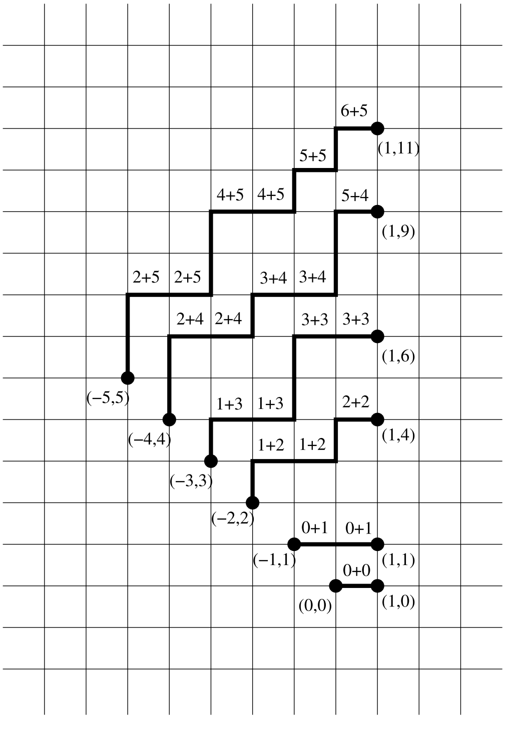

Fig. 11. Non-intersecting lattice paths.

result on the enumeration of non-intersecting lattice paths of Lindström [12, Lemma 1] and of Gessel and Viennot [10, Theorem 1], this number is equal to

$$\det_{1 \leqslant i,j \leqslant n} \binom{k_j + j - 1}{i - 1}, \tag{A.1}$$

which is, by the Vandermonde determinant evaluation, equal to (1.1). (Note that $\binom{k_j+j-1}{i-1}$ is a polynomial in k_j of degree $i-1$.)

Interestingly, another possibility to extend the combinatorial interpretation of (1.1) to all $(k_1, \dots, k_n) \in \mathbb{Z}^n_{\geqslant 0}$ is related to this interpretation in terms of families of non-intersection lattice paths: for arbitrary non-negative integers $k_1, k_2, \dots, k_n$, consider families of n lattice paths with unit steps to the north and to the east (in general, these families are intersecting for the moment) that connect the starting points $(0,0), (-1,1), \dots, (-n+1, n-1)$ to the endpoints $(0,k_1), (0,k_2+1), \dots, (0,k_n+n-1)$, in any order. (Now we omit the vertical steps at the end of the paths.) Suppose that the i-th starting point $(-i+1, i-1)$ is connected to the π_i-th end point $(0, k_{\pi_i} + \pi_i - 1)$ then the sign of the family is defined as the sign of the permutation $(\pi_1, \pi_2, \dots, \pi_n) = \pi$. Then, (1.1) is the signed enumeration of families of lattice paths with these starting points and end points. The merit of the theorem of Lindström and of Gessel and Viennot is the definition of a sign reversion involution on the families of *intersecting* lattice paths, which shows that only the non-intersecting families remain in the signed enumeration. Depending on the relative positions of the numbers $k_1, k_2+1, \dots, k_n+n-1$, there is at most one permutation π for which a family of non-intersecting lattice paths exists at all. (There is no such permutation if $k_i + i = k_j + j$ for distinct $i, j \in \{1, 2, \dots, n\}$.) This implies that the signed enumeration of families of lattice paths reduces essentially (i.e. up to the sign of π) to the plain enumeration of families of non-intersecting lattice paths.

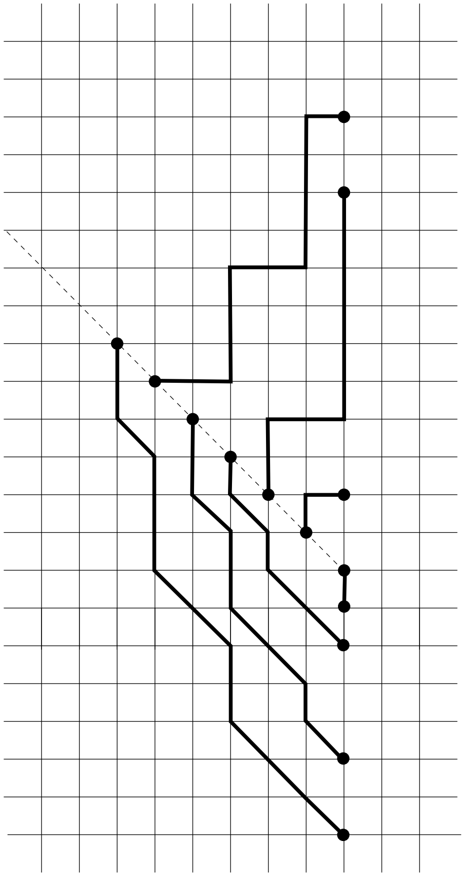

Fig. 12. Non-intersecting lattice paths that go below the x-axis.

Finally, it is worth mentioning (without proof) that the requirement that all k_i are non-negative can be avoided. A close look at the proof shows that this requirement is useful at first place to guarantee that the location of the end points is not "too far" to the south of the starting points. If an end point is south-east of a starting point then there is obviously no lattice path connecting them which only uses steps of the form $(1, 0)$ and $(0, 1)$. However, in such a case it is convenient to allow steps of the form $(1, -1)$ and $(0, -1)$. Moreover if we require these paths to start with a step of the form $(0, -1)$ and let each step of the form $(1, -1)$ contribute a minus sign, we obtain an interpretation of (1.1) for all $(k_1, \dots, k_n) \in \mathbb{Z}^n$. A typical situation is sketched in Fig. 12. Note that in this case there may be more than one permutation π of the end points for which a family of non-intersecting lattice paths exists.

Appendix B. Another proof of the shift-antisymmetry

We sketch a (sort of) combinatorial proof of the shift-antisymmetry of the signed enumeration of Gelfand–Tsetlin *patterns* with prescribed bottom row which does not rely on the notion of Gelfand–Tsetlin tree sequences. The argument is a bit involved and thus shows the merit of the notion of Gelfand–Tsetlin tree sequences. On the other hand, it could be helpful for proving the analogous property for monotone triangles as we have not established a notion that is analogous to that of Gelfand–Tsetlin tree sequences for monotone triangles so far, see Section 6.

The following notion turns out to be extremely useful in order to avoid case distinctions: we define $[x, y] := \{z \in \mathbb{Z} \mid x \leqslant z \leqslant y\}$ if $x \leqslant y$ as usual, $[x, x-1] := \emptyset$ and $[x, y] := [y+1, x-1]$ if $y+1 \leqslant x-1$. The latter situation is said to be an inversion. By considering all possible relative positions of x, y, z, it is not hard to see that

$$[x, y] \triangle [x, z+1] = [y+1, z+1],$$

where $A \triangle B := (A \setminus B) \cup (B \setminus A)$ is the symmetric difference. In fact, concerning this symmetric difference, the following can be observed: either one set is contained in the other or the sets are disjoint. The latter situation occurs iff exactly one of $[x, y]$ and $[x, z+1]$ is an inversion. On the other hand,

$$[z, x] \triangle [y-1, x] = [x+1, z-1] \triangle [x+1, y-2] = [z, y-2] = [y-1, z-1]$$

and we have $[z, x] \setminus [y-1, x] \neq \emptyset$ and $[y-1, x] \setminus [z, x] \neq \emptyset$ (which implies that the two sets are disjoint) iff exactly one of $[z, x]$ and $[y-1, x]$ is an inversion.

Let $\mathcal{L}_n(k_1, \dots, k_n) := \mathcal{L}_n(\mathcal{B}, \mathbf{k})$ denote the set of Gelfand–Tsetlin patterns with bottom row $k_1, k_2, \dots, k_n$ and $L_n(k_1, \dots, k_n) := L_n(\mathcal{B}, \mathbf{k})$ the corresponding signed enumeration. The proof is by induction with respect to n. Nothing is to be done for $n = 1$. Otherwise, it suffices to consider the case $j = i+1$. We fix $i \in \{1, 2, \dots, n-1\}$ and decompose $\mathcal{L}_n(k_1, \dots, k_n)$ into four sets: let $\mathcal{L}^1_{n,i}(k_1, \dots, k_n)$ denote the subset of patterns $(a_{p,q})_{1 \leqslant q \leqslant p \leqslant n} \in \mathcal{L}_n(k_1, \dots, k_n)$ for which the replacement $a_{n,i} \to k_{i+1}+1$ and $a_{n,i+1} \to k_i - 1$ produces another Gelfand–Tsetlin pattern (which is obviously an element of $\mathcal{L}_n(k_1, \dots, k_{i-1}, k_{i+1}+1, k_i-1, k_{i+2}, \dots, k_n)$ then). If we perform this replacement we can either have a contradiction concerning the requirement for $l_{i-1} := a_{n-1,i-1}$ or for $l_{i+1} := a_{n-1,i+1}$. (There cannot be a contradiction for $l_i := a_{n-1,i}$ as $l_i \in [k_i, k_{i+1}]$ if and only if $l_i \in [k_{i+1}+1, k_i-1]$.) We let $\mathcal{L}^2_{n,i}(k_1, \dots, k_n)$ denote the set of patterns, where we have a contradiction for l_{i-1} but not for l_{i+1}, $\mathcal{L}^3_{n,i}(k_1, \dots, k_n)$ denote the set of patterns, where we have a contradiction for l_{i+1} but not for l_{i-1} and $\mathcal{L}^4_{n,i}(k_1, \dots, k_n)$ denote the set of patterns, where we have a contradiction for both l_{i-1} and l_{i+1}. Finally, we let $L^j_{n,i}(k_1, \dots, k_n)$ denote the respective signed enumerations. We aim to show that

$$L^j_{n,i}(k_1, \dots, k_n) = -L^j_{n,i}(k_1, \dots, k_{i-1}, k_{i+1}+1, k_i-1, k_{i+2}, \dots, k_n) \tag{B.1}$$

if $j \in \{1, 2, 3, 4\}$.

The case $j = 1$ is almost obvious, only the sign requires the following thoughts: having no contradiction for both l_{i-1} and l_{i+1} means that $l_{i-1} \in [k_{i-1}, k_i] \cap [k_{i-1}, k_{i+1}+1]$ and $l_{i+1} \in [k_{i+1}, k_{i+2}] \cap [k_i - 1, k_{i+2}]$. This is in fact true for patterns in $\mathcal{L}^1_{n,i}(k_1, \dots, k_n)$ as well as for patterns in $\mathcal{L}^1_{n,i}(k_1, \dots, k_{i+1}+1, k_i-1, \dots, k_n)$. The intersection $[k_{i-1}, k_i] \cap [k_{i-1}, k_{i+1}+1]$ is empty if exactly one of the intervals is an inversion. Thus we may assume that they are either both inversions or both not inversions. This implies that l_{i-1} is an inversion for the patterns on the left if and only if it is an inversion for the patterns on the right. The same is true for l_{i+1}. On the other hand, l_i is obviously an inversion on the left if and only if it is no inversion on the right, which takes care of the minus sign.

We show (B.1) for $j = 2$ (the case $j = 3$ is analogous by symmetry): given an element of $\mathcal{L}^2_{n,i}(k_1, \dots, k_n)$, we have $l_{i-1} \in [k_{i-1}, k_i] \setminus [k_{i-1}, k_{i+1}+1]$, whereas for an element of $\mathcal{L}^2_{n,i}(k_1, \dots, k_{i+1}+1, k_i-1, \dots, k_n)$, we have $l_{i-1} \in [k_{i-1}, k_{i+1}+1] \setminus [k_{i-1}, k_i]$. The conditions for the other elements are the same. (In particular, $l_{i+1} \in [k_{i+1}, k_{i+2}] \cap [k_i - 1, k_{i+2}]$.) If we are in the case that either both sets $[k_{i-1}, k_i]$ and $[k_{i-1}, k_{i+1}+1]$ are no inversions or both sets are inversions then one set is contained in the other, which implies that one of the conditions for l_{i-1} cannot be met. However, then the condition for l_{i-1} in the other set is that it lies in $[k_i+1, k_{i+1}+1]$. As the condition for l_i is that it is contained in $[k_i, k_{i+1}]$ it follows, by the shift-antisymmetry for $n-1$, that the signed enumeration of the patterns in this set must be zero.

If, however, exactly one set of $[k_{i-1}, k_i]$ and $[k_{i-1}, k_{i+1}+1]$ is an inversion then the sets are disjoint and their union is $[k_i+1, k_{i+1}+1]$. We decompose the two sets $\mathcal{L}^2_{n,i}(k_1, \dots, k_n)$ and $\mathcal{L}^2_{n,i}(k_1, \dots,$

$k_{i+1}+1, k_i-1, \dots, k_n)$ further according whether $l_i \in [k_{i-1}-1, k_i-1]$ or $l_i \in [k_{i-1}-1, k_{i+1}]$. (Observe that also $[k_i, k_{i+1}]$ is the disjoint union of $[k_{i-1}-1, k_i-1]$ and $[k_{i-1}-1, k_{i+1}]$.) By the shift-antisymmetry for $n-1$, the signed enumeration of the elements in $\mathcal{L}^2_{n,i}(k_1, \dots, k_n)$ which satisfy $l_i \in [k_{i-1}-1, k_i-1]$ is zero as the requirement for l_{i-1} is that it is contained in $[k_{i-1}, k_i]$. Similarly, the signed enumeration of the elements in $\mathcal{L}^2_{n,i}(k_1, \dots, k_{i+1}+1, k_i-1, \dots, k_n)$ with $l_i \in [k_{i-1}-1, k_{i+1}]$ is zero. Thus, for the first set, we are left with the patterns that satisfy $l_{i-1} \in [k_{i-1}, k_i]$ and $l_i \in [k_{i-1}-1, k_{i+1}]$ and, for the second set, the patterns with $l_{i-1} \in [k_{i-1}, k_{i+1}+1]$ and $l_i \in [k_{i-1}-1, k_i-1]$ remain. By the symmetry of these conditions and the shift-antisymmetry for $n-1$, we see that the signed enumeration of the first set is the negative of the signed enumeration of the second set: as for the sign observe that l_{i-1} is an inversion on the left (which is the case iff $[k_{i-1}, k_i]$ is an inversion) if and only if it is no inversion on the right (which is the case iff $[k_{i-1}, k_{i+1}+1]$ is no inversion). The analog assertion is true for l_i as it is an inversion on the left iff $[k_i, k_{i+1}]$ is an inversion and it is an inversion on the right iff $[k_{i+1}+1, k_i-1]$ is an inversion. Finally, for l_{i+1} we have the situation that it is an inversion on the left iff it is an inversion on the right or the condition $l_{i+1} \in [k_{i+1}, k_{i+2}] \cap [k_i-1, k_{i+2}]$ cannot be met.

The case $j=4$ is similar though a bit more complicated and left to the interested reader.

Appendix C. Two other combinatorial extensions of $\alpha(n; k_1, \dots, k_n)$ to all $(k_1, k_2, \dots, k_n) \in \mathbb{Z}^n$

We provide two additional combinatorial extensions that can be seen as variations of the first combinatorial extension given in Section 6. At the moment it seems to be unclear, which of them is the most convenient to work with.

C.1. Third extension

The summation can also be written in the following more symmetric manner: we choose a subset $I \subseteq [n-1]$ such that $l_i = k_i$ if $i \in I$ and a subset $J \subseteq [n-1]$ such that $l_j = k_{j+1}$ if $j \in J$. The sets I, J have to be disjoint and, moreover, $i \in I$ implies $i-1 \notin J$ (which is equivalent to $(I-1) \cap J = \emptyset$). On the other hand, if $h \in [n-1] \setminus (I \cup J)$ then $k_h < l_h < k_{h+1}$. Equivalently,

$$\sum_{p,q \geqslant 0} \sum_{\substack{I=\{i_1,\dots,i_p\}, J=\{j_1,\dots,j_q\} \subseteq [n-1] \\ I \cap J = \emptyset, (I-1) \cap J = \emptyset}} \sum_{l_{i_1}=k_{i_1}}^{k_{i_1}} \cdots \sum_{l_{i_p}=k_{i_p}}^{k_{i_p}} \sum_{l_{j_1}=k_{j_1+1}}^{k_{j_1+1}} \cdots \sum_{l_{j_q}=k_{j_q+1}}^{k_{j_q+1}} \sum_{l_{h_1}=k_{h_1}+1}^{k_{h_1+1}-1} \cdots \sum_{l_{h_r}=k_{h_r}+1}^{k_{h_r+1}-1},$$

where $[n-1] \setminus (I \cup J) = \{h_1, \dots, h_r\}$. Using this representation, we can deduce the following extension: a *monotone triangle of order n and type* 1*b* is a triangular array $(a_{i,j})_{1 \leqslant j \leqslant i \leqslant n}$ of integers such that the following conditions are fulfilled.

- There is a subset of "left-special" entries $a_{i,j}$ with $i < n$ for which we require $a_{i,j} = a_{i+1,j}$ and we mark them with a star on the left as well as a subset of "right-special" entries $a_{i,j}$ with $i < n$ for which we require $a_{i,j} = a_{i+1,j+1}$ and mark them with a star on the right.
- An entry cannot be a left-special entry and a right-special entry. If a right-special entry and a left-special entry happen to be in the same row then the right-special entry may not be situated immediately to the left of the left-special entry.
- If $a_{i,j}$ is not a special entry then we have $a_{i+1,j} < a_{i,j} < a_{i+1,j+1}$ or $a_{i+1,j+1} \leqslant a_{i,j} \leqslant a_{i+1,j}$, respectively. In the latter case we have an inversion.

Next we give an example of such an array.

$$\begin{array}{ccccccccc}
 & & & & 3 & & & & \\
 & & & 2 & & 4 & & & \\
 & & 3 & & {}^*2 & & 6 & & \\
 & {}^*3 & & 2 & & 5^* & & 7 & \\
3 & & 1 & & 7 & & 5 & & 8
\end{array}$$

The sign of a monotone triangle is again -1 to the number of inversions and $\alpha(n; k_1, \dots, k_n)$ is the signed enumeration of these extended monotone triangles with prescribed $a_{n,i} = k_i$. Although we think that the extension of type 2 is probably the nicest, the extensions of type 1 and type 1b are the only ones where in case that $k_1 < k_2 < \dots < k_n$ the removal of all stars leads to a monotone triangle in the original sense and no array is assigned a minus sign, i.e. we have a plain enumeration.

C.2. Fourth extension

Yet another possibility to write the summation in (6.2) in terms of simple summations is the following:

$$\sum_{p \geqslant 0} (-1)^p \sum_{\substack{2 \leqslant i_1 < i_2 < \dots < i_p \leqslant n-1 \\ i_{j+1} \neq i_j + 1}} \sum_{l_1=k_1}^{k_2} \sum_{l_2=k_2}^{k_3} \dots \sum_{l_{i_1-1}=k_{i_1}}^{k_{i_1}} \sum_{l_{i_1}=k_{i_1}}^{k_{i_1}} \dots \sum_{l_{i_p-1}=k_{i_p}}^{k_{i_p}} \sum_{l_{i_p}=k_{i_p}}^{k_{i_p}} \dots \sum_{l_{n-1}=k_{n-1}}^{k_n} .$$

This leads to the following extension: a *monotone triangle of order n and type* 1c is a triangular array $(a_{i,j})_{1 \leqslant j \leqslant i \leqslant n}$ of integers such that the following conditions are fulfilled. The entries $a_{i-1,j-1}$ and $a_{i-1,j}$ are said to be the parents of $a_{i,j}$.

- Among the entries $(a_{i,j})_{1<j<i \leqslant n}$ we may have special entries such that if two of them happen to be in the same row they must not be adjacent. We mark these entries with a star. For the parents of a special entry $a_{i,j}$ we have require $a_{i-1,j-1} = a_{i,j} = a_{i-1,j}$.
- If $a_{i,j}$ is not the parent of a special entry then $a_{i+1,j} \leqslant a_{i,j} \leqslant a_{i+1,j+1}$ and $a_{i+1,j+1} < a_{i,j} < a_{i+1,j}$, respectively. In the latter case we have an inversion.

In this case, the sign of a monotone triangle is -1 to the number of inversions plus the number of special entries. Then $\alpha(n; k_1, \dots, k_n)$ is the signed enumeration of monotone triangles with $a_{n,i} = k_i$. Next we give an example of such an array.

$$\begin{array}{ccccccccc}
 & & & & 4 & & & & \\
 & & & 5 & & 3 & & & \\
 & & 5 & & 5 & & 2 & & \\
 & 3 & & \overset{*}{5} & & 2 & & 2 & \\
4 & & 1 & & 7 & & \overset{*}{2} & & 5
\end{array}$$

This is the extension that has already appeared in [7]. There we have indicated that the non-adjacency requirement for special entries can also be ignored: suppose that $(a_{i,j})_{1 \leqslant j \leqslant i \leqslant n}$ is an array with the properties given above accept that we allow special entries to be adjacent: suppose $a_{i,j}$ and $a_{i,j+1}$ are two adjacent special entries such that $i + j$ is maximal with this property. Then we have $a_{i-1,j-1} = a_{i,j} = a_{i-1,j} = a_{i,j+1} = a_{i-1,j+1}$. This implies that $a_{i-2,j-1} = a_{i-1,j} = a_{i-2,j}$ whether or not $a_{i-1,j}$ is a special entry, which implies that changing the status of the entry $a_{i-1,j}$ is a sign-reversing involution.

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