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Overrings of primitive factors of quantum sl(2)

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Abstract

We construct certain completely prime Dixmier algebras which are overrings of primitive factors of quantum sl(2). As in the classical case, these algebras are parametrized by infinitesimal characters corresponding to the half-integers. We also show that all completely prime Dixmier algebras for quantum sl(2) are essentially isomorphic to the algebras we have constructed.

0. Introduction

Let G be a semisimple connected algebraic group. In [8] Vogan defined a class of algebras (later called Dixmier algebras) associated with G, and he posed the problem of classifying these algebras. In [6] this problem was solved for the case $G = \mathrm{SL}(2)$. The present paper is a quantized version of [6]. More specifically, we define Dixmier algebras in the context of quantized enveloping algebras, then construct and classify these algebras for quantum sl(2). Thus this paper investigates only the most basic questions about quantum Dixmier algebras, namely their definition and existence. Other questions, perhaps more interesting (e.g., whether the elements of the Dixmier algebra can be realized naturally as quantized differential operators), are not discussed.

It turns out that the results of [6] transfer readily to the quantum case without major modifications. The techniques used in this paper, however, come mostly from [5] (which extended the results of [6] to some other nonquantum Lie algebras). The application of these techniques to the quantum context is straightforward and not difficult. However, since we are here dealing with the simplest quantum algebra, it is possible to be much more concrete than [5]. In fact, we will not need the quantized versions of sophisticated theorems from [2] which were used in [5]; almost everything

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we use can be derived in an elementary fashion from the definitions, similar to [4], Sections 2 and 3. This being the case, we will attempt to make the paper almost self-contained, with occasional references only to [4] for computational details.

1. Preliminaries on Hopf algebras

Here we collect some definitions and results about Hopf algebras. Recall that a Hopf algebra A (over a field K) is an associative algebra with identity 1 equipped with a coproduct $\Delta: A \to A \otimes A$, a counit $\varepsilon: A \to K$, and an antipode $\sigma: A \to A$ satisfying various properties (see, e.g., [4] and [7, Section 1]). When working with Hopf algebras, it is useful to follow the Sweedler notation, which is explained in [4, 7]. For example, in this notation, the coproduct is written $\Delta(a) = a_{(1)} \otimes a_{(2)}$.

Now let X be an A-bimodule. For each $a \in A$ we define an endomorphism $\operatorname{ad} a$ of X by

$$(\operatorname{ad} a)(x) = a_{(1)} x \sigma(a_{(2)}).$$

It is immediate that $(\operatorname{ad} a)(\operatorname{ad} b) = \operatorname{ad}(ab)$, so ad is an algebra homomorphism. We also define the space of $(\operatorname{ad} A)$-finite vectors in X:

$$F(X) = \{x \in X \mid \dim(\operatorname{ad} A)x < \infty\}.$$

When we apply this definition to A itself, we get a subalgebra $F(A)$ of A. (That $F(A)$ is a subalgebra follows immediately from the identity $(\operatorname{ad} a)(bc) = (\operatorname{ad} a_{(1)})b(\operatorname{ad} a_{(2)})c$, which is easy to verify. See [4, 2.2].) It is not hard to check that even though $F(X)$ is not necessarily an A-bimodule, $F(X)$ is in fact an $F(A)$-bimodule.

Next we consider tensor products. Let M and N be (left) A-module. We can form the A-module $M \otimes N$ by defining the action of $a \in A$ on $m \otimes n \in M \otimes N$ as follows:

$$a(m \otimes n) = a_{(1)} m \otimes a_{(2)} n.$$

There are similar definitions for right A-modules and A-bimodules.

We also define an A-module structure on the dual space M^* of linear maps from M to K: for $f \in M^*$ and $a \in A$,

$$(af)(m) = f(\sigma(a)m) \quad m \in M.$$

Proposition 1.1. *Let E, M, and N be A-modules, with E finite-dimensional. Then there are canonical isomorphisms*

$$\hom_A(E, \hom(M, N)) \simeq \hom_A(E \otimes M, N) \simeq \hom_A(M, E^* \otimes N).$$

Proof. This is mostly an exercise in using Hopf algebra notation. We will do a typical calculation here. First, on the level of vector spaces, there is the usual canonical isomorphism $\hom(E, \hom(M, N)) \to \hom(E \otimes M, N)$ which sends the map $\phi: E \to \hom(M, N)$ to the map $\Phi: E \otimes M \to N$, defined by $\Phi(e \otimes m) = (\phi(e))m)$. We then put an A-module structure on $\hom(M, N)$ via the adjoint action. Now we have to show that if ϕ is an A-map, then Φ is also an A-map.

Let $\phi: E \to \hom(M, N)$ be an A-map. Then, for any $a \in A$, $e \in E$, $m \in M$, we have

$$\begin{aligned}\Phi(a(e \otimes m)) &= \Phi(a_{(1)}e \otimes a_{(2)}m) = \phi(a_{(1)}e)(a_{(2)}m)\\ &= (a_{(1)}\phi(e))(a_{(2)}m) = a_{(1)}(\phi(e)(\sigma(a_{(2)})a_{(3)}m))\\ &= a_{(1)}\phi(e)(\varepsilon(a_{(2)}m) = a_{(1)}\varepsilon(a_{(2)})\phi(e)(m)\\ &= u\phi(e)(m) = u\Phi(e \otimes m)\end{aligned}$$

as required.

Other parts of the calculation are similar. □

Now let X be an A-bimodule which is also a K-algebra. Assume that the two structures are compatible, i.e., for $a \in A$ and $x_1, x_2 \in X$, we have $a(x_1x_2) = (ax_1)x_2$ and $(x_1x_2)a = x_1(x_2a)$. Then $F(X)$ is an $F(A)$-bimodule which is also a subalgebra of X.

Lemma 1.2. *Let E be a finite-dimensional A-module. Suppose $\Phi: E \to X$ is a map of A-modules, where X is given an A-module structure via the adjoint action. Furthermore, let E^ℓ be the bimodule we get from E if we let A act as usual on the left and by the counit ε on the right. Then the map $E^\ell \otimes X \to X$ defined by $e \otimes x \mapsto \Phi(e)x$ is a bimodule map.*

Proof. This is another straightforward calculation. If a is in A, e is in E, and x is in X, then the element $a(e \otimes x) = a_{(1)}e \otimes a_{(2)}x$ is sent to the element

$$\begin{aligned}\Phi(a_{(1)}e)a_{(2)}x &= [(\operatorname{ad} a_{(1)})\Phi(e)][(a_{(2)}x)]\\ &= a_{(1)}\Phi(e)\sigma(a_{(2)})(a_{(3)}x)\\ &= a_{(1)}\Phi(e)\varepsilon(a_{(2)})x\\ &= a_{(1)}\varepsilon(a_{(2)})\Phi(e)x\\ &= a\Phi(e)x\end{aligned}$$

as expected, and the element $(e \otimes x)a = e \cdot a_{(1)} \otimes xa_{(2)}$ is sent to $\Phi(e \cdot a_{(1)}))\cdot xa_{(2)} = \Phi(e\varepsilon(a_{(1)})xa_{(2)} = \Phi(e)x\varepsilon(a_{(1)})a_{(2)} = \Phi(e)xa$ as expected. □

Note. Since $\operatorname{ad} a$ is a linear combination of the left and right actions of $a_{(1)}$ and $a_{(2)}$, the proposition above implies that the map $E^\ell \otimes X \to X$ also respects the adjoint action.

2. Quantum sl(2) and its representation theory

Let q be an indeterminate and let $K = k(q^{1/2})$ be the field of rational functions in $q^{1/2}$ over the field k of characteristic 0. (We use $q^{1/2}$ instead of q so that we will not have any trouble working with nonintegral weights later on.) Define quantum sl(2) – which we henceforth denote by U – to be the K-algebra with identity 1 generated by x, y, t, t^{-1} with the relations

$$tt^{-1} = t^{-1}t = 1, \quad txt^{-1} = q^2x, \quad tyt^{-1} = q^{-2}y, \quad xy - yx = \frac{t^2 - t^{-2}}{q^2 - q^{-2}}.$$

U becomes a Hopf algebra with the following structure maps:

$$\Delta(x) = x \otimes t^{-1} + t \otimes x, \quad \Delta(y) = y \otimes t^{-1} + t \otimes y, \quad \Delta(t^{\pm 1}) = t^{\pm 1} \otimes t^{\pm 1}$$

$$\varepsilon(x) = \varepsilon(y) = 0, \quad \varepsilon(t^{\pm 1}) = 1, \quad \sigma(x) = -q^{-2}x, \quad \sigma(y) = -q^2y, \quad \sigma(t^{\pm 1}) = t^{\mp 1}.$$

The theory of U-modules is rather similar to the nonquantum case and is becoming well-known. We describe here parts of the theory that we will need.

Let M be a U-module. If $m \in M$ and $t \cdot m = q^r m$, then we say m is a vector of weight r. If in addition we have $x \cdot m = 0$, then m is called a highest weight vector of weight r. (Note the slight deviation from the classical terminology. We are following [4].) If M is generated by a highest weight vector, then M is said to be a highest weight module. Just as in the classical case, there is a universal highest weight module of weight r: the Verma module $M(r)$. If v is a highest weight vector in $M(r)$, it is easy to see that $M(r)$ has a basis $\{y^i v\}_{i=0}^{\infty}$; each basis vector $y^i v$ has weight $r - 2i$. Using this, it is not hard to prove that $M(r)$ is irreducible unless r is a nonnegative integer.

Every finite-dimensional U-module is completely reducible, i.e., it is a direct sum of irreducible U-modules. For $r = 0, 1, 2, \ldots$, there is a unique finite-dimensional irreducible module of highest weight r; we denote this module by $L(r)$. The weights of $L(r)$ are $r, r-2, \ldots, -(r-2), -r$. The dual module $L(r)^*$ is actually isomorphic to $L(r)$. (This is not true for other quantum algebras.)

Caution. In contrast to the classical case, there are four nonisomorphic irreducible modules of a given (finite) dimension. We will, however, need only one of these four.

The center $Z(U)$ of U is a polynomial ring in one variable $K[z]$, where z can be taken to be the element

$$z := yx + \left(\frac{qt + q^{-1}t^{-1}}{q^2 - q^{-2}}\right)^2 \tag{2.1}$$

(see [4, 3.3]). Therefore an algebra map $\lambda : Z(U) \to K$ is completely determined by the scalar $\lambda(z)$. Quite often we identify the map λ with the scalar $\lambda(z)$.

Let M be a U-module. As usual, we say that M has infinitesimal character λ if every element $c \in Z(U)$ acts as the scalar $\lambda(c)$ on M. Every highest weight module has an infinitesimal character.

Lemma 2.1 (Harish–Chandra's Theorem). *Suppose M and N are highest weight modules with highest weights r and s, respectively. Then M and N have the same infinitesimal character if and only if $r = s$ or $r = -s-2$.*

Proof. This is a simple calculation involving the element z of (2.1). On M, the element $(q^2 - q^{-2})z$ acts as the scalar $(q^{r+1} - q^{-r-1})^2$, while on N the same element acts as $(q^{s+1} - q^{-s-1})^2$. The two are equal if and only if $r = s$ or $r = -s-2$. □

We will also need to know the behaviour of Verma modules when tensored with finite-dimensional modules.

Lemma 2.2. *Let s be a nonnegative integer and r any integer or half-integer. Then there is a Verma flag*

$$L(s) \otimes M(r) = M_{s+1} \supseteq M_s \supseteq \cdots \supseteq M_0 = 0$$

with $M_{i+1}/M_i \simeq M(r+s-2i)$ $(i = 0, 1, \ldots, s)$.

Proof. Let m and e be highest weights of $M(r)$ and $L(s)$, respectively. Define

$$M_{i+1} = \sum_{j=0}^{i} K(y^j e \otimes m) \quad \text{for } i = 0, 1, \ldots, s.$$

An induction argument shows that the canonical image of $y^i \otimes m$ in M_{i+1}/M_i is a highest weight vector; so M_{i+1}/M_i is a highest weight module of highest weight $r+s-2i$. Counting dimensions of various weight spaces shows that M_{i+1}/M_i is isomorphic to $M(r+s-2i)$. As the details are quite similar to the classical case (see [1, Theorem 2.2]), I will say no more here. □

Finally, we will also need the following result (Kostant's separation of variables theorem for U). This can be found in [4, Section 3.11].

Lemma 2.3. *For s a nonnegative integer, define $L_{2s} := (\operatorname{ad} U)(xt^{-1})^s$. Then L_{2s} is isomorphic to $L(2s)$. Define $\mathscr{L}$ to be $\bigoplus_{s \geq 0} L_{2s}$. Then multiplication in $F(U)$ induces a (vector space) isomorphism $\mathscr{L} \otimes Z(U) \to F(U)$.*

The above lemma is useful in determining the annihilators of Verma modules.

Lemma 2.4. *Let M be a Verma module with infinitesimal character λ. Let I_λ be the ideal in $F(U)$ generated by $z - \lambda(z)$. Then $\operatorname{Ann}_{F(U)} M = I_\lambda$.*

Proof. Clearly $\operatorname{Ann}_{F(U)} M \supseteq I_\lambda$; we need to show the reverse inclusion. Let u be in $\operatorname{Ann}_{F(U)} M$; we can assume u is a weight vector. It is easy to calculate that $(\operatorname{ad} u')u$ is also in $\operatorname{Ann}_{F(U)} M$, for any $u' \in U$.

By Lemma 2.3, we can write

$$u = c_0 + c_1(z - \lambda(z)) + c_2(z - \lambda(z))^2 + \cdots + c_n(z - \lambda(z))^n$$

for some $c_0, c_1, \dots, c_n \in \mathscr{L}$. We need to show that $c_0 = 0$. Suppose not. Then

$$c_0 = \alpha_s(\operatorname{ad} y)^r(xt^{-1})^s + \alpha_{s-1}(\operatorname{ad} y)^{r-1}(xt^{-1})^{s-1} + \cdots + \alpha_0,$$

where $s, r \geq 0$ and $\alpha_i \in K$ with $\alpha_s \neq 0$. But $(\operatorname{ad} x)^r c_0$ is also in $\operatorname{Ann}_{F(U)} M$, so $(xt^{-1})^s$ annihilates M, which is definitely not true. So $c_0 = 0$ and u is in I_λ. □

Remark. Lemma 2.3 also implies that $F(U)/I_\lambda$ is isomorphic to $\mathscr{L} = \bigoplus_{s \geq 0} L(2s)$ as $(\operatorname{ad} U)$-modules.

3. Some properties of U-bimodules

Let M and N be U-modules. Then $\hom(M,N)$ is naturally a U-bimodule: for $u_1, u_2 \in U$, $\phi \in \hom(M,N)$, and $m \in M$, we define $(u_1 \phi u_2)(m)$ to be $u_1 \cdot \phi(u_2 m)$. The $(\operatorname{ad} U)$-finite part of $\hom(M,N)$ is denoted by $L(M,N)$. In this section we investigate such $F(U)$-bimodules when M and N are Verma modules. Note that if M and N have infinitesimal characters λ and μ, respectively, then $c \in Z(U)$ acts as the scalar $\lambda(c)$ on the right and as the scalar $\mu(c)$ on the left. Thus $L(M,N)$ has infinitesimal character (μ, λ).

Since $L(M,N)$ is $(\operatorname{ad} U)$-finite, it breaks up into a direct sum of irreducible finite-dimensional modules. The number of times a certain irreducible finite-dimensional module E occurs as a summand of $L(M,N)$ is called the *multiplicity* of E and is denoted by $[L(M,N):E]$.

Proposition 3.1. *If r is not a negative integer, then $[L(M(r), M(s)):E]$ is 1 if $r - s$ is a weight of E and 0 otherwise.*

Proof. The multiplicity of E in $L(M(r), M(s))$ is exactly equal to the dimension of $\hom_U(E, \hom(M(r), M(s)))$, which by Lemma 1.2 is equal to the dimension of $\hom_U(M(r), E^* \otimes M(s))$. By Lemma 2.2 $E^* \otimes M(s)$ has a Verma flag with factors isomorphic to $M(s + t)$, where t ranges over all the weights of E. Since r is not a negative integer, $\hom_U(M(r), M(s + t))) = 0$ unless $r = s + t$, i.e., unless $t = r - s$. Thus $\hom_U(M(r), E^* \otimes M(s))$ is one-dimensional if $r - s$ is a weight of E^* and it is zero-dimensional otherwise. We get the assertion of the theorem since the weights of E^* coincide with the weights of E. □

Corollary 3.2. *Let M be a Verma module with infinitesimal character λ; suppose $M = M(r)$ with r not a negative integer. Then we have an $F(U)$-bimodule isomorphism $F(U)/I_\lambda \simeq L(M,M)$.*

Proof. There is a bimodule map $F(U) \to L(M,M)$ sending $u \in F(U)$ to the map $m \mapsto um$ $(m \in M)$. The kernel of this map is $\operatorname{Ann}_{F(U)} M = I_\lambda$ by Lemma 2.4. Hence we have a bimodule injection $F(U)/I_\lambda \to L(M,M)$. By the above proposition and the remark after Lemma 2.4, both sides have the same multiplicities. Hence the injection is actually a bimodule isomorphism, as desired. □

We now investigate what happens when we tensor $L(M, N)$ with a finite-dimensional module.

Proposition 3.3. *Let E be a finite-dimensional U-module. Recall that E^ℓ is the bimodule we get from E when U acts as usual on the left and by the counit on the right. Then*

$$E^\ell \otimes L(M,N) \simeq L(M, E \otimes N)$$

as $F(U)$-bimodules.

Proof. Since $E^\ell \otimes L(M,N) = F(E^\ell \otimes \hom(M,N)$, we only need to show that $E^\ell \otimes \hom(M,N)$ is isomorphic to $\hom(M, E \otimes N)$ as U-bimodules.

There is a vector space isomorphism $E \otimes \hom(M,N) \to \hom(M, E \otimes N)$ sending the element $e \otimes \phi$ to the map $\phi' : m \mapsto e \otimes \phi(m)$ $(e \in E,\ \phi \in \hom(M,N),\ m \in M)$. It is quite straightforward to verify that this isomorphism respects the bimodule structures. □

Note that the above proof only used the fact that U is a Hopf algebra.

Definition 3.4. Let r be an odd integer. Define the $F(U)$-bimodule $X(r)$ to be

$$X(r) := L(M(r/2 - 1), \quad M(-r/2 - 1)).$$

Note that the left and right infinitesimal characters of $X(r)$ coincide. $X(r)$ and $X(s)$ have the same infinitesimal characters if and only if $r = s$ or $r = -s$. In fact, more is true.

Theorem 3.5. *As $F(U)$-bimodules, $X(r)$ and $X(-r)$ are isomorphic.*

Proof. By Proposition 3.2, the decomposition of $X(r)$ and $X(-r)$ into a direct sum of $(\operatorname{ad} U)$-modules is

$$X(r) \simeq X(-r) \simeq L(r) \oplus L(r+2) \oplus L(r+4) \oplus \cdots;$$

thus $X(r)$ and $X(-r)$ have the same multiplicities. Hence it suffices to show that there is a surjective bimodule map from $X(-r)$ to $X(r)$.

Let $E = L(r)$. I claim that $X(r)$ is generated as a right $F(U)$-module by the copy of E inside it. To be more precise, let $\Phi : E \to X(r)$ be a nonzero $(\operatorname{ad} U)$-module map. (There is only one such map, up to scalar, since $[X(r) : E] = 1$.)

Claim. $X(r) = \Phi(E) F(U)$.

Proof. Let e be a highest weight of E. Then $\Phi(E)F(U)$ is stable under $\operatorname{ad} U$ and contains $\Phi(e)(xt^{-1})^j$, which is a highest weight of $L(r+2j)$, $j = 0, 1, \ldots$ Thus $\Phi(e)F(U)$ contains $L(r) \otimes L(r+2) \otimes \cdots$; hence it contains all of $X(r)$.

This means the multiplication map $\Phi(E) \otimes F(U) \to X(r)$ is surjective. But according to Lemma 1.2, such a map induces a bimodule map $\Psi: E^\ell \otimes F(U) \to X(r)$. Now $F(U) \simeq L(M(-r/2-1), M(-r/2-1))$ (Corollary 3.2), so the left side is isomorphic to $L(M(-r/2-1), E \otimes M(-r/2-1))$, by Proposition 3.3. By looking at the infinitesimal characters of the factors in the Verma flag of $E \otimes M(-r/2-1)$ (cf. Lemma 2.2), we see that $E \otimes M(-r/2-1)$ decomposes into a direct sum of Verma modules $M(r/2-1) \oplus M(r/2-3) \oplus \cdots \oplus M(-3r/2-1)$. Thus we have a surjective map

$$\Psi: X(-r) \oplus X' \to X(r),$$

where $X' = L(M(-r/2-1), M(r/2-3)) \oplus \cdots \oplus L(M(-r/2-1), M(-3r/2-1))$. The infinitesimal characters of the summands of X' do not match the infinitesimal character of $X(r)$, so Ψ is zero on X'. Thus we have a surjective map from $X(-r)$ to $X(r)$ as required. $\square$

4. Dixmier algebras

Here we state formally what a Dixmier algebra is. The definition follows [8, Definition 2.1], adapted to quantized enveloping algebras. Thus a Dixmier algebra is a pair (A, ϕ), where A is an algebra equipped with a locally finite action ad of U on A; and $\phi: F(U) \to A$ is an algebra map which commutes with the adjoint actions on $F(U)$ and on A. We note that A becomes an $F(U)$-bimodule via the map ϕ. We require the $F(U)$-bimodule structure to be compatible with the $(\operatorname{ad} U)$-module structure. A should also be finitely generated under the adjoint and bimodule actions; and any irreducible $(\operatorname{ad} U)$-module should have only finite multiplicity in A.

With this out of the way, we now construct our Dixmier algebras.

Definition 4.1. Let r be an odd integer. We let $B(r)$ denote the algebra (and bimodule)

$$B(r) := L\left(M\left(\frac{r}{2}-1\right) \oplus M\left(\frac{-r}{2}-1\right),\ M\left(\frac{r}{2}-1\right) \oplus M\left(\frac{-r}{2}-1\right)\right).$$

If ϕ is an element of $X(r)$ (Definition 3.4), we can extend ϕ so that it becomes an element ϕ' of $B(r)$ as follows: on $M(r/2-1)$, we define ϕ' to be ϕ, while on $M(-r/2-1)$, we define ϕ' to be $\Psi(\phi)$, where Ψ is the isomorphism $X(r) \xrightarrow{\sim} X(-r)$ described in Theorem 3.5. We can then define the Dixmier algebra $A(r)$ to be the subalgebra (with identity) generated by the set

$$S = \{\phi' \in B(r) \mid \phi \in X(r)\}.$$

In other words, $A(r)$ is generated by maps which act like members of $X(r)$ on $M(r/2-1)$ and like the corresponding members of $X(-r)$ on $M(-r/2-1)$.

Note that S is an $F(U)$-bimodule isomorphic to $X(r)$ (or equivalently, to $X(-r)$). It is also clear that $A(r)$ as we defined it above is really a Dixmier algebra: it is (ad U)-finite with finite multiplicities, and it is finitely generated.

We will spend the rest of this section elucidating the structure or $A(r)$.

Theorem 4.2. *As an $F(U)$-bimodule, $A(r)$ is isomorphic to $X(r) \oplus F(U)/I_\lambda$, where λ is the infinitesimal character of $M(r/2-1)$.*

Proof. We first show that $S \cdot S$ is the diagonal copy of $F(U)/I_\lambda$ inside $B(r)$.

Multiplication (or, in this case, map composition) in $S \cdot S$ induces a bimodule map $L(r) \oplus S \to S \cdot S$. Arguing as in the proof of Theorem 3.5, we obtain from this a surjective bimodule map $F(U)/I_\lambda \to S \cdot S$. Comparing annihilators in $F(U)$, we see that this map is injective. Hence $S \cdot S$ is isomorphic to $F(U)/I_\lambda$ as an $F(U)$-bimodule. We still need to show, however, that $S \cdot S$ is actually the *diagonal copy* of $F(U)/I_\lambda$ in $B(r)$. In other words, we have to prove that if η is a map in $S \cdot S$ corresponding to $u \in F(U)/I_\lambda$, then $\eta(m) = um$ and $\eta(m') = um'$ for any $m \in M(r/2-1)$ and $m' \in M(-r/2-1)$. But this is not too difficult; since $S \cdot S$ is isomorphic to $F(U)/I_\lambda$, we only need to show this for *one* element η of $S \cdot S$.

S, being isomorphic to $X(r)$, has a copy of the (ad U)-module $L(r)$ inside it. Let ϕ' be a lowest weight vector in this copy; then $(\operatorname{ad} y)\phi' = 0$ (so that ϕ' commutes with yt^{-1}), and ϕ' has weight $-r$. We are going to consider the effect of ϕ' on basis vectors of $M(r/2-1)$ and $M(-r/2-1)$.

So let v and w be highest weight vectors of $M(r/2-1)$ and $M(-r/2-1)$, respectively. A basis for $M(r/2-1)$ consists of $\{(yt^{-1})^j v\}_{j=0}^{\infty}$; similarly, a basis for $M(-r/2-1)$ consists of $\{(yt^{-1})^j w\}_{j=0}^{\infty}$. ϕ' is completely determined by $\phi'(v)$ and $\phi'(w)$; since ϕ' has weight $-r, \phi'(v)$ is a multiple of w and $\phi'(w)$ is a multiple of $(yt^{-1})^r v$. We can arrange it so that $\phi'(v) = w$ and $\phi'(w) = yt^{-1})^r v$. Then it is clear that $(\phi')^2 \in S \cdot S$ acts as $(yt^{-1})^r$ on both $M(r/2-1)$ and $M(-r/2-1)$. Thus $S \cdot S$ is indeed the diagonal copy of $F(U)/I_\lambda$.

The rest of the proof is easy. We can now conclude that $S^k = S$ if k is odd, and $S^k = S^2$ if k is even. Thus

$$A(r) = \sum_{k=0}^{\infty} S^k \simeq F(U)/I_\lambda \oplus X(r). \qquad \square$$

We now show that the Dixmier algebras we have constructed are completely prime, i.e., they are integral domains. We begin first with $F(U)/I_\lambda$.

Lemma 4.3. *For any infinitesimal character λ, the ring $F(U)/I_\lambda$ is completely prime.*

Proof. This is proved in full generality in [3, Section 8.1]. Since our algebra is so simple, we do not have to use the full power of this result; below is a sketch of a computational proof.

As a K-algebra, $F(U)/I_\lambda$ is generate by the images of xt^{-1}, yt^{-1}, and $(\operatorname{ad} x)(yt^{-1}) = xy - q^{-4}yx$. Define a filtration on $F(U)/I_\lambda$ by declaring these elements to be of degree 2: it is a matter of computation to verify that we can do this. The associated graded ring $\operatorname{gr} F(U)/I_\lambda$ is again generated by three elements which we still call xt^{-1}, yt^{-1}, and $xy - q^{-4}yx$. It is again straightforward, though rather tedious, computation to verify that $\operatorname{gr} F(U)/I_\lambda$ can be embedded inside a skew polynomial ring $K[\alpha, \beta]$ with $\alpha\beta = q^2\beta\alpha$, as follows:

$$xt^{-1} \mapsto -q^{-4}\alpha^2; \quad yt^{-1} \mapsto \beta^2; \quad xy - q^{-4}yx \mapsto q\alpha\beta + q^{-1}\beta\alpha.$$

Clearly, $K[\alpha, \beta]$ is completely prime, hence so is $\operatorname{gr} F(U)/I_\lambda$; thus so is $F(U)/I_\lambda$. □

Theorem 4.4. *The Dixmier algebra $A(r)$ of Definition* 4.1 *is completely prime.*

Proof. As a bimodule, $A(r)$ can be decomposed as $A_0 \oplus A_1$ with $A_0 = F(U)/I_\lambda$ and $A_1 = X(r)$; and $A_0A_0 \subseteq A_0$, $A_1A_0 = A_0A_1 \subseteq A_1$, and $A_1A_1 \subseteq A_0$. Lemma 4.3 shows that A_0 is completely prime, hence $Q := \operatorname{Fract} A_0$ exists. It is easy to verify that $QA_1 = A_1Q$.

Suppose $(a_0 + a_1)(a_0' - a_1') = 0$, where $a_0, a_0' \in A_0$ and $a_1, a_1' \in A_1$. If $a_0 \neq 0$, then $a_0' \neq 0$ also, hence we can multiply the equation on the left by a_0^{-1} and on the right by $(a_0')^{-1}$ to get $(1 + a)(1 - a') = 0$, where $a, a' \in QA_1$. Thus $a = a'$ and $a^2 = 1$. But the only ad-submodule of $A(r)$ isomorphic to the trivial module is the scalars. Hence a is a scalar, a contradiction. So $a_0 = a_0' = 0$, and we only need to show that A_1 has no nontrivial zero divisors.

Let $0 \neq \phi \in A_1$. Then there is a $\psi \in A_1$ such that $\phi\psi \neq 0$, since otherwise $\phi \circ L(M(r/2 - 1), M(-r/2 - 1)) = \phi(M(-r/2 - 1)) = 0$, contradicting $0 \neq \phi$. Now suppose $\phi'\phi = 0$. Then $\phi'(\phi\psi) = 0$, where $\phi\psi$ is a nonzero element of A_0. Hence $\phi' = 0$. A similar argument shows that if $\phi\phi' = 0$, then $\phi' = 0$ too. Hence $A(r)$ is completely prime, as claimed. □

5. Uniqueness

In this section we let A be a completely prime Dixmier algebra containing $F(U)/I_\lambda$. we will determine what forms A could take. Essentially, A has to be one of the $A(r)$'s constructed in the previous section.

There is one possibility which we wish to take care of immediately. Let K' be any overfield of K generated by algebraic elements. Then $K' \otimes_K F(U)/I_\lambda$ is a completely prime Dixmier algebra. It is larger than $F(U)/I_\lambda$ but not essentially different; we have just extended the ground field. Therefore we will eliminate this possibility in the discussion that follows, by making the assumption that A (and also $Q = \operatorname{Fract} A$) contains nothing algebraic over K other than the elements of K itself.

Lemma 5.1. *Suppose $\alpha \in A$ generates the trivial module under the adjoint action. Then α is a scalar.*

Proof. The elements $1, \alpha, \alpha^2, \alpha^3, \ldots$ each generate the trivial module under the adjoint action. The trivial module has finite multiplicity in A, so there exist scalars $c_0, c_1, \ldots, c_{n-1}$ such that $\alpha^n + c_{n-1}\alpha^{n-1} + \cdots + c_0 = 0$. Hence α is algebraic over K. By our assumption above, this implies that α is a scalar. $\square$

Note that the lemma applies equally well if α is an element of Q, not just of A.

We will now investigate A. Our main weapon is the method of proof in Theorems 3.5 and 4.2; since we will use this technique again and again, we separate it here as a lemma. First note that $F(U)/I_\lambda \subseteq A$ can be written as $L(M(s), M(s))$, where $s \geq -1$ (Corollary 3.2).

Lemma 5.2. *Let E be any irreducible* ($\operatorname{ad} U$)*-submodule of A, and let X be the $F(U)$-subbimodule of A generated by E.*

(a) *If s is not an integer and the highest weight of E is even, then $X \simeq L(M(s), M(s))$.*

(b) *If s is not an integer and the highest weight of E is odd, then $s = r/2 - 1$ for some positive integer r, and $X \simeq L(M(s), M(s-2))$.*

If s is an integer, then the highest weight of E is even, say $2n$. The next two cases describe the possibilities for X for different values of $2n$.

(c) *If $2n < 2s + 2$, then $X \simeq L(M(s), M(s))$.*

(d) *If $2n \geq 2s + 2$ then there is a surjective map $L(M(s), M) \to X$, where $M(s) \subseteq M$ and $M/M(s) \simeq M(-s-2)$.*

Proof. The proofs of (a)–(d) are similar and depend on Lemmas 2.2 and 2.3. We first note that X is precisely $EF(U)$ since $EF(U)$ is closed under the left and right actions of $F(U)$ and the adjoint action of U. Thus by Lemma 1.2, there is a surjective homomorphism $E^\ell \otimes L(M(s), M(s)) \to X$. The left side is isomorphic to $L(M(s), E \otimes M(s))$, which decomposes into a direct sum of bimodules with different infinitesimal characters. The only summand with the same infinitesimal character as X is $L(M(s), M(s))$ (in cases (a) and (c)), or $L(M(s), M(-s-2)$ (in case (b)), or $L(M(s), M)$ (in case (d)). This finishes the proof of (d). In cases (a) and (c), the kernel of the maps is of the form J/I_λ, where J is an ideal (hence a subbimodule) of $F(U)$. If J were strictly bigger than I_λ, then there would be nonzero elements in $F(U)/I_\lambda$ annihilating X, contradicting the complete primality of A. This finishes the proof of (a) and (c). In case (b), we have a surjective map $L(M(r/2-1), M(-r/2-1)) \to X$. The left side can be shown to be irreducible by exactly the same method we are using. Thus we have an isomorphism $L(M(r/2-1), M(-r/2-1)) \simeq X$. This finishes (b). $\square$

In what follows we will write A_0 for the subalgebra (and subbimodule) of A isomorphic to $F(U)/I_\lambda \simeq L(M(s), M(s))$. We reserve the notation "$L(M(s), M(s))$" when we want to consider this bimodule in the abstract.

Theorem 5.3. *Suppose A is a completely prime Dixmier algebra with A_0 isomorphic to $L(M(s), M(s))$, where $s \geq -1$ and s not an integer. Then either $A = A_0$,* or *$s = r/2 - 1$ with r a positive odd integer and $A = A(r)$ (definition* 4.1).

Proof. Let E be any (ad U)-submodule of A. The highest weight of E is either even or odd; assume first that it is even. Then by Lemma 5.2 (a), the $F(U)$-subbimodule X of A generated by E is isomorphic to $L(M(s), M(s))$; in particular, X contains the trivial (ad U)-module, which comprise the scalars by Lemma 5.1. The scalars generate A_0; thus $X = A_0$. We conclude that the only (ad U)-submodules of A that do not lie in A_0 must have odd highest weight.

Now suppose E does have odd highest weight. As before, let X be the subbimodule generated by E. By Lemma 5.2 (b), s is of the form $r/2 - 1$ for some positive integer r, and $X \simeq L(M(r/2 - 1), M(-r/2 - 1))$.

I claim next that A can have at most one copy of $L(M(r/2 - 1), M(-r/2 - 1))$. To prove this we use the same technique as in Lemma 5.2. Multiplication of E with any copy of $L(M(r/2 - 1), M(-r/2 - 1))$ (which must land in A_0 since the (ad U)-submodules with even highest weights are all in A_0) induces a bimodule map $L(M(r/2 - 1), E \otimes M(-r/2 - 1)) \to F(U)/I_\lambda$. This time the left side decomposes into a sum of bimodules where the summand with the correct infinitesimal character is $L(M(r/2 - 1), M(r/2 - 1)) \simeq F(U)/I_\lambda$. Thus there is only one possible map, up to scalar; and hence there is also only one possible multiplication (up to scalar) of E with any copy of $L(M(r/2 - 1), M(-r/2 - 1))$ in A. If X_1 and X_2 are two different copies of $L(M(r/2 - 1), M(-r/2 - 1))$, then we can find elements $\phi_1 \in X_1$ and $\phi_2 \in X_2$ such that $e\phi_1 = e\phi_2$ for all $e \in E$. Thus $e(\phi_1 - \phi_2) = 0$, contradicting the complete primality of A.

We conclude that if $A \neq A_0$, then $s = r/2 - 1$ and $A \simeq F(U)/I_\lambda \oplus X(r)$ as $F(U)$-bimodules. But the argument above already shows that there is only one possible multiplication $X(r) \times X(r) \to F(U)/I_\lambda$ (up to scalar). The algebras generated by choosing different scalars in the multiplication are easily seen to be isomorphic, so there could be only one possible algebra structure on $F(U)/I_\lambda \oplus X(r)$. In Section 4 we have constructed such an algebra structure, calling the result $A(r)$. So $A = A(r)$, as claimed. $\square$

There is only one remaining case to investigate, namely when $A_0 \simeq L(M(s), M(s))$ with s a nonnegative integer. We assume this for the rest of this section.

Since s is a nonnegative integer, the Verma module $M(-s-2)$ is a submodule of $M(s)$. Thus $L(M(s), M(s))$ is not irreducible; it has the subbimodule $L(M(s), M(-s-2))$. As an (ad U)-module, $L(M(s), M(-s-2))$ is a direct sum of the irreducibles $L(2n)$, $n = s+1, s+2, \ldots$ (see Proposition 3.1).

Lemma 5.4. *Let E be an* $(\operatorname{ad} U)$*-submodule of* $L(M(s), M(s))$ *with highest weight* $2n \geq 2s+2$. *Then the* $F(U)$*-subbimodule generated by* E *is* $L(M(s), M(-s-2))$.

Proof. Let Y be the subbimodule generated by E. Then by Proposition 3.1, Y is contained in $L(M(s), M(-s-2))$; also, by Lemma 5.2(d), there is a surjective bimodule map $X \to Y$, where $X = L(M(s), M)$. Now X has a subbimodule X_1 isomorphic to $L(M(s), M(s))$ and by counting multiplicities we see that X/X_1 is isomorphic to $L(M(s), M(-s-2))$. Since Y does not contain any $(\operatorname{ad} U)$-submodule isomorphic to $L(2n)$, $n < s+1$, the kernel of the map $X \to Y$ must contain these $(\operatorname{ad} U)$-modules. But by Lemma 5.2(c) any one of them generate X_1, so the kernel must contain X_1. Comparing annihilators we see that the kernel must be exactly X_1, so Y is isomorphic to $X/X_1 \simeq L(M(s), M(-s-2))$, as claimed □

Lemma 5.4 implies that $L(M(s), M(-s-2))$ is irreducible; and taking Lemma 5.2(c) into account, we see that there is only one nontrivial subbimodule of $L(M(s), M(s))$, namely $L(M(s), M(-s-2))$.

Lemma 5.5. *Suppose* $A \neq A_0$. *Then there exists a subbimodule* $B \subseteq A$ *containing* A_0 *such that* $B/A_0 \simeq L(M(s), M(-s-2))$.

Proof. Let B be the smallest subbimodule of A properly containing A_0. Let E be an $(\operatorname{ad} U)$-submodule of B lying outside A_0; the highest weight of E must be even and larger than $2s+2$, by Lemma 5.2(c). Let Y be the subbimodule of B generated by E. Then by Lemma 5.2(d) we see that Y is quotient of some bimodule X, where X has a subbimodule X_1 isomorphic to A_0 and $X/X_1 \simeq L(M(s), M(-s-2))$. If Y has no $(\operatorname{ad} U)$-submodule with highest weight less than $2s+2$, then all such submodules must be in the kernel of the map $X \to Y$. But by Lemma 5.2(c) such submodules generate X_1, so $Y = X/X_1 \simeq L(M(s), M(-s-2))$; hence $B = A_0 \oplus Y$ and $B/A \simeq L(M(s), M(-s-2))$, as required.

On the other hand, suppose Y does contain an $(\operatorname{ad} U)$-submodule with highest weight less than $2s+2$. Lemma 5.2(c) then implies that Y contains a copy of $L(M(s), M(s))$; by Lemma 5.1 this copy must be A_0 itself. Therefore the minimality of B forces $Y = B$. Since B is larger than A_0 and B is a quotient of X and X/X_1 is irreducible, we must have $B \simeq X$. Thus $B/A_0 \simeq X/X_1 \simeq L(M(s), M(-s-2))$, as required. □

We are now ready for the final result.

Theorem 5.6. *Suppose A is a completely prime Dixmier algebra containing* $A_0 = L(M(s), M(s))$, *with s a nonnegative integer. Then* $A = A_0$.

Proof. The following proof is quite similar to [6, Lemma 2.3]. Let Q be the fraction field of A_0. As usual we can extend any A_0-module X to a Q-module QX (see, e.g., [2, Ch. 11]).

Suppose $A \neq A_0$. Then we can find B as in Lemma 5.5 such that $B/A_0 \simeq L(M(s), M(-s-2))$. Thus $QB/QA_0 \simeq Q(B/A_0) \simeq QL(M(s), M(-s-2)) \simeq Q$. Let b be an element (with weight 0) of QB such that its canonical image $\bar{b}$ in QB/QA_0 corresponds to the unit 1 in Q. In other words, $\bar{b}$ generate the trivial module under the adjoint action; thus $(\operatorname{ad} x)b \in QA_0 = Q$ and $(\operatorname{ad} y)b \in QA_0 = Q$. This implies, in particular, that b commutes with xt^1 and yt^{-1} (modulo Q), hence with all of Q (modulo Q).

Now QB is a finitely generated (left) Q-module, so we can choose a minimal integer m such that

$$b^m + \alpha_{m-1}b^{m-1} + \cdots + \alpha_0 = 0; \quad \alpha_i \in Q.$$

We can of course choose the α's to be of weight zero. Now apply $\operatorname{ad} x$ to the above expression. Recalling that $(\operatorname{ad} x)b \in Q$ and b commutes with Q modulo Q, we get another expression of lower degree with leading term $[m(\operatorname{ad} x)b + (\operatorname{ad} x)\alpha_{m-1}]b^{m-1}$. By minimality of m, we conclude that $(\operatorname{ad} x)(mb + \alpha_{m-1}) = 0$. Similarly, we also conclude that $(\operatorname{ad} y)(mb + \alpha_{m-1}) = 0$. Thus $mb + \alpha_{m-1}$ generate the trivial module under the adjoint action, which implies that b is an element of Q, i.e., $\bar{b} = 0$. This is absurd. We conclude that $A = A_0$. □

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