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FREE TOPOLOGICAL GROUPS OVER METRIZABLE SPACES

A.V. ARHANGEL'SKIĬ

Department of Mathematics and Mechanics, Moscow State University, 119899 Moscow, USSR

O.G. OKUNEV

Department of Mathematics, Kalinin State University, 117002 Kalinin, USSR

V.G. PESTOV

Department of Mathematics and Mechanics, Tomsk State University, 634010 Tomsk, USSR

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Let X be a metrizable space and $F(X)$ and $A(X)$ be the free topological group over X and the free Abelian topological group over X respectively. We establish the following criteria:
(a) tightness of $A(X)$ is countable iff the set X' of all nonisolated points in X is separable,
(b) tightness of $F(X)$ is countable iff X is separable or discrete,
(c) $A(X)$ is a k-space iff X is locally compact and X' is separable,
(d) $F(X)$ is a k-space iff X is locally compact separable or discrete.
We also show that if X is second-countable, then $F(X)$ and $A(X)$ are k_R-spaces iff X is locally compact.

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Introduction

The structure of the free topological group over a topological space X is very simple from the algebraic point of view—it is exactly the free algebraic group over the set X. On the contrary, the topology of $F(X)$ is rather complicated even for very simple spaces X. Indeed, only when X is discrete can the space $F(X)$ have the Baire property or the Fréchet-Urysohn property, only for X countable and discrete the space $F(X)$ is second-countable. Even for the space $\mathbb{Q}$ of rationals $F(\mathbb{Q})$ is not a k-space. It remains unknown whether $F(X)$ is paracompact if X is the product of a compact space with a discrete space.

In this paper we consider the following questions: For which space X is the space $F(X)$ a k-space? When is the tightness of $F(X)$ countable? It is only natural to state similar questions for the free Abelian topological group $A(X)$—more so because the answers turn out to be different.

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The first theorem of this type was obtained by Graev [11] who proved that for a compact space X the spaces $F(X)$ and $A(X)$ are k_ω-spaces. Later this was generalized by Mack, Morris and Ordman [14] who showed that the same is true if X is a k_ω-space. On the other hand, the free topological group over rationals is not a k-space [10] as we already mentioned above. Using [11] one can easily deduce that if X is compact of countable tightness, then the tightness of $F(X)$ is countable. If X is second-countable, then $F(X)$ and $A(X)$ (and, in fact, any topological group generated by its subspace homeomorphic to X) have countable network weight [3] and therefore countable tightness. Thomas noted [22] that the tightness of the free Abelian topological group over the sum of a segment of the real line and a discrete space is countable. The question whether the same is true for the free topological group was raised by Čoban [8]. The criterion obtained in this paper yields the negative answer to Čoban's question.

Some results of this were announced in [7, 6].

1. Notations, terminology and some general facts

All topological spaces in this paper are assumed to be Tychonoff. In notations and terminology we follow [1, 9]. The symbols $F(X)$ and $A(X)$ denote respectively the free topological group and the free Abelian topological group over a space X [15, 11]. The group $F(X)$ without topology is the free algebraic group over the set of generators X—that is, $F(X)$ is the set of all irreducible words $x_1^{\varepsilon_1}\dots x_n^{\varepsilon_n}$ where $x_1, \dots, x_n$ are elements of X and $\varepsilon_i \in \{-1, 1\}$, $i = 1, \dots, n$, equipped with the natural group operation. We denote by e the empty word which is the unity of the group $F(X)$. The length of an irreducible word $x_1^{\varepsilon_1}\dots x_n^{\varepsilon_n}$ is the number n (by definition, the length of e is 0). We denote by $F_n(X)$ the subspace in $F(X)$ consisting of all words the length of which does not exceed n.

The free Abelian topological group $A(X)$ is the topological factor-group of $F(X)$ by its commutant. In dealing with $A(X)$ we use additive notation. The length of an element in $A(X)$ and subspaces $A_n(X)$ are defined similarly to that in $F(X)$.

A covering γ of a space Z is called *generating* if a subset F in Z is closed iff for each $P \in \gamma$ the intersection $F \cap P$ is closed in P. Clearly, Z is a k-space iff its covering $\gamma = \{K \subset Z\colon K \text{ is compact}\}$ is generating. The spaces allowing countable compact covering are called k_ω-spaces. We often use:

Theorem 1.1 [14]. *If X is a k_ω-space, then both $A(X)$ and $F(X)$ are k_ω-spaces.*

The tightness of a space Z is countable iff Z has a generating covering all elements of which are countable.

The following assertions concerning generating coverings are almost obvious.

Proposition 1.2. (i) *Every open covering is generating.*

(ii) *If γ is a generating covering of Z and Z_1 is a closed subspace in Z, then $\gamma_1 = \{Z_1 \cap P\colon P \in \gamma\}$ is a generating covering of Z_1.*

(iii) *If a covering γ is refined by a generating covering γ_1, then γ is generating.*

(iv) *If γ' is a generating covering of Z, γ is a covering of Z and $\gamma_P = \{P \cap A\colon A \in \gamma\}$ is a generating covering of P for each $P \in \gamma'$, then γ is a generating covering of Z.*

For each subspace Y of a space X we consider subgroups $A(Y, X)$ in $A(X)$ and $F(Y, X)$ in $F(X)$ generated by elements of Y. As shown in [11] closedness of Y in X implies closedness of $A(Y, X)$ in $A(X)$ and closedness of $F(Y, X)$ in $F(X)$. The following two assertions [25, 23] play an important role in our reasoning.

Theorem 1.3. *If X is metrizable and Y is a closed subspace in X, then $A(Y, X)$ is naturally topologically isomorphic to $A(Y)$.*

Theorem 1.4. *If X is metrizable and Y is a closed subspace in X, then $F(Y, X)$ is naturally topologically isomorphic to $F(Y)$.*

For each subset Φ in $F(X)$ we define the *carrier* of Φ in X as the set car Φ of all elements of X taking part in irreducible expressions of elements of Φ. In different words, car Φ is the minimal set $B \subset X$ with the property $\Phi \subset F(B, X)$. We use similar definition (and similar denotation) for subsets in $A(X)$.

Recall that a subset Φ in a space Z is called *bounded* (in Z) if every real-valued continuous function on Z is bounded on Φ. If Z is Dieudonné-complete (in particular, if Z is paracompact), the closure of every bounded set in Z is compact. The spaces in which the latter condition holds are called *μ-spaces.*

The next theorem was obtained independently by Arhangel'skii and Čoban. However, the proof presented in [8] is not quite correct, so we give its complete proof here.

Theorem 1.5. *If Φ is a bounded set in $F(X)$ (in $A(X)$), then* car Φ *is bounded in X.*

We prove the theorem for $F(X)$; the proof for $A(X)$ is quite similar.

Lemma 1.6. *If $\{x_n : n \in N\}$ is a sequence of distinct points in X and $f: X \to R$ is a continuous function, then there exists a continuous function $g: X \to R$ such that*

(a) $|f(x) - g(x)| \leq 1$ *for all $x \in X$, and*

(b) $g(x_i) \neq g(x_j)$ *whenever $i \neq j$.*

Proof. Define by induction a sequence $\{g_n : n \in N\}$ of real-valued continuous functions on X satisfying the following conditions:

(1) $|g_n(x)| \leqq \frac{1}{2}^n$ for all $x \in X$ and $n \in N$,

(2) $g_i(x_j) = 0$ whenever $i < j$,

(3) $f_{n+1}(x_i) \neq f_{n+1}(x_{n+1})$ for every $i < n+1$, where

$$f_n(x) = f(x) + \sum_{i=1}^{n} g_i(x), \quad x \in X, \quad n \in N.$$

Put $g_1(x) = \frac{1}{2}$ for all $x \in X$. Assume that functions $g_1, \dots, g_k$ are already defined. Put $f_k(x) = f(x) + \sum_{i=1}^{k} g_i(x)$ and choose a continuous function $g_{k+1}: X \to R$ such that $|g_{k+1}(x)| \leq \frac{1}{2}^{k+1}$ for all $x \in X$, $g_{k+1}(x_i) = 0$ and $g_{k+1}(x_{k+1}) \neq f_k(x_i) - f_k(x_{k+1})$ whenever $i \leq k$. Clearly, the sequence $\{g_i : i \in N\}$ thus obtained satisfies conditions (1)-(3).

Put $g(x) = \sum_{i=1}^{\infty} f_i(x)$ for all $x \in X$. The function $g(x)$ is obviously continuous, $|f(x) - g(x)| \leq 1$ for all $x \in X$ and $g(x_i) \neq g(x_j)$ for all $i \neq j$, $i, j \in N$. □

Proof of Theorem 1.5. Put $B = \text{car}\ \Phi$ and assume that the set B is not bounded in X. Take a continuous function $f: X \to R$ which is not bounded on B and choose a sequence $\{b_n : n \in N\}$ of points in B such that $f(b_n) \geq n$ for all $n \in N$.

For each b_n fix an irreducible word $h_n \in \Phi$ such that $b_n \in K_n$ where $K_n = \text{car}\ \{h_n\}$ is the set of all "letters" participating in h_n. Put $A = \bigcup \{K_n : n \in N\}$. Arrange elements of A into a sequence: $A = \{x_n : n \in N\}$, where $x_i \neq x_j$ whenever $i \neq j$.

We have $\{b_n : n \in N\} \subset A$, hence the function f is unbounded on A. By Lemma 1.6 there exists a continuous function $g: X \to R$ such that $|f(x) - g(x)| \leq 1$ for all $x \in X$ and $g(x_i) \neq g(x_j)$ whenever $i \neq j$.

Clearly, g is unbounded on A. Take the continuous homomorphism $\hat{g}: F(X) \to F(R)$ extending the function g. Put $\Phi_1 = g(\Phi)$ and $A_1 = g(A) = \hat{g}(A)$. The set A_1 is unbounded in R because g is unbounded on A.

The mapping $g|A: A \to A_1$ is a bijection and K_n is a subset in A. Hence the images of distinct points in K_n are distinct points in A_1. Since K_n is the set of all "letters" in an irreducible word $h_n \in \Phi$, $\hat{g}(K_n)$ is the set of all letters in an irreducible word $\hat{g}(h_n) \in \hat{g}(\Phi)$. Hence $A_1 = \bigcup \{g(K_n) : n \in N\}$ is in car Φ_1.

The set Φ_1 is an image of the bounded set Φ in $F(X)$ under continuous mapping g, hence bounded in $F(R)$. The space $F(R)$ is Lindelöf, therefore Dieudonné-complete, which implies compactness of the closure $\bar{\Phi}_1$ of Φ_1 in $F(R)$. Since R is a k_ω-space, one can find $a \in R$ such that $F([-a, a], R) \supset \bar{\Phi}_1 \supset \Phi_1$ (see [14]). Then $A_1 \subset \text{car}\ \Phi_1 \subset [-a, a]$ which contradicts unboundedness of A_1 in R. The proof of Theorem 1.5 is complete. □

The following fact is well known (see e.g. [22]).

Theorem 1.7. *If K is a compact set in $A(X)$ (in $F(X)$), then K is in $A_n(X)$ (in $F_n(X)$) for some $n \in N$.*

Corollary 1.8. *If X is Dieudonné-complete and Φ is a compact set in $A(X)$ (in $F(X)$), then there exist a compact $Z \subset X$ and $n \in N$ such that Φ is a continuous image of a subspace in Z^n.*

The *Fréchet-Urysohn fan* of cardinality $\aleph_1$ [2] is the space $V(\aleph_1) = \{a_{m\alpha}: m \in N, \alpha < \omega_1\} \cup \{\theta\}$ in which all points $a_{m\alpha}$, $m \in N$, $\alpha < \omega_1$ are isolated and a set $U \subset V(\aleph_1)$ is a neighbourhood of point θ iff for every $\alpha < \omega_1$ the set $\{m \in N: a_{m\alpha} \notin U\}$ is finite. In different terms, $V(\aleph_1)$ is the quotient space obtained from the sum of $\aleph_1$ many converging sequences by identifying limit points. It is convenient to assign to each function $\varphi: \omega_1 \to N$ a subset

$$O(\varphi) = \{\theta\} \cup \{a_{m\alpha}: m \geqslant \varphi(\alpha)\}$$

in $V(\aleph_1)$; obviously, $\{O(\varphi): \varphi \in N^{\omega_1}\}$ is a base of $V(\aleph_1)$ at point θ. Clearly, $V(\aleph_1)$ is a paracompact Fréchet space. The following assertion is obvious.

Proposition 1.9. *A space Z of cardinality $\aleph_1$ is homeomorphic to $V(\aleph_1)$ iff there exists a generating covering $\{C_\alpha: \alpha < \omega_1\}$ of Z and a point $z_0 \in Z$ such that*
(a) *$C_\alpha \cap C_\beta = z_0$ whenever $\alpha \neq \beta$, $\alpha, \beta < \omega_1$,*
(b) *each C_α, $\alpha < \omega_1$, is a converging sequence with the limit point z_0.*

Applying Proposition 1.2(ii) and (iv) we get:

Proposition 1.10. *A space Z of cardinality $\aleph_1$ is homeomorphic to $V(\aleph_1)$ iff there exists a covering $\gamma = \{C_\alpha: \alpha < \omega_1\}$ of Z and a point $z_0 \in Z$ such that*
(a) *$C_\alpha \cap C_\beta = z_0$ whenever $\alpha \neq \beta$, $\alpha, \beta < \omega_1$,*
(b) *each C_α, $\alpha < \omega_1$, is a converging sequence with the limit point z_0,*
(c) *the covering $\gamma_0 = \{\bigcup \gamma': \gamma' \subset \gamma, \gamma'$ finite$\}$ is generating.*

The proof for the next fundamental theorem was found by Malyhin (and is presented here with his kind permission).

Theorem 1.11. *Tightness of the product $V(\aleph_1) \times V(\aleph_1)$ is uncountable.*

Proof. As shown in [20, 13], there exist two families $\mathcal{A} = \{A_\alpha: \alpha < \omega_1\}$ and $\mathcal{B} = \{B_\alpha: \alpha < \omega_1\}$ of infinite subsets in N such that
(a) $A_\alpha \cap B_\beta$ is finite for any $\alpha, \beta < \omega_1$, and
(b) there exists no $A \subset N$ such that all sets $A_\alpha \setminus A$, $B_\alpha \cap A$, $\alpha < \omega_1$ are finite.
Fix a pair $\mathcal{A}$, $\mathcal{B}$ of such families and put

$$X = \{(a_{n\alpha}, a_{n\beta}) \in V(\aleph_1) \times V(\aleph_1): m \in A_\alpha \cap B_\beta\}.$$

We are going to verify that the point (θ, θ) is a limit point for X but not for a countable subset in X, which proves uncountability of tightness of $V(\aleph_1) \times V(\aleph_1)$.

Take an arbitrary neighbourhood U of point (θ, θ) in $V(\aleph_1) \times V(\aleph_1)$. It is no loss of generality to assume that $U = O(\varphi) \times O(\varphi)$ where $\varphi: \omega_1 \to N$ is a function. Put

$$A'_\alpha = \{n \in A_\alpha: n \geqslant \varphi(\alpha)\}, \qquad B'_\alpha = \{n \in B_\alpha: n \geqslant \varphi(\alpha)\}.$$

If we had $A'_\alpha \cap B'_\beta = \emptyset$ for all $\alpha < \omega_1$, $\beta < \omega_1$, we would have

$$(\bigcup \{A'_\alpha: \alpha < \omega_1\}) \cap (\bigcup \{B'_\alpha: \alpha < \omega_1\}) = \emptyset$$

and, putting $A = \bigcup\{A'_\alpha: \alpha < \omega_1\}$, come to a contradiction with condition (b). Hence one can find $\alpha < \omega_1$ and $\beta < \omega_1$ such that $A'_\alpha \cap B'_\beta \neq \emptyset$. Choose $n \in A'_\alpha \cap B'_\beta$. Now

$$(a_{n\alpha}, a_{n\beta}) \in X \cap (O(\varphi) \times O(\varphi)) = X \cap U,$$

and $X \cap U \neq \emptyset$. Thus (θ, θ) is a limit point for X.

Let K be an arbitrary countable subset in X. There exists an ordinal $\gamma < \omega_1$ such that

$$K \subset \{(a_{n\alpha}, a_{n\beta}): \alpha, \beta < \gamma, n \in N\}.$$

It is convenient to renumerate families A and B in such a way that $\gamma = \omega_0 = N$.

For each $k \in N$ define a function $\varphi_k : N \to N$ such that for every $k \in N$ there exists $r_k \in N$ such that $\varphi(l) \geq \varphi_k(l)$ for all $l \geq r_k$. Now put

$$\psi(l) = \max\{\varphi(l), \max(\bigcup\{B_m : m < r_l\} \cap A_l) + 1\}$$

and $U = O(\psi) \times O(\psi)$.

Clearly, U is a neighbourhood of the point (θ, θ). We are going to check that $U \cap K = \emptyset$. Note that for all $k, l \in N$, $\psi(l) \geq \max(B_k \cap A_l) + 1$. Hence if $(a_{nl}, a_{nk}) \in K$, then $\psi(l) > n$ ($n \in A_l \cap B_k$ by definition of X) and $(a_{nl}, a_{nk}) \notin U$.

Thus (θ, θ) is not a limit point for arbitrary countable subset K in X and the proof is complete. □

Corollary 1.12. *The product $V(\aleph_1) \times V(\aleph_1)$ is not a k-space.*

Indeed, every compact set in $V(\aleph_1)$ and hence in $V(\aleph_1) \times V(\aleph_1)$ is countable.

2. Countability of tightness and k-property in free Abelian topological groups over metrizable spaces

Proposition 2.1. *Let X be a metrizable space. If the tightness of $A(X)$ is countable, then the set X' of all nonisolated points in X is separable.*

Proof. If X' is nonseparable, one can choose an uncountable discrete in X family $\{U_\alpha : \alpha < \omega_1\}$ of open sets in X each of which contains a point $x_\alpha \in X'$. For each $\alpha < \omega_1$ choose a nontrivial sequence $C_\alpha \subset U_\alpha$ converging to x_α (we assume $x_\alpha \in C_\alpha$) and put

$$Y = \bigcup\{C_\alpha : \alpha < \omega_1\}, \qquad Y_0 = \{x_\alpha : \alpha < \omega_1\}.$$

Clearly, Y is closed in X and is homeomorphic to the product $C \times D(\aleph_1)$ where C is a converging sequence and $D(\aleph_1)$ is the discrete space of cardinality $\aleph_1$. Consider the quotient space Z obtained by identifying to a point the subset Y_0 in X and the subspace $Z_1 = p(Y)$ in Z where $p: X \to Z$ is the projection. Clearly, p is closed. Hence the restriction $p|Y$ is quotient and therefore Z_1 is homeomorphic to $V(\aleph_1)$.

Assume that the tightness of $A(X)$ is countable. The homomorphism $p: X \to Z$ extending quotient mapping $p: X \to Z$ is open [5]. Hence the tightness of $A(Z)$ must be countable. To get the contradiction we need only to refer to Theorem 1.11 and the fact that $A(Z)$ contains a homeomorphic copy of $Z \times Z$ [19]. □

Proposition 2.2. *Let X be a metrizable space, X' the set of all nonisolated points in X. Then the tightness of $A(X)$ does not exceed the weight of X'.*

Proof. Denote by τ the weight of X'. It suffices to prove that if the unity e of $A(X)$ is a limit point for a set $M \subset A(X)$, then e is a limit point for a subset $M' \subset M$ the cardinality of which does not exceed τ.

Choose an external base $\mathscr{B}$ of X' in X of cardinality $\leq \tau$ and put

$$\mathcal{M} = \{\mathscr{A} \subset \mathscr{B}: \bigcup \mathscr{A} \supset X'\}.$$

Assign to each $\mathscr{A} \in \mathcal{M}$ an entourage of the diagonal Δ in $X \times X$ of the form

$$V(\mathscr{A}) = \{U \times U: U \in \mathscr{A}\} \cup \Delta.$$

Clearly, each $V(\mathscr{A})$, $\mathscr{A} \in \mathcal{M}$ is a neighbourhood of Δ in $X \times X$; metrizability of X implies $V(\mathscr{A})$ being an element of the universal uniformity on X. Since $\{V(\mathscr{A}): \mathscr{A} \in \mathcal{M}\}$ is a base for $X \times X$ at the set Δ, this family of sets is a base for the universal uniformity on X.

It is natural to identify M with a subset in the set $D^{\mathscr{B}}$ of all functions from $\mathscr{B}$ to $D = \{0, 1\}$ by assigning to each $\mathscr{A} \in \mathcal{M}$ its characteristic function. Equip D with the discrete topology, $D^{\mathscr{B}}$ with the product topology and M with the topology of subspace in $D^{\mathscr{B}}$. Clearly, the weight of this topology does not exceed τ.

Now consider the space $S = \mathcal{M}^N$ equipped with the product topology. Clearly, the weight of S does not exceed τ. Assign to each element $s = (\mathscr{A}_1, \ldots, \mathscr{A}_n, \ldots)$ of S the set G_s of all elements a in $A(X)$ which can be represented in the form

$$a = x_1 - y_1 + \cdots + x_k - y_k,$$

where $k \in N$, $(x_i, y_i) \in V(\mathscr{A}_i)$ for all $i \leq k$. The description of neighbourhoods of unity in free Abelian topological groups given in [21] (see also [18]) implies the family $\{G_s: s \in S\}$ being a base at unity e in $A(X)$.

Lemma 2.3. *For each $a \in A(X)$ the set $P_a = \{s \in S: a \notin G_s\}$ is closed in S.*

Proof. Fix $a \in A(X)$ and assume $a \in G_{s_0}$ for some $s_0 \in S$, $s_0 = (\mathscr{A}_1^0, \ldots, \mathscr{A}_n^0, \ldots)$. We are going to find a neighbourhood W of s_0 in S such that $W \cap P_a = \emptyset$.

By definition of G_s, a can be represented in the form

$$a = x_1 - y_1 + \cdots + x_k - y_k,$$

where $k \in N$, $(x_i, y_i) \in V(A_i^0)$ for all $i \leq k$. Put $L = \{i \leq k: x_i \neq y_i\}$ and for each $i \in L$ choose a set U_i in $\mathscr{A}_i^0$ containing both x_i and y_i. Thus we obtain a finite set $\{U_i : i \in L\}$ of elements in B. Now put

$$W = \{s = (\mathscr{A}_1, \ldots, \mathscr{A}_n, \ldots) \in S: U_i \in A_i \text{ for each } i \in L\}.$$

Clearly, W is open in S, $s_0 \in W$ and $a \in G_s$ for all $s \in W$, i.e. $W \cap P_a = \emptyset$. □

Lemma 2.4. *For every subset H in $A(X)$ the set $P_H = \{s \in S: G_s \cap H = \emptyset\}$ is closed in S.*

This follows directly from Lemma 2.3 and the equality $P_H = \bigcap \{P_a : a \in H\}$.

Now assume that the tightness of $A(X)$ exceeds τ. Then there exists a set M in $A(X)$ such that e is in the closure of M but is not in the closure of any subset H in M of cardinality $\leq \tau$. Then for each subset H in M of cardinality $\leq \tau$ there exists a neighbourhood of e disjoint with H, hence $P_H \neq \emptyset$. Then the family $\{P_H : H \subset M, |H| \leq \tau\}$ has the τ-intersection property. All sets in this family are closed in S by Lemma 2.4 and the weight of S does not exceed τ. Hence the intersection $P_M = \bigcap \{P_H : H \subset M, |H| \leq \tau\}$ is not empty. Choosing an element s in P_M we obtain a neighbourhood G_s of e disjoint with M which contradicts the assumption that e is in the closure of M. The proof is complete. □

Combining Propositions 2.1 and 2.2 we get the criterion:

Theorem 2.5. *Let X be a metrizable space. Then the tightness of the free Abelian group $A(X)$ is countable if and only if the set X' of all nonisolated points in X is separable.*

Problem 2.6. Is it true that for X metrizable the tightness of $A(X)$ always coincides with the weight of the set X' of all nonisolated points in X?

Now we turn to studying the k-property in $A(X)$. From Corollary 1.8, Proposition 1.2(iv) and Proposition 2.1 readily follows:

Proposition 2.7. *If X is metrizable and $A(X)$ is a k-space, then the set X' of all nonisolated points in X is separable.*

Proposition 2.8. *If X is metrizable and $A(X)$ is a k-space, then X is locally compact.*

Proof. Assume that a point $x_0 \in X$ has no neighbourhood with compact closure in X. Choose a countable base $\{V_n : n \in N\}$ at point x_0 such that all sets $B_n = V_n \setminus V_{n+1}$, $n \in N$ are noncompact. For each $n \in N$ fix a closed discrete in X infinite set $\{x_{mn} : m \in N\} \subset B_n$. Clearly, all points of the set $M = \{x_{mn} : m \in N, n \in N\} \cup \{x_0\}$, except point x_0, are isolated in M. By Theorem 1.2, $A(M)$ is homeomorphic to a closed subgroup in $A(X)$. To end the proof of the proposition we need to apply the following lemma proved by Gul'ko by Pestov's request.

Lemma 2.9. *The free Abelian group $A(M)$ over the defined above space M is not a k-space.*

Proof. Assign to each pair k, l of positive integers an element

$$h_{kl} = x_{(2l)k} - x_{(2l+1)k} + \cdots + x_{1l} - x_{2l} + x_{(2k-1)l} - x_{(2k)l}$$

in $A(M)$ and consider sets $H_k = \{h_{kl}: l > k\}$, $k \in N$ and $H = \bigcup \{H_k : k \in N\}$. We are going to check that intersection of H with any compact set in $A(M)$ is closed (in fact, finite) and nevertheless H is not closed in $A(M)$.

First of all, observe that the length of h_{kl} equals $2k+2$, i.e. $H_k \subset A(M) \backslash A_{2k+2}(M)$. Fix a compact set $K \subset A(M)$. By Theorem 1.7, K is in $A_n(M)$ for some $n \in N$. Hence K intersects only finitely many sets H_k and it suffices to check that $K \cap H_k$ is finite for each $k \in N$.

The set $X_k = \{x_{mk} : m \in N\}$ is closed and discrete in M. Clearly, $X_k \cap \operatorname{car}\{h_{kl}\} = \{x_{(2l)k}, x_{(2l+1)k}\}$—note that these sets are disjoint. Hence for every infinite $H' \subset H_k$ the intersection $X_k \cap \operatorname{car} H'$ is infinite, therefore unbounded in M. Theorem 1.5 implies now that no infinite subset of H is bounded in $A(M)$. Hence $K \cap H_k$ is finite and $K \cap H$ is finite.

Clearly, $e \notin H$. We shall show that e is a limit point for the set H. To that end we are going to use the description of neighbourhoods of unity in free Abelian topological groups given in [21, 18].

For each $n \in N$ put

$$V_n = \{x_{kl} : l > n,\ l, k \in N\} \cup \{x_0\}.$$

The family $\{V_n : n \in N\}$ is a base of M at point x_0. Hence the family $\{U_n : n \in N\}$ where

$$U_n = (V_n \times V_n) \cup \Delta$$

is a base of $M \times M$ at diagonal Δ. As M is metrizable, $\{U_n : n \in N\}$ is a base for the universal uniformity on M.

Assign to each sequence $P = (p_1, \ldots, p_n, \ldots)$ of naturals a set G_P of all elements in $A(M)$ of the form $y_1 - z_1 + \cdots + y_r - z_r$, where $r \in N$, $(y_i, z_i) \in U_{p_i}$, $i = 1, \ldots, r$. The description of neighbourhoods of unity in $A(M)$ [21, 18] implies the family $\{G_P : P \in N^N\}$ being a base of $A(M)$ at unity e. To end the proof it suffices to show that each G_P contains a point from H.

Fix a $P = (p_1, \ldots, p_n, \ldots) \in N^N$. Choose naturals k and l in such a way that $k > p_1$ and $l > \max\{k, p_2, \ldots, p_k\}$. Then $(x_{(2l)k}, x_{(2l+1)k}) \in U_k \subset U_{p_1}$ and $(x_{(2i-1)l}, x_{(2i)l}) \in U_l \subset U_{p_i}$ for each $i \leq k$. Hence h_{kl} is in G_P and the lemma is proved. □

Proposition 2.10. *If X is a locally compact metrizable space and the set X' of all nonisolated points in X is separable, then $A(X)$ is homeomorphic to a product of a k_ω-space with a discrete space and therefore is a k-space.*

Proof. Choose a countable covering γ of X with open sets having compact closures in X and put $X_0 = \bigcup\{\bar{U}: U \in \gamma\}$. Clearly, X_0 is a k_ω-space and $X_1 = X \backslash X_0$ is closed and open in X discrete. Hence we have $A(X) = A(X_0) \times A(X_1)$ [12], $A(X_0)$ is a k_ω-space by Theorem 1.1 and $A(X_1)$ is discrete. □

Combining Propositions 2.7, 2.8 and 2.10 we get:

Theorem 2.11. *If X is metrizable and X' is the set of all nonisolated points in X, then the following conditions are equivalent*:
(a) *$A(X)$ is a k-space,*
(b) *$A(X)$ is homeomorphic to a product of a k_ω-space with a discrete space,*
(c) *X is locally compact and X' is separable.*

Corollary 2.12. *The free Abelian topological group $A(\mathbb{Q})$ over the space of rationals $\mathbb{Q}$ is not a k-space.*

3. Tightness and k-property in free topological groups over metrizable spaces

The free Abelian topological group $A(X)$ is an image of the free topological group $F(X)$ under an open mapping—the canonical projection. Hence $F(X)$ being a k-space implies $A(X)$ being a k-space and the tightness of $A(X)$ does not exceed the tightness of $F(X)$. This together with Propositions 2.1 and 2.8 gives:

Proposition 3.1. *Let X be metrizable and X' be the set of all nonisolated points in X. If the tightness of $F(X)$ is countable, then X' is separable. Moreover, if $F(X)$ is a k-space, then X' is separable and X is locally compact.*

Denote $C = \{x_n : n \in N\} \cup \{x_0\}$ a converging sequence with the limit point x_0 and D the discrete space of cardinality $\aleph_1$.

Proposition 3.2. *The free topological group $F(X)$, where $X = C \oplus D$, is not a k-space and the tightness of $F(X)$ is uncountable.*

Proof. Assume first that the tightness of $F(X)$ is countable. Then the covering $\{F(C \oplus A, X): A \subset D, A \text{ is countable}\}$ of $F(X)$ must be generating because the set car B is countable for every countable set $B \subset F(X)$. Each element of this covering is a k_ω-space (this follows from Theorems 1.4 and 1.1), hence $F(X)$ must be a k-space. Thus, the assumption that tightness of $F(X)$ is countable implies $F(X)$ being a k-space.

Now assume that $F(X)$ is a k-space. Obviously, every bounded set in $X = C \oplus D$ is contained in a set of the form $C \oplus A$, where $A \subset D$ is finite. Applying Theorem 1.5 and Proposition 1.7(iv) we get: the covering

$$\gamma = \{F(C \oplus A, X) : A \subset D, A \text{ is finite}\}$$

is generating.

Put $C_a = \{a^{-1}x_0^{-1}xa : x \in C\}$ for each $a \in D$ and $Y = \bigcup\{C_a : a \in D\}$. Clearly, each C_a is homeomorphic to C and $C_a \cap C_b = \{e\}$ whenever $a \neq b$, $a, b \in D$. For each finite set $A \subset D$ the set $Y_A = Y \cap F(C \oplus A, X) = \bigcup\{C_a : a \in A\}$ is compact, therefore closed in $F(C \oplus A, X)$. The covering γ being generating, Y must be closed in $F(X)$. Hence the covering $\gamma_Y = \{Y_A : A \subset D, A \text{ is finite}\}$ of Y must be generating. Using Proposition 1.10 we claim: Y is homeomorphic to $V(\aleph_1)$. Thus, the assumption that $F(X)$ is a k-space implies the existence of a closed subspace in $F(X)$ homeomorphic to the Fréchet–Urysohn fan $V(\aleph_1)$.

Now Corollary 1.12 implies $F(X) \times F(X)$ not being a k-space. But $F(X)$ is homeomorphic to $F(X \oplus X)$ (this readily follows from the results of [12]) and $F(X \oplus X)$ allows an open mapping onto $F(X) \times F(X)$ (see [16]). Since the k-property is preserved by open mappings we conclude that $F(X)$ is not a k-space.

Thus, the assumption of countability of tightness or a k-property in $F(X)$ leads to contradiction, which proves the proposition. □

Corollary 3.3. *If X is a nondiscrete nonseparable metrizable space, then $F(X)$ is not a k-space and the tightness of $F(X)$ is uncountable.*

Indeed, every nonseparable nondiscrete metrizable space X contains a closed subspace homeomorphic to the sum of the converging sequence C and the uncountable discrete space D, and we have only to refer to Theorem 1.4 and Proposition 3.2.

To get a criterion we are left now to check two simple assertions following directly from results in [14, 3].

Proposition 3.4. *If X is separable metrizable, then the tightness of $F(X)$ is countable.*

Proposition 3.5. *If X is locally compact separable metrizable, then $F(X)$ is a k_ω-space.*

Summing up Propositions 3.1, 3.3–3.5 we get:

Theorem 3.6. *If X is metrizable, then the tightness of $F(X)$ is countable iff X is separable or discrete.*

Theorem 3.7. *If X is metrizable, then the following conditions are equivalent:*

(a) *$F(X)$ is a k-space,*

(b) *$F(X)$ is a k_ω-space or discrete,*

(c) *X is locally compact separable or discrete.*

Corollary 3.8. *The free topological group $F(I \oplus D)$ where I is a segment and D is uncountable discrete has uncountable tightness and no k-property.*

This gives a negative answer to Čoban's question raised in [24].

4. Free topological groups over separable metrizable spaces and k_R-property

If we restrict ourselves to the class of separable metrizable spaces, some results of the previous sections can be improved. The improvements are based on the notion of $\aleph_0$-space introduced by Michael in [16]. A space X is called *$\aleph_0$-space* [16] if there exists a countable *k-network* in X, which is a family S of subsets in X such that for every compact $B \subset X$ and every neighbourhood V of B there exists $P \in S$ such that $B \subset P \subset V$. All separable metrizable spaces are $\aleph_0$-spaces and all $\aleph_0$-spaces are Lindelöf and therefore Dieudonné-complete.

Theorem 4.1. *If X is a $\aleph_0$-space, then $F(X)$ and $A(X)$ are $\aleph_0$-spaces.*

Proof. The product X^n is a $\aleph_0$-space for each $n \in N$ [23]. Fix a countable k-network S_n in X^n. Denote $\psi_n : (X \oplus X^{-1} \oplus \{e\})^n \to F_n(X)$ the canonical mapping (multiplication in $F(X)$). Since X is Dieudonné-complete for every compact $\Phi \subset F_n(X)$ there exists a compact $\Phi_1 \subset (X \oplus X^{-1} \oplus \{e\})^n$ such that $\psi_n(\Phi_1) = \Phi$ (see Corollary 1.8). This together with continuity of ψ_n implies that the family $P_n = \{\psi_n(A): A \in S_n\}$ is a countable k-network in $F_n(X)$. But every compact $\Phi \subset F(X)$ is in $F_n(X)$ for some $n \in N$. Hence $P = \bigcup \{P_n : n \in N\}$ is a countable k-network in $F(X)$ and $F(X)$ is a $\aleph_0$-space.

The proof of the analogous assertion for $A(X)$ is quite similar. □

We also need the following assertion [4].

Proposition 4.2. *If X is metrizable, then $F(X)$ and $A(X)$ are countable unions of their closed metrizable subspaces.*

Recall that a space X is called a *k_R-space* if every real-valued function f on X is continuous provided that its restriction over any compact subspace in X is continuous. Clearly, all k-spaces are k_R-spaces. The space $R^{\aleph_1}$ is an example of topological group which is a k_R- but not a k-space.

Theorem 4.3. *Let X be separable metrizable. Then the group $F(X)$ is a k_R-space iff it is a k-space. The same holds for $A(X)$.*

Proof. Assume that X is separable metrizable and $F(X)$ is a k_R-space. By Propositions 4.2 and 4.3, $F(X)$ is a $\aleph_0$-space which is a countable union of its closed

k-subspaces. By Michael's theorem in [26] this implies $F(X)$ being a k-space. The same reasoning proves the theorem for $A(X)$. □

Theorems 4.3, 2.13 and 3.7 imply:

Theorem 4.4. *If X is separable metrizable, then the following conditions are equivalent:*
(a) *$A(X)$ is a k_R-space,*
(b) *$F(X)$ is a k_R-space,*
(c) *$A(X)$ is a k_ω-space,*
(d) *$F(X)$ is a k_ω-space,*
(e) *X is locally compact.*

Corollary 4.5. *The groups $A(\mathbb{Q})$ and $F(\mathbb{Q})$ where $\mathbb{Q}$ is the space of rationals are not k_R-spaces.*

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