Generic Bifurcation with Application to the von Kármán Equations

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We examine the possible types of generic bifurcation that can occur for a three-parameter family of mappings from a Banach space into itself. Specifically, the general form of the bifurcation equations arising from the von Kármán equations for the buckling of a rectangular plate is investigated. Chow, Hale, and Mallet-Paret (Applications of generic bifurcation. II, Arch. Rational Mech. Anal. 67 (1976)) studied the bifurcation of solutions to these equations in a two-parameter setting. These parameters were related to the normal loading and to the compressive thrust applied at the ends of the plate. We introduce a third bifurcation parameter by considering the length of the plate as variable. The generic hypotheses of Chow et al. no longer apply in this three-parameter setting, but modifications and extensions of these hypotheses permit a characterization of the three-parameter bifurcation diagram. The bifurcation sheets of this diagram appear as a natural generalization of the finite collection of arcs comprising the two-parameter diagram. As an example of this theory, an actual three-parameter bifurcation diagram is constructed for a specific form of the von Kármán equations.

1. INTRODUCTION

The objective of this paper is to characterize the possible types of generic bifurcation that occur for certain families of mappings of a Banach space \( \mathcal{X} \) into itself. The three parameter families considered are motivated by the von Kármán equations for the buckling of a rectangular plate. Once the form of this family of mappings is determined, a mathematical theory of bifurcation is developed and then applied to the specific example of the von Kármán equations.

Chow et al. [5, 6] studied the bifurcation of solutions to these equations in a two-parameter setting. These parameters related to the normal loading and to the compressive thrust applied at the ends of the plate. We introduce a third
bifurcation parameter by treating the length of the plate as variable. The
generic hypotheses of Chow et al. no longer apply in this three-parameter
setting, but modifications and extensions of these hypotheses permit a charac-
terization of the three-parameter bifurcation diagram. The bifurcation sheets
of this diagram constitute a natural generalization of the finite collection of arcs
comprising the original two-parameter diagram. In illustration of our theory,
a three-parameter bifurcation diagram is constructed for a specific form on the
von Kármán equations.

There exist several definitions of the word "bifurcation." In general, our
usage of this term coincides with that of Chow et al. Suppose \( r \) is a normed
family of mappings between two Banach spaces \( X, Z \) and that \( T \in \tau \) possesssees
an isolated zero \( x_0 \). Then, the operator \( T \) is called a bifurcation point for \( \tau \) at \( x_0 \)
if for every neighborhood \( U \) of \( T \) and \( V \) of \( x_0 \), there exist \( S \in U \), and distinct
\( x_1, x_2 \in V \), such that \( Sx_1 = Sx_2 = 0 \). Hence, as the operators \( S \in \tau \) vary near
\( T \), the number of solutions to the equation \( Sx = 0 \) (for \( x \) near \( x_0 \)) does not
remain constant. Since the zeros of \( S \) may represent equilibrium states (some
stable, some unstable) of a physical system, it is important that the bifurcation
points of a family \( \tau \) be determined.

In the case where \( \tau \) is an \( n \)-parameter family of mappings \( S(\cdot, \lambda), \lambda \in \mathbb{R}^n, \lambda_0 \)
is said to be a bifurcation point if \( S(\cdot, \lambda_0) \) is a bifurcation point for \( \tau \) in the
above sense (with respect to the \( \mathbb{R}^n \) topology induced by the parameter space).
Hence, as \( \lambda \) varies near \( \lambda_0 \), the number of solutions \( x \) to the equation

\[
S(x, \lambda) = 0 \quad (1.1)
\]

changes.

In certain instances, say for \( \lambda \in \mathbb{R} \), there might exist a smooth curve of solutions
\( x_0(\lambda) \) for all \( \lambda \) in a neighborhood of \( \lambda_0 \). In this case, for bifurcation to occur at \( \lambda_0 \),
there must exist at least one separate curve of solutions \( x_1(\lambda) (x_0 \neq x_1) \) branching
off from \( x_0(\lambda) \) at \( \lambda = \lambda_0 \). This situation is shown in Fig. 1. The curve \( x_0(\lambda) \)

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{Figure 1}
\end{figure}

often corresponds to the trivial solution. The parameter value \( \lambda_0 \) is commonly
referred to as a primary bifurcation point, and \( x_1(\lambda) \) is called a primary branch
of solutions. If there exists a second parameter value \( \lambda_1 \) at which a curve of
solutions \( x_{\alpha}(\lambda) \) branches off from \( x_1(\lambda) \), then \( \lambda_1 \) is called a secondary bifurcation
point and \( x_2(\lambda) \) a secondary solution branch (see Fig. 2). Obviously, these definitions can be extended to points of tertiary bifurcation, etc.

In many papers, the term "bifurcation point" presupposes the existence of the basic solution branch \( x_0(\lambda) \). Hence, "bifurcation points" are points of primary (secondary, tertiary, etc.) bifurcation, as defined in the preceding paragraph. Parameter values \( \lambda_0 \) at which two (or more) distinct solutions \( x_1(\lambda), x_2(\lambda) \) branch from a single point, as in Fig. 3, are simply called "branching points" (as in Keener [7]).

Our definition of bifurcation point, then, includes both of the above; that is, it includes branching points as well as (primary, secondary,...) bifurcation points. The usefulness of this definition derives from the fact that when \( S(\cdot, \lambda) \in \tau \) depend upon more than one parameter \( (\lambda \in \mathbb{R}^n, n \geq 2) \) there exist many interesting bifurcations as depicted in Fig. 3 (see, for instance, [6, Fig. 2.5]). The perturbation techniques normally used in finding the solution branches \( x_1, x_2 \) in Figs. 1 and 2 are not sufficient for Fig. 3. However, our methods, motivated by those of Chow et al., are applicable to "branching" bifurcations of this form.

In order for \( \lambda_0 \) to be a bifurcation point (with respect to \( \tau \)) at \( x_0 \), the partial derivative \( S_x(x_0, \lambda_0) \) cannot be an isomorphism. Were it an isomorphism, the implicit function theorem would imply the existence of a unique zero \( x(\lambda) \) of (1.1) for all \( \lambda \) in a neighborhood of \( \lambda_0 \). The first [5] of the two papers of Chow et al. contains a thorough discussion of the bifurcation when the kernel of \( S_x(x_0, \lambda_0) \) is one-dimensional. Their second paper [6] treats the more difficult case where \( \dim(\ker S_x(x_0, \lambda_0)) = 2 \). Writing \( S_x(x_0, \lambda_0) = (I_2 - \lambda_0 L) \), Chow et al. completely characterize the bifurcation for (1.1) in the case where \( \lambda_0^{-1} \) varies near \( \lambda^{-1} \), an eigenvalue of \( L \) with multiplicity two.
The theory presented in Section 2 treats the case where there are two eigenvalue terms in the linearized problem associated with (1.1). Through an application of the Lyapunov–Schmidt procedure, (1.1) is transformed into an algebraic equation which has two distinct eigenvalues varying near zero. (Chow et al., after applying the Lyapunov–Schmidt procedure, obtain an algebraic equation with a double eigenvalue near 0).

In applying our theory to the von Kármán equations, a third bifurcation parameter is introduced by considering the length $s$ of the plate as variable about $s = 2^{1/3}$. This third parameter "splits" the double eigenvalue $\lambda_0^{-1} = 2/9\pi^2$ of Chow et al. into two simple eigenvalues $\lambda_1^{-1}, \lambda_2^{-1}$ ($\lambda_1 \neq \lambda_2$ when $s \neq 2^{1/3}$). As a result, the three-parameter bifurcation diagram constructed in Section 3 constitutes a generalization of the original two-parameter diagram.

In Section 4, we consider a particular cross section of our three-parameter diagram, obtained by setting the normal loading $\lambda_3 = 0$. This enables us to study the bifurcation behavior associated with the splitting of the double eigenvalue $\lambda_0^{-1}$ into two simple eigenvalues. It is seen that this splitting transforms a multiple primary bifurcation point into several simple primary and secondary bifurcation points. This is an example of the bifurcation behavior studied in recent papers by Bauer et al. [3], and Keener [7], whose work is related to our own.

2. Theory

We first develop the bifurcation theory for the general three-parameter equation

$$f(u, \lambda) = M(u) + \lambda_1 L_1 u + \lambda_2 L_2 u + \lambda_3 k + \text{h.o.t.} = 0,$$

where

- $f(u, \lambda): \mathbb{R}^2 \times \mathbb{R}^3 \rightarrow \mathbb{R}^2$ is a smooth mapping,
- $M(u): \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a homogeneous cubic polynomial,
- $L_1, L_2: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ are linear operators,
- $k = [k_1, k_2] \in \mathbb{R}^2$ is constant,
- $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ is a parameter triple near $(0, 0, 0)$, and
- \text{h.o.t.} = $O( |u|^4 + (\lambda_1^2 + \lambda_2^2) |u| + (|\lambda_1| + |\lambda_2|) |u|^2 + \lambda_3^2 + |\lambda_3| |u|)$. 

Later we examine, in some detail, a particular application arising from the von Kármán equations. Equation (2.1) differs from the bifurcation equation

$$f(u, \lambda) = M(u) + \lambda_1 L_1 u + \lambda_3 k + \text{h.o.t.} = 0,$$

studied by Chow et al. [6], since it includes an additional eigenvalue parameter $\lambda_2$. 

The bifurcation problem for (2.1) consists of studying simultaneous solutions in $u$ of the equations $f_1 = 0, f_2 = 0$ for $(u, \lambda)$ near $(0, 0)$. The bifurcation points are those parameter values $\lambda$ at which the number of such simultaneous solutions changes; the collection of all bifurcation points $\lambda$, as depicted in $R^3$, is called a bifurcation diagram.

To obtain these values $\lambda$, we define, for fixed $h$, the planar curves

$$F_i(h) \equiv \{u \in R^2 : f_i(u, \lambda) = 0\}, \quad i = 1, 2,$$

and investigate the intersections of $F_i(h)$ with $F_2(h)$. Each of these two curves varies smoothly with respect to $h$. Hence, if all intersections of $F_i(h)$ with $F_2(h)$ are transversal for a particular $\lambda = \lambda$, we expect these intersections to be transversal for $\lambda$ near $\lambda$. Or, equivalently, if $F_i(h)$ and $F_2(h)$ have no points of tangency, then $F_i(h)$ and $F_2(h)$ will have no points of tangency for $\lambda$ near $\lambda$. Hence, in some neighborhood of $\lambda$, the number of solutions remains constant and no bifurcation occurs.

Parameter values $\lambda$ for which $F_1(h)$ and $F_2(h)$ do possess a tangency represent possible bifurcation points. For such values $\lambda$, the curves $F_1(h)$ and $F_2(h)$ might intersect at the point of tangency and then pass through each other as $\lambda$ passes through $\lambda$. This would result in the existence of several new solutions. If, for instance, such tangencies were points of nondegenerate second-order contact, we might expect the number of solutions to increase or decrease by two as $\lambda$ passes through $\lambda$.

The analytic condition describing transversality of $F_1(h)$ with $F_2(h)$ is

$$f(u, \lambda) = 0 \Rightarrow \det(\partial f/\partial u)(u, \lambda) \neq 0. \quad (2.2)$$

If we define $\mathcal{C}$ to be the open set of all $\lambda$ such that (2.2) holds, we must then examine the complement $\sim \mathcal{C}$ for potential bifurcation points. These are the values $\lambda = \lambda$ for which there exists some $u^*$ satisfying

$$f(u^*, \lambda) = 0,$$

$$\det \frac{\partial f}{\partial u}(u^*, \lambda) = 0. \quad (2.3)$$

Hence, in characterizing the bifurcation diagram of (2.1), our immediate objective is to study the simultaneous solutions of (2.3). With several natural hypotheses, the resulting bifurcation diagram consists of a finite number of sheets, each emanating from and encircling (or partially encircling) the origin. Any single sheet divides the parameter space into regions of $n$ and $(n + 2)$ solutions of (2.1). When passing through a sheet, the number of solutions $u$ to (2.1) increases or decreases by two.

Pairs of these sheets might intersect each other transversally, or might coalesce along a curve and vanish. The analytic condition for these situations will be
Given. Aside from these irregularities, the finite collection of sheets comprising the three-parameter bifurcation diagram are obvious analogs of the finite collection of curves, emanating from the origin, which comprise the two-parameter diagram of Chow et al. In fact, the sheets of the three-parameter diagram are generated by the curves of the two-parameter diagram. For this reason, the generic hypotheses used in constructing the three-parameter diagram are motivated by the hypotheses of Chow et al. Modifications must be made, however, since certain of these original hypotheses no longer apply in the three-parameter setting.

Before proceeding, we define the following functions:

\[
g(u, \mu, \lambda_2, \lambda_3) \equiv M(u) + L_1 u + \lambda_2 L_2 u + \lambda_3 k + O(1, \mu),
\]

\[
\tilde{g}(u, \lambda_3) \equiv k_2 g_1(u, 0, \lambda_2, \lambda_3) - k_1 g_2(u, 0, \lambda_2, \lambda_3),
\]

\[
\Delta_1(u, \mu, \lambda_2, \lambda_3) \equiv \det \left( \frac{\partial g}{\partial u} \right)_{(u, \lambda_2, \lambda_3)},
\]

\[
\Delta_3(u, \lambda_2) \equiv \Delta_1(u, 0, \lambda_2, \lambda_3),
\]

\[
\Delta_2(u, \lambda_2) \equiv \det \frac{\partial (g, A_1)}{\partial (u, \lambda_2)}_{(u, 0, \lambda_2, \lambda_3)},
\]

\[
\Delta^*(u, \lambda_2) \equiv \det \frac{\partial (\tilde{g}, A_1)}{\partial (u_1, u_2, \lambda_2)}_{(u, \lambda_2)}.
\]

With these definitions, we assume the following hypotheses:

(J1) \( M(u) = 0 \Rightarrow u = 0. \)

(J2) The two curves \( (M_i(u) + k_i = 0), i = 1, 2 \) intersect transversally, in that \( (M(u) + k = 0) \Rightarrow \partial M/\partial u \) is nonsingular.

(J3) If \( (u, \lambda_3) = (u^*, \lambda_3^*) \) satisfies

\[
\tilde{g}(u^*, \lambda_3^*) = 0,
\]

\[
\Delta_1(u^*, \lambda_3^*) = 0,
\]

\[
\Delta^*(u^*, \lambda_3^*) = 0,
\]

then

\[
\Delta^0(u^*, \lambda_3^*) \neq 0.
\]

(J4) Define, for \( u \in \mathbb{R}^2, \ v \in \mathbb{R}^3, \ (\mu, \lambda_2, \lambda_3) \in \mathbb{R}^3, \)

\[
F_1(u, v, \mu, \lambda_2, \lambda_3) \equiv g_1(u, \mu, \lambda_2, \lambda_3),
\]

\[
F_2(u, v, \mu, \lambda_2, \lambda_3) \equiv g_1(v, \mu, \lambda_2, \lambda_3),
\]

\[
F_3(u, v, \mu, \lambda_2, \lambda_3) \equiv g_2(u, \mu, \lambda_2, \lambda_3),
\]

\[
F_4(u, v, \mu, \lambda_2, \lambda_3) \equiv g_2(v, \mu, \lambda_2, \lambda_3),
\]

\[
F_5(u, v, \mu, \lambda_2, \lambda_3) \equiv \Delta_1(u, \mu, \lambda_2, \lambda_3),
\]

\[
F_6(u, v, \mu, \lambda_2, \lambda_3) \equiv \Delta_1(v, \mu, \lambda_2, \lambda_3).
\]
If there exist $u^*, v^* \in \mathbb{R}^2$, $\tilde{\lambda}_2 \in \mathbb{R}$, $\tilde{\lambda}_3 \in \mathbb{R}$ such that $F_1 = F_2 = \cdots = F_6 = 0$, evaluated at $(u, v, \mu, \lambda_2, \lambda_3) = (u^*, v^*, 0, \lambda_2, \lambda_3)$, then
\[
\det \frac{\partial (F_1, \ldots, F_6)}{\partial (u, v, \lambda_2, \lambda_3)} \bigg|_{(u^*, v^*, 0, \lambda_2, \lambda_3)} \neq 0.
\]

Hypotheses (J1), (J2) are identical with the generic hypotheses (H1), (H2) of Chow et al.

**Lemma 2.1.** Suppose (J1) is satisfied. For all solutions of (1) with $(u, \lambda)$ near $(0, 0)$,
\[
|u| \leq C(|\lambda_1|^{1/2} + |\lambda_2|^{1/2} + |\lambda_3|^{1/3}). \tag{2.5}
\]

**Proof.** The proof of this lemma (for two parameters) may be found in Chow et al. and is reproduced here, for the case of three-parameters. If (2.5) is not satisfied, there exists a sequence of solutions
\[
(u^n, \lambda^n_1, \lambda^n_2, \lambda^n_3) \to 0
\]
with
\[
\frac{|\lambda^n_1|^{1/2}}{|u^n|} \to 0, \quad \frac{|\lambda^n_2|^{1/2}}{|u^n|} \to 0, \quad \frac{|\lambda^n_3|^{1/3}}{|u^n|} \to 0.
\]
Dividing $f(u^n, \lambda^n)$ by $|u^n|^3$, (2.1) implies
\[
M \left[ \frac{u^n}{|u^n|} \right] = O \left[ \frac{|\lambda^n_1|}{|u^n|^3} + \frac{|\lambda^n_2|}{|u^n|^2} + \frac{|\lambda^n_3|}{|u^n|^3} \right] = o(1),
\]
as $n \to \infty$. Hence, taking a convergent subsequence $u^n/|u^n| \to u^0 \neq 0$, one obtains $M(u^0) = 0$, contradicting (J1).

According to the following theorem, there exist “sectorial” regions about the positive and negative $\lambda_3$-axis, within which no bifurcation may occur.

**Theorem 2.2.** Suppose (J1), (J2) are satisfied. There exist $\alpha > 1$, $\beta > 0$ such that no bifurcation occurs in the region
\[
\mathcal{R}_{\alpha, \beta} = \{(\lambda_1, \lambda_2, \lambda_3) : |\lambda_3| \leq \beta, (|\lambda_1|^{1/3} + |\lambda_3|^2) \leq |\lambda_4|^{2/\alpha}\}.
\]

**Proof.** Lemma (2.1) implies that in the region $\mathcal{R}_{\alpha, \beta}$
\[
|u| \leq C(|\lambda_1|^{1/2} + |\lambda_2|^{1/2} + |\lambda_3|^{1/3}) \leq C[\alpha/\alpha^{1/6} + 1] |\lambda_3|^{1/3} \leq 3C |\lambda_3|^{1/3}.
\]
Hence, we may apply the scaling $u = \eta \mu$, $\lambda_1 = m_1 \mu^2$, $\lambda_2 = m_2 \mu^2$, $\lambda_3 = \mu^2$, where $\eta$ lies in a bounded region, $(m_1^3 + m_2^3) \leq \alpha^{-1}$, and $\mu$ is near 0. With this scaling, the bifurcation equation (2.1) assumes the form
\[
h(\eta, m_1, m_2, \mu) = M(\eta) + m_1 L_1 \eta + m_2 L_2 \eta + k + O(|\mu|) = 0. \tag{2.6}
\]
When \( m_1 = m_2 = \mu = 0 \), all zeros of \( h \) are simple, by (J2), since
\[
(0 = h(\eta, 0, 0, 0) = M(\eta) + k) = [(\partial h/\partial \eta)(\eta, 0, 0, 0) = \partial M/\partial \eta \neq 0].
\]
Hence, all intersections of \( (h_1(\eta, m_1, m_2, \mu) = 0) \) with \( (h_2(\eta, m_1, m_2, \mu) = 0) \) are
transversal for \( (m_1, m_2, \mu) \) in some neighborhood \( V \) of \( (0, 0, 0) \). Within \( V \),
then, the number of solutions \( \eta \) of (2.6) remains constant. Picking \( \alpha \) sufficiently
large and \( \beta \) sufficiently small so as to ensure \( (m_1, m_2, \mu) \) remains in \( V \), the
region \( \Re_{a,\beta} \) is determined.

Q.E.D.

We now consider the region which is the local complement of \( \Re_{a,\beta} \) : namely,
\[
\Re_{a,\beta}^c = \{(\lambda_1, \lambda_2, \lambda_3): (|\lambda_1|^2 + |\lambda_2|^2) \leq \beta, |\lambda_3|^2 \leq \alpha(|\lambda_1|^2 + |\lambda_2|^2)\}.
\]

By the preceding theorem, the complete bifurcation diagram must lie within
\( \Re_{a,\beta}^c \), which may be decomposed into four regions of identical size and shape:
\[
\begin{align*}
\Re_{++} &= \Re_{a,\beta}^c \cap \{(\lambda_1, \lambda_2, \lambda_3): \lambda_1 > 0, |\lambda_2| \leq |\lambda_1|\}, \\
\Re_{+-} &= \Re_{a,\beta}^c \cap \{(\lambda_1, \lambda_2, \lambda_3): \lambda_2 > 0, |\lambda_1| \leq |\lambda_3|\}, \\
\Re_{-+} &= \Re_{a,\beta}^c \cap \{(\lambda_1, \lambda_2, \lambda_3): \lambda_1 < 0, |\lambda_2| \leq |\lambda_1|\}, \\
\Re_{--} &= \Re_{a,\beta}^c \cap \{(\lambda_1, \lambda_2, \lambda_3): \lambda_2 < 0, |\lambda_1| \leq |\lambda_2|\}.
\end{align*}
\]

For convenience, we restrict our analysis to the regions \( \Re_{++} \), with the under-
standing that identical methods and conclusions apply with respect to \( \Re_{+-}, \Re_{-+}, \Re_{--} \).

In scaling for the region \( \Re_{++} \), we again utilize Lemma 2.1, which now
provides the inequality \(|u| \leq C(|\lambda_1|^{1/2})\) for some constant \( C \). This motivates
our choice of scaling
\[
u = \eta \mu, \quad \lambda_1 = \mu^2, \quad \lambda_2 = m \mu^2, \quad \lambda_3 = \eta \mu^2,
\]
for \( \eta \) in a bounded region, \(|\nu|^2 \leq 2\alpha, -1 \leq m \leq 1, \) and \( \mu \) near 0. With this
scaling, the bifurcation equation (2.1) becomes
\[
g(\eta, \mu, m, \nu) = M(\eta) + L_1\eta + mL^2\eta + \nu k + O(|\mu|) = 0. \quad (2.7)
\]

If \( \eta^* \) is a simple zero of (2.7) for \((\mu, m, \nu) = (0, \overline{m}, \nu^*)\) then, by the implicit
function theorem, there exists a unique zero \( \eta \) near \( \eta^* \), for \((\mu, m, \nu) \) near
\((0, \overline{m}, \nu^*)\). When \( \eta^* \) is a nonsimple zero, the tangency equations
\[
\begin{align*}
g(\eta^*, 0, \overline{m}, \nu^*) &= 0, \\
\det \frac{\partial g}{\partial \eta}(\eta^*, 0, \overline{m}, \nu^*) &= \Delta(\eta^*, \overline{m}) = 0
\end{align*}
\]
are satisfied.
Remark. In solving (2.8) for \((\eta^*, \nu^*)\) (with \(\bar{m}\) fixed), it is convenient to first solve the system

\[
\begin{align*}
\tilde{g}(\eta^*, \bar{m}) &= 0, \\
\Delta_1(\eta^*, \bar{m}) &= 0
\end{align*}
\]  

(2.8')

for \(\eta^*\), and then substitute into the equation \((g = 0)\) to obtain \(\nu^*\).

Regarding \(\bar{m}\) as fixed, the analysis of (2.8) corresponds to that undertaken by Chow et al. The condition

\[
\Delta^*(\eta^*, \bar{m}) = \det \left. \frac{\partial(g, A_1)}{\partial(n_1, n_2)} \right|_{(n_1, n_2)} \neq 0
\]

corresponds exactly to their generic condition

\[
\Delta_2(\eta^*) = \det \left. \frac{\partial(g, A_1)}{\partial(\eta, \lambda_3)} \right|_{\eta^*} \neq 0.
\]

In Chow et al. the equations

\[
\begin{align*}
g(\eta^*, 0, \nu^*) &= M(\eta^*) + L\eta^* + \nu^*k = 0, \\
\Delta_1(\eta^*) &= \det \left. \frac{\partial g}{\partial \eta} (\eta^*, 0, \nu^*) \right|_{\eta^*} = 0, \\
\Delta_2(\eta^*) &= \det \left. \frac{\partial g}{\partial (\eta, \nu)} (\eta^*, 0, \nu^*) \right|_{\eta^*} \neq 0
\end{align*}
\]

indicate that at \((\eta^*, \nu^*)\) there exists a tangential second-order contact of the curves \((g_1 = 0)\) and \((g_2 = 0)\), which is transversal with respect to \(\nu\). They obtain a finite collection of pairs \((\eta^*, \nu^*)\) satisfying (2.9). Each solution \((\eta^*, \nu^*)\) represents a bifurcation arc \(A_{\eta^*}\) of the form

\[
A_{\eta^*}: \lambda_2 = \gamma(\lambda_1^{1/2}) \lambda_1^{3/2} \sim \nu^* \lambda_1^{3/2}, \quad \lambda_1 \to 0^\pm
\]

\((\lambda_1, \lambda_2)\) are parameters of Eq. (2.1'), with bifurcation occurring along a curve \(\eta = \Psi(\mu) \ (\eta^* = \Psi(0))\) or, in unscaled coordinates,

\[
u = \Psi(\lambda_1^{1/3}) \lambda_1^{1/3} \sim \eta^* \lambda_1^{1/3}, \quad \lambda_1 \to 0^+.
\]

Crossing an arc \(A_{\eta^*}\) results in the gain (loss) of two solutions, branching from (coalescing into) \(u = \Psi(\lambda_1^{1/3}) \lambda_1^{1/3}\).

In our three-parameter setting, the functions \(g, \Delta_1, \Delta_2(\Delta^*)\) depend on \(\bar{m}\). For those values \(\bar{m}\) such that condition (2.8') implies \(\Delta^*(\eta^*, \bar{m}) \neq 0\), the methods of Chow et al. are directly applicable. In solving (2.8) (for a fixed value \(\bar{m}\)), we obtain finitely many solutions \((\eta^*(\bar{m}), \nu^*(\bar{m}))\). The solutions \(\nu^*(\bar{m})\) represent bifurcation arcs

\[
A_{\nu^*}(\bar{m}): \nu = \gamma(\mu) \quad (\nu^* = \gamma(0)),
\]

(2.9)
which, in the half-plane $P_m = \{(\lambda_1, \lambda_2, \lambda_3): \lambda_1 > 0, \lambda_2/\lambda_1 = m\}$, assume the form
\[ A_{\nu}(m): \lambda_3 \sim \nu^*(m) \lambda_1^{3/2}, \quad \lambda_1 \to 0^+. \tag{2.10} \]

Bifurcation occurs along a curve
\[ \eta = \Psi_m(\mu) \quad (\eta^* = \Psi_m(0)) \tag{2.10'} \]

which, in unscaled coordinates, is of the form
\[ u \sim \eta^*(m) \lambda_1^{1/2} \quad \lambda_1 \to 0^+. \]

In general, as $\mu$ varies, the bifurcation arcs (2.10) sweep out a finite collection of sheets whose intersection with any half-plane $P_m$ consists of the arcs $A_{\nu^*}(m)$.

In fact, by the implicit function theorem, the condition
\[ \Delta^*(\eta^*, m) = \det \left[ \frac{\partial (\tilde{F}, A_1)}{\partial (n_1, n_2)} \right]_{(\nu^*, m)} \neq 0 \tag{2.11} \]
guarantees that for $m$ near $m$, there exist unique solutions $(\eta(m), \nu(m))$ of (2.8), with $(\eta, \nu)$ near $(\eta^*, \nu^*)$. These solutions $(\eta, \nu)$ vary smoothly with $m$. Hence, as $m$ passes through values of $m$ such that (2.11) is satisfied for all solutions of (2.8), a family of smooth bifurcation sheets is generated. When crossing such a sheet in the bifurcation diagram, a gain (or loss) of two solutions occurs.

For some values of $m$, there exist solutions $(\eta^*, \nu^*)$ of (2.8), with $\Delta^*(\eta^*, m) = 0$. However, before investigating these exceptional points we summarize the preceding remarks in a theorem.

**Theorem 2.3.** For all but a finite number of exceptional values $m = \tilde{m}$, condition (2.8) implies $\Delta^*(\eta^*, \tilde{m}) \neq 0$, and the results of Chow et al. are immediately applicable. Hence, excepting finitely many $m (-1 \leq m \leq 1)$, the half-plane $P_m = \{(\lambda_1, \lambda_2, \lambda_3): \lambda_1 > 0, \lambda_2/\lambda_1 = m\}$ satisfies the following:

There exists a finite collection of curves
\[ A_{\nu^*}(m): \lambda_3 \sim \nu^*(m) \lambda_1^{3/2}, \quad \lambda_1 \to 0^+ \tag{2.12} \]
on $P_m$, with bifurcation occurring at
\[ u \sim \eta^*(m) \lambda_1^{1/2}, \quad \lambda_1 \to 0^+, \]
where $(\nu^*, \eta^*)$ satisfy
\[ g(\eta^*, 0, m, \nu^*) = 0, \]
\[ A_1(\eta^*, 0, m, \nu^*) = 0. \]
In crossing one of these curves $A_2(m)$, moving upward ($\lambda_0$ increases) in the plane $P_m$, the number of solutions to (2.1) increases or decreases by two, depending on whether $A_2(\eta^*, m)$ is positive or negative, respectively. The two new solutions branch off from (2.13). One branch is an unstable saddle and the other is a node, stable or unstable, depending on whether the quantity $\lambda^* = (\text{trace } A_1(\eta^*, m))$ is negative or positive. In each region between the curves (2.12) the number of solutions of (2.1) remain constant.

As $m$ varies, the curves (2.12) generate a finite collection of bifurcation sheets encircling the origin. The intersection of this collection with any halfplane $P_m$ consists of the arcs (2.12). In passing through one of these sheets, the number of solutions to (2.1) increases (decreases) by two if $A_2(\eta^*, m)$ is positive (negative) for those values $\eta^*(m)$ associated with the generative arcs (2.12).

Proof. First, we show there are a finite number of exceptional points. It is easily seen that

$$A_2(\eta^*, m) = 0 \iff A^*(\eta^*, m) = 0$$

(in fact, $A_2$ and $A^*$ are of opposite sign). Hence, we examine those values $(\eta^*, m)$ for which

$$\tilde{g}(\eta^*, m) = 0, \quad A_1(\eta^*, m) = 0, \quad A_2(\eta^*, m) = 0. \quad (2.14)$$

The solutions $(\eta^*, m) = (\eta^*_1, \eta^*_2, m)$ of the system (2.14) are points of intersection, in $R^3$, of three sheets defined by the above equations. Solving for $m$ as a function of $(\eta^*_1, \eta^*_2)$ in the cubic equation $\tilde{g} = 0$, and noting that $A_1, A^*_2$ are quartic and quintic polynomials, respectively, it follows by Bezout's theorem that there exist finitely many intersection points. This proves the first statement. The inset portion of the theorem is a rephrasing of the basic results of Chow et al., which are proved in their paper. Finally, the manner in which smooth bifurcation sheets are generated by the arcs (2.12) (as a consequence of the implicit function theorem) has already been described.

Q.E.D.

Remark. Solutions $\eta$ of (2.7) are said to be stable if they are stable with respect to the linearized problem, i.e., in terms of the eigenvalues of $A_1(\eta, m) = \det(\partial g/\partial \eta)|_{(\eta, m)}$. If both eigenvalues of $A_1$ are negative (positive), the solution $\eta$ is called a stable (unstable) node. If these eigenvalues are of opposite sign, $\eta$ is said to be a saddle point.

Using an argument from degree theory, Chow et al. showed that at a point of bifurcation $\eta$ (along a curve (2.10'), the solution branches into (i) a saddle point, and (ii) a node, either stable or unstable. As long as $\partial g/\partial \eta$ remains non
singular along a solution branch, the stability cannot change. Moreover, if the
nonsingular Jacobian \( \frac{\partial g}{\partial \eta_{(n,u,m,v)}} \) is symmetric for \( \mu = 0 \), it follows that in
some neighborhood of \( \mu = 0 \), no eigenvalues cross the imaginary axis. This
prevents a Hopf bifurcation from occurring and permits a complete charac-
terization of the stability properties of all solutions "between" two bifurcation
sheets. In the specific example of the von Kármán bifurcation equations, this
Jacobian condition (symmetry) is satisfied.

We now examine those exceptional values \( m = \bar{m} \) for which equation (2.8')
does not imply \( A^*(\eta^*, \bar{m}) \neq 0 \). Even at these values, some of the results in
Theorem 2.3 remain valid. That is, there exist a finite number of "possible"
bifurcation arcs of the form (2.12), with "possible" bifurcation occurring
along the curves (2.13).

However, at values \( \bar{m}, \eta^*, \nu^* \) satisfying both (2.8') and the condition
\( (A^*(\eta^*, \bar{m}) = 0) \), the arc (2.12) exists only in a formal sense and does not
represent bifurcation. In crossing (2.12) (within the plane \( P_{\bar{m}} \)), the number of
solutions of (2.1) remains constant. In this situation, the point \( (\bar{m}, \eta^*(\bar{m}), \nu^*(\bar{m})) \)
represents a coalescence of two separate bifurcation sheets. At \( \bar{m} \), two distinct
values of \( \nu^*(m) \), representing two separate bifurcation arcs (2.12), coalesce into
the single value \( \nu^*(\bar{m}) \) and vanish. In other words as \( m \) passes through \( \bar{m} \), two
distinct arcs \( A_{1}\nu^*(m), A_{2}\nu^*(m) \) \( (\nu^*_1(m) \neq \nu^*_2(m)) \) collapse into arcs \( A_{1}\nu^*(\bar{m}), A_{2}\nu^*(\bar{m}) \) of
identical asymptotic behavior \( (\nu^*_1(\bar{m}) = \nu^*_2(\bar{m})) \), and then disappear (there exist
no continuations of \( \nu^*_1(m), \nu^*_2(m) \)). Hence, along some curve "close" to the arc
\( A_{s}(\eta^*) \), two distinct bifurcation sheets coalesce and vanish.

For this phenomenon to occur, it is obvious that the number of solutions
above the "top" sheet must equal the number of solutions below the "bottom"
sheet. In this sense, the curve of coalescence of the two sheets might be pictured
as a "fold" in a larger sheet. Crossing through this larger sheet changes the
number of solutions by two. Pictorially, this situation might be represented as
in Fig. 4. The "fold" in Fig. 4 need not be smooth. It could form a cusp as in

\[ \begin{array}{c}
\nu \\
\nu^*_1(m) \\
\nu^*_2(m) \\
\nu^*(\bar{m}) \\
\text{n solutions} \\
\{n+2\} \text{ or } \{n-2\} \text{ solutions} \\
(m=0) \\
\bar{m} \\
\end{array} \]

\text{FIGURE 4}

Fig. 5. Also, there will generally exist other solutions \( \nu^*(\bar{m}) \) corresponding to \( \bar{m} \),
so that a complete cross-sectional diagram might appear as in Fig. 6.

Making use of hypothesis (J3), we summarize these results in the following
theorem.
Theorem 2.4. At those values \( \bar{m} \) for which there exist \((\nu^*(\bar{m}), \eta^*(\bar{m}))\) satisfying both (2.8') and the condition \( \Delta^0(\eta^*, \bar{m}) = 0 \), two distinct values of \( \nu^*(m) \) coalesce and vanish. If \( \Delta^0(\eta^*, \bar{m}) > 0 \), these values vanish for \( m < \bar{m} \); if \( \Delta^0(\eta^*, \bar{m}) < 0 \), they vanish for \( m > \bar{m} \). In either case, the distinct bifurcation generating arcs \( A_{\nu_1^*}(m), A_{\nu_2^*}(m) \) (\( \nu_1^*(m) \neq \nu_2^*(m) \)) coalesce into arcs \( A_{\nu_1^*}(\bar{m}), A_{\nu_2^*}(\bar{m}) \) of identical asymptotic behavior (\( \nu_i^*(\bar{m}) = \nu_i^*(\bar{m}) \)) and then vanish. Hence, two distinct bifurcation sheets, generated by the arcs \( A_{\nu_1^*}(m) \) and \( A_{\nu_2^*}(m) \), coalesce along a curve \( \gamma = (\dot{m}(\mu), \mu, \dot{\mu}(\mu)) \), where \( \dot{m}(0) = \bar{m} \) and \( \dot{\mu}(0) = \nu^*(\bar{m}) \).

Proof. We recall that for fixed \( m \), the solutions \( \nu^*(m) \) are obtained by first solving the system (2.8') for \((\eta_1^*(m), \eta_2^*(m))\), and then substituting these values into the equation \( g(\eta^*, 0, m, \nu) = 0 \). By (13), we have

\[
\Delta^0(\eta^*, \bar{m}) \equiv \det \left( \frac{\partial (\tilde{g}, A_1, A_2)}{\partial (\eta_1, \eta_2, m)} \right) \bigg|_{(\eta^*, \bar{m})} = 0.
\]

Hence, we may assume that one of the elements of \( \Delta^0(\eta^*, \bar{m}) \) is nonzero; say, \( (\partial \tilde{g}/\partial \eta_1)(\eta^*, \bar{m}) \neq 0 \). As a result of the implicit function theorem, we may then solve \( \tilde{g}(\eta_1, \eta_2, m) = 0 \) locally for \( \eta_1 \) as a function of \((\eta_2, \bar{m})\). We write \( \eta_1 = x(\eta_2, m) \), where

\[
x(\eta_2, \bar{m}) = \eta_1^*, \quad \tilde{g}(x(\eta_2, m), \eta_2, m) = 0.
\]
We are left with the one-dimensional problem

\[ G(\eta_2, m) = \Delta_1(\chi(\eta_2, m), \eta_2, m) = 0 \]

for \( \eta_2 \) near \( \eta_2^* \). Clearly, \( G(\eta_2^*, m) = 0 \) and

\[
\frac{\partial G}{\partial \eta_2} (\eta_2^*, m) = \left[ \frac{\partial \Delta_1}{\partial \eta_2} \frac{\partial \chi}{\partial \eta_2} + \frac{\partial \Delta_1}{\partial \eta_2} \right] \bigg|_{(\eta_2^*, m)} = 0
\]

since \( \Delta^*(\eta_2^*, m) = 0 \). However,

\[
\frac{\partial^2 G}{\partial \eta_2^2} (\eta_2^*, m) = \left[ \frac{1}{\left| \frac{\partial \chi}{\partial \eta_1} \right|^2} \right] \left[ \frac{\partial}{\partial \eta_2} \Delta^*(\chi(\eta_2, m), \eta_2, m) \right] \bigg|_{\eta_2 = \eta_2^*} = 0
\]

by (J3) and Lemma 2.5 which also imply

\[
\frac{\partial G}{\partial m} (\eta_2^*, m) = \left[ \frac{\partial \Delta_1}{\partial \eta_1} \frac{\partial x}{\partial m} + \frac{\partial \Delta_1}{\partial m} \right] \bigg|_{(\eta_2^*, m)} = 0.
\]

Since \( \partial G(\eta_2^*, m)/\partial \eta_2^* \neq 0 \), \( \partial G(\eta_2, m)/\partial \eta_2 \) has a unique zero \( \eta_2 \) near \( \eta_2^* \), for all values \( m \) near \( \bar{m} \). This means that for all \( m \) near \( \bar{m} \), \( G(\eta_2, m) \) has a unique critical point \( \eta_2 \) near \( \eta_2^* \). This critical point is a minimum (maximum) if \( \det[\partial (\tilde{g}, \Delta^*)/\partial (\eta_1, \eta_2)] \) is positive (negative). To obtain this critical point, we again use the implicit function theorem, solving locally for \( \eta_2 \) as \( \eta_2 = y(m) \), where

\[
y(m) = \eta_2^*, \quad \partial G/\partial \eta_2 (y(m), m) = 0.
\]

Denoting the value of \( G \) at its critical point by the function \( H \), i.e., \( H(m) = G(y(m), m) \), we have \( H(\bar{m}) = 0 \) and \( dH/\partial m(\bar{m}) = \partial G/\partial m(\eta_2^*, \bar{m}) \neq 0 \).

If the critical point of \( G \) is a minimum (maximum) and if \( \partial H/\partial m(\bar{m}) < 0 \) \( (\partial H/\partial m(\bar{m}) > 0) \), then two new solutions \( \eta^1(m), \eta^2(m) \) of the bifurcation equation (2.8') branch from \( \eta^* \) as \( m \) increases through \( \bar{m} \). If the critical point is a maximum (minimum) and if \( \partial H/\partial m(\bar{m}) < 0 \) \( (\partial H/\partial m(\bar{m}) > 0) \) then two solutions \( \eta^1(m), \eta^2(m) \) coalesce into \( \eta^* \) and vanish as \( m \) increases through \( \bar{m} \). Substituting into Eq. (2.8), we obtain distinct values \( \nu^1(m), \nu^2(m) \) coalescing into \( \nu^*(\bar{m}) \) and then vanishing as \( m \) passes through \( \bar{m} \) (in the appropriate direction). These values represent distinct bifurcation arcs \( A_{\nu^1(m)} \) and \( A_{\nu^2(m)} \), generating distinct
bifurcation sheets which, by the implicit function theorem, coalesce along a curve

\[ \gamma = (\dot{m}(\mu), \mu, \dot{\nu}(\mu)), \quad \dot{m}(0) = \bar{m}, \quad \dot{\nu}(0) = \nu^*(\bar{m}) \]

near the arc \( A_{r-}(\bar{m}) \) (parametrized with respect to \( (m, \mu, \nu) \)). (Actually the implicit function theorem cannot be directly applied to the condition

\[
\begin{bmatrix}
\partial g(\eta^*, \bar{m}) = 0 \\
\partial A_1(\eta^*, \bar{m}) = 0 \\
\partial A^*(\eta^*, \bar{m}) = 0
\end{bmatrix}
\Rightarrow \Delta^0(\eta^*, \bar{m}) \equiv \det \frac{\partial (g, A_1, A^*)}{\partial (\eta_1, \eta_2, m)} \bigg|_{(\eta^*, \bar{m})} \neq 0
\]

since the brackets contain a system of three equations in three variables. However, when applied to the equivalent condition

\[
\begin{bmatrix}
\partial g(\eta^*, 0, \bar{m}, \nu^*) = 0 \\
\partial A_1(\eta^*, 0, \bar{m}, \nu^*) = 0 \\
\partial A^*(\eta^*, 0, \bar{m}, \nu^*) = 0
\end{bmatrix}
\Rightarrow \Delta^0(\eta^*, 0, \bar{m}, \nu^*) \equiv \det \frac{\partial (g, A_1, A^*)}{\partial (\eta, m, \nu)} \bigg|_{(\eta^*, 0, \bar{m}, \nu^*)} \neq 0,
\]

where

\[ \Delta^0(\eta^*, \bar{m}) = \det \frac{\partial (g, A_1)}{\partial (\eta, \nu)} \bigg|_{(\eta^*, 0, \bar{m}, \nu^*)} \]

the implicit function theorem yields the existence of \( \gamma \).)

Finally, we consider the direction of bifurcation in relation to the sign of \( \Delta^0(\eta^*, \bar{m}) \). Writing

\[ \Delta^0(\eta^*, \bar{m}) = \left| \begin{array}{ccc}
\frac{\partial g}{\partial \eta_1} & \frac{\partial g}{\partial \eta_2} & \frac{\partial g}{\partial m} \\
\frac{\partial A_1}{\partial \eta_1} & \frac{\partial A_1}{\partial \eta_2} & \frac{\partial A_1}{\partial m} \\
\frac{\partial A^*}{\partial \eta_1} & \frac{\partial A^*}{\partial \eta_2} & \frac{\partial A^*}{\partial m}
\end{array} \right|_{(\eta^*, \bar{m})}
\]

and observing that \( \Delta^0(\eta^*, \bar{m}) = 0 \), we have

\[
\Delta^0(\eta^*, \bar{m}) = \left| \begin{array}{ccc}
\frac{\partial g}{\partial m} & \frac{\partial A_1}{\partial \eta_1} & \frac{\partial A^*}{\partial \eta_1} \\
\frac{\partial g}{\partial \eta_1} & \frac{\partial A_1}{\partial \eta_2} & \frac{\partial A^*}{\partial \eta_2} \\
\frac{\partial g}{\partial \eta_2} & \frac{\partial A^*}{\partial m} & \frac{\partial g}{\partial \eta_2}
\end{array} \right|_{(\eta^*, \bar{m})}
\]

\[
= -\left| \begin{array}{ccc}
\frac{\partial g}{\partial \eta_1} & \frac{\partial g}{\partial \eta_2} & \frac{\partial g}{\partial m} \\
\frac{\partial A_1}{\partial \eta_1} & \frac{\partial A_1}{\partial \eta_2} & \frac{\partial A_1}{\partial m} \\
\frac{\partial A^*}{\partial \eta_1} & \frac{\partial A^*}{\partial \eta_2} & \frac{\partial A^*}{\partial m}
\end{array} \right|_{(\eta^*, \bar{m})} \cdot \det \frac{\partial (g, A_1, A^*)}{\partial (\eta_1, \eta_2, m)} \bigg|_{(\eta^*, \bar{m})}.
\]
Hence, when $\partial H/\partial m$ and $\partial^2 G/\partial \eta_2^2$ are of the same sign, $A^0$ is negative; by an earlier remark, this corresponds to the situation where there exist two arcs $A_{\nu(m), \nu(m)}$ bifurcating from $A_{*,v(m)}$ for $(m - \bar{m})$ positive. Similarly, for $\partial H/\partial m$ and $\partial^2 G/\partial \eta_2^2$ of opposite sign $(A^0$ positive), this bifurcation occurs for $(m - \bar{m})$ negative. This concludes the proof of Theorem 2.4.

**Lemma 2.5.** Let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

be a matrix with nonzero determinant. If

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = 0$$

then

$$\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \neq 0 \quad \text{and} \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \neq 0.$$  

**Proof.** Obvious

**Remark.** In the proof of Theorem 2.4, we assumed that distinct solution values $\eta_1^1(m), \eta_2^1(m)$ corresponded to distinct parameter values $\nu^1(m), \nu^2(m)$. As one of their hypotheses (H4), Chow et al. assumed that to distinct $\eta^*\nu$'s there corresponded distinct $\nu^*\eta$'s. Since this assumption was generic in the case of two parameters, it remains so when considering the bifurcation equation for any fixed value $m = \bar{m}$. Hence, in a generic sense, our assumption of unique correspondence in the preceding proof is valid.

Those values $m = \bar{m}$ for which two distinct solution values $\eta_1, \eta_2$ correspond to a single parameter value $\nu^*$ may be obtained by solving the system of equations

$$g(\eta_1, 0, \bar{m}, \nu^*) = 0,$$

$$g(\eta_2, 0, \bar{m}, \nu^*) = 0,$$

$$\Delta_1(\eta_1, \bar{m}) = 0,$$

$$\Delta_1(\eta_2, \bar{m}) = 0.$$  

(2.15)

Values $\bar{m}$ for which there exist $(\eta_1^1, \eta_2^2, \nu^*)$ satisfying (2.15) represent intersections of distinct bifurcation sheets. The hypothesis (J4) merely ensures that such intersections are transversal. Also, the determinant condition in (J4) guarantees, by the implicit function theorem, the local existence of the actual curve

$$\Psi: (\hat{m}(\mu), \mu, \hat{\nu}(\mu))$$  

(2.16)
GENERIC BIFURCATION

of intersection, where \( \hat{m}(0) = \bar{m}, \hat{v}(0) = v^*(\bar{m}) \), and \( \hat{v}(\mu) = \gamma^*_{m(\mu)}(\mu) \), where \( \gamma \) denotes the bifurcation arc (2.10). In crossing such an intersection curve \( \Psi \), it is possible for the number of solutions of (2.1) to increase (decrease) by four. For instance, a cross-sectional view of a transversal intersection of two bifurcation sheets might appear as in Fig. 7.

These comments are summarized in:

**Theorem 2.6.** Each value of \( m = \bar{m} \) for which there exist \( (\eta^1, \eta^2, v^*) \) satisfying (2.15) corresponds to the transversal intersection of two distinct bifurcation sheets, with intersection occurring along the curve \( \Psi \) in (2.16). Any neighborhood of \( \Psi \) contains three distinct regions, in which there exist \( n - 2 \), \( n \), and \( n + 2 \) solutions of (2.1).

The theorems of this chapter provide a characterization of the bifurcation diagram of the three-parameter equation (2.1). Generically, this diagram consists of a finite collection of sheets emanating from, and encircling, the origin. Pairs of these sheets might coalesce and vanish, or might intersect each other transversally, in the manners we have described.

The word "generic" refers to the fact that hypotheses (J1)-(J4) may be expected to hold for all \( f \) within some subset (of functions of the form (2.1)), which is open and dense with respect to any reasonable function space topology. For example, with regard to the specific example of the von Kármán equations (which shall be considered in the next section), the hypotheses (J1)-(J4) are satisfied for \( (k_1, k_4) \) in an open dense subset of \( R^2 \). In the two-parameter setting of Chow et al., the hypotheses (H1)-(H4) were "generic" in an identical sense.

### 3. The von Kármán Equation

The buckled states of a simply supported flat rectangular plate

\[ D = \{(x, y) : 0 < x < s, 0 < y < 1\} \]
subject to normal loading and to constant compressive thrust applied at the ends \( x = 0, s \) are assumed to be described by the von Kármán equations

\[
\begin{align*}
\Delta^2 f &= -\frac{1}{2}[w, w] \quad \text{in } D \\
\Delta^2 w &= [w, f] - \lambda \nu w + \nu p \quad \text{in } D \\
f &= \Delta f = w = \Delta w = 0 \quad \text{on } \delta D,
\end{align*}
\]  

(3.1)

where \( \Delta \) is the harmonic operator,

\[
[u, v] = u_{xx}v_{yy} + u_{yy}v_{xx} - 2u_{xy}v_{xy},
\]

\( \lambda \) is the magnitude of thrust at \( x = 0, s \),

\( \nu(x, y) \) is the normal loading,

\( \nu \) is the magnitude of normal loading,

\( f \) is related to the (excess) stress on the plate (resulting from the plate's deformation),

\( w \) is related to the displacement of the plate.

Bauer and Reiss [2] first formulated this dimensionless model (3.1) of the plate's behavior. They and other, including Berger and Fife [4], Knightly and Sather [8], and Chow et al. [6] have studied the bifurcation of solutions to (3.1) near critical parameter values. The paper by Chow et al. contains additional references.

Knightly and Sather [8], using techniques similar to those of Berger and Fife [4], transformed (3.1) into an operator equation in an appropriate real Hilbert space \( \mathcal{H} \). Specifically, they select \( \mathcal{H} \) as the closure, in the Sobolev space \( W^{2,2}(D) \), of the set of smooth functions defined on \( \bar{D} \) and vanishing on \( \delta D \). Then, by Sobolev's Embedding Theorem, \( \mathcal{H} \) consists of all functions in \( W^{2,2}(D) \) which are continuous in \( \bar{D} \) and vanish on \( \delta D \). A coercive estimate (see [4]) implies the equivalence, in \( \mathcal{H} \), of the norms \( \| \cdot \|_{a,2} \) and \( \| \cdot \|_{\mathcal{H}} \), where \( \| \cdot \|_{a,2} \) is the original Sobolev norm and \( \| \cdot \|_{\mathcal{H}} \) is the norm

\[ \| u \|_{\mathcal{H}} = \left[ \int_D (\Delta u)^2 \right]^{1/2} \]

associated with the inner product

\[ \langle u, v \rangle_{\mathcal{H}} = \int_D (\Delta u) \cdot (\Delta v). \]

In this setting, Eqs. (3.1) assume the form

\[
\begin{align*}
(i) \quad f &= -\frac{1}{2}B(w, w), \\
(ii) \quad w &= B(w, f) + \lambda \nu w + \nu \tilde{p},
\end{align*}
\]  

(3.2)
where

\[ B(u, v) = \Delta^{-1}[u, v] \]

is a bilinear form mapping \( \mathcal{H} \times \mathcal{H} \) into \( \mathcal{H} \),

\[ Lu = -\Delta^{-2}u_{xx} \]

is a compact self-adjoint linear operator mapping \( \mathcal{H} \) into \( \mathcal{H} \),

\[ \hat{p} = \Delta^{-2}p, \]

\( \Delta^{-1} \) is the inverse of the Laplacian with zero Dirichlet data.

By substituting (3.2i) into (3.2ii), one obtains the single equation

\[
S(zu, A, v) = (I - XL) w + C(w) - v\hat{p} = 0, \quad (3.3j)
\]

\[
C(w) = \frac{1}{2}B(w, B(w, w)).
\]

Equation (3.3) is the exact form of the operator equation studied by Chow, Hale, and Mallet-Paret [6]. (Knightly and Sather [8] did not consider normal loading; hence, they obtained the abbreviated equation \((I - hL)w + C(w) = 0\).)

Since \( S_\infty(0, \lambda, v)(w) = (I - \lambda L)w, \lambda^{-1} \) must be near a characteristic value \( \lambda_0^{-1} \) of \( L \) if bifurcation is to occur for small \((w, v)\). When the null space associated with \( \lambda_0 \) is one-dimensional (and \( v = 0 \)), Sather [10] has demonstrated the existence of exactly one pair of nontrivial buckled states associated with \( \lambda_0 \).

Here, as in Knightly and Sather and Chow et al., we treat the more difficult situation where \( \lambda_0 \) is of multiplicity two. (For a thorough discussion (in a general setting) of the case where \( \lambda_0 \) is of multiplicity one, see Chow et al. [5].)

For a plate of length \( s = 2l^2/\pi^2 \), the first characteristic value \( \lambda_1^{-1} = 2/9\pi^2 \) is of multiplicity two. Knightly and Sather [8] and Matkowsky and Putnick [9] have completely determined the bifurcation in this case (with zero normal loading \( (v = 0) \)), obtaining the unique solution \( u^* = 0 \) for \( \lambda_1 = (\lambda/\lambda_0 - 1) < 0 \), and nine solutions (four stable nodes, four saddles, and the unstable node \( u^* = 0 \)) for \( \lambda_1 > 0 \). Chow et al. examined the same equation with loading \( (v \neq 0) \) and obtained the bifurcation diagram for solutions to (3.3) in terms of the small parameters \( \lambda_1 = (\lambda/\lambda_0 - 1), \nu \), for fixed \( s = 2^{1/2} \). This two-parameter diagram was of the form described in Section 2 (a finite collection of arcs \( A_{s,\nu} \)), and appears as a cross section of the particular three-parameter diagram we construct.

In the present study, \( s \) is allowed to vary near \( s = 2^{1/2} \). This variation in \( s \) introduces a third bifurcation parameter. For \( s \neq 2^{1/2} \), the double eigenvalue \( \lambda_0^{-1} \) splits into two simple eigenvalues. More precisely, if we take as an orthonormal basis for \( \mathcal{H} \) the set of eigenfunctions

\[
\phi_{m,n} = C_{mn} \sin m\pi x/s \sin n\pi y, \quad C_{mn} = 2s^{3/2}/[\pi^2(m^2 + n^2)],
\]

with associated eigenvalues

\[
\lambda_{mn}^{-1} = s^2m^2/[\pi^2(m^2 + s^2n^2)^2],
\]

\[
\lambda_{mn}^{-1} = s^2n^2/[\pi^2(m^2 + s^2n^2)^2].
\]
then for \( s \neq 2^{1/2} \), the double eigenvalue \( \lambda_0^{-1} \) splits into the simple eigenvalues \((\lambda_{11})^{-1}, (\lambda_{21})^{-1}\) given by

\[
\lambda_{11} = \pi^2(1/s + s)^2, \quad \lambda_{21} = \pi^2(2/s + s/2)^2
\]

with corresponding eigenfunctions

\[
\phi_{11} = C_{11} \sin(\pi x/s) \sin(\pi y), \quad \phi_{21} = C_{21} \sin(2\pi x/s) \sin(\pi y).
\]

Note that for \( s = 2^{1/2} \), \( \lambda_{11} = \lambda_{21} = \lambda_0 \) and \( \{\phi_{11}, \phi_{21}\} \) spans the two-dimensional kernel of \((I - \lambda_0 L)\).

Writing \( w = u_1\phi_{11} + u_2\phi_{21} \), \((u_1, u_2) \in \mathbb{R}^2\), we follow Knightly and Sather and Chow et al. in applying the Liapunov–Schmidt procedure (see, for instance, Vainberg and Trenogin [11]) to (3.3). This procedure reduces the problem of finding solutions of (3.3) to the equivalent problem of solving the two-dimensional bifurcation equation

\[
\begin{align*}
    f_1(u, \lambda) &\equiv -a u_1^2 - b u_1 u_2^2 + \lambda_3 k_1 = \text{h.o.t.}, \\
    f_2(u, \lambda) &\equiv -b u_1^2 u_2 - a u_2^3 + \lambda_2 u_2 + \lambda_3 k_2 = \text{h.o.t.},
\end{align*}
\]

where

\[
\begin{align*}
    \lambda_1 &= \left(\frac{\lambda}{\lambda_{11}} - 1\right), \quad \lambda_2 = \left(\frac{\lambda}{\lambda_{21}} - 1\right), \quad \lambda_3 \text{ near } 0, \\
    k_1 &= \langle \bar{P}, \phi_{11} \rangle_{\mathcal{W}}, \quad k_2 = \langle \bar{P}, \phi_{21} \rangle_{\mathcal{W}}
\end{align*}
\]

and, h.o.t. designates the higher-order terms, which are of the order

\[O(|u|^4 + (\lambda_1^2 + \lambda_2^2) |u| + (|\lambda_1| + |\lambda_2|) |u|^2 + \lambda_3^2 + |\lambda_3| |u|).
\]

The coefficients \(a, b, c\) are obtained by solving

\[
\begin{align*}
    a &= \frac{1}{2} \|B(\phi_{11}, \phi_{11})\|_{\mathcal{W}}^2, \\
    b &= \|B(\phi_{11}, \phi_{21})\|_{\mathcal{W}}^2 + \frac{1}{2} \langle B(\phi_{11}, \phi_{11}), B(\phi_{21}, \phi_{21}) \rangle_{\mathcal{W}}, \\
    c &= \frac{1}{2} \|B(\phi_{21}, \phi_{21})\|_{\mathcal{W}}^2.
\end{align*}
\]

at \( s = 2^{1/2} \). Chow et al. obtained the approximate values

\[
\begin{align*}
    a &\approx 3.945001 \times 10^{-4}, \\
    b &\approx 5.007428 \times 10^{-4}, \\
    c &\approx 1.623543 \times 10^{-4}.
\end{align*}
\]
Note the manner in which $s$ affects the parameters of Eq. (3.4). An expansion of $\lambda_1, \lambda_2$ in terms of $\tilde{s} = (s - 21/2)$, $\tilde{\lambda} = (\lambda - \lambda_0)$ yields

$$
\lambda_1 = -21/2 \tilde{s}/3 + 2\tilde{\lambda}/9\pi^2 + \text{h.o.t.,}
$$

$$
\lambda_2 = 21/2 \tilde{s}/3 + 2\tilde{\lambda}/9\pi^2 + \text{h.o.t.}
$$

For $s$ near $21/2$ and $\lambda$ near $\lambda_0$, we have

$$
\lambda_j + 1 = \lambda/\lambda_j \approx \lambda_0/\lambda_0 = 1, \quad \text{for } j = 1, 2.
$$

For fixed $s = 21/2$, $\lambda_1 = \lambda_2 = (\lambda/\lambda_0 - 1)$ and Eq. (3.4) assumes the two-parameter form studied by Chow et al.

Equation (3.4) satisfies the form (2.1) of Section 2, with

$$
M_1(u) = -au_1^3 - bu_1u_2^2, \quad M_2(u) = -bu_1^2u_2 - cu_2^3,
$$

$$
L_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad L_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.
$$

Hence, the results of Section 2 may be applied in constructing the bifurcation diagram for (3.4). This diagram is constructed for (fixed) loading parameters $k_1 = k_2 = 1$. Since Chow et al. constructed the two-parameter bifurcation diagram for the specific case $k_1 = k_2 = 1$, their diagram appears as a cross section of our three-parameter diagram.

The generic conditions (J1), (J2) are satisfied with $k_1 = k_2 = 1$. Hence, there exists a “sectorial” region

$$
\mathcal{R}_{\alpha, \beta} = \left\{ (\lambda_1, \lambda_2, \lambda_3) : |\lambda_3| \leq \beta, (|\lambda_1|^3 + |\lambda_2|^3) \leq \frac{1}{\alpha} |\lambda_3|^2 \right\}
$$

within which no bifurcation may occur. Investigating the complement of $\mathcal{R}_{\alpha, \beta}$, we search the regions $\mathcal{R}_{++}, \mathcal{R}_{+-}, \mathcal{R}_{-+}, \mathcal{R}_{--}$ (defined in Section 2) for bifurca-
tion points. The intersection of these four regions with the plane \( \{ \lambda_3 = 0 \} \) are shown in Fig. 8. The respective scaling for these regions are:

<table>
<thead>
<tr>
<th>( R_{++} )</th>
<th>( R_{+-} )</th>
<th>( R_{-+} )</th>
<th>( R_{-+} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( u = \eta \mu )</td>
<td>( u = \eta \mu )</td>
<td>( u = \eta \mu )</td>
<td>( u = \eta \mu )</td>
</tr>
<tr>
<td>( \lambda_1 = \mu^2 )</td>
<td>( \lambda_1 = m \mu^2 )</td>
<td>( \lambda_1 = -\mu^2 )</td>
<td>( \lambda_1 = m \mu^2 )</td>
</tr>
<tr>
<td>( \lambda_2 = m \mu^2 )</td>
<td>( \lambda_2 = \mu^2 )</td>
<td>( \lambda_2 = m \mu^2 )</td>
<td>( \lambda_2 = -\mu^2 )</td>
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<tr>
<td>( \lambda_3 = \eta \mu^3 )</td>
<td>( \lambda_3 = \eta \mu^3 )</td>
<td>( \lambda_3 = \eta \mu^3 )</td>
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</tr>
</tbody>
</table>

\((\eta, \nu \text{ bounded, } -1 \leq m \leq 1, \mu \text{ near } 0)\) which, when applied to (3.4), yield the scaled bifurcation equations

\[
\begin{align*}
R_{++} : & \quad g_1(\eta, \mu, m, \nu) = -a_{11} \eta^3 + b_{11} \eta \gamma_2^2 + \eta_1 + \nu k_1 + O(|\mu|) = 0 \\
& \quad g_2(\eta, \mu, m, \nu) = -b_{11} \eta \gamma_2^2 + c_{11} \gamma_2^3 + m \eta \gamma_2 + \nu k_2 + O(|\mu|) = 0, \\
R_{+-} : & \quad g_1(\eta, \mu, m, \nu) = -a_{11} \eta^3 - b_{11} \eta \gamma_2^2 + m \eta \gamma_2 + \nu k_2 + O(|\mu|) = 0 \\
& \quad g_2(\eta, \mu, m, \nu) = -b_{11} \eta \gamma_2^2 - c_{11} \gamma_2^3 + \eta_2 + \nu k_2 + O(|\mu|) = 0, \\
R_{-+} : & \quad g_1(\eta, \mu, m, \nu) = -a_{11} \eta^3 - b_{11} \eta \gamma_2^2 - \eta_1 + \nu k_1 + O(|\mu|) = 0 \\
& \quad g_2(\eta, \mu, m, \nu) = -b_{11} \eta \gamma_2^2 - c_{11} \gamma_2^3 + \eta_2 + \nu k_2 + O(|\mu|) = 0, \\
R_{-+} : & \quad g_1(\eta, \mu, m, \nu) = -a_{11} \eta^3 - b_{11} \eta \gamma_2^2 - \eta_1 + \nu k_1 + O(|\mu|) = 0 \\
& \quad g_2(\eta, \mu, m, \nu) = -b_{11} \eta \gamma_2^2 - c_{11} \gamma_2^3 - \eta_2 + \nu k_2 + O(|\mu|) = 0.
\end{align*}
\]

As \( m \) varies between \(-1 \) and 1 in each of these four regions, we seek simultaneous solutions \((\eta^*, \nu^*)\) of the system

\[
g(\eta^*, 0, m, \nu^*) = 0,
\]

\[
\Delta_1(\eta^*, m) \equiv \det(\partial g / \partial \eta) \mid_{(0, 0, m, \eta^*)} = 0.
\]

Using Newton’s method, a finite number of solutions \((\eta^*(m), \nu^*(m))\) are obtained for each fixed \( m \). The value \( \nu^*(m) \) represents a two-dimensional bifurcation arc \( A_{\nu^*(m)} \) in the half-plane \( P_m = \{ (\lambda_1, \lambda_2, \lambda_3) : \lambda_1 > 0, \lambda_2 / \lambda_1 = m \} \) (see 2.10). As \( m \) varies, this arc generates a sheet of bifurcation points in the manner described in Section 2. From the symmetry of the von Kármán bifurcation equations, \((\eta^*, \nu^*)\) satisfies (3.5) if and only if \((-\eta^*, \nu^*)\) is also a solution. Hence the three-parameter bifurcation diagram is symmetrical: to each bifurcation sheet lying above the plane \( \{ \lambda_3 = 0 \} \) \((\nu^*(m) > 0)\), there corresponds a “dual” sheet (corresponding to \(-\nu^*(m)\)) lying below this plane.

In particular, for \( \lambda_3 / \lambda_1 = 1 \), we obtain for \( \lambda_1 > 0 \) (i.e., \( m = 1 \) in either of the regions \( R_{++}, R_{+-} \)) the following solutions:
in which \( \lambda^* = \text{trace } A_\star (q^*, W_0) \). As stated in Theorem 2.3, for \( \Delta_\star (q^*, m) \) positive (negative), the number of solutions to (3.4) increases (decreases) by two when crossing through the curve \( A^* (m) \) (\( \lambda_3 \) increasing) in the plane \( P_m \). One of these solutions is unstable and the other is either stable or unstable, depending on whether \( \lambda^* \) is negative or positive.

In the two-parameter case studied by Chow et al., \( \lambda_1 = \lambda_1 \). Hence, Table (3.6), which represents a cross section of our three-parameter diagram, is an exact copy of the original two-parameter diagram (see [6, Table 3.2]). The cross section represented by (3.6) is depicted in Fig. 9. Figure 9 is the complete two-parameter bifurcation diagram obtained by Chow et al. (see [6, Fig. 2.2]). As \( m = \lambda_0 / \lambda_1 \) varies (\( m \) decreasing in either \( R_{+-} \) or \( R_{++} \)), these arcs generate the bifurcation sheets of the three-parameter diagram.

For \( \lambda_0 / \lambda_1 = 1 \), \( \lambda_1 < 0 \) (i.e., \( m = -1 \) in either of the regions \( R_{+-} \), \( R_{--} \)), the only solution to (3.5) is \( (q^*, v^*) = (0, 0) \). Hence, for \( \lambda_1 < 0 \) in the plane \( \{ \lambda_1 = \lambda_3 \} \), the zero solution is unique. This information is contained in Fig. 9.
Fig. 10. Region $R_{++} (m, n)$. $m =$ number of stable solutions; $n =$ number of unstable solutions.

As $m$ varies through each of the four regions $R_{++}, R_{+-}, R_{-+}, R_{--}$, the entire three-parameter bifurcation diagram is constructed. Figure 10 contains a cross-sectional representation $v^*(m)$ (positive values) plotted versus $m$ of the bifurcation diagram in $R_{++}$, the most interesting of the four regions. In the complete diagram, the curves of each region extend naturally into those of adjoining regions. Points representing the coalescence of two distinct bifurcation sheets $d^*(\eta^*, m) = 0$ are marked with a dot. At each of these points the generic condition $d^*(\eta^*, m) \neq 0$ is satisfied. Those points at which two separate sheets intersect transversally are apparent. The number of solutions of (3.4) in any region bounded by two or more sheets is given by $(m, n)$, where $m$ denotes the number of stable solutions, $n$ the number of unstable solutions.

Owing to the symmetry of the von Kármán equations, there exist two points $P_1, P_2$ (corresponding to $m = a/b$ in $R_{++}$ and $m = a/b$ in $R_{-+}$) at which two distinct pairs of sheets simultaneously coalesce and vanish. This behavior is an inherent feature of the von Kármán equations and occurs for all $(k_1, k_2)$. It does not constitute a violation of any of the generic hypotheses (J1)–(J4).

Qualitatively, the particular three-parameter diagram we have constructed assumes the anticipated generic structure. The loading parameter values $(k_1, k_2) = (1, 1)$ are not significant, since the construction of the bifurcation diagram for other $(k_1, k_2)$ would be similar (excepting those nongeneric $(k_1, k_2)$ for which one of the hypotheses (J2)–(J4) is violated.)
4. CROSS-SECTIONAL BIFURCATION DIAGRAM: ZERO NORMAL LOADING

As a final illustration, we consider a different cross section of the bifurcation diagram associated with \( k_1 = k_2 = 1 \). Namely, we consider the intersection of this diagram with the \( \lambda_1 - \lambda_2 \) plane. This corresponds to the case of zero normal loading (i.e., \( \lambda_3 = 0 \) in (3.4)), and assumes the form shown in Fig. 11.

In Fig. 11, there are six bifurcation curves asymptotic to rays at the origin. The two curves \( C_1, C_2 \) (asymptotic to the rays \( \lambda_2 = c\lambda_1/b, \lambda_2 = b\lambda_1/a \)) correspond, respectively, to the points \( P_1, P_2 \) mentioned in Section 3. Along these curves, two pairs of bifurcation sheets simultaneously coalesce, and the number of solutions of (3.4) changes by four. Owing to the symmetry in the physical situation represented by the von Kármán equations, the curves \( C_1, C_2 \) cannot "split" into distinct pairs of bifurcation curves when higher-order terms are considered in (3.4). Hence, there exist (locally) no regions in the \( \lambda_1 - \lambda_2 \) plane for which there are seven solutions to (3.4).

The diagram in Fig. 11 might be directly obtained by computing simultaneous solutions to the pair of equations

\[
\begin{align*}
-\alpha u_1^2 - bu_1 u_2^2 + \lambda_2 u_1 &= 0, \\
-bu_1 u_2^2 - cu_2^3 + \lambda_2 u_2 &= 0.
\end{align*}
\]

These equations correspond to the bifurcation equation (3.4) with zero normal loading and with higher-order terms eliminated. In solving (4.1), one obtains
$u = (0, 0)$ is always a solution,

(ii) if $\lambda_1 > 0$, $(u_1, u_2) = (0, \pm (\lambda_1/\alpha)^{1/2})$ are solutions,

(iii) if $\lambda_1 > 0$, $(u_1, u_2) = (\pm (\lambda_1/\alpha)^{1/2}, 0)$ are solutions,

(iv) if $c\lambda_1/b < \lambda_2 < b\lambda_1/a$,

$$ (u_1, u_2) = \left[ \pm \left( \frac{b\lambda_2 - c\lambda_1}{b^2 - ac} \right)^{1/2}, \pm \left( \frac{b\lambda_2 - a\lambda_1}{b^2 - ac} \right)^{1/2} \right] $$

are solutions.

By calculating the Jacobian of (4.1), the stability properties of the solutions (4.2) are obtained:

(i) the zero solution $u = (0, 0)$ is stable only for $\lambda_1 < 0, \lambda_2 < 0$,

(ii) the solutions $u = (0, \pm (\lambda_1/\alpha)^{1/2})$ are stable only for $\lambda_2 > c\lambda_1/b$,

(iii) the solutions $u = (\pm (\lambda_1/\alpha)^{1/2}, 0)$ are stable only for $\lambda_2 < b\lambda_1/a$,

(iv) the remaining four solutions (4.2iv) are always unstable.

The dotted line ($\lambda_2 = \lambda_1$) corresponds to the case of a single eigenvalue parameter in (3.3); as mentioned earlier, the bifurcation behavior in this case is well known. The emergence of eight new solutions of (3.4) for $\lambda_1 = \lambda_2 > 0$ is consistent with previous results (cf. Matkowsky and Putnick [9], Knightly and Sather [8], and Chow, Hale, and Mallet-Paret [6]).

Note that when crossing the curve $C_1$ (moving towards the right), the four solutions (4.2iv) coalesce into the two (already existing) solutions (4.2ii). Similarly, when crossing the curve $C_2$ (moving towards the left), the four solutions (4.2iv) coalesce into the two solutions (4.2iii). Hence, the curves $C_1, C_2$ may be regarded as points of secondary bifurcation, whereas the curves asymptotic to the coordinate axes represent primary bifurcation (i.e., bifurcation from the zero solution). In effect, the introduction of a second eigenvalue parameter “splits” a multiple primary bifurcation point $((\lambda_1, \lambda_2) = (0, 0))$ into curves of primary and secondary bifurcation points of lesser multiplicity.

The corresponding physical situation is better understood by reconstructing Fig. 11 in terms of the original bifurcation parameters $\tilde{s} = (s - 2^{1/2})$ and $\tilde{\lambda} = (\lambda - \lambda_0)$. Doing this, we obtain Fig. 12.

We first consider the case $\tilde{s} < 0$, which corresponds to a rectangular plate with ratio of length to width less than $2^{1/2}$. As $\tilde{\lambda}$ increases through the curve $\tilde{\lambda} = (\lambda_{11}(s) - \lambda_0)$ associated with the first eigenvalue $\lambda_{11}(s)$, the plate buckles. A pair of stable solutions asymptotic to (4.2iii) bifurcate from the zero solution, which becomes unstable. Recalling that the variable $u_1$ is associated with the eigenfunction

$$ \phi_{11} = c_{11} \sin(\pi x/\tilde{s}) \sin(\pi y) $$
FIG. 12. \((m, n)\). \(m = \) number of stable solutions; \(n = \) number of unstable solutions.

\[
\begin{align*}
C_1: \lambda^* \sim \frac{b+1}{(b-1)} & \frac{\sqrt{3}}{2} n^2 \\
C_2: \lambda^* \sim \frac{b-1}{b+1} & \frac{\sqrt{3}}{2} n^2
\end{align*}
\]

\[
\lambda = \lambda^* \sim \frac{3}{2} \frac{\lambda^*}{a^2}
\]

\[
\lambda^* = (\lambda_{zz}(\epsilon) - \lambda_0)^{-\frac{2}{3}}
\]

\[
\lambda = (\lambda_{zz}(\epsilon) - \lambda_0)^{-\frac{2}{3}} \frac{3}{2} a^2
\]

\[
\hat{\lambda} = (\lambda_{zz}(\epsilon) - \lambda_0)^{-\frac{2}{3}} \frac{3}{2} a^2
\]

\[
\frac{\hat{\lambda}}{10} = (\lambda_{zz}(\epsilon) - \lambda_0)^{-\frac{2}{3}} \frac{3}{2} a^2
\]

\[
(2,3)
\]

\[
(1,0)
\]
while $u_2$ is associated with

$$\phi_{21} = c_{21} \sin(2\pi x/s) \sin(\pi y),$$

we observe that the two bifurcating solutions asymptotic to (4.2iii) represent buckled states whose approximate shape is depicted in Fig. 15iii. As $\lambda$ further increases, passing through the curve $\lambda = (\lambda_{21}(s) - \lambda_0)$, a pair of unstable solutions asymptotic to (4.2ii) bifurcate from the unstable zero solution. These solutions represent buckled states whose approximate shape appears in Fig. 15ii. Finally, as $\lambda$ passes through the curve $C_1$, the solutions near (4.2ii) become stable, as four new unstable solutions bifurcate from (4.2ii). These solutions are asymptotic to (4.2iv), and represent “hybrid” buckled states whose approximate shape is given in Fig. 15iv. These unstable solutions represent positions of the plate which are “balanced” between pairs of stable solutions (i.e., one stable solution of type (ii) and another of type (iii)). Since there are two symmetric positions corresponding to each of the shapes in Figs. 15ii, 15iii, there exist four symmetric positions for the unstable “hybrids” in Fig. 15iv.

![Fig. 15. (i) Flat plate: $z = 0$. (ii) Buckled plate: $z = a_2 \sin(2\pi x/s) \sin(\pi y)$ (here, $a_2 = 0.1$). (iii) Buckled plate: $z = a_1 \sin(\pi x/s) \sin(\pi y)$ (here $a_1 = 0.1$). (iv) Buckled plate: $z = a_1 \sin(\pi x/s) \sin(\pi y) + a_2 \sin(2\pi x/s) \sin(\pi y)$ (here $a_1 = a_2 = 0.05$).]
From the above discussion, the bifurcations of the plate for \( s < 2^{1/2} \) might be represented in Fig. 13. In this diagram, \( s \) denotes stability and \( u \) instability, while the indices refer to those in (4.2) and in Fig. 15.

For \( s > 2^{1/2} \), another diagram similar to Fig. 13 is obtained, with solutions (ii) and (iii) interchanged. That is, as \( \lambda \) passes through \( (\lambda = (\lambda_{01}(s) - \lambda_0)) \) a pair of stable solutions asymptotic to (4.2ii) (and corresponding to the shape in Fig. 15ii) bifurcate from zero, while the unstable solutions (iv) now bifurcate from (iii) as \( \lambda \) passes through \( C_2 \).

At \( s = 2^{1/2} \), Fig. 13 collapses into Fig. 14 with a single multiple primary bifurcation point at \( \lambda = 0 \).

Remarks. As mentioned earlier (cf. Sather [10]), the eigenvalue curves \( \lambda_{11}(s), \lambda_{21}(s) \) are actual bifurcation curves for (3.4) (as opposed to being mere asymptotic approximations of these curves). Since these curves correspond, respectively, to the axes \( (\lambda_1 = 0) \) and \( (\lambda_2 = 0) \) in Fig. 11, it follows that the coordinate axes in Fig. 11 represent the actual bifurcation curves for (3.4).

The "splitting" of multiple bifurcation points into several simple primary and secondary bifurcation points (near a multiple eigenvalue of the linearized equation), was first discovered by Bauer et al. [3] in a numerical study of
secondary axisymmetric buckling of spherical shells subjected to uniform external pressure. Other examples of this behavior are contained in recent papers by Bauer et al. [1], and Keener [7]. Using perturbation methods, Bauer et al. [1] analyzed secondary bifurcation near a multiple primary bifurcation point, then applied their results to the buckling of spherical shells. Keener [7], obtained necessary and sufficient conditions for secondary (as well as primary) bifurcation, and investigated the relationship between bifurcation at multiple eigenvalues and the existence of secondary bifurcation.

References