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Maximal dissipativity of Kolmogorov operators with Cahn–Hilliard type drift term

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ABSTRACT

We prove the existence of invariant measures μ for Kolmogorov operators L_F associated with semilinear stochastic partial differential equations with Cahn–Hilliard type drift term. Based on gradient estimates on the pseudo-resolvent associated with L_F and a priori estimates for the moments of μ we prove maximal dissipativity of L_F in the space $L^1(\mu)$.

1. Introduction and preliminaries

Consider the following semilinear stochastic differential equation with additive noise

$$\begin{cases} dX(t) = \big(AX(t) + (-A)^{\frac{1}{2}} F\big(X(t)\big)\big)\,dt + dW_t, & t \geqslant 0, \\ X(0) = x \end{cases} \tag{1.1}$$

on a separable real Hilbert space H. Here, $(A, D(A))$ is a self-adjoint operator of negative type, $F: D(F) \subset H \to H$ is a vector field and $(W_t)_{t\geqslant 0}$ is a cylindrical Wiener process on H, defined on

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a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, \mathbb{P})$. The precise assumptions on A and F are given in the hypotheses (H_0)–(H_3) below. For general reference on stochastic differential equations with additive noise, together with their corresponding Kolmogorov equations, let us mention the monographs by Da Prato [4], Da Prato, Zabczyk [10–12] and Cerrai [2]. Eq. (1.1) can be seen as an abstract generalization of the stochastic Cahn–Hilliard equation

$$du(t,x) = \left(-\frac{d^4}{dx^4}u(t,x) - \frac{d^2}{dx^2} f\big(u(t,x)\big)\right)dt + dW_t(x), \quad (t,x) \in \mathbb{R}_+ \times [0,\pi], \tag{1.2}$$

on the Hilbert space $H = \{u \in L^2([0,\pi]) \mid \int u\,dx = 0\}$ considered in [6].

Let L_F denote the Kolmogorov operator associated with (1.1). Clearly, L_F admits the representation

$$L_F\varphi(x) = L\varphi(x) + \big\langle F(x), (-A)^{\frac{1}{2}} D\varphi(x)\big\rangle, \quad \varphi \in \mathcal{F}C_b^2\big(D(A)\big),\ x \in H, \tag{1.3}$$

where

$$L\varphi(x) := \frac{1}{2}\operatorname{Tr} D^2\varphi(x) + \big\langle x, AD\varphi(x)\big\rangle$$

denotes the Ornstein–Uhlenbeck operator associated with the linear operator A,

$$\mathcal{F}C_b^2\big(D(A)\big) := \big\{\varphi \in C_b^2(H),\ \varphi(x) = f\big(\langle x, e_1\rangle, \ldots, \langle x, e_m\rangle\big),\ f \in C_b^2\big(\mathbb{R}^m\big)\big\}$$

is the associated space of cylindrical test functions, and $D\varphi$ (resp. $D^2\varphi$) denotes the first (resp. second) Fréchet derivative.

The first goal of this paper is to prove the existence of infinitesimally invariant measures for L_F, i.e., probability measures μ on the Borel subsets $\mathcal{B}(H)$ of H satisfying $L_F\varphi \in L^1(H,\mu)$ and $\int L_F\varphi\,d\mu = 0$ for all $\varphi \in \mathcal{F}C_b^2(D(A))$. This implies in particular that L_F is dissipative on the space $L^1(H,\mu)$, hence in particular closable, and it is then our main purpose to give sufficient conditions on the coefficients A and F that imply that L_F is essentially m-dissipative, i.e., the closure of $(L_F, \mathcal{F}C_b^2(D(A)))$ generates a C_0-semigroup (of contractions) on the space $L^1(H,\mu)$. In this case we say that L_F is L^1-unique.

In the case of Kolmogorov operators associated with semilinear stochastic differential equations with Lipschitz continuous nonlinearity, L^1-uniqueness is well known (see [12, Section 11.2], [3]). Recently, in the paper [16], the global Lipschitz assumption was relaxed to local Lipschitz continuous nonlinearities. Similar uniqueness results have also been obtained in [8] for Kolmogorov operators associated with dissipative nonlinearity. However, none of these assumptions are satisfied for the Kolmogorov operator associated with the stochastic Cahn–Hilliard equation (1.2). Nevertheless, using the fact that in this particular example the drift term is of gradient type, hence an infinitesimally invariant measure μ is explicitly known, Da Prato et al. prove in [6] the $L^{1+\beta}$-uniqueness of the associated Kolmogorov operator for all $\beta \in [0,1]$. In [17], a similar result is shown for the Kolmogorov operator associated with the semilinear stochastic differential equation

$$du(t,x) = \left(\frac{d^2}{dx^2}u(t,x) - u(t,x) + \frac{d}{dx}F(u)(t,x)\right)dt + dW_t(x), \quad (t,x) \in \mathbb{R}_+ \times [0,2\pi], \tag{1.4}$$

for some continuously differentiable vector field F.

Since both equations are of type (1.1), our main result concerning L^1-uniqueness of L_F (see Theorem 3.2) complements the above mentioned results. We emphasize that in (H_3) we only assume that F is dissipative, but not $(-A)^{\frac{1}{2}}F$, and that our method does not require the explicit knowledge of an infinitesimally invariant measure μ. Only a priori estimates on certain moments of μ and square integrability of F w.r.t. μ are needed. Both can be checked in applications (see the example of the stochastic Cahn–Hilliard equation in Section 4). Another key ingredient for our uniqueness result is the gradient estimate for the pseudo-resolvent associated with smooth approximations $L_{F_{\alpha,\beta}}$ of L_F that are uniform w.r.t. α, β (see Theorem 2.2 below).

Results on existence and uniqueness of strong solutions for stochastic Cahn–Hilliard equation have been obtained by Elezović and Mikelić in [13] with nuclear noise and by Da Prato and Debussche in [5] with space–time white noise. For Eq. (1.2) with multiplicative noise see the work [1] by Cardon-Weber. Whenever an invariant probability measure μ exists for the stochastic Cahn–Hilliard equation (1.2), the associated transition probabilities induce a strongly continuous semigroup on the space $L^1(H,\mu)$. The corresponding infintesimal generator has L_F as a realization on the space $\mathcal{F}C_b^2(D(A))$. However, for the abstract equation (1.1) it is not known in general, whether, and in which sense, Eq. (1.1) has a solution. Nevertheless, in the case of L^1-uniqueness of the Kolmogorov operator L_F, the abstract Cauchy problem corresponding to L_F in $L^1(H,\mu)$ is well-posed. In addition, the L^1-uniqueness of L_F implies the uniqueness of the solution of the martingale problem in the sense of Definition 2.5 in [19]. In this case the uniquely determined semigroup, which is Markovian, is the one induced by transition probabilities of the martingale solution of (1.1).

Let us now fix some notations and our main assumptions. Let $(e_n)_{n\in\mathbb{N}}$ be an orthonormal basis in H consisting of eigenfunctions of A with eigenvalues $\{-\lambda_n, \lambda_n > 0\}$. For $\gamma \in \mathbb{R}$ let

$$V_\gamma := \big(D\big((-A)^\gamma\big), \langle\cdot,\cdot\rangle_\gamma\big), \quad \text{where } \langle x, y\rangle_\gamma = \big\langle(-A)^\gamma x, (-A)^\gamma y\big\rangle \quad \text{for } x, y \in V_\gamma.$$

Note that, since A has a compact resolvent, the embedding $V_\gamma \hookrightarrow H$ is compact. In the following $\|\cdot\|_{HS}$ denotes the Hilbert–Schmidt operator norm on the space H. We shall formulate our assumptions:

(H_0) A is self-adjoint and $\|e^{tA}\| \leqslant e^{-\omega t}$ for certain $\omega > 0$.
(H_1) There exists $\nu \in]\frac{1}{4}, \frac{1}{2}[$ such that for all $t > 0$

$$\int_0^t s^{-2\nu} \|e^{sA}\|_{HS}^2 \, ds < \infty.$$

(H_2) F is a continuous map from $V_{\frac{1}{4}}$ into $V_{\frac{1}{4}}$ and leaves $V_{\frac{1}{2}}$ invariant. There exist $a > 0$, $r > 1$ such that

$$\big\|F(x)\big\|_{\frac{1}{4}} \leqslant a\big(1 + \|x\|_{\frac{1}{4}}^r\big), \quad x \in V_{\frac{1}{4}}.$$

(H_3) F is m-dissipative:

$$\mathrm{rg}(I - F) = H, \qquad \big\langle F(u) - F(v), u - v\big\rangle \leqslant 0, \quad u, v \in V_{\frac{1}{4}}.$$

We remark that hypothesis (H_3) can be replaced by a weaker assumption on F (see Remark 3.3 for precise conditions). Let us set up now the framework for our investigation. For $\alpha > 0$, consider the Yosida approximation of F defined by

$$F_\alpha(x) = F\big(J_\alpha(x)\big), \quad \text{where } J_\alpha(x) = (\mathrm{Id} - \alpha F)^{-1}(x), \quad x \in V_{\frac{1}{4}}.$$

For the sequence F_α we have the following:

(i) For any $\alpha > 0$, F_α is dissipative and Lipschitz continuous.
(ii) $|F_\alpha(x)| \leqslant |F(x)|$ for any $x \in V_{\frac{1}{4}}$.

Note that the function F_α is not differentiable in general. Therefore we shall consider a C^1-approximation as in [12]. For $\alpha,\ \beta > 0$ we set

$$F_{\alpha,\beta}(x) := \int_H e^{\beta A} F_\alpha\left(e^{\beta A}x + y\right) \mathcal{N}_{0,Q_\beta}(dy)$$

where $\mathcal{N}_{0,Q_\beta}$ is the Gaussian measure on H with mean 0 and covariance operator defined by $Q_\beta := \int_0^\beta e^{2sA}\,ds$. Then, $F_{\alpha,\beta}$ is dissipative and by the Cameron–Martin formula it is C^∞ differentiable. Moreover, as $\alpha,\ \beta \to 0$, $F_{\alpha,\beta} \to F$ pointwise. The following lemma gives an estimate for $F_{\alpha,\beta}$ which will be useful later.

Lemma 1.1. *There exists a positive constant $c_{\alpha,\beta} > 0$ such that*

$$\left\| F_{\alpha,\beta}(x) \right\|_{\frac{1}{4}} \leqslant c_{\alpha,\beta}\left(1 + |x|\right), \quad x \in V_{\frac{1}{4}}. \tag{1.5}$$

Proof. Let $x \in V_{\frac{1}{4}}$. We have

$$\begin{aligned}\left\| F_{\alpha,\beta}(x) \right\|_{\frac{1}{4}} &= \left\| (-A)^{\frac{1}{4}} \int_H e^{\beta A} F_\alpha\left(e^{\beta A}x + y\right) \mathcal{N}_{0,Q_\beta}(dy) \right\| \leqslant \frac{c_\beta}{\beta^{\frac{1}{4}}} \int_H \left\| F_\alpha(y) \right\| \mathcal{N}_{e^{\beta A}x, Q_\beta}(dy) \\ &\leqslant \frac{c_\beta c_\alpha}{\beta^{\frac{1}{4}}} \int_H \left(1 + |y|\right) \mathcal{N}_{e^{\beta A}x, Q_\beta}(dy) \quad (F_\alpha \text{ is Lipschitz on } H) \\ &\leqslant c_{\alpha,\beta}\left(1 + |x|\right).\end{aligned}$$

This yields the proof. □

We now introduce the following approximation problem

$$\begin{cases} dX_{\alpha,\beta}(t) = \left(AX_{\alpha,\beta}(t) + (-A)^{\frac{1}{2}} F_{\alpha,\beta}\left(X_{\alpha,\beta}(t)\right)\right) dt + dW_t, & t \geqslant 0, \\ X_{\alpha,\beta}(0) = x. \end{cases} \tag{1.6}$$

For $T > 0$ and $\frac{1}{\alpha} > p > 2$ we denote by $\mathcal{H}_{p,T}$ the Banach space of all adapted processes in $L^p(\Omega, C([0,T], H))$ endowed with the norm

$$\|Y\|_{p,T}^p = \mathbb{E} \sup_{t\in[0,T]} \left(\left|Y(t)\right|^p\right).$$

Definition 1.2. A mild solution of Eq. (1.6) is an $\mathcal{F}_t$-adapted process in $\mathcal{H}_{p,T}$ which satisfies the following integral equation

$$X_{\alpha,\beta}(t) = e^{tA}x + \int_0^t (-A)^{\frac{1}{2}} e^{(t-s)A} F_{\alpha,\beta}\left(X(s)\right) ds + \int_0^t e^{(t-s)A}\,dW_s, \quad t \geqslant 0. \tag{1.7}$$

Note that in our case hypothesis (H_1) implies that the stochastic convolution

$$W_A(t) := \int_0^t e^{(t-s)A}\,dW_s$$

is well defined in H and by [10, Proposition 7.9] there exists a constant $c_p(T) > 0$ such that

$$\mathbb{E}\left(\sup_{t\in[0,T]}\left(\int_0^t e^{(t-s)A}\,dW_s\right)^p\right) \leqslant c_p(T)\left|\int_0^T s^{-2\nu}\left\|e^{sA}\right\|_{HS}^2 ds\right|^{\frac{p}{2}} < \infty.$$

Since $F_{\alpha,\beta}$ is global Lipschitz one can prove by using a standard Banach fixed point theorem on the space $\mathcal{H}_{p,T}$ the following theorem.

Theorem 1.3. *Under hypotheses* (H_0), (H_1) *and* (H_2), *for any* $x \in V_{\frac{1}{4}}$, *Eq.* (1.6) *has a unique mild solution* $X_{\alpha,\beta}(\cdot, x) \in \mathcal{H}_{p,T}$.

The *transition semigroup* $P_t^{\alpha,\beta}$ corresponding to (1.6) is defined by

$$P_t^{\alpha,\beta} f(x) := \mathbb{E}\varphi\big(X_{\alpha,\beta}(t,x)\big), \quad t \geqslant 0,\ \varphi \in \mathcal{B}_b(H),$$

where $\mathcal{B}_b(H)$ denotes the space of bounded real valued Borel functions on H. An *invariant measure* for (1.6) is a Borel probability measure $\mu_{\alpha,\beta}$ on H such that

$$\big(P_t^{\alpha,\beta}\big)^*\mu_{\alpha,\beta} = \mu_{\alpha,\beta} \quad \text{for all } t \geqslant 0,$$

where $(P_t^{\alpha,\beta})^*$ denotes the adjoint of $P_t^{\alpha,\beta}$. We recall that the semigroup $(P_t^{\alpha,\beta})_{t\geqslant 0}$ has the Feller property if for any $\varphi \in C_b(H)$ and $t > 0$, we have that $P_t^{\alpha,\beta}\varphi \in C_b(H)$. Here, $C_b(H)$ is the space of all bounded real valued continuous functions on H equipped with the supremum norm $\|\cdot\|_\infty$. Since $F_{\alpha,\beta}$ is Lipschitz it is straightforward to prove that the process $X_{\alpha,\beta}(t,\cdot)_{t\geqslant 0}$ associated to (1.6) is Feller, hence to obtain the existence of an invariant measure $\mu_{\alpha,\beta}$ for $(P_t^{\alpha,\beta})_{t\geqslant 0}$ it is sufficient to check tightness of the set of probability measures $\{\mu_T := \frac{1}{T}\int_0^T \mu_{X_{\alpha,\beta}(t,x)}\,dt,\ T \geqslant 1\}$. Here, $\mu_{X_{\alpha,\beta}(t,x)}$ denotes the distribution of $X_{\alpha,\beta}(t,x)$, $t \geqslant 0$. Indeed, using [11, Theorem 3.1.1] any limit point $\mu_{\alpha,\beta}$ of some weakly convergent subsequence of $(\mu_T)_{T\geqslant 1}$ will be an invariant measure for (1.6).

The following lemma is essential for the proof of the existence and a priori estimate of an invariant measure for (1.6).

Lemma 1.4. *Assume hypotheses* (H_0), (H_1) *hold. Then we have*

$$M := \sup_{t\geqslant 0}\mathbb{E}\big(\left\|W_A(t)\right\|_{\frac{1}{4}}^2\big) = \int_0^\infty \left\|(-A)^{\frac{1}{4}}e^{tA}\right\|_{HS}^2 dt < \infty.$$

In particular,

$$\sup_{t\geqslant 0}\mathbb{E}\big(\left\|W_A(t)\right\|_\delta^2\big) < \infty, \quad \textit{and} \quad \sup_{t\geqslant 0}\mathbb{E}\big(e^{\varepsilon\|W_A(t)\|_\delta^2}\big) < \infty, \quad \textit{for all } \delta \in \left[0, \frac{1}{4}\right],\ \varepsilon < \frac{1}{2M}. \tag{1.8}$$

Proof. We write

$$\int_0^\infty \left\|(-A)^{\frac{1}{4}}e^{tA}\right\|_{HS}^2 dt = \sum_{k=0}^\infty \int_k^{k+1} \left\|(-A)^{\frac{1}{4}}e^{tA}\right\|_{HS}^2 dt$$

$$= \sum_{k=0}^{\infty} \int_0^1 \left\| (-A)^{\frac{1}{4}} e^{tA} e^{kA} \right\|_{HS}^2 dt \leqslant \sum_{k=0}^{\infty} \left\| e^{kA} \right\|^2 \int_0^1 \left\| (-A)^{\frac{1}{4}} e^{tA} \right\|_{HS}^2 dt$$

$$\leqslant \sum_{k=0}^{\infty} e^{-2\omega k} \int_0^1 \left\| (-A)^{\frac{1}{4}} e^{tA/2} \right\|^2 \left\| e^{tA/2} \right\|_{HS}^2 dt$$

$$\leqslant \sum_{k=0}^{\infty} e^{-2\omega k} \int_0^1 \sqrt{\frac{2}{t}} \left\| e^{tA/2} \right\|_{HS}^2 dt < \infty. \quad \big(\text{Using hypothesis } (\mathrm{H}_1).\big)$$

This finishes the proof of the first estimate in (1.8). The second estimate follows from [10, Proposition 2.16]. □

From the estimates in (1.8) we have in particular $\sup_{t \geqslant 0} \mathbb{E}|W_A(t)|^2 < \infty$. This gives the following estimate for the process $X_{\alpha,\beta}(\cdot)$

$$\sup_{0 \leqslant s \leqslant t} \mathbb{E}\left|X_{\alpha,\beta}(s,x)\right|^2 \leqslant C_{t,\alpha,\beta}\big(1 + |x|^2\big), \quad \text{for any } t > 0, \tag{1.9}$$

for some positive constant $C_{t,\alpha,\beta}$ depending on t, α and β.

Indeed, for $t \geqslant 0$ we use (1.8), to obtain

$$\mathbb{E}\left|X_{\alpha,\beta}(t,x)\right|^2 \leqslant 2|x|^2 + 4\mathbb{E}\left| \int_0^t (-A)^{\frac{1}{2}} e^{(t-s)A} F_{\alpha,\beta}\big(X_{\alpha,\beta}(s)\big)\, ds \right|^2 + 4 \sup_{t \geqslant 0} \mathbb{E}\left|W_A(t)\right|^2$$

$$\leqslant 2|x|^2 + 8\sqrt{t} \int_0^t \mathbb{E}\left\| F_{\alpha,\beta}\big(X_{\alpha,\beta}(s)\big)\right\|_{\frac{1}{4}}^2 ds + 4 \sup_{t \geqslant 0} \mathbb{E}\left|W_A(t)\right|^2$$

$$\big(\text{use estimate } (1.5)\big) \leqslant 2|x|^2 + 8t^{\frac{3}{2}} c_{\alpha,\beta}\Big(1 + \sup_{0 \leqslant s \leqslant t} \mathbb{E}\left\|X_{\alpha,\beta}(s,x)\right\|^2\Big) + 4 \sup_{t \geqslant 0} \mathbb{E}\left|W_A(t)\right|^2.$$

Hence for $t_0 > 0$ small enough we find

$$\sup_{0 \leqslant s \leqslant t_0} \mathbb{E}\left|X_{\alpha,\beta}(s,x)\right|^2 \leqslant C_{t_0,\alpha,\beta}\big(1 + |x|^2\big).$$

For general $t > 0$ the estimate now follows by iteration.

Using again estimate (1.8) we can find a positive constant c, independent of α, and β such that

$$\left\| F_{\alpha,\beta}(x) \right\| \leqslant c\big(1 + \|x\|_{\frac{1}{4}}^r\big), \quad x \in V_{\frac{1}{4}}. \tag{1.10}$$

Indeed, we have by (1.8) that the Gaussian measure $\mathcal{N}_{e^{\beta A}x, Q_\beta}$ has finite moments of any order with respect to the $\|\cdot\|_{\frac{1}{4}}$-norm. Hence hypothesis (H_2) and a straightforward computation yields the estimate.

Proposition 1.5. *Under hypotheses* (H$_0$), (H$_1$), (H$_2$) *and* (H$_3$), *there exists a unique invariant measure* $\mu_{\alpha,\beta}$ *for the transition semigroup* $(P_t^{\alpha,\beta})_{t\geqslant 0}$. *Moreover there exist some constants* $\theta > 0$ *and* $\kappa > 0$, *independent of* α *and* β, *such that*

$$\int_H \|x\|_{\frac{1}{4}}^2 \, \mu_{\alpha,\beta}(dx) \leqslant \theta \quad \textit{and} \quad \int_H \|x\|^4 \, \mu_{\alpha,\beta}(dx) \leqslant \kappa. \tag{1.11}$$

Proof. The existence and uniqueness of $\mu_{\alpha,\beta}$ was proved in [14]. Let us next prove the moment estimates.

We set

$$Y_{\alpha,\beta}(t) := X_{\alpha,\beta}(t) - W_A(t).$$

Uniqueness of $\mu_{\alpha,\beta}$ implies ergodicity of $\mu_{\alpha,\beta}$ and hence

$$\lim_{T\to+\infty} \frac{1}{T} \int_0^T \mathbb{E}\big(\varphi\big(X_{\alpha,\beta}(s,x)\big)\big)\, ds = \int_H \varphi(x)\, \mu_{\alpha,\beta}(dx) \quad \text{for all } \varphi \in L^2(H, \mu_{\alpha,\beta}). \tag{1.12}$$

If we set $\mu_T := \frac{1}{T}\int_0^T \mu_{X_{\alpha,\beta}(t,x)}\, dt$, $T \geqslant 1$, where $\mu_{X_{\alpha,\beta}(t,x)}$ denotes the distribution of $X_{\alpha,\beta}(t,x)$, $t \geqslant 0$, then (1.12) can be written as

$$\lim_{T\to+\infty} \int_H \varphi(x)\, \mu_T(dx) = \int_H \varphi(x)\, \mu_{\alpha,\beta}(dx) \quad \text{for all } \varphi \in L^2(H, \mu_{\alpha,\beta}).$$

For $p \geqslant 1$ define $\zeta_p(x) := \mathbf{1}_{\{\|x\|_{\frac{1}{4}}^2 \leqslant p\}} \cdot \|x\|_{\frac{1}{4}}^2$. Then for any $p \geqslant 1$, $\zeta_p \in L^2(H, \mu_{\alpha,\beta})$ and $\zeta_p(x) \leqslant \|x\|_{\frac{1}{4}}^2$ for $x \in H$. Hence for a sequence $(T_k)_{k\in\mathbb{N}}$ with $T_k \xrightarrow{k\to+\infty} +\infty$ it holds

$$\int_H \zeta_p(x)\, \mu_{\alpha,\beta}(dx) = \lim_{k\to\infty} \int_H \zeta_p(x)\, \mu_{T_k}(dx) \leqslant \lim_{k\to\infty} \int_H \|x\|_{\frac{1}{4}}^2\, \mu_{T_k}(dx). \tag{1.13}$$

On the other hand

$$\begin{aligned} \int_H \|x\|_{\frac{1}{4}}^2\, \mu_{T_k}(dx) &= \frac{1}{T_k} \int_0^{T_k} \mathbb{E}\big(\big\|X_{\alpha,\beta}(s)\big\|_{\frac{1}{4}}^2\big)\, ds \\ &\leqslant \frac{2}{T_k} \int_0^{T_k} \mathbb{E}\big(\big\|Y_{\alpha,\beta}(s)\big\|_{\frac{1}{4}}^2\big)\, ds + \frac{2}{T_k} \int_0^{T_k} \mathbb{E}\big(\big\|W_A(s)\big\|_{\frac{1}{4}}^2\big)\, ds. \end{aligned} \tag{1.14}$$

We now claim that there exists $C > 0$ such that for large $k \geqslant 1$

$$\mathbb{E}\left(\frac{1}{T_k}\int_0^{T_k} \big\|Y_{\alpha,\beta}(s)\big\|_{\frac{1}{4}}^2\, ds\right) \leqslant C\left(1 + \frac{1}{T_k}\right).$$

Indeed, since the semigroup generated by A is analytic, $(Y_{\alpha,\beta}(t))_{t\geqslant 0}$ is differentiable on $V_{\frac{1}{4}}$ for $t>0$ with derivative

$$Y'_{\alpha,\beta}(t) = AY_{\alpha,\beta}(t) + (-A)^{\frac{1}{2}} F_{\alpha,\beta}\big(Y_{\alpha,\beta}(t) + W_A(t)\big), \quad t>0.$$

Let $t>0$. Using the dissipativity of $F_{\alpha,\beta}$ and (1.10) we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\big\|(-A)^{-\frac{1}{4}}\big(Y_{\alpha,\beta}(t)\big)\big\|^2 &= \big\langle AY_{\alpha,\beta}(t) + (-A)^{\frac{1}{2}} F_{\alpha,\beta}\big(Y_{\alpha,\beta}(t)+W_A(t)\big), (-A)^{-\frac{1}{2}}\big(Y_{\alpha,\beta}(t)\big)\big\rangle \\ &= \big\langle -(-A)^{\frac{1}{2}} Y_{\alpha,\beta}(t) + F_{\alpha,\beta}\big(Y_{\alpha,\beta}(t)+W_A(t)\big), Y_{\alpha,\beta}(t)\big\rangle \\ &\leqslant -\big\|Y_{\alpha,\beta}(t)\big\|^2_{\frac{1}{4}} + \big\langle F_{\alpha,\beta}\big(W_A(t)\big), Y_{\alpha,\beta}(t)\big\rangle \\ &\leqslant -\big\|Y_{\alpha,\beta}(t)\big\|^2_{\frac{1}{4}} + \sigma\big\|Y_{\alpha,\beta}(t)\big\|^2 + \frac{1}{2\sigma}c^2\big(1+\big\|W_A(t)\big\|^{2r}_{\frac{1}{4}}\big).\end{aligned}$$

Since $\|y\|^2_{\frac{1}{4}} \geqslant \sqrt{\omega}\|y\|^2$, we obtain

$$\frac{1}{2}\frac{d}{dt}\big\|(-A)^{-\frac{1}{4}}\big(Y_{\alpha,\beta}(t)\big)\big\|^2 \leqslant -\left(1-\frac{\sigma}{\sqrt{\omega}}\right)\big\|Y_{\alpha,\beta}(t)\big\|^2_{\frac{1}{4}} + \frac{1}{2\sigma}c^2\big(1+\big\|W_A(t)\big\|^{2r}_{\frac{1}{4}}\big).$$

Therefore

$$\begin{aligned}&\frac{1}{2}\big\|(-A)^{-\frac{1}{4}}\big(Y_{\alpha,\beta}(t)\big)\big\|^2 + \left(1-\frac{\sigma}{\sqrt{\omega}}\right)\int_0^t \big\|Y_{\alpha,\beta}(s)\big\|^2_{\frac{1}{4}}\,ds \\ &\quad\leqslant \frac{1}{2}\big\|(-A)^{-\frac{1}{4}}x\big\|^2_{\frac{1}{4}} + \eta\int_0^t \big\|W_A(s)\big\|^{2r}_{\frac{1}{4}}\,ds + \eta t \end{aligned}\tag{1.15}$$

where $\eta := \frac{1}{2\sigma}c^2$. Using the estimate

$$\|w\|^{2r}_{\frac{1}{4}} \leqslant \big(3rMe^{\frac{1}{r}\frac{1}{3M}\|w\|^2_{\frac{1}{4}}}\big)^r = (3rM)^r e^{\frac{1}{3M}\|w\|^2_{\frac{1}{4}}}$$

and (1.8) we obtain

$$\sup_{t\geqslant 0}\mathbb{E}\big(\big\|W_A(t)\big\|^{2r}_{\frac{1}{4}}\big) \leqslant K, \quad \text{for some } K>0,$$

so that

$$\mathbb{E}\left(\int_0^t \big\|W_A(s)\big\|^{2r}_{\frac{1}{4}}\,ds\right) = \int_0^t \mathbb{E}\big(\big\|W_A(s)\big\|^{2r}_{\frac{1}{4}}\big)\,ds \leqslant K\cdot t. \tag{1.16}$$

Choosing $\sigma>0$ such that $\rho := 1-\frac{\sigma}{\sqrt{\omega}}>0$ and taking expectation in (1.15), we obtain that

$$\frac{1}{2}\mathbb{E}\left(\big\|(-A)^{-\frac{1}{4}}\big(Y_{\alpha,\beta}(t)\big)\big\|^2 + \rho\int_0^t \big\|Y_{\alpha,\beta}(s)\big\|^2_{\frac{1}{4}}\,ds\right) \leqslant \tilde{C}(t+1)$$

for $t\geqslant 0$, and some constant $\tilde{C}>0$.

Thus

$$\mathbb{E}\left(\frac{1}{T_k}\int_0^{T_k}\left\|Y_{\alpha,\beta}(s)\right\|_{\frac{1}{4}}^2\,ds\right) \leqslant C\left(1+\frac{1}{T_k}\right) \quad \text{for } k \geqslant 1, \text{ and some constant } C>0. \tag{1.17}$$

Hence putting estimates (1.16) and (1.17) together in (1.14) we get

$$\int_H \|x\|_{\frac{1}{4}}^2\,\mu_{T_k}(dx) \leqslant c_1\cdot C\left(1+\frac{1}{T_k}\right)+c_2\cdot K.$$

Therefore by (1.13)

$$\int_H \zeta_p(x)\,\mu_{\alpha,\beta}(dx) \leqslant \lim_{k\to\infty}\int \|x\|_{\frac{1}{4}}^2\,\mu_{T_k}(dx) = c_1\cdot C + c_2\cdot K.$$

The right-hand side does not depend on p, so that Fatou's lemma implies

$$\int_H \|x\|_{\frac{1}{4}}^2\,\mu_{\alpha,\beta}(dx) \leqslant \liminf_{p\to+\infty}\int_H \zeta_p(x)\,\mu_{\alpha,\beta}(dx) \leqslant c_1\cdot C + c_2\cdot K.$$

To prove that $\int_H \|x\|^4\,\mu_{\alpha,\beta}(dx) \leqslant \kappa$ we use again the dissipativity of $F_{\alpha,\beta}$ and a similar argument as above, so we write

$$\begin{aligned}
&\frac{1}{4}\frac{d}{dt}\left\|(-A)^{-\frac{1}{4}}\left(Y_{\alpha,\beta}(t)\right)\right\|^4\\
&\quad= \left\|Y_{\alpha,\beta}(t)\right\|_{-\frac{1}{4}}^2\left\langle AY_{\alpha,\beta}(t)+(-A)^{\frac{1}{2}}F_{\alpha,\beta}\left(Y_{\alpha,\beta}(t)+W_A(t)\right),(-A)^{-\frac{1}{2}}\left(Y_{\alpha,\beta}(t)\right)\right\rangle\\
&\quad= \left\|Y_{\alpha,\beta}(t)\right\|_{-\frac{1}{4}}^2\left\langle -(-A)^{\frac{1}{2}}Y_{\alpha,\beta}(t)+F_{\alpha,\beta}\left(Y_{\alpha,\beta}(t)+W_A(t)\right),Y_{\alpha,\beta}(t)\right\rangle\\
&\quad\leqslant -\left\|Y_{\alpha,\beta}(t)\right\|_{-\frac{1}{4}}^2\left\|Y_{\alpha,\beta}(t)\right\|_{\frac{1}{4}}^2+\left\|Y_{\alpha,\beta}(t)\right\|_{-\frac{1}{4}}^2\left\langle F_{\alpha,\beta}\left(W_A(t)\right),Y_{\alpha,\beta}(t)\right\rangle\\
&\quad\leqslant -\left\|Y_{\alpha,\beta}(t)\right\|^4+\omega\left\|F_{\alpha,\beta}\left(W_A(t)\right)\right\|\left\|Y_{\alpha,\beta}(t)\right\|^3 \quad \left(\text{using } \|u\|^2 \leqslant \|u\|_{-\frac{1}{4}}\|u\|_{\frac{1}{4}}\right).
\end{aligned}$$

Hence by using (1.10) and Young's inequality we obtain

$$\frac{1}{4}\frac{d}{dt}\left\|(-A)^{-\frac{1}{4}}\left(Y_{\alpha,\beta}(t)\right)\right\|^4 \leqslant -\left\|Y_{\alpha,\beta}(t)\right\|^4+\sigma\left\|Y_{\alpha,\beta}(t)\right\|^4+C(\sigma,\omega)\left(1+\left\|W_A(t)\right\|_{\frac{1}{4}}^{4r}\right).$$

Thus,

$$\frac{1}{4}\frac{d}{dt}\left\|(-A)^{-\frac{1}{4}}\left(Y_{\alpha,\beta}(t)\right)\right\|^4 \leqslant -(1-\sigma)\left\|Y_{\alpha,\beta}(t)\right\|^4+C(\sigma,\omega)\left(1+\left\|W_A(t)\right\|_{\frac{1}{4}}^{4r}\right).$$

Now the same argument, used to estimate $\int_H \|x\|_{\frac{1}{4}}^2\,\mu_{\alpha,\beta}(dx)$, yields

$$\int_H \|x\|^4\,\mu_{\alpha,\beta}(dx) \leqslant \kappa \quad \text{for some } \kappa>0 \text{ independent of } \alpha \text{ and } \beta. \qquad \square$$

According to estimate (1.11) and the compact embedding $V_{\frac{1}{4}} \hookrightarrow H$ we have that the family of measures $(\mu_{\alpha,\beta})_{\alpha,\beta\geqslant 0}$ is tight. Hence we have in particular the tightness of $(\mu_\alpha)_{\alpha>0}$ corresponding to Eq. (1.6) with $\beta = 0$ and $F_{\alpha,\beta} = F_\alpha$. Therefore there exists a subsequence $(\mu_{\alpha_n})_{n\geqslant 1}$ that converges weakly to some probability measure μ. Since the estimates in (1.11) are independent of α and β, a straightforward application of Fatou's lemma implies that

$$\int_H \|x\|_{\frac{1}{4}}^2 \,\mu(dx) \leqslant \theta \quad \text{and} \quad \int_H \|x\|^4 \,\mu(dx) \leqslant \kappa. \tag{1.18}$$

We will show in the next proposition the infinitesimal invariance of the measure μ for our starting operator L_F under the additional hypothesis:

(H_4) There exists $G : V_{\frac{1}{4}} \to H$ such that for all $\alpha > 0$

$$\left|F(x) - F_\alpha(x)\right| \leqslant \alpha\left|G(x)\right|, \quad x \in V_{\frac{1}{4}},$$

and

$$\int_H \left|G(x)\right| \mu(dx) + \sup_{\alpha>0} \int_H \left|G(x)\right| \mu_\alpha(dx) < \infty.$$

Proposition 1.6. *Let μ be any limit of some weakly convergent subsequence $(\mu_{\alpha_n})_{n\geqslant 1}$ for $\lim_{n\to\infty} \alpha_n = 0$. Under hypothesis ($H_4$) it follows that μ is infinitesimally invariant for L_F, i.e., $L_F\varphi \in L^1(\mu)$ and $\int L_F\varphi \, d\mu = 0$ for all $\varphi \in \mathcal{F}C_b^2(D(A))$.*

For the proof of the proposition we need some preparation. To this end let us define the Banach space $C_{b,2}(H)$ to be the space of all $\varphi : H \to \mathbb{R}$ such that the mapping $x \mapsto \frac{\varphi(x)}{1+|x|^2}$ is uniformly continuous and bounded endowed with the norm

$$\|\varphi\|_{b,2} := \sup_{x\in H} \frac{|\varphi(x)|}{1+|x|^2}.$$

The following lemma shows the weak convergence of the sequence $(\mu_{\alpha_n})_{n\geqslant 1}$ to μ on the space $C_{b,2}(H)$.

Lemma 1.7. *For any $\varphi \in C_{b,2}(H)$ we have that*

$$\lim_{\alpha\to 0} \int_H \varphi(x)\,\mu_\alpha(dx) = \int_H \varphi(x)\,\mu(dx).$$

Proof. Let $\varphi \in C_{b,2}(H)$. We write

$$\begin{aligned}
\int_H \varphi(x)\,\mu_\alpha(dx) &= \int_H \frac{\varphi(x)}{1+|x|^2} \cdot \left(1+|x|^2\right) \mu_\alpha(dx) \\
&= \int_H \frac{\varphi(x)}{1+|x|^2}\,\mu_\alpha(dx) + \int_H \frac{\varphi(x)}{1+|x|^2}|x|^2\,\mu_\alpha(dx).
\end{aligned}$$

Hence it is enough to show that

$$\lim_{\alpha\to 0}\int_H \psi(x)|x|^2\,\mu_\alpha(dx)=\int_H \psi(x)|x|^2\mu(dx),\quad \psi\in C_b(H).$$

Take $\psi\in C_b(H)$ and $\varepsilon>0$. We write

$$\begin{aligned}
&\left|\int_H \psi(x)|x|^2\,\mu_\alpha(dx)-\int_H \psi(x)|x|^2\,\mu(dx)\right| \\
&\quad\leqslant \left|\int_H \psi(x)\frac{|x|^2}{1+\varepsilon|x|^2}\,\mu_\alpha(dx)-\int_H \psi(x)\frac{|x|^2}{1+\varepsilon|x|^2}\,\mu(dx)\right| \\
&\qquad+\varepsilon\left|\int_H \psi(x)\frac{|x|^4}{1+\varepsilon|x|^2}\,\mu_\alpha(dx)\right|+\left|\int_H \psi(x)\frac{|x|^4}{1+\varepsilon|x|^2}\,\mu(dx)\right| \\
&\quad\leqslant \left|\int_H \psi(x)\frac{|x|^2}{1+\varepsilon|x|^2}\,\mu_\alpha(dx)-\int_H \psi(x)\frac{|x|^2}{1+\varepsilon|x|^2}\,\mu(dx)\right|+2\varepsilon\kappa\|\psi\|\infty.
\end{aligned}$$

Hence the conclusion follows by letting α and $\varepsilon\to 0$. □

We are now ready to proof Proposition 1.6.

Proof of Proposition 1.6. We show first that (H_4) implies $L_F\varphi\in L^1(H,\mu)$ for all $\varphi\in\mathcal{F}C_b^2(D(A))$. Indeed, suppose that φ admits the representation

$$\varphi(x)=g\big(\langle x,e_1\rangle,\dots,\langle x,e_n\rangle\big),\quad g\in C_b^2\big(\mathbb{R}^n\big),$$

then

$$(-A)^{\frac12}D\varphi(x)=\sum_{k=1}^n\sqrt{-\lambda_k}\,\frac{\partial g}{\partial x_k}\big(\langle x,e_1\rangle,\dots,\langle x,e_n\rangle\big)e_k$$

which implies that

$$(-A)^{\frac12}D\varphi\in C_b(H)\quad\text{and}\quad \sup_{x\in H}\big\|D\varphi(x)\big\|_{\frac12}<+\infty. \tag{1.19}$$

Since in addition F_α is Lipschitz for $\alpha>0$, hence at most of linear growth, we have that

$$\big|F(x)\big|\leqslant\big|F(x)-F_\alpha(x)\big|+\big|F_\alpha(x)\big|\leqslant\alpha\big|G(x)\big|+C\big(1+|x|\big)\in L^1(H,\mu).$$

Consequently,

$$\big|L_F\varphi(x)\big|\leqslant C+\big|\big\langle(-A)^{\frac12}F(x),D\varphi(x)\big|\leqslant C+\sup_{x\in H}\big\|D\varphi(x)\big\|_{\frac12}\big|F(x)\big|\in L^1(H,\mu).$$

To show the infinitesimal invariance fix $\varphi\in\mathcal{F}C_b^2(D(A))$. We can write

$$\int_H L_F\varphi(x)\,\mu(dx) = \int_H L_F\varphi(x) - L_{F_{\alpha_n}}\varphi(x)\,\mu(dx) + \int_H L_{F_{\alpha_n}}\varphi(x)\,\mu(dx) = J_1 + J_2,$$

where

$$J_1 = \int_H L_F\varphi(x) - L_{F_{\alpha_n}}\varphi(x)\,\mu(dx) \quad \text{and} \quad J_2 = \int_H L_{F_{\alpha_n}}\varphi(x)\,\mu(dx).$$

Since F_α is Lipschitz, it follows that $L_{F_\alpha}\varphi \in C_{b,2}(H)$, hence

$$J_2 = \lim_{m\to\infty} \int L_{F_{\alpha_n}}\varphi\, d\mu_{\alpha_m}$$

by Lemma 1.7. Invariance of μ_{α_m} for the semigroup $(P_t^{\alpha_m})_{t\geqslant 0}$ implies infinitesimal invariance for $L_{F_{\alpha_m}}$, so that

$$\begin{aligned}\left|\int L_{F_{\alpha_n}}\varphi\, d\mu_{\alpha_m}\right| &= \left|\int L_{F_{\alpha_n}}\varphi - L_{F_{\alpha_m}}\varphi\, d\mu_{\alpha_m}\right| = \left|\int \left\langle F_{\alpha_n} - F_{\alpha_m}, (-A)^{\frac{1}{2}} D\varphi\right\rangle d\mu_{\alpha_m}\right| \\ &\leqslant (\alpha_n + \alpha_m) \sup_{x\in H}\left\| D\varphi(x)\right\|_{\frac{1}{2}} \sup_{\beta>0}\int \left|G(x)\right| \mu_\beta(dx),\end{aligned}$$

thus

$$|J_2| \leqslant \alpha_n \sup_{x\in H}\left\| D\varphi(x)\right\|_{\frac{1}{2}} \sup_{\beta>0}\int \left|G(x)\right| \mu_\beta(dx)$$

which implies that $\lim_{n\to\infty} J_2 = 0$. Similarly,

$$|J_1| = \left|\int \left\langle F - F_{\alpha_n}, (-A)^{\frac{1}{2}} D\varphi\right\rangle d\mu\right| \leqslant \alpha_n \sup_{x\in H}\left\| D\varphi(x)\right\|_{\frac{1}{2}} \int \left|G(x)\right| \mu(dx),$$

hence $\lim_{n\to\infty} J_1 = 0$. □

As consequence of the above proposition we obtain that the operator $(L_F, \mathcal{F}C_b^2(D(A)))$ is dissipative on $L^1(H,\mu)$, in particular closable. In Section 3 we will discuss its maximal dissipativity.

2. Gradient bound for the resolvent

In this section we prove an estimate for the resolvent associated to (1.6). This estimate will be important to prove the well-posedness of the Kolmogorov equation associated with our starting operator (1.3) in Section 3. We start by proving a simple lemma for the process $(X_{\alpha,\beta}(t))_{t\geqslant 0}$.

Lemma 2.1. *Assume that hypotheses* (H_0), (H_1) *and* (H_3) *hold. Then for the mild solution of* (1.6) *we have*

$$\left(\int_0^t \left\| X_{\alpha,\beta}(s,x) - X_{\alpha,\beta}(s,y)\right\|_{\frac{1}{4}}^2 ds\right)^{\frac{1}{2}} \leqslant \frac{1}{\sqrt{2}}\|x-y\|_{-\frac{1}{4}} \quad \text{for all } x, y \in V_{\frac{1}{4}}. \tag{2.1}$$

Proof. We write

$$\begin{aligned}\frac{1}{2}\left(\frac{d}{dt}\left\|Y_{\alpha,\beta}(t,x) - Y_{\alpha,\beta}(t,y)\right\|_{-\frac{1}{4}}^2\right) &= \left\langle A\big(Y_{\alpha,\beta}(t,x) - Y_{\alpha,\beta}(t,y)\big), Y_{\alpha,\beta}(t,x) - Y_{\alpha,\beta}(t,y)\right\rangle_{-\frac{1}{4}} \\ &\quad + \left\langle F_{\alpha,\beta}\big(X_{\alpha,\beta}(t,x)\big) - F_{\alpha,\beta}\big(X_{\alpha,\beta}(t,y)\big), X_{\alpha,\beta}(t,x) - X_{\alpha,\beta}(t,y)\right\rangle_H \\ &\leqslant -\left\|Y_{\alpha,\beta}(t,x) - Y_{\alpha,\beta}(t,y)\right\|_{\frac{1}{4}}^2.\end{aligned}$$

Consequently

$$\int_0^t \left\| Y_{\alpha,\beta}(s,x) - Y_{\alpha,\beta}(s,y) \right\|_{\frac{1}{4}}^2 \, ds \leqslant \frac{1}{2} \|x-y\|_{-\frac{1}{4}}.$$

Since $X_{\alpha,\beta}(\cdot,x) - X_{\alpha,\beta}(\cdot,y) = Y_{\alpha,\beta}(\cdot,x) - Y_{\alpha,\beta}(\cdot,y)$, the proof of the lemma follows. □

Theorem 2.2. *Let $\Phi : V_{\frac{1}{4}} \to \mathbb{R}$ be a bounded Lipschitz continuous function with Lipschitz constant L_Φ. Then the function $G_\zeta \Phi$ defined by*

$$G_\zeta \Phi(x) := \int_0^{+\infty} e^{-\zeta t} \mathbb{E}\big(\Phi\big(X_{\alpha,\beta}(t,x)\big)\big)\, dt, \quad \zeta > 0,$$

is Fréchet differentiable and it holds

$$\left\| DG_\zeta \Phi(x) \right\|_{\frac{1}{4}} \leqslant \frac{1}{2\sqrt{2}} \sqrt{\frac{\pi}{\zeta}} L_\Phi.$$

Proof. Since the function $F_{\alpha,\beta}$ is regular, it follows by a result from [7] that $P_t^{\alpha,\beta}\Phi \in C_b^1(H)$. Hence by taking the Laplace transform, the differentiability of $G_\zeta \Phi$ follows. Let us prove the gradient estimate for $G_\zeta \Phi$. For $x \in V_{\frac{1}{4}}$, $h \in H$ we write

$$\big\langle DG_\zeta \Phi(x), h \big\rangle = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \big(G_\zeta \Phi(x+\varepsilon h) - G_\zeta \Phi(x)\big).$$

From the estimate in (2.1) we obtain

$$\begin{aligned}
\big|G_\zeta \Phi(x+\varepsilon h) - G_\zeta \Phi(x)\big| &= \left| \int_0^\infty e^{-\zeta t} \mathbb{E}\big(\Phi\big(X_{\alpha,\beta}(t,x+\varepsilon h)\big) - \Phi\big(X_{\alpha,\beta}(t,x)\big)\big)\, dt \right| \\
&\leqslant L_\Phi \int_0^\infty e^{-\zeta t} \mathbb{E} \left\| X_{\alpha,\beta}(t,x+\varepsilon h) - X_{\alpha,\beta}(t,x) \right\|_{\frac{1}{4}} dt \\
&= L_\Phi \mathbb{E}\left(\zeta \int_0^\infty e^{-\zeta t} \int_0^t \left\| X_{\alpha,\beta}(s,x+\varepsilon h) - X_{\alpha,\beta}(s,x) \right\|_{\frac{1}{4}} ds\, dt \right) \\
&\leqslant L_\Phi \mathbb{E}\left(\zeta \int_0^\infty e^{-\zeta t} \sqrt{t} \left(\int_0^t \left\| X_{\alpha,\beta}(s,x+\varepsilon h) - X_{\alpha,\beta}(s,x) \right\|_{\frac{1}{4}}^2 ds \right)^{\frac{1}{2}} \right) \\
&\leqslant L_\Phi \frac{1}{\sqrt{2}} \zeta \int_0^\infty e^{-\zeta t} \sqrt{t}\, dt \cdot \varepsilon \|h\|_{-\frac{1}{4}}.
\end{aligned}$$

Consequently

$$\big|\big\langle DG_\zeta \Phi(x), h \big\rangle\big| \leqslant \frac{1}{2\sqrt{2}} \sqrt{\frac{\pi}{\zeta}} L_\Phi \cdot \|h\|_{-\frac{1}{4}}.$$

Hence we deduce

$$\left\| DG_\zeta \Phi(x) \right\|_{\frac{1}{4}} \leqslant \frac{1}{2\sqrt{2}} \sqrt{\frac{\pi}{\zeta}} L_\Phi. \qquad \square$$

Remark 2.3. The proof of Theorem 2.2 shows also that $G_\zeta \Phi$ is in addition Lipschitz continuous w.r.t. the $V_{-\frac{1}{4}}$-norm.

3. The Kolmogorov operator L_F

Throughout the whole section let μ be an infinitesimally invariant measure for L_F satisfying the moment estimates (1.18) and the following additional hypothesis

(H_5) $\|F\|_{\frac{1}{4}} \in L^2(H, \mu)$ and

$$\lim_{\alpha,\beta \to 0} \left\| F(x) - F_{\alpha,\beta}(x) \right\|_{\frac{1}{4}} = 0 \quad \text{in } L^2(H, \mu).$$

Here, F_α denotes the Yosida approximation of F introduced in Section 1. We will also need its regular approximation $F_{\alpha,\beta}$ of Section 1.

Our main goal in this section is to prove that the operator $(L_F, \mathcal{F}C_b^\infty 2(D(A)))$ is essentially m-dissipative on $L^1(H, \mu)$. This is equivalent to prove that L_F is dissipative, hence in particular closable, and its closure $(\overline{L_F}, D(\overline{L_F}))$ generates a strongly continuous semigroup on $L^1(H, \mu)$. Dissipativity of L_F follows from the infinitesimal invariance of μ (see for example Lemma 3.2 in [16]).

Recall the definition of the Banach space $C_{b,2}(H)$ from Section 1. Let $(R_t)_{t \geqslant 0}$ be the Ornstein–Uhlenbeck semigroup corresponding to $(L, D(L))$ on the space $C_{b,2}(H)$. For any $h > 0$ we set

$$\Delta_h \varphi = \frac{1}{h}(R_h \varphi - \varphi), \quad \varphi \in C_{b,2}(H).$$

It is then well known from [18] that the generator $D(L)$ can be defined as

$$D(L) := \Big\{ \varphi \in C_{b,2}(H)\colon\ \exists \psi \in C_{b,2}(H),\ \lim_{h \to 0^+} \Delta_h \varphi(x) = \psi(x),$$
$$\forall x \in H,\ \sup_{h \in (0,1]} \|\Delta_h \varphi\|_{b,2} < +\infty \Big\}.$$

We now fix $\alpha, \beta > 0$ and for $f \in \mathcal{F}C_b^2(D(A))$ consider the following elliptic problem

$$(\lambda - L_{F_{\alpha,\beta}}) \varphi_{\alpha,\beta} = f, \quad \lambda > 0.$$

It is well known that this equation has a solution $\varphi_{\alpha,\beta}$ and can be writing in the form $\varphi_{\alpha,\beta} = R(\lambda, L_{F_{\alpha,\beta}}) f$, where

$$R(\lambda, L_{F_{\alpha,\beta}}) = \int_0^{+\infty} e^{-\lambda t} \mathbb{E}\big(f\big(X_{\alpha,\beta}(t,x)\big)\big)\, dt$$

is the pseudo-resolvent associated with $L_{F_{\alpha,\beta}}$. Thus we have

$$\|\lambda \varphi_{\alpha,\beta}\|_\infty \leqslant \|f\|_\infty, \tag{3.1}$$

and using Theorem 2.2 we also have

$$D := \sup_{\alpha,\beta>0} \left\|(-A)^{\frac{1}{4}} D\varphi_{\alpha,\beta}(x)\right\|^2 < +\infty, \quad x \in H. \tag{3.2}$$

The following lemma will be useful in the sequel.

Lemma 3.1. *We have $\varphi_{\alpha,\beta} \in D(\overline{L_F}) \cap D(L) \cap C_b^1(H)$ and*

$$\overline{L_F}\varphi_{\alpha,\beta}(x) = L\varphi_{\alpha,\beta}(x) + \left\langle F(x), (-A)^{\frac{1}{2}} D\varphi_{\alpha,\beta}(x)\right\rangle, \quad x \in V_{\frac{1}{4}}.$$

Proof. We first prove that $\varphi_{\alpha,\beta} \in D(L) \cap C_b^1(H)$. Note that by Theorem 2.2 we have $\varphi_{\alpha,\beta} \in C_b^1(H)$. This implies in particular $\varphi_{\alpha,\beta} \in C_{b,2}(H)$. For the semigroup $(R_t)_{t\geqslant 0}$ we can write

$$R_t(\varphi_{\alpha,\beta})(x) = \mathbb{E}\varphi_{\alpha,\beta}\big(Z(t,x)\big),$$

where

$$Z(t,x) = e^{tA}x + \int_0^t e^{(t-s)A}\,dW_s = X_{\alpha,\beta}(t,x) - \int_0^t e^{(t-s)A}(-A)^{\frac{1}{2}} F_{\alpha,\beta}\big(X_{\alpha,\beta}(s,x)\big)\,ds.$$

For any $h > 0$, we have

$$\begin{aligned}
&\frac{1}{h}\big(R_h\varphi_{\alpha,\beta}(x) - \varphi_{\alpha,\beta}(x)\big)\\
&\quad= \frac{1}{h}\mathbb{E}\big(\varphi_{\alpha,\beta}\big(Z(h,x)\big) - \varphi_{\alpha,\beta}(x)\big)\\
&\quad= \frac{1}{h}\left(\varphi_{\alpha,\beta}\left(X_{\alpha,\beta}(h,x) - \int_0^h e^{(h-s)A}(-A)^{\frac{1}{2}} F_{\alpha,\beta}\big(X_{\alpha,\beta}(s,x)\big)\,ds\right) - \varphi_{\alpha,\beta}(x)\right)\\
&\quad= \frac{1}{h}\mathbb{E}\big(\varphi_{\alpha,\beta}\big(X_{\alpha,\beta}(h,x)\big) - \varphi_{\alpha,\beta}(x)\big)\\
&\qquad - \frac{1}{h}\mathbb{E}\left(\left\langle (-A)^{\frac{1}{4}} D\varphi_{\alpha,\beta}\big(X_{\alpha,\beta}(h,x)\big), \int_0^h e^{(h-s)A}(-A)^{\frac{1}{4}} F_{\alpha,\beta}\big(X_{\alpha,\beta}(s,x)\big)\,ds\right\rangle\right) + o(h)\\
&\quad= \frac{1}{h}\mathbb{E}\big(\varphi_{\alpha,\beta}\big(X_{\alpha,\beta}(h,x)\big) - \varphi_{\alpha,\beta}(x)\big) + I(h,x) + o(h).
\end{aligned}$$

Consider now $0 < h \leqslant 1$. By estimates (1.5), (1.9) and (3.2) we have

$$\begin{aligned}
\big\|I(h,x)\big\| &\leqslant \frac{1}{2}D^2 + \mathbb{E}\frac{1}{2h^2}\left\|\int_0^h e^{(h-s)A}(-A)^{\frac{1}{4}} F_{\alpha,\beta}\big(X_{\alpha,\beta}(s,x)\big)\,ds\right\|^2\\
&\leqslant \frac{1}{2}D^2 + \frac{1}{2}\mathbb{E}\sup_{0\leqslant s\leqslant 1}\big\|F_{\alpha,\beta}\big(X_{\alpha,\beta}(s,x)\big)\big\|_{\frac{1}{4}}^2\\
&\leqslant \frac{1}{2}D^2 + \frac{1}{2}c_{\alpha,\beta}^2\Big(1 + \mathbb{E}\sup_{0\leqslant s\leqslant 1}\big\|X_{\alpha,\beta}(s,x)\big\|^2\Big)\\
&\leqslant \frac{1}{2}D^2 + \frac{1}{2}c_{\alpha,\beta}^2\big(1 + C_{1,\alpha,\beta}\big(1 + |x|^2\big)\big).
\end{aligned}$$

Consequently, we have

$$\sup_{h\in(0,1]} \left\| I(h,\cdot)\right\|_{b,2} < +\infty$$

and letting h tend to 0 we find that $\varphi_{\alpha,\beta} \in D(L)$ and

$$L\varphi_{\alpha,\beta} = L_{F_{\alpha,\beta}}\varphi_{\alpha,\beta} - \left\langle (-A)^{\frac{1}{2}} F_{\alpha,\beta}(x), D\varphi_{\alpha,\beta}(x)\right\rangle, \quad x \in V_{\frac{1}{4}}.$$

Let us now prove that $\varphi_{\alpha,\beta} \in D(\overline{L_F})$. We consider the space $\mathcal{E}_A(H)$ consisting of real and imaginary part of all exponential functions $e^{i\langle h,x\rangle}$, $x \in H$, with $h \in D(A)$. Since $\varphi_{\alpha,\beta} \in D(L)$, it follows by a result from [9, Proposition 2.7] that there exists a sequence $\varphi_n \in \mathcal{E}_A(H)$ such that

$$\lim_{n\to+\infty} \varphi_n(x) = \varphi_{\alpha,\beta}(x), \qquad \lim_{n\to+\infty} L\varphi_n(x) = L\varphi_{\alpha,\beta}(x), \qquad \lim_{n\to+\infty} D\varphi_n(x) = D\varphi_{\alpha,\beta}(x) \tag{3.3}$$

and a positive constant $C(\varphi_{\alpha,\beta})$ independent of n such that

$$\sup_{x\in H}\left\{\frac{|\varphi_n(x)| + |L\varphi_n(x)| + |D\varphi_n(x)|}{1+|x|^2}\right\} \leqslant C(\varphi_{\alpha,\beta}). \tag{3.4}$$

We recall from [15, Lemmas 3.1–3.3] that for $\varphi \in C_b(H)$ we have $DR_t\varphi(x) \in D(-A)^{\frac{1}{4}}$ and $D(\lambda - L)^{-1}\varphi(x) \in D((-A)^{\frac{1}{4}})$ for any $\lambda > 0$, $x \in H$. Moreover, using the Cameron–Martin formula and similar calculation as in [12, Proposition 11.2.5] we have

$$\frac{|(-A)^{\frac{1}{4}} DR_t\varphi(x)|}{1+|x|^2} \leqslant c_1 t^{-3/4} \sup_{x\in H} \frac{|\varphi(x)|}{1+|x|^2},$$

$$\frac{|(-A)^{\frac{1}{4}} D(\lambda - L)^{-1}\varphi(x)|}{1+|x|^2} \leqslant c_2 \lambda^{\frac{1}{4}} \Gamma(1/4) \sup_{x\in H} \frac{|\varphi(x)|}{1+|x|^2}.$$

This implies in particular that $D\varphi_{\alpha,\beta}(x) \in D((-A)^{\frac{1}{4}})$ for all x. Set $f_n := \lambda\varphi_n - L\varphi_n$. By using the estimate above we have

$$\sup_{n\geqslant 1}\left|(-A)^{\frac{1}{4}} D\varphi_n(x)\right| \leqslant C(\varphi_{\alpha,\beta})\left(1+|x|^2\right). \tag{3.5}$$

Hence for any $x \in V_{\frac{1}{4}}$, $((-A)^{\frac{1}{4}} D\varphi_n(x))_{n\geqslant 1}$ converges weakly to $(-A)^{\frac{1}{4}} D\varphi_{\alpha,\beta}(x)$ in H, by using (3.3) and the fact that $(-A)^{\frac{1}{4}}$ is a closed operator, hence in particular weakly closed. In particular,

$$\lim_{n\to+\infty} \left\langle (-A)^{\frac{1}{4}} F(x), (-A)^{\frac{1}{4}} D\varphi_n(x)\right\rangle = \left\langle (-A)^{\frac{1}{4}} F(x), (-A)^{\frac{1}{4}} D\varphi_{\alpha,\beta}(x)\right\rangle, \quad x \in V_{\frac{1}{4}},$$

and consequently

$$\lim_{n\to\infty} L_F\varphi_n(x) = L_F\varphi_{\alpha,\beta}(x). \tag{3.6}$$

The assertion $\varphi_{\alpha,\beta} \in D(\overline{L_F})$ now follows from dominated convergence, since estimates (3.4) and (3.5) yield

$$\left|L_F\varphi_n(x)\right| \leqslant C(\varphi_{\alpha,\beta})\left(1+|x|^2\right)\left(1+\left\|F(x)\right\|_{\frac{1}{4}}\right) \in L^1(H,\mu)$$

by (1.18) and hypothesis (H_5). □

The following theorem is our main result in this section.

Theorem 3.2. *The closure $(\overline{L_F}, D(\overline{L_F}))$ of $(L_F, \mathcal{F}C_b^2(D(A)))$ generates a C_0-semigroup of contractions $(\overline{T}_t)_{t\geqslant 0}$ on $L^1(H,\mu)$.*

Proof. For the proof it suffices to show that $(\lambda - \overline{L_F})(D(\overline{L_F}))$ is dense in $L^1(H,\mu)$ for some $\lambda > 0$. Let $\alpha, \lambda > 0$, $f \in \mathcal{F}C_b^2(D(A))$ and consider the approximation equation

$$(\lambda - L_{F_{\alpha,\beta}})\varphi_{\alpha,\beta} = f, \quad \lambda > 0. \tag{3.7}$$

By Lemma 3.1 we have $\varphi_{\alpha,\beta} \in D(L) \cap C_b^1(H)$ and hence using the approximating sequence $(\varphi_n)_{n\in\mathbb{N}}$ as in the proof of Lemma 3.1 and the identity $L\varphi_n^2 = 2\varphi_n L\varphi_n$, we can see that $\varphi_{\alpha,\beta}^2 \in D(L) \cap C_b^1(H)$ and

$$\overline{L_F}\big(\varphi_{\alpha,\beta}^2\big) = 2\varphi_{\alpha,\beta}\overline{L_F}\varphi_{\alpha,\beta} + \langle D\varphi_{\alpha,\beta}, D\varphi_{\alpha,\beta}\rangle.$$

Integrating this inequality with respect to the invariant measure implies

$$\frac{1}{2}\int_H \big\langle D\varphi_{\alpha,\beta}(x), D\varphi_{\alpha,\beta}(x)\big\rangle \,\mu(dx) = -\int_H \overline{L_F}\varphi_{\alpha,\beta}(x)\varphi_{\alpha,\beta}(x)\,\mu(dx). \tag{3.8}$$

Let us prove that $(\varphi_{\alpha,\beta})_{\alpha>0}$ is a Cauchy sequence in $L^2(H,\mu)$. To shorten notation let $\varphi_{\alpha,\beta}^{\gamma,\delta} = \varphi_{\alpha,\beta} - \varphi_{\gamma,\delta}$ for $\alpha, \beta, \gamma, \delta > 0$.

Eq. (3.8) implies that

$$\lambda \int_H \big(\varphi_{\alpha,\beta}^{\gamma,\delta}(x)\big)^2 \mu(dx) + \frac{1}{2}\int_H \big\langle D\varphi_{\alpha,\beta}^{\gamma,\delta}(x), D\varphi_{\alpha,\beta}^{\gamma,\delta}(x)\big\rangle \,\mu(dx) = \int_H (\lambda - \overline{L_F})\varphi_{\alpha,\beta}^{\gamma,\delta}(x)\varphi_{\alpha,\beta}^{\gamma,\delta}(x)\,\mu(dx).$$

But

$$\begin{aligned}\int_H (\lambda - \overline{L_F})\varphi_{\alpha,\beta}^{\gamma,\delta}(x)\,\varphi_{\alpha,\beta}^{\gamma,\delta}(x)\,\mu(dx) &= \int_H \big\langle (-A)^{\frac{1}{4}}\big(F_{\alpha,\beta}(x) - F(x)\big), (-A)^{\frac{1}{4}} D\varphi_{\alpha,\beta}(x)\big\rangle \varphi_{\alpha,\beta}^{\gamma,\delta}(x)\,\mu(dx) \\ &\quad - \int_H \big\langle (-A)^{\frac{1}{4}}\big(F_{\gamma,\delta}(x) - F(x)\big), (-A)^{\frac{1}{4}} D\varphi_{\gamma,\delta}(x)\big\rangle \varphi_{\alpha,\beta}^{\gamma,\delta}(x)\,\mu(dx) \\ &\leqslant 2D\frac{\|f\|_\infty}{\lambda}\big\|(-A)^{\frac{1}{4}}(F_{\alpha,\beta} - F)\big\|_{L^2(H,\mu)} \\ &\quad + 2D\frac{\|f\|_\infty}{\lambda}\big\|(-A)^{\frac{1}{4}}(F_{\gamma,\delta} - F)\big\|_{L^2(H,\mu)}\end{aligned}$$

and due to hypothesis (H_5) the right-hand side of the above inequality converges to 0 as $\alpha, \beta, \gamma, \delta \to 0$.

Let $\varphi \in L^2(H,\mu)$ be the limit of $(\varphi_{\alpha,\beta})_{\alpha,\beta>0}$. Clearly, $\lim_{\alpha,\beta\to 0}\varphi_{\alpha,\beta} = \varphi$ in $L^1(H,\mu)$ and thus

$$\begin{aligned}\overline{L_F}\varphi_{\alpha,\beta} &= L_{F_{\alpha,\beta}}\varphi_{\alpha,\beta} + \big\langle F - F_{\alpha,\beta}, (-A)^{\frac{1}{2}} D\varphi_{\alpha,\beta}\big\rangle \\ &= \lambda\varphi_{\alpha,\beta} - f + \big\langle F - F_{\alpha,\beta}, (-A)^{\frac{1}{2}} D\varphi_{\alpha,\beta}\big\rangle \xrightarrow[L^1(H,\mu)]{[\alpha,\beta\to 0]} \lambda\varphi - f.\end{aligned}$$

It follows that $\varphi \in D(\overline{L_F})$ and $(\lambda - \overline{L_F})\varphi = f$. We have thus shown that

$$\mathcal{F}C_b^2\big(D(A)\big) \subset (\lambda - \overline{L_F})\big(D(\overline{L_F})\big).$$

Since $\mathcal{F}C_b^2(D(A))$ is dense in $L^1(H,\mu)$, the proof is complete. □

We close this section by the following remark.

Remark 3.3. All the proofs and results of the previous sections hold if the Yosida approximation F_α is replaced by any sequence of dissipative Lipschitz continuous vector fields F_α such that $\|F_\alpha(x)\| \leqslant a(1+\|x\|_{\frac{1}{4}}^r)$, for some constant $a>0$ independent of α, $x \in V_{\frac{1}{4}}$. For the existence of an infinitesimal invariant measure for the operator $(L_F, \mathcal{F}C_b^2(D(A)))$ only the first approximation by F_α is needed. The sequence $F_{\alpha,\beta}$ was introduced to prove the regularity of the resolvent of transition semigroups corresponding to (1.6).

4. Stochastic Cahn–Hilliard equations

Let $I=[0,1] \subset \mathbb{R}$ be a bounded interval and $A = -\frac{d^4}{dx^4}$ be the bi-Laplacian with Neumann boundary conditions. Clearly, A is a negative definite self-adjoint operator on $\dot{H}$ with domain

$$D(A) := \left\{ v \in \dot{H} \cap H^4(I)\colon\ \frac{dv}{dx}(x) = \frac{d^3 v}{dx}(x) = 0,\ x = 0,1 \right\},$$

where

$$\dot{H} := \left\{ v \in L^2(I)\colon\ \int_0^1 v(x)\,dx = 0 \right\}.$$

Denote by $\mu_k = \pi^2 k^2$, $k \geqslant 1$, the eigenvalues of the negative Laplacian $-\Delta$ subject to Neumann boundary conditions and let $(\psi_k)_{k\in\mathbb{N}}$ denote the complete orthonormal set of corresponding eigenfunctions, $\psi_k = \cos k\pi x$. Then the sequence $-(\mu_k^2)_{k\geqslant 1}$ is the sequence of eigenvalues of the operator A with eigenfunctions $(\psi_k)_{k\in\mathbb{N}}$. Let $f(t) = -t^3$ and consider the following stochastic Cahn–Hilliard equation

$$du(t,x) = \left(-\frac{d^4}{dx^4}u(t,x) - \frac{d^2}{dx^2} f\big(u(t,x)\big)\right)dt + \eta(t,x), \quad (t,x) \in \mathbb{R}_+ \times I, \tag{4.1}$$

where $\eta(t,x) = dW_t(x)$ and (W_t) is a cylindrical Wiener process on $\dot{H}$. It is well known that $D((-A)^{\frac{1}{2}}) = V_{\frac{1}{2}} = \{u \in \dot{H}^2(I),\ \frac{dv}{dx}(0) = \frac{dv}{dx}(1) = 0\}$ and hence $V_{\frac{1}{4}} = \dot{H}^1(I)$ (domain of the square root of the Neumann Laplacian on $\dot{H}$) endowed with the norm

$$\|u\|_{\frac{1}{4}}^2 = \|u'\|_{L^2(I)}^2.$$

Define the nonlinear operator on the space $V_{\frac{1}{4}}$

$$F(u)(x) = -u^3(x), \quad u \in V_{\frac{1}{4}}.$$

This definition makes sense since the fractional space $V_{\frac{1}{4}}$ is a Banach algebra under the pointwise multiplication. This property is also true for $V_{\frac{1}{2}}$ (see [20]). Hence F satisfies (H_2). Clearly the map F

is dissipative with respect to the $\|\cdot\|_H$-norm and hence to use Remark 3.3, we introduce the following sequence

$$F_\alpha(u) := f_\alpha(u), \quad u \in V_{\frac{1}{4}}, \quad \text{where } f_\alpha(t) = \frac{-t^3}{1+\alpha t^2}.$$

Note that this definition makes sense on $V_{\frac{1}{4}}$ and $V_{\frac{1}{2}}$ is invariant under F_α. Indeed, this follows from the fact that the real function f_α is Lipschitz with $f_\alpha(0) = 0$ and its derivatives f'_α and f''_α are bounded. Moreover, there exists a constant $c_\alpha > 0$ such that

$$\|F_\alpha(x)\|_{\frac{1}{4}} \leqslant c_\alpha \|x\|_{\frac{1}{4}}, \quad x \in V_{\frac{1}{4}}. \tag{4.2}$$

On the other hand, it is straightforward to see that F_α is global Lipschitz on H and dissipative. Moreover by the Sobolev embedding theorem it is not difficult to see that $\|F_\alpha(x)\| \leqslant a(1 + \|x\|^r_{\frac{1}{4}})$, for some constant $a > 0$ independent of α, $x \in V_{\frac{1}{4}}$. Thus the sequence F_α satisfies Remark 3.3.

We shall apply our previous result to the Kolmogorov operator associated with (4.1). To this purpose we consider the approximate equation

$$du_\alpha(t,x) = \left(-\frac{d^4}{dx^4}u_\alpha(t,x) - \frac{d^2}{dx^2}F_\alpha\big(u_\alpha(t,x)\big)\right)dt + dW_t, \quad (t,x) \in \mathbb{R}_+ \times I. \tag{4.3}$$

To check hypothesis (H_1) we have $\|e^{sA}\|^2_{HS} = \sum_{n=1}^\infty e^{-2\mu_n^2 s}$, so that

$$\int_0^t s^{-2\nu}\|e^{sA}\|^2_{HS}\,ds \leqslant \sum_{n=1}^\infty \int_0^\infty s^{-2\nu}e^{-2\mu_n^2 s}\,ds = \frac{\Gamma(1-2\nu)}{2^{1-2\nu}}\sum_{n=1}^\infty \frac{1}{\mu_n^{2-4\nu}} \leqslant c\sum_{n=1}^\infty \frac{1}{n^{4-8\nu}} < \infty$$

if $\nu < \frac{3}{8}$ and hence hypothesis (H_1) is satisfied.

It remains now to check hypotheses (H_4) and (H_5).

Lemma 4.1. *Let μ_α be the invariant measure for Eq.* (4.3)*. Then for any $m \geqslant 1$ we have*

$$\int \|v^m\|^2_{\frac{1}{4}}\,\mu_\alpha(dv) < \delta, \quad \textit{for all } v \in V_{\frac{1}{4}}, \tag{4.4}$$

for some positive constant δ independent of α.

Proof. We write

$$X_\alpha(\cdot) = Y_\alpha(\cdot) + W_A(\cdot).$$

For the process $Y_\alpha(\cdot)$ we have the following

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\big(\|Y_\alpha(t)^m\|^2_{V_{-\frac{1}{4}}}\big) &= -m\big\langle(-A)^{\frac{1}{2}}Y_\alpha(t), Y_\alpha(t)^{2m-1}\big\rangle + m\big\langle F_\alpha\big(X_\alpha(t)\big), Y_\alpha^{2m-1}(t)\big\rangle \\
&\leqslant -m(2m-1)\int\big(\partial_x Y_\alpha(t)\big)^2\cdot Y_\alpha(t)^{2m-2}\,dx + m\int F_\alpha\big(X_\alpha(t)\big)Y_\alpha^{2m-1}(t)\,dx \\
&= -\frac{2m-1}{m}\int\big(\partial_x Y_\alpha^m(t)\big)^2\,dx + m\int F_\alpha\big(X_\alpha(t)\big)Y_\alpha^{2m-1}(t)\,dx.
\end{aligned}$$

We now claim that there exists a constant $C_m > 0$, independent of α, such that

$$\int F_\alpha\big(X_\alpha(t)\big)Y_\alpha^{2m-1}(t)\,dx \leqslant C_m \big\|W_A(t)\big\|_{\frac{1}{4}}^{2m+2}. \tag{4.5}$$

Indeed, we have

$$\begin{aligned}
\int F_\alpha\big(X_\alpha(t)\big)Y_\alpha^{2m-1}(t)\,dx &= -\int \frac{X_\alpha(t)^3}{1+\alpha X_\alpha(t)^2}Y_\alpha^{2m-1}(t)\,dx \\
&= -\int \frac{Y_\alpha^{2m+2}(t)}{1+\alpha X_\alpha(t)^2}\,dx - \int \frac{3Y_\alpha(t)^{2m+1}W_A(t)}{1+\alpha X_\alpha(t)^2}\,dx \\
&\quad - \int \frac{3Y_\alpha(t)^{2m}W_A^2(t)}{1+\alpha X_\alpha(t)^2}\,dx - \int \frac{Y_\alpha(t)^{2m-1}W_A^3(t)}{1+\alpha X_\alpha(t)^2}\,dx \\
\text{(Young's inequality)} &\leqslant -\frac{1}{2}\int \frac{Y_\alpha^{2m+2}(t)}{1+\alpha X_\alpha(t)^2}\,dx + C_m\int \frac{W_A^{2m+2}(t)}{1+\alpha X_\alpha(t)^2}\,dx \\
&\leqslant C_m \int W_A^{2m+2}(t)\,dx.
\end{aligned}$$

Hence by the Sobolev embedding theorem we deduce (4.5).

Consequently,

$$\frac{1}{2}\frac{d}{dt}\big(\big\|Y_\alpha(t)^m\big\|_{V_{-\frac{1}{4}}}\big) \leqslant -\frac{2m-1}{m}\int \big(\partial_x Y_\alpha^m(t)\big)^2\,dx + C_m\big\|W_A(t)\big\|_{\frac{1}{4}}^{2m+2}.$$

From

$$\big\|Y_\alpha^m(t)\big\|_{\frac{1}{4}}^2 = \int \big(\partial_x Y_\alpha^m(t)\big)^2\,dx$$

we deduce that for $T > 0$

$$\frac{1}{T}\int_0^T \big\|Y_\alpha^m(t)\big\|_{\frac{1}{4}}^2\,dt \leqslant \frac{1}{T}\big\|Y_\alpha^m(0)\big\|_{V_{-\frac{1}{4}}} + C_m\frac{1}{T}\int_0^T \big\|W_A(t)\big\|_{\frac{1}{4}}^{2m+2}\,dt. \tag{4.6}$$

From the multiplicative property of the space $V_{\frac{1}{4}}$ it follows

$$\begin{aligned}
\big\|X_\alpha^m(t)\big\|_{\frac{1}{4}} &\leqslant \sum_{k=0}^m c_{mk}\big\|Y_\alpha^k(t)W_A^{m-k}(t)\big\|_{\frac{1}{4}} \\
&\leqslant C_{\frac{1}{4}}\sum_{k=0}^m c_{mk}\big\|Y_\alpha^k(t)\big\|_{\frac{1}{4}}\cdot\big\|W_A(t)\big\|_{\frac{1}{4}}^{m-k} \\
&\leqslant C\Bigg(\sum_{k=0}^m \big\|Y_\alpha^k(t)\big\|_{\frac{1}{4}}^2 + \big\|W_A(t)\big\|_{\frac{1}{4}}^{2m+2}\Bigg)
\end{aligned}$$

for some constant $C > 0$. Hence by making use of (4.6) we obtain

$$\frac{1}{T}\int_0^T \|X_\alpha^m(t)\|_{\frac{1}{4}}^2\,dt \leqslant C\sum_{k=0}^m \frac{1}{T}\int_0^T \|Y_\alpha^k(t)\|_{\frac{1}{4}}^2\,dt + C\frac{1}{T}\int_0^T \big(\|W_A(t)\|_{\frac{1}{4}}^{2m+2}+1\big)\,dt$$

$$\leqslant C\frac{1}{T}\sum_{k=0}^m \|Y_\alpha^k(0)\|_{-\frac{1}{4}}^2 + C\frac{1}{T}\int_0^T \big(\|W_A(t)\|_{\frac{1}{4}}^{2m+2}+1\big)\,dt.$$

By taking the expectation in the last inequality we obtain

$$\int \|x^m\|_{\frac{1}{4}}^2\,\mu_{T,\alpha}(dx) \leqslant C\frac{1}{T}\sum_{k=0}^m \mathbb{E}\big(\|Y_\alpha^k(0)\|_{-\frac{1}{4}}^2\big) + C(M+1)$$

$$= C\frac{1}{T}\sum_{k=0}^m \|x^k\|_{-\frac{1}{4}}^2 + C(M+1).$$

Recall from Section 1 that μ_α can be obtained as a limit point of some weakly convergent subsequence $(\mu_{\alpha,T_n})_{n\geqslant 1}$. Hence an application of Fatou's lemma yields

$$\int_H \|x^m\|_{\frac{1}{4}}^2\,\mu_\alpha(dx) = \lim_{K\to+\infty}\int \|x^m\|_{\frac{1}{4}}^2 \wedge K\,\mu_\alpha(dx)$$

$$\leqslant \liminf_{K\to+\infty}\lim_{n\to+\infty}\int_H \|x^m\|_{\frac{1}{4}}^2 \wedge K\,\mu_{T_n,\alpha}(dx) \leqslant C(M+1) := \delta.$$

This finishes the proof of the lemma. □

Now it is not difficult to check hypothesis (H_4). In fact, an application of the estimate (1.11) in Proposition 1.5 (with $\beta = 0$) and the compact embedding $V_{\frac{1}{4}} \hookrightarrow H$ gives the tightness of the family of measure $(\mu_\alpha)_{\alpha>0}$. Let μ be the limit point of any subsequent of $(\mu_\alpha)_{\alpha>0}$. Then by (4.4) in the above lemma and Fatou's lemma as in the proof above we deduce

$$\int_H \|x^m\|_{\frac{1}{4}}^2\,\mu(dx) < \infty. \tag{4.7}$$

This estimate yields

$$\int_H \|F(x)\|_{\frac{1}{4}}^2\,\mu(dx) < +\infty.$$

We now write

$$\big|F_\alpha(u) - F(u)\big| = \left|\frac{\alpha u^5}{1+\alpha u^2}\right| \leqslant \alpha\big|G(u)\big|,$$

where $G(u) = u^5$. Therefore estimates (4.4) and (4.7) yield hypothesis (H_4).

Let us now prove hypothesis (H_5). Using (4.2) we write

$$\big\|F_{\alpha,\beta}(u)\big\|_{\frac{1}{4}} = \left\|(-A)^{\frac{1}{4}}\int_H e^{\beta A}F_\alpha(v)\mathcal{N}_{e^{\beta A}u,Q_\beta}(dv)\right\| \leqslant \int_H \big\|F_\alpha(v)\big\|_{\frac{1}{4}}\mathcal{N}_{e^{\beta A}u,Q_\beta}(dv)$$
$$\leqslant \int_H c_\alpha\|v\|_{\frac{1}{4}}\mathcal{N}_{e^{\beta A}u,Q_\beta}(dv) \leqslant \tilde{c}_\alpha\big(1+\|u\|_{\frac{1}{4}}\big) \tag{4.8}$$

for a positive constant $\tilde{c}_\alpha$ independent of β. Since $\lim_{\beta\to 0}(-A)^{\frac{1}{4}}F_{\alpha,\beta}(u) = (-A)^{\frac{1}{4}}F_\alpha(u)$ we have by the dominated convergence theorem that $\lim_{\beta\to 0}(-A)^{\frac{1}{4}}F_{\alpha,\beta} = (-A)^{\frac{1}{4}}F_\alpha$ in $L^2(H,\mu)$. This yields in particular

$$\lim_{\beta\to 0}\int_H \big\|F_{\alpha,\beta}(u)-F(u)\big\|^2_{\frac{1}{4}}\,\mu(du) = \int_H \big\|F_\alpha(u)-F(u)\big\|^2_{\frac{1}{4}}\,\mu(du).$$

But

$$F_\alpha(u)-F(u) = \frac{\alpha u^5}{1+\alpha u^2} = \frac{\alpha u^3}{1+\alpha u^2}\cdot u^2 = h_\alpha(u)\cdot u^2,$$

where $h_\alpha(z) = \frac{\alpha z^3}{1+\alpha z^2}$. Since h_α has a bounded derivative, independent of α, we can find a constant $K>0$, independent of $\alpha>0$, such that

$$\big\|h_\alpha(u)\big\|_{\frac{1}{4}} \leqslant K\|u\|_{\frac{1}{4}}, \quad u\in V_{\frac{1}{4}}.$$

Now by the multiplicative property of $V_{\frac{1}{4}}$ and Lebesgue's theorem hypothesis (H_5) holds.

Hence making use of the result from the previous sections we deduce the following theorem.

Theorem 4.2. *The family of measure $(\mu_\alpha)_{\alpha>0}$ is tight. Let μ be any limit point of some weakly convergent subsequence of $(\mu_\alpha)_{\alpha>0}$. Then μ is an infinitesimal invariant measure for the operator L_F. In particular, $(L,\mathcal{F}C_b^2(D(A)))$ is closable on $L^1(H,\mu)$. Moreover, μ satisfies the moment estimates* (1.18) *and the closure of $(L,\mathcal{F}C_b^2(D(A)))$ in $L^1(H,\mu)$ generates a Markov C_0-contraction semigroup.*

Remark 4.3. The last example can be generalized to any nonlinearity F of the form $F(u) = -u^{2p+1}$, $p\geqslant 1$, by just considering the sequence $F_\alpha u = \frac{-u^{2p+1}}{1+\alpha u^{2p}}$, $\alpha>0$, and using the same argument as above.

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