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Transversal Theory and the Study of Abstract Independence*

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My subject is combinatorial analysis, which may be regarded (to a first approximation) as a branch of the theory of sets. To make the exposition more tractable, I shall confine my remarks strictly to finite sets, without recalling this convention every time. (For an account including the discussion of the transfinite theory, see the survey article [6].)

Let $\mathfrak{A} = (A_1, \ldots, A_n)$ be a family of (not necessarily different) subsets of a set E. A subset T of E is called a *transversal* of $\mathfrak{A}$ if it consists of just n distinct elements, say $x_1, \ldots, x_n$, and if, for a suitable permutation $i_1, \ldots, i_n$ of $1, \ldots, n$, we have

$$x_1 \in A_{i_1}, \ldots, x_n \in A_{i_n}.$$

Thus, to assert that $\mathfrak{A}$ possesses a transversal is equivalent to saying that there exist n distinct elements $y_1, \ldots, y_n$ such that $y_1 \in A_1, \ldots, y_n \in A_n$. We then say that y_i "represents" A_i. The transversal $\{y_1, \ldots, y_n\}$ is sometimes known as a "system of distinct representatives" of $\mathfrak{A}$.

A family of sets need not, of course, possess a transversal; and our first objective is to formulate necessary and sufficient conditions for a transversal to exist. Suppose, in the first place, that $\mathfrak{A} = (A_1, \ldots, A_n)$ has a transversal. Then there exist distinct elements $x_1, \ldots, x_n$ such that $x_1 \in A_1, \ldots, x_n \in A_n$. Let $1 \leqslant k \leqslant n$ and consider any k sets in $\mathfrak{A}$, say $A_{i_1}, \ldots, A_{i_k}$, where $1 \leqslant i_1 < \cdots < i_k \leqslant n$. Then

$$A_{i_1} \cup \cdots \cup A_{i_k} \supseteq \{x_{i_1}, \ldots, x_{i_k}\}$$

and so, denoting by $|X|$ the cardinal of X, we have

$$|A_{i_1} \cup \cdots \cup A_{i_k}| \geqslant k.$$

In other words, for each k with $1 \leqslant k \leqslant n$, any k sets among the A's contain between them at least k elements. It will be convenient to express this condi-

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tion in a slightly different notation. Writing $I = \{i_1, ..., i_k\}$, we have $|I| = k$ and so

$$\left| \bigcup_{i \in I} A_i \right| \geqslant |I| \tag{1}$$

for each $I \subseteq \{1, ..., n\}$. This obviously necessary condition for the existence of a transversal turns out to be unobviously sufficient: this is the content of P. Hall's classical theorem [3].

THEOREM 1 (P. Hall, 1935). *The family* $(A_1, ..., A_n)$ *of sets possesses a transversal if and only if* (1) *holds for each* $I \subseteq \{1, ..., n\}$.

Once this criterion has been formulated, a number of further problems concerned with the existence of transversals readily suggest themselves. I shall consider briefly a few representative examples.

(a) We say that a subset X of E is a *partial transversal* (PT) of the family $\mathfrak{A} = (A_1, ..., A_n)$ (of subsets of E) if X is a transversal of some subfamily of $\mathfrak{A}$. Thus, if $X = \{x_1, ..., x_k\}_{\neq}$[1] (where, of course, $1 \leqslant k \leqslant n$), then there exist distinct integers $i_1, ..., i_k$ among $1, ..., n$ such that $x_1 \in A_{i_1}, ..., x_k \in A_{i_k}$. (It is then clear that a PT of cardinal n is simply a transversal.) We shall ultimately recognize that the notion of a PT is much more fundamental than that of a transversal. For the moment, however, we merely ask the limited question: when does $\mathfrak{A}$ possess a PT of preassigned cardinal? The answer is given by a theorem due to Ore [9].

THEOREM 2 (O. Ore, 1955). *The family* $\mathfrak{A} = (A_1, ..., A_n)$ *of subsets of* E *possesses a partial transversal of cardinal k if and only if, for each* $I \subseteq \{1, ..., n\}$,

$$\left| \bigcup_{i \in I} A_i \right| \geqslant |I| + k - n. \tag{2}$$

We observe that, for $k = n$, this result reduces to Hall's theorem. The idea of the proof is as follows (cf. [5]). Let D be a set of elements—"dummy" elements since we shall get rid of them eventually—such that $D \cap E = \varnothing$ and $|D| = n - k$. We now consider the family

$$\mathfrak{A}^* = (A_1 \cup D, ..., A_n \cup D).$$

It is easily verified that $\mathfrak{A}$ possesses a PT of cardinal k if and only if $\mathfrak{A}^*$ possesses a transversal. But it follows almost at once from Hall's theorem

[1] We denote by $\{x_1, ..., x_k\}_{\neq}$ the set consisting of the elements $x_1, ..., x_k$ and at the same time express the fact that all x's are distinct.

that $\mathfrak{A}^*$ possesses a transversal if and only if (2) holds for each $I \subseteq \{1,..., n\}$. This establishes the desired conclusion.

Let us note a useful consequence of the result just proved.

THEOREM 3. *Let* $\mathfrak{A} = (A_1, ..., A_n)$ *be a family of subsets of a set* E, *and let* $X \subseteq E$. *Then* X *contains a partial transversal of* $\mathfrak{A}$ *of cardinal* k *if and only if, for each* $I \subseteq \{1,..., n\}$,

$$\left|\left(\bigcup_{i\in I} A_i\right) \cap X\right| \geqslant |I| + k - n.$$

It is clear that X contains a PT of $\mathfrak{A}$ of cardinal k if and only if $(A_i \cap X : 1 \leqslant i \leqslant n)$ has a PT of cardinal k. The assertion therefore follows by Theorem 2.

(b) The results just considered are rather obvious: the next question has a bit more top spin. Let $\mathfrak{A} = (A_1, ..., A_n)$ be a family of subsets of E, and let $M \subseteq E$. When does $\mathfrak{A}$ possess a transversal which contains M as a subset? This problem was solved by Hoffman and Kuhn [4].

THEOREM 4 (Hoffman and Kuhn, 1956). *Let* $\mathfrak{A} = (A_1, ..., A_n)$ *be a family of subsets of* E, *and let* $M \subseteq E$. *Then* $\mathfrak{A}$ *possesses a transversal which contains* M *as a subset if and only if both the following conditions are satisfied.*

(i) $|\bigcup_{i\in I} A_i| \geqslant |I|$ *for all* $I \subseteq \{1,..., n\}$;

(ii) $|(\bigcup_{i\in I} A_i) \cap M| \geqslant |I| + |M| - n$ *for all* $I \subseteq \{1,..., n\}$.

Moreover, conditions (i) *and* (ii) *are respectively equivalent to the following statements:* (i′) $\mathfrak{A}$ *has a transversal;* (ii′) M *is a partial transversal of* $\mathfrak{A}$.

The equivalence (i) $\Leftrightarrow$ (i′) holds by Theorem 1, and (ii) $\Leftrightarrow$ (ii′) by Theorem 3. We note that, for $M = \varnothing$, the theorem (stated in terms of conditions (i) and (ii)) reduces to Hall's theorem.

To prove the assertion, we use an argument suggested by Hazel Perfect. We may clearly assume that $|E| \geqslant n$. Let $\mathfrak{A}^*$ denote the family consisting of the sets $A_1, ..., A_n$ together with $|E| - n$ copies of $E - M$. A moment's reflexion will show that $\mathfrak{A}$ has a transversal containing M if and only if $\mathfrak{A}^*$ has a transversal. An appeal to Hall's theorem (for $\mathfrak{A}^*$) and a short calculation now complete the proof.

(c) Next, we consider two families $\mathfrak{A}$ and $\mathfrak{B}$ of subsets of E. If a subset X of E is a transversal of both $\mathfrak{A}$ and $\mathfrak{B}$, then it is called a *common transversal* (CT) of these two families. A criterion for the existence of a CT is contained in the following theorem [2].

THEOREM 5 (Ford and Fulkerson, 1958). *The families* $\mathfrak{A} = (A_1, \ldots, A_n)$ *and* $\mathfrak{B} = (B_1, \ldots, B_n)$ *of subsets of* E *possess a common transversal if and only if, for all* I, J $\subseteq \{1,\ldots, n\}$,

$$\left| \left(\bigcup_{i\in I} A_i\right) \cap \left(\bigcup_{j\in J} B_j\right) \right| \geqslant |I| + |J| - n.$$

Hall's theorem is the special case of this result which corresponds to the choice $B_j = E$ $(1 \leqslant j \leqslant n)$.

To establish Theorem 5, we employ Hazel Perfect's very neat construction [10]. Assuming (as may be done without loss of generality) that $\{1,\ldots, n\} \cap E = \varnothing$, we consider the family

$$\mathfrak{C} = (C_k : k \in \{1,\ldots, n\} \cup E),$$

where

$$C_k = \begin{cases} A_k & (k \in \{1,\ldots, n\}) \\ \{k\} \cup \{j : 1 \leqslant j \leqslant n, k \in B_j\} & (k \in E). \end{cases}$$

It can be verified that $\mathfrak{A}$ and $\mathfrak{B}$ possess a CT if and only if $\mathfrak{C}$ possesses a transversal; and an application of Hall's theorem is then used to show that this is the case precisely when the inequalities stated in Theorem 5 are valid.

By now the scope of transversal theory should be tolerably clear: in this branch of combinatorial analysis we study conditions which secure the existence of transversals satisfying certain additional requirements. The three examples I have sketched so far—as well as many others—have a salient feature in common: in each case, starting from a given family $\mathfrak{A}$ (or two such families), we construct a new and more complicated family $\mathfrak{A}^*$ to which we then apply Hall's theorem. The derivation of $\mathfrak{A}^*$ from $\mathfrak{A}$ almost always depends on one or other of the following procedures, which I call "elementary constructions".

(i) *Adjunction.* The sets in $\mathfrak{A}$ are enlarged by the adjunction of "dummy" elements which do not belong to the ground set E.

(ii) *Extension.* Here we extend the family $\mathfrak{A}$ by the addition of further sets.

(iii) *Replication.* We obtain a new family by taking a suitable number of copies of each set in $\mathfrak{A}$.

(iv) *Proliferation.* Each element x in a set X is replaced by the set of pairs $(x, 1), (x, 2),\ldots, (x, k_x)$. If k_x is independent of x and is equal to, say, k, then proliferation simply reduces to the formation of the cartesian product $X \times \{1,\ldots, k\}$.

Of these four elementary constructions, the first three have actually been used in the treatment of problems (a), (b), (c).

By a combination of devices such as those I have described, the entire transversal theory of finite sets can be exhibited as a series of corollaries to Hall's theorem. This claim is not tautologous: I do not define transversal theory as the class of all statements derivable from Hall's theorem. Indeed, many results in transversal theory were originally discovered in quite different ways (e.g., by ad hoc arguments, through graph theory, or by means of linear programming) and were only much later shown to be consequences of Hall's theorem.

The method of elementary constructions is thus extremely effective, but too rigid an insistence on its adequacy would not be helpful. In many cases, of which example (a) is characteristic, the use of this method is simple and appropriate. In other instances, such as (b), the argument is still easy though not particularly illuminating; while the proof of (c) indicated above is essentially a tour de force.

To gain fuller insight into transversal theory, we have to cast round for a new idea, and this is found in the study of abstract independence. It was inevitable that sooner or later the very pervasive notion of linear independence in vector spaces should become the subject of axiomatization. The first steps on this road were taken by Whitney [14] and by van der Waerden [12] a little over thirty years ago. By now, a great deal of work has been done in this field; but here we shall need little more than a few definitions.

Let, then, E be a given (finite) set and let $\mathscr{E}$ be a collection of subsets of E. Suppose that $\mathscr{E}$ satisfies the following axioms.

(I1) $\varnothing \in \mathscr{E}$.

(I2) If $X \in \mathscr{E}$ and $Y \subseteq X$, then $Y \in \mathscr{E}$.

(I3) If $\{x_1, \ldots, x_k\}_{\neq} \in \mathscr{E}$ and $\{y_1, \ldots, y_{k+1}\}_{\neq} \in \mathscr{E}$, then, for some i with $1 \leqslant i \leqslant k+1$, $\{x_1, \ldots, x_k, y_i\}_{\neq} \in \mathscr{E}$.

Then $\mathscr{E}$ is called an *independence structure* on E, and the elements of $\mathscr{E}$ are called the *independent subsets* of E.

The meaning of the axioms is plain. They are chosen to imitate some of the salient features of *linear* independence, and the most obvious model of an independence structure is therefore taken from vector space theory: if E is a subset of a vector space and $\mathscr{E}$ denotes the collection of all linearly independent subsets of E, then $\mathscr{E}$ is an independence structure on E. The most telling axiom is (I3); it is called the "replacement axiom" and reflects Steinitz's exchange lemma in the theory of vector spaces.

Again, let E be any set and let $\mathscr{E}$ denote the collection of *all* subsets of E. Then $\mathscr{E}$ is clearly an independence structure. It will be referred to as the *universal structure* on E.

We note the following obvious result. Let $\mathscr{E}$ be an independence structure on E, and let $X \in \mathscr{E}$. Then there exists a set $Y \subseteq E$ such that $X \subseteq Y$, $Y \in \mathscr{E}$

but Y is not a proper subset of any independent set. In other words, *every independent set is contained in a maximal independent set.* (The specialization of this result to vector spaces is, of course, very familiar.)

With E and $\mathscr{E}$ as before, we shall denote by $\rho(X)$ (where $X \subseteq E$) the maximum cardinal of all independent subsets of X. Then the mapping ρ from the subsets of E into the non-negative integers is called the *rank function* of $\mathscr{E}$. If $\mathscr{E}$ is the universal structure on E, then, of course, $\rho(X) = |X|$ for all $X \subseteq E$.

Next, let $\mathfrak{A} = (A_1, ..., A_n)$ be a family of subsets of E and let $\mathscr{E}$ be an independence structure on E. If $\mathfrak{A}$ possesses a transversal X which at the same time is an independent set (i.e. $X \in \mathscr{E}$), then X is called an *independent transversal* of $\mathfrak{A}$. We owe to R. Rado [11] the following criterion (which, for the case of a universal structure, reduces to Hall's theorem).

THEOREM 6 (R. Rado, 1942). *Let $\mathscr{E}$ be an independence structure, with rank function ρ, on a set* E. *Further, let $\mathfrak{A} = (A_1, ..., A_n)$ be a family of subsets of* E. *Then $\mathfrak{A}$ possesses an independent transversal if and only if, for each* $I \subseteq \{1, ..., n\}$,

$$\rho\left(\bigcup_{i \in I} A_i\right) \geqslant |I|.$$

The demonstration of this remarkable result is not difficult, as fortunately at least two of the proofs of Hall's theorem can be adapted to the more general situation considered here.

At this point it may be useful to distinguish between "first-order" and "second-order" transversal theory. The former is the discussion of the kind of problems I have already indicated; the latter is the study of similar problems subject to the additional requirement that the transversals whose existence we seek to establish are to be independent. On the face of it, Rado's theorem is only relevant in second-order theory; in reality, it enables us to clear up a number of questions left obscure in the first-order theory. This phenomenon is largely accounted for by the following theorem due to Edmonds and Fulkerson [1].

THEOREM 7 (Edmonds and Fulkerson, 1965). *Let $\mathfrak{A} = (A_1, ..., A_n)$ be a family of subsets of* E. *Then the class of all partial transversals of $\mathfrak{A}$ is an independence structure on* E.

Axiom (I1) is purely conventional and axiom (I2) holds trivially. The gist of the theorem lies in the validity of the replacement axiom (I3). To establish this conclusion, we shall associate a vector with each element of E, and then invoke a well-known result in the theory of vector spaces. Let, then,

$E = \{x_1, \ldots, x_m\}_{\neq}$ and consider the $m \times n$ matrix $\Gamma = (\gamma_{ij})$, where γ_{ij} is 0 if $x_i \notin A_j$ and γ_{ij} is an indeterminate if $x_i \in A_j$, it being understood that all the indeterminates (say $z_1, \ldots, z_r$) are independent over the field Q of rational numbers. The matrix Γ is then called the *formal incidence matrix* of $\mathfrak{A}$. Now it is almost immediate that a subset of E, say $\{x_{i_1}, \ldots, x_{i_k}\}_{\neq}$, is a PT of $\mathfrak{A}$ if and only if the rows of Γ with suffixes $i_1, \ldots, i_k$ are linearly independent vectors over the field $Q(z_1, \ldots, z_r)$ of rational functions (with rational coefficients) in the z's. But the rows of Γ satisfy, of course, the replacement axiom: the same is therefore true of PTs of $\mathfrak{A}$.

Now that Theorems 6 and 7 have been added to our armory, we can reconsider some of the earlier questions. Let us begin by looking at Theorem 4. Our previous treatment involved a certain amount of calculation and was, in any case, not very transparent. In the light of Theorem 7, however, Theorem 4 becomes at once obvious. For denote by $\mathscr{E}$ the independence structure on E consisting of all PTs of $\mathfrak{A}$. Then statement (ii′) simply means that $M \in \mathscr{E}$. Now every independent set can be extended to a maximal independent set and, by virtue of (i′), the maximal independent sets in $\mathscr{E}$ are precisely the transversals of $\mathfrak{A}$. Thus M can be extended to a transversal, and Theorem 4 is therefore proved. I think it will be agreed that this way of looking at the problem is more enlightening than recourse to an elementary construction.

Next, let us turn to Theorem 5. We shall denote by $\mathscr{E}$ the independence structure on E consisting of all PTs of $\mathfrak{B}$, and by ρ the rank function of $\mathscr{E}$. Then $\mathfrak{A}$ and $\mathfrak{B}$ possess a CT precisely if $\mathfrak{A}$ has an independent transversal and, by Rado's theorem (Theorem 6), this is the case if and only if

$$\rho\Big(\bigcup_{i \in I} A_i\Big) \geqslant |I| \qquad \text{(all I).} \tag{3}$$

Now the statement $\rho(X) \geqslant k$ means that X contains an independent subset of cardinal k, i.e., X contains a PT of $\mathfrak{B}$ of cardinal k. By Theorem 3, this is the case if and only if

$$\Big|\Big(\bigcup_{j \in J} B_j\Big) \cap X\Big| \geqslant |J| + k - n \qquad \text{(all J).}$$

It follows that (3) holds if and only if

$$\Big|\Big(\bigcup_{j \in J} B_j\Big) \cap \Big(\bigcup_{i \in I} A_i\Big)\Big| \geqslant |J| + |I| - n \qquad \text{(all I, J).}$$

The proof (which is taken from [7]) is therefore complete.

It is possible to adapt this argument so as to settle a harder question of the same kind. This line of reasoning is due to D. J. A. Welsh [13]. If $M \subseteq E$, when do the families $\mathfrak{A}$ and $\mathfrak{B}$ possess a CT which contains M as a subset? (Both the preceding questions deal with special cases of this problem). Let $\mathscr{E}$ be again the independence structure consisting of the class of all PTs of $\mathfrak{B}$. Denote by $\mathscr{E}^*$ the class of all subsets $X \subseteq E$ such that $X \cup M \in \mathscr{E}$. It can be shown that, if $M \in \mathscr{E}$, then $\mathscr{E}^*$ is again an independence structure on E and that the rank functions ρ, ρ^* of $\mathscr{E}$, $\mathscr{E}^*$ respectively are linked by the equation

$$\rho^*(F) = \rho(F \cup M) - |M| + |F \cap M| \qquad (F \subseteq E). \tag{4}$$

Now, as is easily seen, $\mathfrak{A}$ and $\mathfrak{B}$ possess a CT containing M if and only if $\mathfrak{A}$ possesses a transversal which is a member of $\mathscr{E}^*$. By Rado's theorem this is the case if and only if

$$\rho^*\Big(\bigcup_{i \in I} A_i\Big) \geqslant |I| \qquad (\text{all } I).$$

Using (4) and the method employed in the previous question, we can conclude without difficulty that the required necessary and sufficient condition is that, for all I, $J \subseteq \{1, \ldots, n\}$, the inequality

$$\Big|\Big(\bigcup_{i \in I} A_i\Big) \cap \Big(\bigcup_{j \in J} B_i\Big)\Big| + \Big|\Big\{\Big(\bigcup_{i \in I} A_i\Big) \cup \Big(\bigcup_{i \in J} B_j\Big)\Big\} \cap M\Big|$$

$$\geqslant |I| + |J| + |M| - n$$

should be valid. This theorem, too, can be established by means of a suitable elementary construction and an application of Hall's theorem (see [8]), but the details of the argument are undoubtedly heavy.

Rado's theorem has been known for just over a quarter of a century, but for the greater part of that time it has lain fallow. It is only in the last three or four years that its power and adaptability in first-order transversal theory have begun to be recognized, and there is every reason to think that this process is still in its early stages. More generally, my impression is that ideas derived from the investigation of independence structures will be drawn more and more intimately into the treatment of combinatorial problems.

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