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A general iterative method for obtaining an infinite family of strictly pseudo-contractive mappings in Hilbert spaces

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ABSTRACT

In this work, we consider a general composite iterative method for obtaining an infinite family of strictly pseudo-contractive mappings in Hilbert spaces. It is proved that the sequence generated by the iterative scheme converges strongly to a common point of the set of fixed points, which solves the variational inequality $\langle(\gamma f-\mu F)q, p-q\rangle \leq 0$, for $p \in \cap_{i=1}^{\infty} F(T_i)$. Our results improve and extend corresponding ones announced by many others.

1. Introduction

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. A self-mapping $f : C \longrightarrow C$ is a contraction on C if there exists a constant $\alpha \in (0, 1)$ such that $\|f(x)-f(y)\| \leq \alpha\|x-y\|$, $\forall x, y \in C$. We use Π_C to denote the collection of mappings f verifying the above inequality. That is, $\Pi_C = \{f : C \to C \mid f \text{ is a contraction with constant } \alpha\}$. Note that each $f \in \Pi_C$ has a unique fixed point in C.

A mapping $T : C \to C$ is said to be λ-strictly pseudo-contractive if there exists a constant $\lambda \in [0, 1)$ such that

$$\|Tx-Ty\|^2 \leq \|x-y\|^2 + \lambda\|(I-T)x-(I-T)y\|^2, \quad x, y \in C,$$

and $F(T)$ denotes the set of fixed points of the mapping T; that is, $F(T) = \{x \in C : Tx = x\}$.

Note that the class of λ-strictly pseudo-contractive mappings includes the class of nonexpansive mappings T on C (that is, $\|Tx-Ty\| \leq \|x-y\|$, $x, y \in C$) as a subclass. That is, T is nonexpansive if and only if T is 0-strictly pseudo-contractive.

A mapping $F : C \to C$ is called k-Lipschitzian if there exists a positive constant k such that

$$\|Fx-Fy\| \leq k\|x-y\|, \quad \forall x, y \in C. \tag{1.1}$$

F is said to be η-strongly monotone if there exists a positive constant η such that

$$\langle Fx-Fy, x-y\rangle \geq \eta\|x-y\|^2, \quad \forall x, y \in C. \tag{1.2}$$

Let A be a strongly positive bounded linear operator on H, that is, there exists a constant $\tilde{\gamma} > 0$ such that

$$\langle Ax, x\rangle \geq \tilde{\gamma}\|x\|^2 \quad \text{for all } x \in H.$$

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A typical problem is that of minimizing a quadratic function over the set of the fixed points of a nonexpansive mapping on a real Hilbert space H:

$$\min_{x\in F(T)} \frac{1}{2}\langle Ax, x\rangle - \langle x, b\rangle,$$

where b is a given point in H.

Remark 1.1. *From the definition of A, we note that a strongly positive bounded linear operator A is a $\|A\|$-Lipschitzian and $\tilde{\gamma}$-strongly monotone operator.*

In 2006, Marino and Xu [1] introduced and considered the following iterative scheme: for $x_1 = x \in C$,

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)Tx_n, \quad n \geq 1. \tag{1.3}$$

They proved that if the sequence $\{\alpha_n\}$ of parameters satisfies appropriate conditions, then the sequence $\{x_n\}$ generated by (1.3) converges strongly to the unique solution of the variational inequality $\langle (\gamma f - A)q, p - q\rangle \leq 0, p \in F(T)$.

Recently, Jung [2] extended the results of Marino and Xu [1] to the class of k-strictly pseudo-contractive mappings and introduced an iterative scheme as follows: for the k-strictly pseudo-contractive mapping $T : C \to H$ with $F(T) \neq \emptyset$ and $x_1 = x \in C$,

$$\begin{cases} y_n = \beta_n x_n + (1 - \beta_n)P_C S x_n, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)y_n, \quad n \geq 1, \end{cases} \tag{1.4}$$

where $S : C \to H$ is a mapping defined by $Sx = kx + (1 - k)Tx$. He proved that $\{x_n\}$ defined by (1.4) converges strongly to a fixed point q of T, which is the unique solution of the variational inequality $\langle \gamma f(q) - Aq, p - q\rangle \leq 0, p \in F(T)$.

Very recently, Tian [3] considered the following iterative method: for a nonexpansive mapping $T : H \to H$ with $F(T) \neq \emptyset$,

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \mu\alpha_n F)Tx_n, \quad n \geq 1, \tag{1.5}$$

where F is a k-Lipschitzian and η-strongly monotone operator. He obtained that the sequence $\{x_n\}$ generated by (1.5) converges to a point q in $F(T)$, which is the unique solution of the variational inequality $\langle (\gamma f - \mu F)q, p - q\rangle \leq 0, p \in F(T)$.

In this work, motivated and inspired by the above results, we introduce a new iterative scheme: for $x_1 = x \in C$,

$$\begin{cases} y_n = \beta_n x_n + (1 - \beta_n)W_n x_n, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \mu\alpha_n F)y_n, \quad n \geq 1, \end{cases} \tag{1.6}$$

where W_n is a mapping defined by (2.3), and F is a k-Lipschitzian and η-strongly monotone operator with $0 < \mu < 2\eta/k^2$. We will prove that if the parameters satisfy appropriate conditions, then $\{x_n\}$ generated by (1.6) converges strongly to a common element of the fixed points of an infinite family of λ_i-strictly pseudo-contractive mappings, which is a unique solution of the variational inequality $\langle (\gamma f - \mu F)q, p - q\rangle \leq 0, p \in \cap_{i=1}^{\infty} F(T_i)$. Our results also extend and improve the corresponding results of Marino and Xu [1], Jung [2], Tian [3] and many others.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot\rangle$ and norm $\|\cdot\|$. For the sequence $\{x_n\}$ in H, we write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x. $x_n \to x$ implies that $\{x_n\}$ converges strongly to x. In a real Hilbert space H, we have

$$\|x - y\|^2 = \|x\|^2 + \|y\|^2 - 2\langle x, y\rangle \quad \forall x, y \in H. \tag{2.1}$$

In order to prove our main results, we need the following lemmas.

Lemma 2.1. *In a Hilbert space H, the following inequality holds:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \quad x, y \in H.$$

Lemma 2.2. *Let F be a k-Lipschitzian and η-strongly monotone operator on a Hilbert space H with $k > 0$, $\eta > 0$, $0 < \mu < 2\eta/k^2$ and $0 < t < 1$. Then $S = (I - t\mu F) : H \to H$ is a contraction with contractive coefficient $1 - t\tau$ and $\tau = \frac{1}{2}\mu(2\eta - \mu k^2)$.*

Proof. From (1.1), (1.2) and (2.1), we have

$$\begin{aligned} \|Sx - Sy\|^2 &= \|(x - y) - t\mu(Fx - Fy)\|^2 \\ &= \|x - y\|^2 + t^2\mu^2\|Fx - Fy\|^2 - 2\mu t\langle Fx - Fy, x - y\rangle \\ &\leq \|x - y\|^2 + t^2\mu^2 k^2\|x - y\|^2 - 2\mu t\eta\|x - y\|^2 \\ &\leq \|x - y\|^2 + t\mu^2 k^2\|x - y\|^2 - 2\mu t\eta\|x - y\|^2 \\ &= [1 - t\mu(2\eta - \mu k^2)]\|x - y\|^2 \\ &\leq (1 - t\tau)^2\|x - y\|^2, \end{aligned}$$

where $\tau = \frac{1}{2}\mu(2\eta - \mu k^2)$, and

$$\|Sx - Sy\| \leq (1 - t\tau)\|x - y\|.$$

Hence S is a contraction with contractive coefficient $1 - t\tau$. □

Let C be a nonempty closed convex subset of H such that $C \pm C \subset C$. Let F be a k-Lipschitzian and η-strongly monotone operator on C with $k > 0$, $\eta > 0$, and $T : C \to C$ be a nonexpansive mapping. Now given $f \in \Pi_C$ with $0 < \alpha < 1$, let us have $t \in (0, 1)$, $0 < \mu < 2\eta/k^2$, $0 < \gamma < \mu(\eta - \frac{\mu k^2}{2})/\alpha = \tau/\alpha$ and $\tau < 1$, and consider a mapping S_t on H defined by

$$S_t x = t\gamma f(x) + (I - t\mu F)Tx, \quad x \in C.$$

It is easy to see that S_t is a contraction. Indeed, from Lemma 2.2, we have

$$\begin{aligned}\|S_t x - S_t y\| &\leq t\gamma\|f(x) - f(y)\| + \|(I - t\mu F)Tx - (I - t\mu F)Ty\|\\ &\leq t\gamma\alpha\|x - y\| + (1 - t\tau)\|Tx - Ty\|\\ &\leq [1 - t(\tau - \gamma\alpha)]\|x - y\|,\end{aligned}$$

for all $x, y \in H$. Hence it has a unique fixed point, denoted as x_t, which uniquely solves the fixed point equation

$$x_t = t\gamma f(x_t) + (I - t\mu F)Tx_t, \quad x_t \in C. \tag{2.2}$$

Lemma 2.3 (*[3]*). *Let H be a Hilbert space and C be a nonempty closed convex subset of H such that $C \pm C \subset C$. Assume that $\{x_t\}$ is defined by (2.2); then x_t converges strongly as $t \to 0$ to a fixed point q of T which solves the variational inequality $\langle(\mu F - \gamma f)q, q - p\rangle \leq 0$, $f \in \Pi_C, p \in F(T)$. Equivalently, we have $P_{F(T)}(I - \mu F + \gamma f)q = q$.*

Lemma 2.4 (*[4]*). *Let H be a Hilbert space, C a closed convex subset of H, and $T : C \to C$ a nonexpansive mapping with $F(T) \neq \emptyset$; if $\{x_n\}$ is a sequence in C weakly converging to x and if $\{(I - T)x_n\}$ converges strongly to y, then $(I - T)x = y$.*

Lemma 2.5 (*[5]*). *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space E and $\{\gamma_n\}$ be a sequence in $[0, 1]$ which satisfies the following condition:*

$$0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1.$$

Suppose that $x_{n+1} = \gamma_n x_n + (1-\gamma_n)z_n$, $n \geq 0$, and $\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then $\lim_{n\to\infty}\|z_n - x_n\| = 0$.

Lemma 2.6 (*[6,7]*). *Let $\{s_n\}$ be a sequence of non-negative real numbers satisfying*

$$s_{n+1} \leq (1 - \lambda_n)s_n + \lambda_n\delta_n + \gamma_n, \quad n \geq 0,$$

where $\{\lambda_n\}$, $\{\delta_n\}$ and $\{\gamma_n\}$ satisfy the following conditions: (i) *$\{\lambda_n\} \subset [0, 1]$ and $\sum_{n=0}^{\infty}\lambda_n = \infty$,* (ii) *$\limsup_{n\to\infty}\delta_n \leq 0$ or $\sum_{n=0}^{\infty}\lambda_n\delta_n < \infty$,* (iii) *$\gamma_n \geq 0$ $(n \geq 0)$, $\sum_{n=0}^{\infty}\gamma_n < \infty$. Then $\lim_{n\to\infty} s_n = 0$.*

Lemma 2.7 (*[8]*). *Let C be a nonempty closed convex subset of a real Hilbert space H and $T : C \to C$ be a λ-strictly pseudo-contractive mapping. Define a mapping $S : C \to C$ by $Sx = \alpha x + (1-\alpha)Tx$ for all $x \in C$ and $\alpha \in [\lambda, 1)$. Then S is a nonexpansive mapping such that $F(S) = F(T)$.*

In this work, we consider the mapping W_n defined by

$$\begin{cases} U_{n,n+1} = I\\ U_{n,n} = \gamma_n T_n' U_{n,n+1} + (1 - \gamma_n)I,\\ U_{n,n-1} = \gamma_{n-1}T_{n-1}'U_{n,n} + (1 - \gamma_{n-1})I,\\ \cdots\\ U_{n,k} = \gamma_k T_k' U_{n,k+1} + (1 - \gamma_k)I,\\ U_{n,k-1} = \gamma_{k-1}T_{k-1}'U_{n,k} + (1 - \gamma_{k-1})I,\\ \cdots\\ U_{n,2} = \gamma_2 T_2' U_{n,3} + (1 - \gamma_2)I,\\ W_n = U_{n,1} = \gamma_1 T_1' U_{n,2} + (1 - \gamma_1)I, \end{cases} \tag{2.3}$$

where $\gamma_1, \gamma_2, \ldots$ are real numbers such that $0 \leq \gamma_n \leq 1$, $T_i' = \theta_i I + (1 - \theta_i)T_i$ where T_i is a λ_i-strictly pseudo-contractive mapping of C into itself and $\theta_i \in [\lambda_i, 1)$. By Lemma 2.7, we know that T_i' is a nonexpansive mapping and $F(T_i) = F(T_i')$. As a result it can be easily seen that W_n is a nonexpansive mapping.

As regards W_n, we have the following lemmas which are important for proving our main results.

Lemma 2.8. *[Shimoji et al. [9]] Let C be a nonempty closed convex subset of a strictly convex Banach space E. Let $T_1', T_2', \ldots$ be nonexpansive mappings of C into itself such that $\cap_{i=1}^{\infty} F(T_i') \neq \emptyset$ and $\gamma_1, \gamma_2, \ldots$ be real numbers such that $0 < \gamma_i \leq b < 1$, for every $i = 1, 2, \ldots$. Then, for any $x \in C$ and $k \in N$, the limit $\lim_{n\to\infty} U_{n,k}x$ exists.*

Using Lemma 2.8, one can define the mapping W of C into itself as follows:

$$Wx := \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x, \quad x \in C.$$

Such a mapping W is called the modified W-mapping generated by $T_1, T_2, \ldots, \gamma_1, \gamma_2, \ldots$ and $\theta_1, \theta_2, \ldots$.

Lemma 2.9. *[Shimoji et al. [9]] Let C be a nonempty closed convex subset of a strictly convex Banach space E. Let $T_1', T_2', \ldots$ be nonexpansive mappings of C into itself such that $\cap_{i=1}^{\infty} F(T_i') \neq \emptyset$ and $\gamma_1, \gamma_2, \ldots$ be real numbers such that $0 < \gamma_i \leq b < 1$ for every $i = 1, 2, \ldots$. Then $F(W) = \cap_{i=1}^{\infty} F(T_i')$.*

Combining Lemmas 2.7–2.9, one can get that $F(W) = \cap_{i=1}^{\infty} F(T_i') = \cap_{i=1}^{\infty} F(T_i)$.

Lemma 2.10 (*[10]*). *Let C be a nonempty closed convex subset of a Hilbert space H, $\{T_i' : C \to C\}$ be a family of infinite nonexpansive mappings with $\cap_{i=1}^{\infty} F(T_i') \neq \emptyset$, $\{\gamma_i\}$ be a real sequence such that $0 < \gamma_i \leq b < 1$, $\forall i \geq 1$. If K is any bounded subset of C, then*

$$\lim_{n\to\infty} \sup_{x\in K} \|Wx - W_n x\| = 0.$$

3. Main results

Now, we study the strong convergence results for an infinite family of strictly pseudo-contractive mappings in Hilbert spaces.

Theorem 3.1. *Let H be a Hilbert space, and C be a nonempty closed convex subset of H such that $C \pm C \subset C$. Let $T_i : C \to C$ be a λ_i-strictly pseudo-contractive mapping with $\cap_{i=1}^{\infty} F(T_i) \neq \emptyset$ and $\{\gamma_i\}$ be a real sequence such that $0 < \gamma_i \leq b < 1$, $\forall i \geq 1$. Let F be a k-Lipschitzian and η-strongly monotone operator on C with $0 < \mu < 2\eta/k^2$ and $f \in \Pi_C$ with $0 < \gamma < \mu(\eta - \frac{\mu k^2}{2})/\alpha = \tau/\alpha$ and $\tau < 1$. Let $\{\alpha_n\}$ and $\{\beta_n\} \subset (0, 1)$ be sequences which satisfy the following conditions:*

(A1) $\lim_{n\to\infty} \alpha_n = 0$;
(A2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(A3) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n \leq a < 1$ *for some constant* $a \in (0, 1)$.

Then $\{x_n\}$ defined by (1.6) converges strongly to $q \in \cap_{i=1}^{\infty} F(T_i)$, which is the unique solution of the following variational inequality: $\langle(\gamma f - \mu F)q, p - q\rangle \leq 0$, $p \in \cap_{i=1}^{\infty} F(T_i)$.

Proof. We proceed with the following steps.

Step 1. We claim that $\{x_n\}$ is bounded. In fact, let $p \in \cap_{i=1}^{\infty} F(T_i)$; from (1.6), we have

$$\begin{aligned} \|y_n - p\| &= \|\beta_n(x_n - p) + (1 - \beta_n)(W_n x_n - p)\| \\ &\leq \beta_n\|x_n - p\| + (1 - \beta_n)\|W_n x_n - p\| \\ &\leq \|x_n - p\|. \end{aligned} \tag{3.1}$$

Then from (1.6) and (3.1) and Lemma 2.2, we obtain

$$\begin{aligned} \|x_{n+1} - p\| &= \|\alpha_n\gamma f(x_n) + (I - \mu\alpha_n F)y_n - p\| \\ &= \|\alpha_n(\gamma f(x_n) - \mu Fp) + (I - \mu\alpha_n F)y_n - (I - \mu\alpha_n F)p\| \\ &\leq (1 - \alpha_n\tau)\|y_n - p\| + \alpha_n(\|\gamma f(x_n) - \gamma f(p)\| + \|\gamma f(p) - \mu Fp\|) \\ &\leq (1 - \alpha_n\tau)\|x_n - p\| + \alpha_n\gamma\alpha\|x_n - p\| + \alpha_n\|\gamma f(p) - \mu Fp\| \\ &\leq [1 - \alpha_n(\tau - \gamma\alpha)]\|x_n - p\| + \alpha_n(\tau - \gamma\alpha)\frac{\|\gamma f(p) - \mu Fp\|}{\tau - \gamma\alpha} \\ &\leq \max\{\|x_n - p\|, \frac{\|\gamma f(p) - \mu Fp\|}{\tau - \gamma\alpha}\}, \quad n \geq 1. \end{aligned}$$

By induction, we have

$$\|x_n - p\| \leq \max\{\|x_1 - p\|, \frac{\|\gamma f(p) - \mu Fp\|}{\tau - \gamma\alpha}\}, \quad n \geq 1.$$

Therefore, $\{x_n\}$ is bounded. We also obtain that $\{y_n\}$, $\{W_nx_n\}$, $\{\mu Fy_n\}$ and $f(x_n)$ are all bounded. Without loss of generality, we may assume that $\{x_n\}, \{y_n\}, \{W_nx_n\}, \{\mu Fy_n\}, f(x_n) \subset K$, where K is a bounded set of C.

Step 2. We claim that $\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0$. To this end, define a sequence $\{z_n\}$ by $z_n = (x_{n+1}-\beta_nx_n)/(1-\beta_n)$, such that $x_{n+1} = \beta_nx_n + (1-\beta_n)z_n$. We now observe that

$$\begin{aligned}
z_{n+1}-z_n &= \frac{x_{n+2}-\beta_{n+1}x_{n+1}}{1-\beta_{n+1}} - \frac{x_{n+1}-\beta_nx_n}{1-\beta_n}\\
&= \frac{\alpha_{n+1}\gamma f(x_{n+1}) + (I-\mu\alpha_{n+1}F)y_{n+1} - \beta_{n+1}x_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n\gamma f(x_n) + (I-\mu\alpha_nF)y_n - \beta_nx_n}{1-\beta_n}\\
&= \frac{\alpha_{n+1}}{1-\beta_{n+1}}(\gamma f(x_{n+1}) - \mu Fy_{n+1}) + \frac{y_{n+1}-\beta_{n+1}x_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}(\gamma f(x_n) - \mu Fy_n) - \frac{y_n-\beta_nx_n}{1-\beta_n}\\
&= \frac{\alpha_{n+1}}{1-\beta_{n+1}}(\gamma f(x_{n+1}) - \mu Fy_{n+1}) + \frac{[\beta_{n+1}x_{n+1} + (1-\beta_{n+1})W_{n+1}x_{n+1}] - \beta_{n+1}x_{n+1}}{1-\beta_{n+1}}\\
&\quad - \frac{\alpha_n}{1-\beta_n}(\gamma f(x_n) - \mu Fy_n) - \frac{\beta_nx_n + (1-\beta_n)W_nx_n - \beta_nx_n}{1-\beta_n}\\
&= \frac{\alpha_{n+1}}{1-\beta_{n+1}}(\gamma f(x_{n+1}) - \mu Fy_{n+1}) - \frac{\alpha_n}{1-\beta_n}(\gamma f(x_n) - \mu Fy_n) + W_{n+1}x_{n+1} - W_nx_n.
\end{aligned} \tag{3.2}$$

It follows from (3.2) that

$$\|z_{n+1}-z_n\| \le \frac{\alpha_{n+1}}{1-\beta_{n+1}}(\|\gamma f(x_{n+1})\| + \|\mu Fy_{n+1}\|) + \frac{\alpha_n}{1-\beta_n}(\|\gamma f(x_n)\| + \|\mu Fy_n\|) + \|W_{n+1}x_{n+1} - W_nx_n\|, \tag{3.3}$$

for all $n \ge 1$. From (2.3), we have

$$\begin{aligned}
\|W_{n+1}x_n - W_nx_n\| &= \|\gamma_1T_1'U_{n+1,2}x_n - \gamma_1T_1'U_{n,2}x_n\|\\
&\le \gamma_1\|U_{n+1,2}x_n - U_{n,2}x_n\|\\
&= \gamma_1\|\gamma_2T_2'U_{n+1,3}x_n - \gamma_2T_2'U_{n,3}x_n\|\\
&\le \gamma_1\gamma_2\|U_{n+1,3}x_n - U_{n,3}x_n\|\\
&\le \cdots\\
&\le \gamma_1\gamma_2\cdots\gamma_n\|U_{n+1,n+1}x_n - U_{n,n+1}x_n\|\\
&\le M_1\prod_{i=1}^n\gamma_i,
\end{aligned}$$

where $M_1 \ge 0$ is a constant such that $\|U_{n+1,n+1}x_n - U_{n,n+1}x_n\| \le M_1$, for all $n \ge 1$.

So, we obtain

$$\begin{aligned}
\|W_{n+1}x_{n+1} - W_nx_n\| &\le \|W_{n+1}x_{n+1} - W_{n+1}x_n\| + \|W_{n+1}x_n - W_nx_n\|\\
&\le \|x_{n+1}-x_n\| + \|W_{n+1}x_n - W_nx_n\|\\
&\le \|x_{n+1}-x_n\| + M_1\prod_{i=1}^n\gamma_i.
\end{aligned} \tag{3.4}$$

Substituting (3.4) into (3.3), we obtain

$$\|z_{n+1}-z_n\| \le M_2\left(\frac{\alpha_{n+1}}{1-\beta_{n+1}} + \frac{\alpha_n}{1-\beta_n}\right) + \|x_{n+1}-x_n\| + M_1\prod_{i=1}^n\gamma_i, \tag{3.5}$$

where $M_2 = \sup\{\|\gamma f(x_n)\| + \|\mu Fy_n\|, n \ge 1\}$. It follows from (3.5) that

$$\|z_{n+1}-z_n\| - \|x_{n+1}-x_n\| \le M_2\left(\frac{\alpha_{n+1}}{1-\beta_{n+1}} + \frac{\alpha_n}{1-\beta_n}\right) + M_1\prod_{i=1}^n\gamma_i. \tag{3.6}$$

Observing condition (A1), (A3), (3.6) and $0 < \gamma_i \le b < 1$, it follows that

$$\limsup_{n\to\infty}(\|z_{n+1}-z_n\| - \|x_{n+1}-x_n\|) \le 0.$$

Hence by Lemma 2.5 we can obtain

$$\lim_{n\to\infty}\|z_n - x_n\| = 0. \tag{3.7}$$

It follows from (A3) and (3.7) that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = \lim_{n\to\infty}(1-\beta_n)\|z_n-x_n\| = 0.$$

Step 3. We claim that $\lim_{n\to\infty}\|x_n - Wx_n\| = 0$. Observe that

$$\begin{aligned}\|x_n - W_nx_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1}-y_n\| + \|y_n - W_nx_n\| \\ &= \|x_n - x_{n+1}\| + \|x_{n+1}-y_n\| + \beta_n\|x_n - W_nx_n\|.\end{aligned}$$

From (A1), (A3) and using Step 2, we have

$$\begin{aligned}(1-a)\|x_n - W_nx_n\| &\le (1-\beta_n)\|x_n - W_nx_n\| \le \|x_n - x_{n+1}\| + \|x_{n+1}-y_n\| \\ &\le \|x_n - x_{n+1}\| + \alpha_n\|\gamma f(x_n) - \mu Fy_n\| \to 0 \ (\text{as } n\to\infty).\end{aligned}$$

This implies that

$$\|x_n - W_nx_n\| \to 0 \ (\text{as } n\to\infty). \tag{3.8}$$

On the other hand, we have

$$\begin{aligned}\|x_n - Wx_n\| &\le \|x_n - W_nx_n\| + \|W_nx_n - Wx_n\| \\ &\le \|x_n - W_nx_n\| + \sup_{x\in K}\|W_nx - Wx\|.\end{aligned} \tag{3.9}$$

By (3.8), (3.9) and using Lemma 2.10, we obtain

$$\lim_{n\to\infty}\|x_n - Wx_n\| = 0.$$

Step 4. We claim that $\limsup_{n\to\infty}\langle(\gamma f-\mu F)q, x_n - q\rangle \le 0$, where $q = \lim_{t\to 0}x_t$ with $x_t = t\gamma f(x_t) + (I - t\mu F)Wx_t$.

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges weakly to z. From $\|x_n - Wx_n\| \to 0$, we obtain $Wx_{n_k} \rightharpoonup z$. From Lemma 2.4, we have $z \in F(W)$. Hence by Lemma 2.3, we have

$$\limsup_{n\to\infty}\langle(\gamma f-\mu F)q, x_n - q\rangle = \lim_{k\to\infty}\langle(\gamma f-\mu F)q, x_{n_k} - q\rangle = \langle(\gamma f-\mu F)q, z - q\rangle \le 0.$$

Step 5. We claim that $\{x_n\}$ converges strongly to q. From (1.6), we have

$$\begin{aligned}\|x_{n+1}-q\|^2 &= \|\alpha_n\gamma f(x_n) + (I-\mu\alpha_nF)y_n - q\|^2 \\ &= \|(I-\mu\alpha_nF)y_n - (I-\mu\alpha_nF)q + \alpha_n(\gamma f(x_n) - \mu Fq)\|^2 \\ &\le \|(I-\mu\alpha_nF)y_n - (I-\mu\alpha_nF)q\|^2 + 2\alpha_n\langle\gamma f(x_n) - \mu Fq, x_{n+1}-q\rangle \\ &\le (1-\alpha_n\tau)^2\|y_n - q\|^2 + 2\alpha_n\langle\gamma f(x_n) - \gamma f(q), x_{n+1}-q\rangle + 2\alpha_n\langle\gamma f(q) - \mu Fq, x_{n+1}-q\rangle \\ &\le (1-\alpha_n\tau)^2\|x_n - q\|^2 + \alpha_n\gamma\alpha(\|x_n-q\|^2 + \|x_{n+1}-q\|^2) + 2\alpha_n\langle\gamma f(q) - \mu Fq, x_{n+1}-q\rangle.\end{aligned}$$

It then follows that

$$\begin{aligned}\|x_{n+1}-q\|^2 &\le \frac{(1-\alpha_n\tau)^2 + \alpha_n\gamma\alpha}{1-\alpha_n\gamma\alpha}\|x_n-q\|^2 + \frac{2\alpha_n}{1-\alpha_n\gamma\alpha}\langle\gamma f(q) - \mu Fq, x_{n+1}-q\rangle \\ &\le \left(1 - \frac{2\alpha_n(\tau-\gamma\alpha)}{1-\alpha_n\gamma\alpha}\right)\|x_n-q\|^2 + \frac{2\alpha_n(\tau-\gamma\alpha)}{1-\alpha_n\gamma\alpha}\left[\frac{1}{\tau-\gamma\alpha}\langle\gamma f(q)\right. \\ &\quad \left. -\ \mu Fq, x_{n+1}-q\rangle + \frac{\alpha_n\tau^2}{2(\tau-\gamma\alpha)}M_3\right],\end{aligned}$$

where $M_3 = \sup_{n\ge 1}\|x_n - q\|^2$. Put $\lambda_n = \frac{2\alpha_n(\tau-\gamma\alpha)}{1-\alpha_n\gamma\alpha}$ and $\delta_n = \frac{1}{\tau-\gamma\alpha}\langle\gamma f(q) - \mu Fq, x_{n+1}-q\rangle + \frac{\alpha_n\tau^2}{2(\tau-\gamma\alpha)}M_3$. From (A1), (A2) and Step 4, it follows that $\sum_{n=1}^{\infty}\lambda_n = \infty$ and $\limsup_{n\to\infty}\delta_n \le 0$. Hence, by Lemma 2.6, the sequence $\{x_n\}$ converges strongly to $q \in \cap_{i=1}^{\infty}F(T_i)$. From $q = \lim_{t\to 0}x_t$ and Lemma 2.3, we have that q is the unique solution of the following variational inequality: $\langle(\gamma f-\mu F)q, p-q\rangle \le 0,\ p \in \cap_{i=1}^{\infty}F(T_i)$. □

Remark 3.2. *Theorem 3.1 extends the corresponding results of Jung [2] from one strictly pseudo-contractive mapping to an infinite family of strictly pseudo-contractive mappings.*

Remark 3.3. *Theorem 3.1 generalizes the results of Marino and Xu [1] and Theorem 2.1 of Jung [2] from a strongly positive bounded linear operator A to a k-Lipschitzian and η-strongly monotone operator F.*

Remark 3.4. *If $\beta_n = 0$, $\gamma_i = 1$ and $T_i = T$ with $\lambda_i = 0$ in Theorem 3.1, we can obtain Theorem 3.2 of Tian [3]. That is, Theorem 3.1 extends Theorem 3.2 of Tian [3] from one nonexpansive mapping to an infinite family of strictly pseudo-contractive mappings.*

Setting $F = A$ and $\mu = 1$ in Theorem 3.1, we can obtain the following result.

Corollary 3.5. *Let H be a Hilbert space, and C be a nonempty closed convex subset of H such that $C \pm C \subset C$. Let $T_i : C \to C$ be a λ_i-strictly pseudo-contractive mapping with $\cap_{i=1}^{\infty} F(T_i) \neq \emptyset$ and $\{\gamma_i\}$ be a real sequence such that $0 < \gamma_i \leq b < 1, \forall i \geq 1$. Let A be a strongly positive bounded linear operator on C with coefficient $0 < \tilde{\gamma} < 1$ and $f \in \Pi_C$ such that $0 < \gamma < \tilde{\gamma}/\alpha$. Let $\{\alpha_n\}$ and $\{\beta_n\} \subset (0, 1)$ be sequences which satisfy the conditions (A1), (A2) and (A3). Let $\{x_n\}$ be a sequence generated by $x_1 = x \in C$:*

$$\begin{cases} y_n = \beta_n x_n + (1 - \beta_n) W_n x_n, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A) y_n, \quad n \geq 1. \end{cases}$$

Then $\{x_n\}$ converges strongly to $q \in \cap_{i=1}^{\infty} F(T_i)$, which is the unique solution of the following variational inequality: $\langle (\gamma f - A)q, p - q\rangle \leq 0, p \in \cap_{i=1}^{\infty} F(T_i)$.

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