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The Research of Affine Bivariate Dual Frames Associated with a Generalized Multiresolution Analysis and Filter Banks

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Abstract

The rise of frame theory in applied mathematics is due to the flexibility and redundancy of frames. In the work, the notion of bivariate affine pseudoframes is introduced and the notion of a bivariate generalized multiresolution analysis (GMRA) is introduced. A novel approach for designing one GMRA of Paley Wiener subspaces of $L^2(\mathbb{R}^2)$ is proposed. The sufficient condition for the existence of a sort of affine pseudoframes with filter banks is obtained by virtue of a generalized multiresolution analysis. The pyramid decomposition scheme is established based on such a generalized multiresolution analysis. An approach for designing a sort of affine bivariate dual frames in two-dimensional space is presented.

Keywords: bivariate, the pyramid decomposition scheme, pseudoframes, dual frames, oblique frames, filter banks

1. Introduction

Wavelet transform is a signal processing technique and has been widely applied in the field of image processing, data compression, signal processing, noise removal and time series analysis. In general, multiresolution analysis is the most application of wavelet transform[1]. According to MRA theory, wavelet transform can be processed iteratively. The wavelet frame is one of the most widely applied wavelet transform. The structured frames are much easier to be constructed than structured orthonormal bases. The notion of frames was introduced by Duffin and Schaeffer [2] and popularized greatly by the work of Daubechies and her co-authors[3]. After this ground breaking work, the theory of frames began to be more widely studied both in theory and in applications[4-7], such as data

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compression and sampling theory, and so on. Every frame (or Bessel sequence) determines an analysis operator, the range of which is important for a number of applications. The notion of Frame multiresolution analysis as described by [5] generalizes the notion of multiresolution analysis by allowing non-exact affine frames. However, subspaces at different resolutions in a FMRA are still generated by a frame formed by translates and dilates of a single function. This article is motivated from the observation that standard methods in sampling theory provide examples of multiresolution structure which are not FMRAs. Inspired by [5] and [7], we introduce the notion of a generalized multiresolution structure (GMRA) of \( L^2(R^2) \) generated by several functions of integer grid translations in space \( L^2(R^2) \).

By \( \Omega \), we denote a separable Hilbert space. We recall that a sequence \( \{\varphi_i : i \in \mathbb{Z}^2 \subseteq \Omega \} \) is a frame for \( \Omega \), if there exist positive real numbers \( C_1, C_2 \) such that

\[
\forall \lambda \in \Omega, \quad C_1 \langle \lambda, \lambda \rangle \leq \sum_{i \in \mathbb{Z}^2} |\langle \lambda, \varphi_i \rangle|^2 \leq C_2 \langle \lambda, \lambda \rangle. \tag{1}
\]

A sequence \( \{\varphi_i \} \subseteq \Omega \) is a Bessel sequence if (only) the upper inequality of (1) follows. If only for all \( \lambda \in \Lambda \subseteq \Omega \), the upper inequality of (1) holds, the sequence \( \{\varphi_i \} \subseteq \Omega \) is a Bessel sequence with respect to (w.r.t.) the subspace \( \Lambda \). If \( \{\varphi_i \} \) is a frame of a Hilbert space \( \Omega \), there exist a dual frame \( \{\varphi_i^* \} \) such that

\[
\forall h \in \Omega, \quad h = \sum_{i \in \mathbb{Z}^2} \langle h, \varphi_i \rangle \varphi_i^* = \sum_{i \in \mathbb{Z}^2} \langle h, \varphi_i^* \rangle \varphi_i. \tag{2}
\]

2. The generalized multiresolution analysis

In what follows, we consider the case of generators, which yield affine pseudoframes of integer grid translates for subspaces of \( L^2(R^2) \). Let \( \{\tau_u h\} \) and \( \{\widetilde{\tau}_u h\} \ (v \in \mathbb{Z}^2) \) be two sequences in \( L^2(R^2) \). Let \( \Omega \) be a closed subspace of \( L^2(R^2) \).

We say that \( \{\tau_u h\} \) forms a dual bivariate pseudoframe for the subspace \( \Omega \) with respect to (w.r.t.) \( \{\tau_u h\} \) \( (u \in \mathbb{Z}^2) \) if

\[
\forall g(s) \in \Omega, \quad g(s) = \sum_{u \in \mathbb{Z}^2} \langle g, \tau_u h \rangle \tau_u h(s). \tag{3}
\]

where we define a translate operator, \( (\tau_v \phi)(s) = \phi(s-v) \ v \in \mathbb{Z}^2 \), for an any function \( \phi(s) \in L^2(R^2) \). It is important to note that \( \{\tau_u h(s)\} \) and \( \{\tau_u h(s)\} \ (v \in \mathbb{Z}^2) \) need not be contained in \( \Omega \).

Definition 1. A bivariate generalized multiresolution analysis \( \{V_n, \phi(s), \tilde{\phi}(s)\} \) is a sequence of closed linear subspaces \( \{V_n\}_{n \in \mathbb{Z}} \) of \( L^2(R^2) \) and elements \( \phi(s), \tilde{\phi}(s) \in L^2(R^2) \), such that (a) \( V_n \subseteq V_{n+1} \forall n \in \mathbb{Z} \); (b) \( \bigcap_{n \in \mathbb{Z}} V_n = \{0\} \); (c) \( \bigcup_{n \in \mathbb{Z}} V_n \) is dense in \( L^2(R^2) \); (c) \( g(s) \in V_n \) if and only if \( Dg(s) \in V_{n+1} \) \quad \forall n \in \mathbb{Z} \), where \( Dg(s) = 2g(2s) \), for \( \forall g(s) \in L^2(R^2) \); (d)
(s) \in V_0 \text{ implies } \tau_v g(s) \in V_0, \text{ for all, } v \in \mathbb{Z}^2; \text{ (e) } \{\tau_v \psi(s), v \in \mathbb{Z}^2\} \text{ constitutes a pseudoframe for } V_0 \text{ with respect to } \{\tau_v \tilde{\psi}(s), v \in \mathbb{Z}^2\}.

3. A gamma of Paley-Wiener subspaces

A necessary and sufficient condition for the construction of pseudoframe of translates for Paley-Wiener subspaces is presented as follows.

**Theorem 1.** Let \( \psi(s) \in L^2(\mathbb{R}^2) \) be such that \(|\hat{\psi}(\omega)| > 0 \) a.e. on a connected neighbourhood of \( 0 \) in \([-1/2, 1/2)^2\) and \(|\hat{\phi}(\omega)| = 0 \) a.e. otherwise. Define \( \Delta = \{\omega \in \mathbb{R}^2: |\hat{\phi}(\omega)| \geq c > 0\} \), and

\[ V_0 = PW_\Delta = \{g(s) \in L^2(\mathbb{R}^2): \sup p(\hat{g}) \subseteq \Delta\}. \]

Then, for \( \tilde{\psi} \in L^2(\mathbb{R}^2) \), \( \{\tau_v \psi, v \in \mathbb{Z}^2\} \) is a pseudoframe of translates for \( V_0 \) w.r.t. \( \{\tau_v \tilde{\psi}, v \in \mathbb{Z}^2\} \) if and only if

\[ \tilde{\phi}(\omega)\hat{\psi}(\omega) = \chi_\Delta(\omega) \quad \text{a.e.,} \quad (4) \]

where \( \chi_\Delta \) is the characteristic function on \( \Delta \), and the Fourier transform of an integrable function \( f(s) \) is defined by

\[ (Ff)(\omega) = \hat{f}(\omega) = \int_{\mathbb{R}^2} f(s)e^{-2\pi i s\omega} ds, \quad \omega \in \mathbb{R}^2, \quad (5) \]

which, as usual, can be naturally extended to functions in space \( \hat{L}(\mathbb{R}^2) \). For a sequence \( c = \{c(v)\} \in \ell^2(\mathbb{Z}^2) \), we define the discrete-time Fourier transform as the function in \( L^2(0,1)^2 \) given by

\[ Fc(\omega) = C(\omega) = \sum_{v \in \mathbb{Z}^2} c(v)e^{-2\pi i v\omega}. \quad (6) \]

Note that the discrete-time Fourier transform is \( \mathbb{Z}^2 \)-periodic. Moreover, if \( \hat{\phi} \) is also such that \(|\hat{\phi}| > 0 \) a.e. on a connected neighbourhood of \( 0 \) in \([-1/2, 1/2)^2\) and \(|\hat{\phi}| = 0 \) a.e. otherwise, that (6) holds, then \( \{\tau_n \phi(s), v \in \mathbb{Z}^2\} \) and \( \{\tau_n \tilde{\psi}(s), v \in \mathbb{Z}^2\} \) are a pair of commutative pseudoframes for \( \Omega \), i.e.,

\[ \forall g(s) \in \Omega, \quad g(s) = \sum_{v \in \mathbb{Z}^2} \langle g, \tau_v \tilde{\psi} \rangle \tau_v \phi(s) \]

\[ = \sum_{v \in \mathbb{Z}^2} \langle g, \tau_v \phi \rangle \tau_v \tilde{\psi}(s). \quad (7) \]

**Proof of Theorem 1.** For all \( f(s) \in PW_\Delta \), consider

\[ F(\sum_{v \in \mathbb{Z}^2} \langle f, \tau_v \tilde{\psi} \rangle \tau_v h) = \sum_{v \in \mathbb{Z}^2} \langle f, \tau_v \tilde{\psi} \rangle F(\tau_v \phi) \]

\[ = \sum_{v \in \mathbb{Z}^2} \int_{\mathbb{R}^2} \hat{f}(\xi)\tilde{\phi}(\xi)e^{2\pi i v\xi} d\xi \hat{\phi}(\omega) e^{-2\pi i v\omega} \]

\[ = \sum_{u \in \mathbb{Z}^2} \int_{[0,1]} \int_{v \in \mathbb{Z}^2} \hat{f}(\xi + v)\tilde{\phi}(\xi + v)e^{2\pi i v\xi} d\xi \hat{\phi}(\omega) e^{-2\pi i v\omega} \]
\[ = \hat{\phi}(\omega) \sum_{v \in \mathbb{Z}} \hat{f}(\omega + v) \hat{\phi}(\omega + v) = \hat{f}(\omega) \hat{\phi}(\omega) \hat{\phi}(\omega) \]

where we use the fact that \( \hat{\phi} \neq 0 \) only on \([-1/2, 1/2]^2\), and that \( \sum_{v \in \mathbb{Z}^2} \hat{f}(\omega + v) \hat{\phi}(\omega + v) \) is \( \mathbb{Z}^2 \)-periodic. Therefore

\[ \hat{\phi}(\omega) \hat{\phi}(\omega) \cdot \chi_{\Delta} = \chi_{\Delta}, \text{ a.e. }, \quad (8) \]

is a necessary and sufficient condition for \( \{ \tau, \phi, v \in \mathbb{Z}^2 \} \) to be a pseudoframe for \( V_0 \) with respect to \( \{ \tau, \tilde{\phi}, v \in \mathbb{Z}^2 \} \).

Direct calculation also shows that (7) is satisfied if \( \hat{h} \) and \( \hat{h} \) satisfy supported conditions specified in Theorem 1.

Thereof, \( \{ \tau, \phi, v \in \mathbb{Z}^2 \} \) and \( \{ \tau, \tilde{\phi}, v \in \mathbb{Z}^2 \} \) are a pair of multiple pseudoframes for \( \Omega \), where \( \Lambda = \{ 1, 2, 3 \} \).

Theorem 2 \cite{7}. Let \( \phi(s), \tilde{\phi}(s) \in L^2(\mathbb{R}^2) \) have the properties specified in Theorem 1 such that the condition (6) is satisfied. Assume that \( V_n \) is defined by (8). Then \( \{ V_n, \phi(s), \tilde{\phi}(s) \} \) forms a GMRA.

The familiar scaling relationships associated with MRAs between dilates of the function \( \phi(s) \) as well as that of \( \tilde{\phi}(s) \) still hold in GMRAs. Define filter functions \( B(\omega) \) and \( \tilde{B}(\omega) \), respectively by

\[ B(\omega) = \sum_{v \in \mathbb{Z}^2} b(v)e^{-2\pi iv\omega} \quad \text{and} \quad \tilde{B}(\omega) = \sum_{n \in \mathbb{Z}^2} \tilde{p}(n)e^{-2\pi in\omega} \]

of the two sequences \( B = \{ b(n) \} \) and \( \tilde{B} = \{ \tilde{b}(n) \} \), respectively, wherever the sum is defined.

Proposition 1 \cite{7} Let sequ. \( \{ b(v) \}_{v \in \mathbb{Z}^2} \) be such that \( B(0) = 2 \) and \( B(\omega) \neq 0 \) in a neighborhood of \( 0 \). Assume also that \( |B(\omega)| \leq 2 \). Then there exist \( \phi(s) \in L^2(\mathbb{R}^2) \) such that

\[ \phi(s) = 2 \sum_{v \in \mathbb{Z}^2} b(v)\phi(2s - v). \quad (9) \]

Similarly, there exist one scaling relationship for \( \tilde{\phi}(t) \) under the same conditions as that of \( p \) for a seq. \( p \), i.e.,

\[ \tilde{\phi}(s) = 2 \sum_{v \in \mathbb{Z}^2} \tilde{b}(n)\tilde{\phi}(2s - v). \quad (10) \]

\[ \overline{B}(\omega)\tilde{B}(\omega)\chi_{\mathbb{H}/2}(\omega) = 4\chi_{\mathbb{H}/2}(\omega) \quad \text{a.e..} \quad (11) \]

4. Affine dual frames of space \( L^1(\mathbb{R}^2) \)

Definition 2. Let \( \{ V_n, \phi(s), \tilde{\phi}(s) \} \) be a given GMRA, and let \( Y_1(s) \) and \( \tilde{Y}_1(s) \) be 2 band-pass functions in \( L^1(\mathbb{R}^2) \). We say \( \{ \tau_n\phi(s), \tau_nY_1(s), \tau_1, 2, 3 \} \) forms a pseudoframe of translates for \( V_1 \).
r. t. \{\tau_\nu \tilde{\phi}(s), \tau_\nu \tilde{Y}_i(s), i \in \{1, 2, 3\}\} for \forall f(s) \in V_1,
\[ f(s) = \sum_{v \in \mathbb{Z}} \left( f, \tau_\nu \tilde{\phi} \right) \tau_\nu \phi(s) + \sum_{v \in \mathbb{Z}} \sum_{t \in \mathbb{A}} \left( f, \tau_\nu \tilde{Y}_i \right) \tau_\nu \tilde{Y}_i(s). \]

**Proposition 2** [8]. Let \{V_n, \phi(s), \tilde{\phi}(s)\} be a given GMRA, and let \{\tau_\nu \phi(s), \tau_\nu \tilde{Y}_i(s), i = 1, 2, 3\} \ be a pseudoframe of translates for \(V_1\) with respect to \{\tau_\nu \phi(s), \tau_\nu \tilde{Y}_i(s), i = 1, 2, 3\}. Then, for every \(n \in \mathbb{Z}\), the family of binary functions \{\phi_{n,k}, \tilde{Y}_{i,n,k}, t \in \Lambda\} forms a pseudoframe of translates for \(V_{n+1}\) with respect to \{\phi_{n,k}, \tilde{Y}_{i,n,k}, t \in \Lambda\}, i.e., \(\forall f(s) \in V_{n+1}\),
\[ f(s) = \sum_{k \in \mathbb{Z}^2} \left( f, \phi_{n,k} \right) \phi_{n,k} + \sum_{k \in \mathbb{Z}^2} \sum_{t \in \mathbb{A}} \left( f, \tilde{Y}_{i,n,k} \right) \tilde{Y}_{i,n,k}(s). \]

To characterize the condition for which \{\tau_\nu \phi(s), \tau_\nu \tilde{Y}_i(s): t \in \Lambda\} forms a pseudoframe of translates for \(V_1\) with respect to \{\tau_\nu \phi(s), \tau_\nu \tilde{Y}_i(s): t \in \Lambda\}, we begin with developing the “wavelet equations” associated with “band-pass” functions \(Y_i(s) (t \in \Lambda)\) and \(\tilde{Y}_i(s) (t \in \Lambda)\) based on a GMRA, namely,
\[ Y_i(s) = 2 \sum_{v \in \mathbb{Z}^2} q_i(v) \phi(2s - v) \text{ in } L^2(R^2), \quad (13) \]
\[ \tilde{Y}_i(s) = 2 \sum_{v \in \mathbb{Z}^2} q_i(v) \tilde{\phi}(2s - v) \text{ in } L^2(R^2). \quad (14) \]

Similar to Proposition 3, we have the following fact.

**Proposition 4** [7]. Let \{q_i(k), t \in \Lambda\} be such that \(Q,(0) = 0\) and \(Q_i(\omega) \in L^\omega(T)\), where \(T = [0, 1]^2\). Let \(\phi(s) \in L^r(R^2)\) be defined by (10). Assume that \{b(k)\} satisfies the conditions in Proposition 3. Then there exist \(Y_i(s) \in L^r(R^2), t \in \Lambda\) generated from (14).

Let \(\chi_\Delta(\omega)\) be the characteristic function of the interval \(\Delta\) defined in Proposition 1.
\(\Gamma_\Delta(\omega) \equiv \sum_{k \in \mathbb{Z}^2} \chi_\Delta(\omega + k)\).

**Theorem 3** [8]. Let \(\Delta\) be the bandwidth of the subspace \(V_0\) defined in Theorem 1. \{\tau_\nu \phi(s), \tau_\nu Y_i(s), t \in \Lambda\} forms a pseudoframe of translates for \(V_1\) with respect to \{\tau_\nu \phi(s), \tau_\nu \tilde{Y}_i(s), t \in \Lambda\} if and only if there exist functions \(P(\omega)\) and \(\{D_i(\omega), t \in \Lambda\}\) in \(L^2([0, 1]^2)\) such that
\[ \overline{P(\omega)\tilde{B}(\omega)} + \sum_{t \in \Lambda} D_i(\omega)\tilde{Q}_i(\omega)\Gamma_\Delta(\omega) = 4\Gamma_\Delta(\omega); \quad (16.1) \]
\[ \overline{P(\omega)\tilde{B} \left( \omega + \frac{1}{2} \right)} + \sum_{t \in \Lambda} D_i(\omega)\tilde{Q}_i \left( \omega + \frac{1}{2} \right)\Gamma_\Delta(\omega) = 0. \quad (16.2) \]

**Theorem 4.** Let \(\phi(s), \tilde{\phi}(s), Y_i(s)\) and \(\tilde{Y}_i(s), t \in \Lambda\) be functions in \(L^2(R^2)\) defined by (14), (15), (18) and (19), respectively. Assume that conditions in Theorem 3 are satisfied. Then, for any function \(\Gamma(s) \in L^2(R^2)\), and any integer \(n\),
\[
\sum_{k \in \mathbb{Z}^2} \left\langle \Gamma, \tilde{\phi}_{n,k} \right\rangle \phi_{n,k} (s) = \sum_{i=1}^{n-1} \sum_{v=-\infty}^{\infty} \sum_{k \in \mathbb{Z}^2} \left\langle \Gamma, \tilde{Y}_{cv,k} \right\rangle Y_{cv,k} (s). 
\] (18)

Furthermore, for any \( \Gamma(s) \in L^2(R^2) \),
\[
\Gamma(s) = \sum_{i=1}^{\infty} \sum_{v=-\infty}^{\infty} \sum_{k \in \mathbb{Z}^2} \left\langle \Gamma, \tilde{Y}_{cv,k} \right\rangle Y_{cv,k} (s). 
\] (19)

Consequently, if \{ \text{Y}_{cv,k} \} and \{ \tilde{\text{Y}}_{cv,k} \}, \( (i \in \Lambda, v \in \mathbb{Z}, k \in \mathbb{Z}^2) \) are also Bessel sequences, they are a pair of affine frames for \( L^2(R^2) \).

**Proof.** (i) Consider, for \( \sigma \geq 0 \), \( \sigma \in \mathbb{Z} \), the operator \( E_\sigma : L^2(R^2) \rightarrow L^2(R^2) \) such that
\[
E_\sigma \Gamma(s) = \sum_{k \in \mathbb{Z}^2} \left\langle \Gamma, \tilde{\phi}_{(-\sigma),k} \right\rangle \phi_{(-\sigma),k} (s).
\]

Then the operator \( E_\sigma \) are well defined and uniformly bounded in the norm on \( L^2(R^2) \). To show that the limit \( E_\sigma \rightarrow 0 \) as \( \sigma \rightarrow \infty \), it is sufficient to show that, for all \( g(s) \) in any dense subspace of bandlimited functions in \( L^2(R^2) \),
\[
\sum_{k \in \mathbb{Z}^2} \left\langle g, \tilde{\phi}_{(-\sigma),k} \right\rangle \phi_{(-\sigma),k} \rightarrow 0 \text{ as } \sigma \rightarrow \infty.
\]

In particular, we may choose the dense set of functions \( g(t) \), whose Fourier transform have compact support, is continuous, and vanishes in a neighborhood of 0.

\[
\left\| \sum_{k} \left\langle g, \tilde{\phi}_{(-\sigma),k} \right\rangle \phi_{(-\sigma),k} \right\| \leq B^{\frac{1}{2}} \left( \sum_{k} \left\| \left\langle g, \tilde{\phi}_{(-\sigma),k} \right\rangle \phi_{(-\sigma),k} \right\|^2 \right)^{\frac{1}{2}}
\] (20)

where \( B \) is the Bessel bound of \{ \phi_{(-\sigma),k} \}. Taking standard calculation of the right-hand side of (20),
\[
\sum_{k} \left\| \left\langle g, \tilde{\phi}_{(-\sigma),k} \right\rangle \phi_{(-\sigma),k} \right\|^2 \leq \left( B^* \right)^{\frac{1}{2}} \left( \int \left| \hat{f}(\omega + 4^n u) \right|^2 \right)^{\frac{1}{2}} \cdot \left( \int \left| \hat{\phi}(4^n \omega) \right|^2 g(\omega) d\omega \right) \text{ where } B^* \text{ is the Bessel bound of } \{ \phi_{(-\sigma),k} \}. \]

Following the lead of [8] and since \( \hat{f}(\omega) \) is continuous with compact support, the term \( \int \left| \hat{f}(4^n \omega) \right|^2 \right\|^2 \leq C^2 \leq +\infty \), being a Riemann sum to the finite integral \( \int [\hat{g}(\omega + t)]^2 dt \).

Moreover, since \( \hat{g}(\omega) \) vanishes in a neighborhood of 0 for all \( \|\omega\| < \delta_\omega \), we get that
\[
\sum_{k} \left\| \left\langle g, \tilde{\phi}_{(-\sigma),k} \right\rangle \phi_{(-\sigma),k} \right\|^2 \leq \left( B^* \right)^{\frac{1}{2}} \sum_{n} \left( \int \left| \hat{\phi}(4^n \omega) \right|^2 g(\omega) d\omega \right)^{\frac{1}{2}}
\]

\[
\leq \left( B^* \right)^{\frac{1}{2}} \int |g|^2 \left( \int_{|\omega| > \delta_\omega} \left| \hat{\phi}(4^n \omega) \right|^2 d\omega \right)^{\frac{1}{2}}
\]

Note that the last integral at the right-hand side tends to 0 as \( \sigma \rightarrow +\infty \). This proves the first part of the theorem since, by using (18) recursively, we have \( \Gamma_\sigma (t) = \)
\[
\sum_{k \in \mathbb{Z}^2} \langle \Gamma, \tilde{\phi}_{n,k} \rangle \phi_{n,k}(s) - \sum_{l=1}^{n-1} \sum_{v \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^2} \langle \Gamma, \tilde{Y}_{tv,k} \rangle Y_{tv,k}(t).
\]

(ii) Since \( \overline{W} = L^2(R^2) \), for any \( h \in L^2(R^2) \) and any \( \varepsilon > 0 \) there exists \( n_0 = n_0(\varepsilon) > 0 \), and for any \( n > n_0 \) there exists \( g \in V_{n_0} \subset V_n \) s.t. \( g(s) = \sum_{k \in \mathbb{Z}^2} \langle g, \tilde{\phi}_{v,k} \rangle \phi_{v,k}(t) \). Moreover, for \( C = \sqrt{BB^*} \),
\[
\|h - g\|_2 < (1 + C)^{-1} \varepsilon.
\]
Now, by (18), for all \( n > n_0 \), we have
\[
\|h - g\|_2 \leq \|h - g\|_2 + C\|h - g\|_2 = \|h - g\|_2 (1 + C) < \varepsilon.
\]
If \( \{Y_{tv,k}\} \) and \( \{	ilde{Y}_{tv,k}\} \) are Bessel sequences, then equation (19) implies that both \( \{Y_{tv,k}\} \) and \( \{	ilde{Y}_{tv,k}\} \) will be affine frames. In fact, the lower frame bound of \( \{Y_{tv,k}\} \) and \( \{	ilde{Y}_{tv,k}\} \) is implied by the upper Bessel bound of the other. Therefore, this completes the proof of the second part of Theorem 4.

**Theorem 5.** If \( \{\Gamma_\beta(x), \beta \in \mathbb{Z}^n_+\} \) be a family of multivariate vector-valued wavelet packets with respect to the orthogonal vector-valued scaling function \( \Phi(x) \), then, for \( \beta, \gamma \in \mathbb{Z}^n_+ \), we have
\[
\langle \Gamma_\beta(), \Gamma_\gamma(-k) \rangle = \delta_{\beta, \gamma} \delta_{\beta, k} I_{\mathbb{Z}^n} , k \in \mathbb{Z}^n .
\]

5.Conclusion

A sort of pseudoframes for the subspaces of \( L^p(R^2) \) are characterized, and a method for constructing a GMRA of Paley-Wiener subspaces of \( L^p(R^2) \) is presented. The pyramid decomposition scheme is derived based on such a GMRA.

References


