The Near Frattini Subgroups of Infinite Groups

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1. INTRODUCTION

The object of this paper is to introduce a new canonical subgroup, the near Frattini subgroup, defined for all members of some relatively large classes of groups. As the name suggests, the near Frattini subgroup closely resembles the Frattini subgroup in many respects. However, in contrast to the situation for the Frattini subgroup, the various subgroups used to define the near Frattini subgroup need not always coincide. Consequently, we begin with a study of these defining subgroups in Section 3. This section also contains fundamental results on the relations between the near Frattini subgroup of a group and arbitrary subgroups of the group, homomorphisms of the group and decompositions of the group into direct products of subgroups. A corollary of these results is that the near Frattini subgroup of a finitely generated abelian group \( A \) coincides with the torsion subgroup of \( A \).

In Section 4, we study relationships between conditions on the near Frattini subgroup of a group and nilpotence of the group. In particular, the near Frattini subgroup of a finitely generated nilpotent group \( G \) contains both the derived subgroup of \( G \), and therefore the Frattini subgroup of \( G \), and the torsion subgroup of \( G \). In the latter case, the near Frattini subgroup need not coincide with the torsion subgroup, and we study the manner in which the factor group of the near Frattini subgroup by the torsion subgroup is embedded in the factor group of the whole group by the torsion subgroup. With regard to nilpotence of the near Frattini subgroup itself, it is shown that the near Frattini subgroup of every polycyclic group contains a nilpotent normal subgroup of finite index.

Section 5 exhibits some of the groups that can be near Frattini subgroups. In particular, we show that every soluble group with the minimal condition for subgroups that has a near Frattini subgroup in fact coincides with its near Frattini subgroup. On the other hand, this class of soluble groups provides examples of groups that have no near Frattini subgroups. The study of a
particular supersoluble group leads to the fact that every finitely generated abelian group occurs both as a near Frattini subgroup and as a Frattini subgroup of suitably chosen finitely generated nilpotent groups. In contrast, there are supersoluble groups that do not occur as near Frattini subgroups of the elements of a wide class of groups.

The main result of this paper occupies most of Section 6. We introduce the notions of near supplementation and of near complementation of a normal subgroup, as well as the notion of complete \( \mathbb{Z} \)-reducibility of a free abelian normal subgroup of finite \( \mathbb{Z} \)-rank of a group. Next, we obtain a criterion for near supplementation of a normal subgroup in terms of the near Frattini subgroup. Finally, we prove a theorem on near complementation and complete \( \mathbb{Z} \)-reducibility of abelian normal subgroups.

In Section 7, some possibilities for near Frattini subgroups of split extensions are given, and in Section 8, the relationships between three classes of groups that play important roles in properties of the near Frattini subgroup and of the Frattini subgroup are briefly considered.

2. Notation

\( X \subseteq Y \) means that \( X \) is a subset of \( Y \), while \( X \subset Y \) means that \( X \) is a proper subset of \( Y \). Similarly, \( H \leq G \) means that \( H \) is a subgroup of \( G \), while \( H < G \) means that \( H \) is a proper subgroup of \( G \). \( Y \setminus X \) denotes the set theoretic complement of \( X \subseteq Y \) in \( Y \). \( H_G \) denotes the core of \( H \leq G \) in \( G \); i.e., the intersection of all conjugates of \( H \) by elements of \( G \). \( |G : H| \) denotes the index of \( H \leq G \) in \( G \). \( \text{Aut } G \) and \( \text{Inn } G \) denote respectively the group of automorphisms and the group of inner automorphisms of the group \( G \). We say that \( G \) has Max (Min) if the group \( G \) satisfies the maximal (minimal) condition for subgroups. \( G_1 \times G_2 \) and \( \prod_{i \in I} G_i \) denote the respective direct products of the groups \( G_1 \) and \( G_2 \) and of the groups \( G_i \) for \( i \in I \). \( A \wr B \) denotes the wreath product of the groups \( A \) and \( B \). \( \Phi(G) \) denotes the Frattini subgroup of \( G \).

\[
\begin{align*}
C(n) & \quad \text{Cyclic group of order } n. \\
C(\infty) & \quad \text{Infinite cyclic group.} \\
\mathbb{Z}(p^n) & \quad \text{Group of type } p^n. \\
\mathbb{P} & \quad \text{Set of all prime numbers.} \\
\mathbb{N} & \quad \text{Set of all natural numbers.} \\
\mathbb{Z} & \quad \text{Set of all integers.} \\
\mathbb{Q} & \quad \text{Set of all rational numbers.}
\end{align*}
\]

\( (\Lambda : \mathbb{Z}) \) denotes the \( \mathbb{Z} \)-rank of a free \( \mathbb{Z} \)-module \( \Lambda \).
For soluble groups $G$ with $\text{Min}$, the letter $A$ shall reserved for the smallest subgroup of finite index in $G$.

An elementary abelian group is a direct product of elementary abelian $p$-groups, for various primes $p$.

3. **Fundamental Properties**

An element $x$ of a group $G$ is a **near generator** of $G$ if there is a subset $S$ of $G$ such that $|G : \langle S \rangle|$ is infinite, but $|G : \langle x, S \rangle|$ is finite. Thus, an element $x$ of $G$ is a non-near generator of $G$ precisely when $S \subseteq G$ and $|G : \langle x, S \rangle|$ finite always imply that $|G : \langle S \rangle|$ is finite. If $x$ and $y$ are non-near generators of $G$, then so are $xy^{-1}$ and $x^a$, for every automorphism $a$ of $G$. Consequently, the set $U(G)$ of non-near generators of a group $G$ is a characteristic subgroup of $G$.

A subgroup $M$ of a group $G$ is **nearly maximal** in $G$ if $|G : M|$ is infinite, but $|G : N|$ is finite, whenever $M < N \leq G$. That is, $M$ is maximal with respect to being of infinite index in $G$. The intersection $V(G)$ of all nearly maximal subgroups of a group $G$ is clearly a characteristic subgroup of $G$.

For every group $G$, $U(G) \subseteq V(G)$, since the elements of $G$ that do not belong to a nearly maximal subgroup of $G$ are near generators of $G$. However, $U(G)$ and $V(G)$ need not coincide. An example of a group $G$ in which $U(G)$ is a proper subgroup of $V(G)$ will be given in Section 5. If $G$ is such that $U(G)$ and $V(G)$ do coincide, then we call this subgroup the **near Frattini subgroup** of $G$, and denote it $\Phi(G)$.

If $F$ is any finite group, then clearly $U(F) = V(F) = F$. A considerably wider class of groups for which the subgroups $U$ and $V$ coincide is the class $\mathcal{C'}$. A group $G$ is an **$\mathcal{C'}$-group** if every subgroup of infinite index in $G$ is contained in a nearly maximal subgroup of $G$. If $G$ is an $\mathcal{C'}$-group, then every near generator of $G$ must lie outside some nearly maximal subgroup, so that $V(G)$ consists precisely of the non-near generators of $G$.

**Proposition 1.** *Every finitely generated group is an $\mathcal{C'}$-group.*

**Proof.** Let $G$ be a finitely generated group and $H \leq G$ be such that $|G : H|$ is infinite. By Zorn's lemma, there is a subgroup $M$ maximal with respect to containing $H$ and being of infinite index in $G$. $M$ is nearly maximal in $G$, since $M < N \leq G$ implies that $|G : N|$ must be finite.

We note that every homomorphic image of an $\mathcal{C'}$-group is again an $\mathcal{C'}$-group.

A set $S$ of elements of a group $G$ is an **irreducible set of near generators** of $G$ if $|G : \langle S \rangle|$ is finite, but $|G : \langle T \rangle|$ is infinite for every $T \subset S$. We denote by $W(G)$ the set of those elements of $G$, each of which belongs to no irreducible set of near generators of $G$. Clearly, $U(G) \subseteq W(G)$ for every group $G$. 


Lemma. If $G$ is finitely generated, then every irreducible set $S$ of near generators of $G$ is finite. If $T < G$ is such that $|G : T|$ is finite, then every set of generators of $T$ contains an irreducible set of near generators of $G$.

Proof. $\langle S \rangle$ is finitely generated, say by $x_1, \ldots, x_n$. Each $x_i$ can be written as a product of a finite number of elements of $S \cup S^{-1}$, hence $S^2$ is generated by a finite subset of $S$, which cannot be a proper subset.

Every set of generators of $T$ contains a finite subset that also generates $T$. Let $t(1), \ldots, t(m)$ generate $T$. We form a subset $T'$ of $\{t(1), \ldots, t(m)\}$ as follows: consider each $t(i)$ in turn, and retain or omit it, according to whether or not not the subgroup generated by the already retained $t(j)$ and the $t(j)$ that follow $t(i)$ is of infinite index in $G$. Thus, if $|G : t(2), \ldots, t(m)|$ is infinite, we retain $t(1)$; otherwise, we omit $t(1)$. If $t(i_1), \ldots, t(i_k)$ have been retained and if

$$|G : \langle t(i_1), \ldots, t(i_k), t(i_k + 2), \ldots, t(m) \rangle|$$

is infinite, then we retain $t(i_k + 1)$; otherwise, we omit $t(i_k + 1)$. The subset $T'$ of $\{t(1), \ldots, t(m)\}$ obtained is an irreducible set of near generators of $G$.

Proposition 2. If $G$ is a group with Max, then $\psi(G) = W(G)$.

Proof. $\psi(G) \subseteq W(G)$, since $G$ is an $\mathfrak{E}$-group. If $x \in G \setminus \psi(G)$, then there is a subset $S$ of $G$ such that $|G : \langle S \rangle|$ is infinite, but $|G : \langle x, S \rangle|$ is finite. If $s_1, \ldots, s_n$ generate $\langle S \rangle$, then $|G : \langle x, s_1, \ldots, s_n \rangle|$ is finite. When the construction in the proof of the preceding Proposition is applied to $\{x, s_1, \ldots, s_n\}$, the subset $S'$ obtained contains $x$. Thus, $x \in G$ if $\psi(G)$.

A subset $S$ of a group $G$ is nearly eliminable if whenever $T' \subseteq G$ and $|G : S, T'|$ is finite, then $|G : \langle T' \rangle|$ is finite. Every subset of a nearly eliminable subset is itself nearly eliminable, and every nearly eliminable subset of a group $G$ is contained in $U(G) \cup \psi(G)$, when it exists, need not be nearly eliminable. For, a group $A$ of type $\mathfrak{p}^{\omega}$ has no near generators, hence $U(A) = \psi(A) = A$. $\psi(A)$ is clearly not nearly eliminable. If $\psi(G)$ is finitely generated, or if $G$ is an $\mathfrak{E}$-group, then $\psi(G)$ is nearly eliminable. When a group $G$ has a nearly eliminable near Frattini subgroup, it is clear that a normal subgroup $N$ of $G$ is contained in $\psi(G)$ if, and only if, $H < G$ and $|G : NH|$ finite always imply that $|G : H|$ is finite.

3.1 Subgroups

If $H \leq G$ and both $H$ and $G$ have near Frattini subgroups, it does not in general follow that $\psi(H) \leq \psi(G)$. For, the nearly maximal subgroups of the infinite dihedral group $D_\infty = \langle a, b \mid b^2 = 1, bab = a^{-1} \rangle$ are the subgroups
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of $D_{\infty}$ of order 2. Consequently, $\psi(D_{\infty}) = 1$, but $\psi(H) = H$ for every subgroup $H$ of $D_{\infty}$ of order 2.

**XYZ Lemma.** Let $Y$ and $Z \leq Y$ be $C'$-groups, and let $X \triangleleft Y$ be such that $X \leq \psi(Z)$. Then $X \leq \psi(Y)$.

**Proof.** $\psi(Y)$ is nearly eliminable, hence we must verify that if $H \leq Y$ is such that $|Y : XH| = \infty$, then $|Y : H| = \infty$ finite. Since $XH \leq \langle Z, XH \rangle \leq Y$, $|\langle Z, XH \rangle : XH|$ is finite. Hence, $|Z : XH \cap Z| = |Z : X(H \cap Z)|$ is finite. Since $X \leq \psi(Z)$, $|Z : H \cap Z| = \infty$. *A fortiori*, $|X(H \cap Z) : H \cap Z| = \infty$. Now $X(H \cap Z) \cdot H = XH$ and

$$(XH \cap Z) \cap H = H \cap Z,$$

hence $|XH : H|$ must be finite.

**Corollary 1.** If $G$ and $N \triangleleft G$ are $C'$-groups, then $\psi(N) \leq \psi(G)$.

**Corollary 2.** If $G$ is an $C'$-group and $F$ is a finite normal subgroup of $G$, then $F \leq \psi(G)$.

### 3.2 Homomorphisms

If $G$ is a group with near Frattini subgroup $\psi(G)$, if $\gamma$ is a homomorphism of $G$ and if $G^\gamma$ has near Frattini subgroup $\psi(G^\gamma)$, then $\psi(G)^\gamma \leq \psi(G^\gamma)$. For, every nearly maximal subgroup of $G^\gamma$ is the image under $\gamma$ of a nearly maximal subgroup of $G$. If $G$ is infinite cyclic and $\gamma \neq 0, 1$, then $\psi(G) = 1$, but $\psi(G^\gamma) = G^\gamma > 1$. Consequently, the inclusion $\psi(G)^\gamma \leq \psi(G^\gamma)$ can be proper.

Two useful properties of the near Frattini subgroup are

(i) $N \triangleleft G$, $N \leq \psi(G)$ and existence of $\psi(G/N)$ imply that $\psi(G/N) = \psi(G)/N$

and

(ii) $N \triangleleft G$, existence of $\psi(G)$ and $\psi(G/N) = 1$ imply that $\psi(G) \leq N$.

### 3.3 Direct Products

**Proposition 3.** If $G$ is an $C'$-group and $G = \text{d.p.}_{i \in I} G_i$, then $\psi(G) = \text{d.p.}_{i \in I} \psi(G_i)$.

**Proof.** For each $i \in I$, $G_i$ is an $C'$-group, hence Corollary 1 of the XYZ Lemma implies that $\psi(G_i) \leq \psi(G)$. Consequently, $\text{d.p.}_{i \in I} \psi(G_i) \leq \psi(G)$. Conversely, if $M_j$ is nearly maximal in $G_j$, then $\text{d.p.}_{i \in I} H_i$ is nearly maximal.
in $G$, where $H_i = G_i$ for $i \neq j$ and $H_j = M_j$. If, for fixed $j \in I$, we intersect all such subgroups $d.p._{i \neq j} H_i$ obtained, as $M_j$ varies over the nearly maximal subgroups of $G_j$, then we conclude that $K_j^* = d.p._{i \neq j} K_i \geq \psi(G)$, where $K_i = G_i$ for $i \neq j$ and $K_j = \psi(G)$. Consequently $\bigcap_{i \neq j} K_i^* = d.p._{i \neq j} \psi(G) \geq \psi(G).

**Corollary.** If

$$A = C(p_1^{l_1}) \times \cdots \times C(p_t^{l_t}) \times C(\infty) \times \cdots \times C(\infty)$$

is a finitely generated abelian group, where $C(\infty) \cong C(\infty)$ for $1 \leq i \leq t$, then $\psi(A) = C(p_1^{l_1}) \times \cdots \times C(p_t^{l_t}) = \text{Tor } A$, the torsion subgroup of $A$.

4. **Nilpotence**

**Proposition 1.** $V(G) \cong G'$ if, and only if, every nearly maximal subgroup of $G$ is normal in $G$.

**Proof.** If $M$ is nearly maximal and normal in $G$, then $G/M$ is an infinite group, every nontrivial subgroup of which is of finite index. Such a group is cyclic [2].

**Proposition 2.** If $G$ is a nilpotent $\mathcal{E}'$-group, then $\psi(G) \cong G'$.

This Proposition follows easily (by induction on the class of $G$) from the

**Lemma.** If $G$ is a nilpotent $\mathcal{E}'$-group of class $c$ with lower central series $G = G_1 \geq G_2 \geq \cdots \geq G_c \geq G_{c+1} = 1$, then $\psi(G) \cong G_c$.

**Proof.** $G_c$ is generated by all simple commutators $z = [x_1, \ldots, x_c]$ of weight $c$, where $x_1, \ldots, x_c$ lie in $G$. We show that each $z$ lies in $\psi(G)$. $G_c \leq Z(G)$, hence $\langle z \rangle \triangleleft G$. Let $H \triangleleft G$ be such that $|G : \langle z \rangle H|$ is finite. If $\langle z \rangle \cap H > 1$, then $|\langle z \rangle : \langle z \rangle \cap H| = |\langle z \rangle H : H|$ is finite. Hence, $|G : H|$ is finite and $\langle z \rangle \leq \psi(G)$. Thus, we may assume that $\langle z \rangle \cap H = 1$. Now $|G : \langle z \rangle H|$ is finite, so that for each $i = 1, 2, \ldots, c$, there is a natural number $m_i$ such that $x_i^{m_i} \in \langle z \rangle H$. We choose a transversal to $\langle z \rangle$ in $\langle z \rangle H$ consisting of elements of $H$, and have $x_i^{m_i} = z h_i$ for each $i = 1, 2, \ldots, c$, where $z_i \in \langle z \rangle$, $h_i \in H$. Then

$$[h_1, \ldots, h_c] = [x_1^{m_1}, \ldots, x_c^{m_c}] = [x_1, \ldots, x_c]^{m_1 \cdots m_c} = z^{m_1 \cdots m_c} \in \langle z \rangle \cap H.$$

Hence, $z$ must be of finite order, i.e., $|\langle z \rangle : \langle z \rangle \cap H|$ is finite. As before, $|G : H|$ is finite.
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COROLLARY. If $G$ is a finitely generated nilpotent group, then $\psi(G) \supseteq \varphi(G)$, $G/\psi(G)$ is free abelian of finite $\mathbb{Z}$-rank and $\psi(G)/\varphi(G)$ is an elementary abelian group.

The infinite dihedral group $D_\infty$ is a supersoluble group $G$ for which $\psi(G) \supseteq G'$.

PROPOSITION 3. If $G$ is a finitely generated nilpotent group, then $\psi(G) \supseteq \text{Tor } G$.

This Proposition follows from the fact [3] that Tor $G$ is a finite fully invariant subgroup of $G$, and from Corollary 2 of the XYZ Lemma.

Propositions 2 and 3 imply that if $F$ is a free nilpotent group of finite rank and of class larger than 1, then $\psi(F) > \text{Tor } F$. This inequality is also true of the nilpotent group

$$E = \langle a, b, c : b^{-1}ab = a, c^{-1}ac = ab, c^{-1}bc = b \rangle$$

discussed in the next section. That the torsion subgroup is trivial in these examples is of no particular significance. For, if $G$ is the direct product of any one of these examples with a nontrivial finite nilpotent group, then $G$ is a finitely generated nilpotent group such that $\psi(G) > \text{Tor } G > 1$.

If $G$ is a finitely generated nilpotent group for which $\psi(G) > \text{Tor } G$, then $\psi(G)/\text{Tor } G$ must have an invariant series with infinite cyclic factors [4]. To describe how $\psi(G)/\text{Tor } G$ lies in $G/\text{Tor } G$, we recall the definition of root groups [5]. A group $X$ is a root group if $x, y \in X$, $n \in \mathbb{N}$ and $x^n = y^n$ always imply that $x = y$. A subgroup $A$ of a root group $X$ is pure in $X$ if whenever $x^n = a$ is soluble in $X$, for $a \in A, n \in \mathbb{N}$, the solution lies in $A$. Mal'cev [6] has shown that every torsion-free nilpotent group is a root group. It follows that $\psi(G)/\text{Tor } G$ is a pure subgroup of $G/\text{Tor } G$.

PROPOSITION 4. The near Frattini subgroup $\psi(G)$ of a polycyclic group $G$ contains a nilpotent normal subgroup of finite index.

Proof. $G$ is nilpotent-by-free abelian-by-finite [7]: let $G \triangleright B \triangleright N \triangleright 1$ be such that $N$ is nilpotent, $B/N$ is free abelian and $G/B$ is finite. If $B/N \cong C_1(\infty) \times \cdots \times C_t(\infty)$, then for each $i = 1, 2, \ldots, t$,

$$M_i/N \cong C_1(\infty) \times \cdots \times C_{i-1}(\infty) \times C_{i+1}(\infty) \times \cdots \times C_t(\infty)$$

is nearly maximal in $B/N$. Consequently, $\psi(B) \leq N$, so that $\psi(B)$ is nilpotent. By Corollary 1 of the XYZ Lemma, $\psi(B) \leq \psi(G)$. If $x \in \psi(G)$, then $x^n \in B$ for some $n \in \mathbb{N}$. In fact, $x^n \in \psi(B)$, for if $S \subseteq B$ is such that $|B : \langle x^n, S \rangle|$ is finite, then $|G : \langle x^n, S \rangle|$ is also finite. It follows that $|G : \langle S \rangle|$ is finite; a fortiori, $|B : \langle S \rangle|$ is finite. Thus, $\psi(G)/\psi(B)$ is a periodic polycyclic group, and is therefore finite.
5. Groups that Occur as Near Frattini Subgroups

Every finite group occurs as its own near Frattini subgroup, thus we wish to consider the occurrence of infinite groups as near Frattini subgroups. We begin by noting that if $A$ is a divisible abelian group, then $\phi(A) = A$. For, $A$ has no proper subgroups of finite index, and $A\phi(A) = A$. Hence, $\cap\phi(A) = A$. Not every infinitely generated abelian group coincides with its near Frattini subgroup. In fact, if $\mathbb{Q}^*$ is the multiplicative group of positive rational numbers, then $\phi(\mathbb{Q}^*) = 1$. For, $\mathbb{Q}^*$ is isomorphic to a free abelian group $B = \mathbb{Z}_a(\infty) \times \mathbb{Z}_a(\infty) \times \mathbb{Z}_a(\infty) \times \cdots$ of countable Z-rank, and for each prime $p$, $M_p = \mathbb{Z}_{p^\infty} \times \mathbb{Z}_{p^\infty} \times \cdots$ is nearly maximal in $B$. Consequently, $V(B) \subseteq \cap_p M_p = 1$.

**Proposition 1.** If $G$ is a soluble group with Min, then $U(G)$ contains the smallest subgroup $A$ of finite index in $G$.

**Proof.** If $x \in A$, then $x$ has only finitely many conjugates in $G$, say $x, x^g_2, \ldots, x^g_k$.

The normal closure $\langle x \rangle_G$ of $\langle x \rangle$ in $G$ is

$$\langle x \rangle_G = \langle x^i(x^g_j)^l \cdots (x^g_k)^h \mid 1 \leq i \leq |x|, 1 \leq j \leq k, g_1 = 1 \rangle.$$

That is, $\langle x \rangle_G$ is finite.

Let $S \subseteq G$ be such that $|G : \langle x, S \rangle|$ is finite. $|\langle x \rangle_G : \langle S \rangle : \langle x \rangle_G|$ is finite, so that $\langle x \rangle_G \cap S$ is also finite. Since $\langle x, S \rangle \leq \langle x \rangle_G \langle S \rangle$, $|G : \langle x \rangle_G \langle S \rangle|$ must be finite, so that $x \in U(G)$.

**Corollary.** If a soluble group $G$ with Min has a near Frattini subgroup, then $\psi(G) = G$.

**Proposition 2.** If $G$ is a soluble group with Min such that $A$ is a group of type $p^\infty$, then $\psi(G) = G$.

**Proof.** Suppose that $x \in G$ and $S \subseteq G$ are such that $|G : \langle x, S \rangle|$ is finite, but $|G : \langle S \rangle|$ is infinite. Then $|\langle A \rangle : \langle S \rangle : \langle A \rangle : \langle S \rangle|$ is finite, so that $\langle A \rangle \cap \langle S \rangle$ must be finite. Since $|\langle A \rangle : A \langle S \rangle : A\langle S \rangle : A|$ is finite, it follows that $\langle S \rangle$ is finite. A fortiori, $S$ is finite, so that $\langle x, S \rangle$ is finitely generated. As $|\langle x, S \rangle : A|$ is finite, $A$ must be finitely generated, a contradiction. Hence, $U(G) = G$.

We give next an example of a group $G$ for which $U(G) \subsetneq V(G)$.

**Example 1.** Let $p$ and $q$ be primes, not necessarily distinct, and let
Let \( G = \mathbb{Z}(p^{\infty}) \triangleleft C(q) \) be the wreath product of a group of type \( p^{\infty} \) by a cyclic group of order \( q \). \( G \) is soluble with Min. If \( C(q) = \langle c \rangle \), then the base group \( A \) of \( G \) may be expressed as
\[
A = \mathbb{Z}(p^{\infty})_1 \times \mathbb{Z}(p^{\infty})_2 \times \cdots \times \mathbb{Z}(p^{\infty})_{q-1}.
\]
By Proposition 1, \( U(G) \supseteq A \). On the other hand, if \( 0 < i < q \), then \( c^i \not\in U(G) \). For \( \langle \mathbb{Z}(p^{\infty})_1, c^i \rangle = G \), but \( |G : \mathbb{Z}(p^{\infty})_1| \) is infinite. Hence, \( U(G) = A \).

If we restrict our attention to the class \( E' \), then \( G \in E' \) implies that \( \psi(G) \) is a proper subgroup of \( G \), unless \( G \) is finite. We consider next a supersoluble group that will enable us to show that every finitely generated abelian group occurs both as a Frattini subgroup and as a near Frattini subgroup of suitably chosen finitely generated nilpotent groups.

**Example 2.** Let \( A \) be a free abelian group of \( \mathbb{Z} \)-rank 2 with \( \mathbb{Z} \)-basis \( \{a, b\} \), and let \( f \) and \( g \) be automorphisms of \( A \), having the respective matrices
\[
\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}
\]
with respect to \( \{a, b\} \). We set \( H = \langle f, g \rangle \leq \text{Aut } G \), and define \( D^* \) to be the split extension of \( A \) by \( H \). Both \( f \) and \( g \) are of order 2, \( fg \) is of infinite order and
\[
H = \langle fg, g \mid g^2 = 1, g(fg)g^{-1} \rangle
\]
is isomorphic with the infinite dihedral group, \( D_\infty \). Setting \( fg = c \) and \( g = d \), \( D^* \) has the finite presentation \( \langle a, b, c, d \mid d^2 = 1, b^{-1}ab = a, c^{-1}ac = ab, dad = ab^2, c^{-1}bc = b, dbd = b^{-1}, dcd = c^{-1} \rangle \).

In addition to \( A \), \( D^* \) has \( \langle b, c \rangle \) as a maximal free abelian normal subgroup, and \( D^*/\langle b, c \rangle \cong C(\infty) \times C(2) \).

\([D^*, D^*] = \langle b, c^2 \rangle \). For, \( D^*/\langle b, c^2 \rangle \cong C(\infty) \times C(2) \times C(2) \) is abelian. On the other hand, \( [a, c] = b \) and \( [c^{-1}, d] = c^2 \).

\( D^* \) is supersoluble, since
\[
1 \triangleleft \langle b \rangle \triangleleft \langle a, b \rangle \triangleleft \langle a, b, c \rangle \triangleleft D^*
\]
is an invariant series of \( D^* \) with cyclic factors. \( D^* \) is not nilpotent, since \( D^*/A \cong D_\infty \) is not nilpotent.

The last isomorphism implies that \( \varphi(D^*) \leq A \) and that \( \psi(D^*) \leq A \). For every prime \( p \), \( \langle a^p, b, c, d \rangle \) is maximal in \( D^* \). Hence, \( \varphi(D^*) \leq \bigcap \langle a^p, b, c, d \rangle \). Consequently, \( \varphi(D^*) \leq \langle b \rangle \). Also, \( \langle b, c, d \rangle \) is nearly maximal in \( D^* \); consequently \( \psi(D^*) \leq \langle b \rangle \).

We consider now the normal subgroup \( E = \langle a, b, c \rangle \) of \( D^* \), and show that \( E' : Z(E) = \varphi(E) = \psi(E) = \langle b \rangle \). For, \( E/\langle b \rangle \) is free abelian of \( \mathbb{Z} \)-rank 2,
and we have already noted that \([a, c] = b\). Hence, \(E' = \langle b \rangle\), and the lower central series of \(E\) is \(E \supset \langle b \rangle \supset 1\). \(\langle a \rangle \cap Z(E) = \langle c \rangle \cap Z(E) = 1\), and no element of the form \(a^r b^t c^t\), with both \(r\) and \(t\) nonzero, lies in \(Z(E)\). Hence \(Z(E) = \langle b \rangle\). Thus, \(E\) is finitely generated and nilpotent of class 2, so that \(\varphi(E) \supset \varphi(E) \supset E'\). But \(E/\langle a, b \rangle \cong E/\langle b, c \rangle \cong C(\infty)\), so that

\[
\varphi(E) = \langle a, b \rangle \cap \langle b, c \rangle = \langle b \rangle.
\]

Similarly, \(\psi(E) \leq \langle b \rangle\).

Since \(E \leq D^t\), \(\varphi(E) \leq \varphi(D^t)\) and \(\psi(E) \leq \psi(D^t)\). Consequently, \(\varphi(D^t) = \psi(D^t) = \langle b \rangle\).

**Proposition 3.** If \(A\) is a finitely generated abelian group, then there are finitely generated groups \(G_1\) and \(G_2\), each nilpotent of class 2, and such that \(\varphi(G_1) = \psi(G_2) = A\).

*Proof.* If \(A = C(p_1^{(i)}) \times \cdots \times C(p_t^{(i)}) \times C_i(\infty) \times \cdots \times C_i(\infty)\), where \(C_i(\infty) \simeq C(\infty)\) for \(1 \leq i \leq t\), we need only take

\[
G_1 = C(p_1^{(i+1)}) \times \cdots \times C(p_t^{(i+1)}) \times E_1 \times \cdots \times E_t
\]

and

\[
G_2 = C(p_1^{(i)}) \times \cdots \times C(p_t^{(i)}) \times E_1 \times \cdots \times E_t,
\]

where \(E_i \simeq E \leq D^t\) for \(1 \leq i \leq t\).

The example \(G_1\) is in a sense complementary to the work of Dlab [1], who has shown that every abelian group \(A\) is the Frattini subgroup of a suitably chosen abelian group \(B\). For, if \(A\) is finitely generated but not finite, then the group \(B\) constructed by Dlab is not finitely generated.

Not all supersoluble groups occur among the near Frattini subgroups of \(C\)-groups. This is a consequence of the next Proposition.

**Proposition 4.** Let \(G\) be an \(C\)-group, and let \(H\) be a normal subgroup of \(G\) such that \(H \leq \psi(G)\) and such that \(\text{Aut } H\) is an \(C\)-group. Then \(\text{Inn } H \leq \psi(\text{Aut } H)\).

*Proof.* Let \(\sigma : G \rightarrow \text{Aut } H\) be defined by \(h^{\sigma(g)} = g^{-1}hg\). Then

\[
H^u = \text{Inn } H \leq \psi(G)^u \leq \psi(G') \leq \psi(G') \leq \text{Aut } H.
\]

Setting \(Y = \text{Aut } H\), \(Z = G^u\) and \(X = \text{Inn } H\) in the XYZ Lemma, we conclude that \(\text{Inn } H \leq \psi(\text{Aut } H)\).

**Corollary.** The infinite dihedral group \(D_\infty\) is not the near Frattini subgroup of any \(C\)-group.
Proof. Aut $D_\infty \cong \text{Inn } D_\infty \cong D_\infty$ and $\psi(D_\infty) = 1$. If there were an $\mathfrak{E}$-group $G$ with $\psi(G) = D_\infty$, then, setting $H = \psi(G)$ in Proposition 4, it would follow that $\text{Inn } D_\infty \subseteq \psi(\text{Aut } D_\infty)$, a contradiction.

6. Near Supplementation and Near Complementation

A normal subgroup $N$ of a group $G$ is nearly supplemented in $G$ (or $G$ nearly reduces over $N$) if there is a subgroup $H$ of $G$ with $|G : H|$ infinite and $|G : NH|$ finite. A normal subgroup $N$ of a group $G$ is nearb complemented in $G$ (or $G$ nearly splits over $N$) if there is a subgroup $H$ of $G$ nearly supplementing $N$ in $G$ and such that $N \cap H \neq 1$.

With this definition of near supplementation, the last remark in the discussion of near eliminability may be rephrased as the next Proposition.

PROPOSITION. Let $G$ be such that $\psi(G)$ is nearly eliminable. A normal subgroup $N$ of $G$ is nearly supplemented in $G$ if, and only if, $N \leq \psi(G)$.

A group $G$ having near Frattini subgroup $\psi(G)$ always nearly reduces over a normal subgroup $N$ satisfying $N \leq \psi(G)$. The converse, however, requires some hypothesis on $G$. For, if $G$ is an infinite soluble group with $\text{Min}$ having near Frattini subgroup $\psi(G)$, then $A$ is a normal subgroup of $G$ which is contained in $\psi(G)$ and over which $G$ nearly reduces.

LEMMA. If $G$ is an $\mathfrak{E}$-group, and if $A$ is a finitely generated abelian normal subgroup of $G$ such that $A \cap \psi(G)$ is finite, then $A \cap \psi(G) = \psi(A)$.

Proof. By Corollary 2 of the XYZ Lemma, with $F = \psi(A)$, it follows that $\psi(A) \leq A \cap \psi(G)$.

When the hypotheses of the Lemma are satisfied and $A \cap \psi(G) = 1$, then $A$ is a free abelian group of finite $\mathbb{Z}$-rank. That is, $A$ is a free $\mathbb{Z}$-module of finite $\mathbb{Z}$-rank, and is also a $\mathbb{Z}G$-module. We may ask whether such a module can be in some sense completely reducible. Evidently, $A$ need not be completely reducible as a $\mathbb{Z}G$-module. As an example, we need only take $G = A = C(\infty)$, or $G = D_\infty = \langle a, b \mid b^2 = 1, bab = a^{-1}\rangle$, and $A = \langle a \rangle$.

Let $G$ be a group with a free abelian normal subgroup $A$ of finite $\mathbb{Z}$-rank. The following three notions of complete reducibility of $A$ are equivalent.

1. We form $Q_A = Q \otimes \mathbb{Z} A$, which is a vector space over $Q$ of dimension $(A : \mathbb{Z})$. $Q_A$ is also a $QG$-module, and as such, we require that $Q_A$ be completely reducible.

2. We say that $A$ is completely $\mathbb{Z}$-reducible as a $\mathbb{Z}G$-module if for every $\mathbb{Z}G$-submodule $B$ of $A$, there is a $\mathbb{Z}G$-submodule $C$ of $A$ such that $B \cap C = 0$ and $(B \oplus C : \mathbb{Z}) = (A : \mathbb{Z})$. 
3. For every normal subgroup \( B \) of \( G \), such that \( 1 < B \leq A \), we require a near complement \( C \) in \( A \) that is normal in \( G \). That is a subgroup \( C \) of \( A \) with \( C \leq G \), \( |A : C| \) infinite, \( |A : BC| \) finite and \( B \cap C \leq B \cap C = 1 \).

**Remark.** In 2, there is no loss of generality if \( B \) is assumed to be \( \mathbb{Z} \)-pure in \( A \).

**Theorem.** If \( G \) is a group with Max and \( A \) is a nontrivial abelian normal subgroup of \( G \) such that \( A \cap \psi(G) = 1 \), then \( A \) is nearly complemented in \( G \). If, in addition, \( G \) is nilpotent, then \( A \) is completely \( \mathbb{Z} \)-reducible as a \( \mathbb{Z}G \)-module.

**Proof.** By the preceding Lemma, \( A \) is a free abelian group of finite \( \mathbb{Z} \)-rank. If \( A \) is of finite index in \( G \), then \( 1 \) is a near complement for \( A \) in \( G \). Hence, let \( A \) be of infinite index in \( G \), so that there are nearly maximal subgroups of \( G \) that contain \( A \) and nearly maximal subgroups of \( G \) that do not contain \( A \). We proceed by induction on the \( \mathbb{Z} \)-rank of \( A \). If \( (A : \mathbb{Z}) = 1 \) and \( H_1 \) is a nearly maximal subgroup of \( G \) such that \( A \cap H_1 = 1 \) and \( A \cap H_1 < \psi(G) \). For, by the XYZ Lemma, \( A \cap H_1 \leq \psi(G) \). Since \( A \cap H_1 \leq A \), \( A \cap H_1 \) must be 1. Failing the existence of a nearly maximal subgroup \( H_1 \) of \( G \) such that \( A \cap H_1 = 1 \), suppose that there is a nearly maximal subgroup \( H_1 \) of \( G \) such that \( A \cap H_1 = 1 \) and \( \psi(H_1) = 1 \).

(*) If there is a nearly maximal subgroup \( H_1 \) of \( G \) such that \( H_1 \nmid A \) and \( A \cap H_1 \), then \( H_1 \) is a near complement for \( A \) in \( G \). This will be the case, for example, if there is a nearly maximal subgroup \( H_1 \) of \( G \) such that \( H_1 \nmid A \) and \( A \cap H_1 \leq \psi(H_1) \). For, by the XYZ Lemma, \( A \cap H_1 \leq \psi(G) \). Since \( A \cap H_1 \leq A \), \( A \cap H_1 \) must be 1. Failing the existence of a nearly maximal subgroup \( H_1 \) of \( G \) such that \( A \cap H_1 = 1 \), suppose that there is a nearly maximal subgroup \( H_1 \) of \( G \) such that \( H_1 \nmid A \) and \( \psi(H_1) = 1 \).

Then \( A \cap H_1 \) is a nontrivial abelian normal subgroup of \( H_1 \) which is of infinite index in \( H_1 \) (since \( |AH_1 : A| \) is infinite), and \( (A \cap H_1 : \mathbb{Z}) < (A : \mathbb{Z}) \) (since \( |AH_1 : H_1| \) is infinite). Hence, by induction, there is a subgroup \( C \) of \( H_1 \) such that \( |H_1 : C| \) is infinite, \( |H_1 : (A \cap H_1) C| \) is finite and \( (A \cap H_1) \cap C \cong 1 \). Consequently, \( |G : C| \) is infinite, \( |H_1 : AC \cap H_1| = |AH_1 : AC| \) is finite, so that \( G : AC \) is finite, and \( A \cap C_G \) is 1. That is, \( C \) is a near complement for \( A \) in \( G \). (*)

There remains the case in which every nearly maximal subgroup \( H_1 \) of \( G \) that does not contain \( A \) is such that \( A \cap H_1 \), and \( A \cap \psi(H_1) \). In this situation, for each subgroup \( H_1 \), there exist nearly maximal subgroups \( H_2 \) of \( H_1 \) that do not contain \( A \cap H_1 \) (otherwise \( A \cap H_1 \leq \psi(H_1) \), a fortiori \( A \cap H_1 \leq \psi(H_1) \)). For each such \( H_2 \), \( (A \cap H_1) \cap H_2 \). Choose such
a subgroup $H_1$, and proceed with $H_1$, $A \cap H_1$ and the nearly maximal subgroups $H_2$ of $H_1$ such that $H_2 \supseteq A \cap H_1$ as in (*) with $G$, $A$ and the nearly maximal subgroups $H_1$ of $G$ such that $H_1 \supseteq A$ (Figure 1). Thus, if there is a nearly maximal subgroup $H_2$ of $H_1$ such that $H_2 \supseteq A \cap H_1$ and $A \cap H_2 \cap H_1 = 1$, then $H_2$ is a near complement for $A$ in $G$. For,

$$|H_1 : (A \cap H_1) H_2| = |H_1 : AH_2 \cap H_1| = |AH_1 : AH_2|$$

![Fig. 1](image-url)

is finite, hence $|G : AH_2|$ is finite. Otherwise, $H_2$ nearly maximal in $H_1$ and $H_2 \supseteq A \cap H_1$ imply that $A \cap H_2 \cap H_1 = 1$. If, now, there is a nearly maximal subgroup $H_2$ of $H_1$ such that $H_2 \supseteq A \cap H_1$ and $(A \cap H_2) \cap \phi(H_2) = 1$, then we note that $A \cap H_2$ is a nontrivial abelian normal subgroup of $H_2$, that $A \cap H_2$ is of infinite index in $H_2$ (for $|H_2 : A \cap H_2| = |(A \cap H_2) H_2 : A \cap H_2|$), and, since $|H_1 : A \cap H_1| = 1$ (since $|H_1 : (A \cap H_1) H_2| = |(A \cap H_1) H_2 : H_2| = 1$), it follows that $H_2 : D$ is infinite, $|H_2 : (A \cap H_2) D| = 1$. Consequently, $|G : D|$ is finite, $|H_2 : AD \cap H_2| = |AH_2 : AD|$ is finite and

$$|AH_1 : AH_2| = |H_1 : AH_2 \cap H_1| = |H_1 : (A \cap H_1) H_2|$$

is finite, so that $|G : AD|$ is finite, and $A \cap D_G = 1$. That is, $D$ is a near complement for $A$ in $G$.

We have left the case in which every nearly maximal subgroup $H_2$ of $H_1$ that does not contain $A \cap H_1$ is such that $A \cap H_2 > 1$ and $A \cap \phi(H_2) > 1$.  

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In this situation, there exist nearly maximal subgroups $H_3$ of $H_2$ that do not contain $A \cap H_2$, and each such $H_3$ must have $(A \cap H_2) \cap H_3 = A \cap H_3 > 1$. Choose such a subgroup $H_2$ and proceed with $H_2$, $A \cap H_2$ and the nearly maximal subgroups $H_3$ of $H_2$ such that $H_3 \nmid A \cap H_2$ as in (*) \cdots (*)

We continue this process until we obtain a chain

$$H_k < H_{k-1} < \cdots < H_0 = G$$

with $H_i$ nearly maximal in $H_{i-1}$ ($i = 1, \ldots, k$), with $k \leq (A : \mathbb{Z}) - 1$ and with either $A \cap H_{k,G} = 1$, so that $H_k$ is a near complement for $A$ in $G$, or $(A \cap H_k) \cap H_{k,G} = 1$, so that by induction there is a near complement $E$ for $A \cap H_k$ in $H_{k,G}$, which is in fact a near complement for $A$ in $G$. Otherwise, setting $(A : \mathbb{Z}) \rightarrow 1 \rightarrow r$, we obtain $H_r$ with nearly maximal subgroups $H$ that do not contain $A \cap H_r$ and are such that $A \cap H_r > 1$. For, if the process has failed to yield a near complement for $A$ in $G$ by step $r - 1$, then we choose $H_{r-1}$ nearly maximal in $H_{r-2}$ and such that $H_{r-1} \nmid A \cap H_{r-2}$, and proceed with $H_{r-1}, A \cap H_{r-1}$ and the nearly maximal subgroups $H_i$ of $H_{r-1}$ such that $H_{r-1} \nmid A \cap H_{r-1}$. Either there is a subgroup $H_{r-1}$ nearly maximal in $H_{r-2}$ such that $H_{r-1} \nmid A \cap H_{r-1}$ and $A \cap H_{r-1} = 1$, or, failing this, such that $H_{r-1} \nmid A \cap H_{r-1}$ and $A \cap \phi(H_{r-1}) = 1$, or $H_r$ nearly maximal in $H_{r-1}$ and $H_r \nmid A \cap H_{r-1}$ imply that $A \cap H_{r,G} = 1$ and $A \cap \phi(H_{r,G}) = 1$. In the latter case, there are subgroups $H$ nearly maximal in $H_r$ such that $H \nmid A \cap H_r$ and $A \cap H_r > 1$.

We note that $(A \cap H_i : \mathbb{Z}) \leq (A : \mathbb{Z}) - i$, so that $(A \cap H_r : \mathbb{Z}) \leq 1$. If $(A \cap H_r : \mathbb{Z}) = 0$, then $A \cap H_r = 1$, a contradiction. If $(A \cap H_r : \mathbb{Z}) = 1$, then since there is a subgroup $H_r$ nearly maximal in $H_{r-1}$ with $A \cap H > 1$, $(A \cap H_{r+1}, A \cap H : \mathbb{Z})$ must be 0 for such $H$. That is $|A \cap H_r : A \cap H| = |A \cap H_{r,G}| = |H|$. Hence, the process must yield a near complement for $A$ in $G$ after at most $(A : \mathbb{Z}) - 1$ repetitions.

To establish the complete $\mathbb{Z}$-reducibility of $A$ when $G$ is nilpotent, we note that if $G$ is a finitely generated abelian group, if $A$ is a nontrivial (abelian normal) subgroup of $G$ for which $A \cap \psi(G) = 1$ and if $B$ is a nontrivial (normal) subgroup of $G$ contained in $A$, then $A$ is free abelian of finite $\mathbb{Z}$-rank, and there is a $\mathbb{Z}$-basis $a_1, \ldots, a_n$ for $A$ and integers $z_1, \ldots, z_m$, with $m \leq n$, such that $z_1a_1, \ldots, z_ma_m$ is a $\mathbb{Z}$-basis for $B$. $C = \langle a_{m+1}, \ldots, a_n \rangle$ is a near complement for $B$ in $A$ (which is normal in $G$). If, now, $G$ is a finitely generated nilpotent group and $A$ is as before, we establish the complete $\mathbb{Z}$-reducibility of $A$ as a $\mathbb{Z}G$-module by induction on the class of $G$. Assume the result true for groups of class smaller than $c$, and let

$$G = G_1 \supset \cdots \supset G_{c+1} = 1$$

be the lower central series of $G$. Consider the natural homomorphism $G \rightarrow G/G_c$ of $G$, under which $A$ maps to $AG_c/G_c \cong A/A \cap G_c = A$. By
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As $A$ is $G$-isomorphic with $AG_e/G_e$, $A$ is similarly completely $\mathbb{Z}$-reducible as a $\mathbb{Z}G$-module.

7. Split Extensions

Suppose that a group $G$ is a split extension of a normal subgroup $N$ by a subgroup $H$, so that $G = NH$ and $N \cap H = 1$. We illustrate some of the possibilities for $\psi(G)$, $\psi(N)$ and $\psi(H)$.

Perhaps the simplest situation is that $\psi(G)$ is a split extension of $\psi(N)$ by $\psi(H)$, which is of course the case if $G$ is an $\mathcal{E}'$-group and $G = N \times H$. This situation also occurs trivially if $G$ is soluble with $\text{Min}$, has a near Frattini subgroup and splits over its smallest subgroup $A$ of finite index.

If we restrict our attention to $\mathcal{E}'$-groups $G$ and assume that $N$ is also an $\mathcal{E}'$-group, then the $XYZ$ Lemma implies that $\psi(N) \subset \psi(G)$. However, $\psi(H)$ need not be contained in $\psi(G)$. For,

$$D_6 = \langle a, b \mid b^2 = 1, b^{-1}ab = a^{-1} \rangle$$

is a split extension of $\langle a \rangle$ by $\langle b \rangle$, and $\psi(\langle b \rangle) = \langle b \rangle$ while $\psi(D_6) = 1$. Similarly, the group $D^*$ of Example 2, Section 5, is a split extension of $E = \langle a, b, c \rangle$ by $\langle d^* \rangle$, and $\psi(\langle d^* \rangle) = \langle d \rangle$ but $\psi(D^*) = \langle b \rangle$. If we do not insist that $G$ split over $N$, then we can obtain a group $G$ with $\psi(G/N)$ cyclic of any desired order and $\psi(G) = 1$. For, the factor group $D/A_r$ of the generalized dihedral group $D = \langle a, b \mid b^{-1}ab = a^{-1} \rangle$ by the normal subgroup $A_r = \langle a^r \rangle$ is a split extension of $\langle A_r, a \rangle \cong C(r)$ by $\langle A_r, b \rangle \cong C(\infty)$. Hence $\psi(D/A_r) = \langle A_r, a \rangle \cong C(r)$.

Finally, we note that there is a split extension $G$ of $N$ by $H$ with $\psi(N) = \psi(H) = 1$, but $\psi(G) \cong C(\infty)$. We need only set $G = D^*$, $N = \langle a, b \rangle$ and $H = \langle c, d \rangle$.

8. The Classes $\mathcal{E}$, $\mathcal{E}'$ and $\mathcal{E}$

The class $\mathcal{E}'$ of groups plays an important role in the properties of the near Frattini subgroup established in this paper. An analogous role is played by the class $\mathcal{E}$ of groups with respect to properties of the Frattini subgroup. We say that a group $G$ is an $\mathcal{E}$-group if every proper subgroup of $G$ is contained in a maximal subgroup of $G$. It is easy to see that every finitely generated group is an $\mathcal{E}$-group, and that every homomorphic image of an $\mathcal{E}$-group is again an $\mathcal{E}$-group. If, in the established properties of the near Frattini subgroup that involve the class $\mathcal{E}'$, one replaces "$\mathcal{E}'$" by "$\mathcal{E}$" and "$\psi(G)$" by "-$\phi(G)$", there results, in most cases, a true statement about the Frattini subgroup.
If we denote by $\mathcal{E}$ the class of finitely generated groups, then we have $\mathcal{E} \subset \mathcal{E}'$ and $\mathcal{E} \subset \mathcal{E}$.

**Proposition.** $\mathcal{E}' \subset \mathcal{E}$.

**Proof.** Let $G \in \mathcal{E}'$ and $H < G$. If $| G : H |$ is finite, and if we choose $M \supseteq H$ so that $| G : M |$ is smallest possible, then $M$ is maximal in $G$. If $| G : H |$ is infinite, let $N \supseteq H$ be nearly maximal in $G$. If $N$ is maximal, we are through. Otherwise, we choose $M \supseteq N$ so that $| G : M |$ is smallest possible, whence $M$ is maximal in $G$. Thus, $G \in \mathcal{E}$.

Let $n$ be a natural number, let $p_1, p_2, \ldots$ be countably many distinct primes and let

$$G = C(p_1^n) \times C(p_2^n) \times \cdots.$$

To each maximal subgroup $M$ of $G$ there corresponds uniquely an index $j$ such that $M \cap C(p_j^n) < C(p_j^n)$. If $H$ is a proper subgroup of $G$, then for some index $i$ we must have $H \cap C(p_i^n) < C(p_i^n)$. $H$ cannot contain any element of order $p_i^n$, consequently the components of elements of $H$ that lie in $C(p_i^n)$ in fact lie in the unique maximal subgroup of $C(p_i^n)$. Thus, $H$ is contained in that maximal subgroup $M$ of $G$ for which $M \cap C(p_j^n) < C(p_j^n)$, and we have shown that $G \in \mathcal{E}$. On the other hand, $G$ has no nearly maximal subgroups. For, if $K \leq G$ is such that $| G : K |$ is infinite, then

$$K \cap C(p_i^n) < C(p_i^n)$$

for infinitely many $i$. If $k$ is a natural number such that $K \cap C(p_k^n) < C(p_k^n)$, then

$$\langle C(p_k^n), C(p_i^n) | j \text{ such that } K \cap C(p_j^n) = C(p_j^n) \rangle$$

contains $K$ properly and is of infinite index in $G$. Thus, $G \notin \mathcal{E}'$.

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