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Two-fluid oscillatory flow in a channel

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Abstract The validity of Navier's partial slip condition is investigated by studying the oscillatory flow in a coated channel. The two-fluid model is used to solve the unsteady viscous equations exactly. Partial slip is experienced by the core fluid. It is found that Naviers condition does not hold for an unsteady core fluid. [doi:10.1063/2.1103207]

Keywords Navier's condition, slip, two-fluid, oscillatory

Apparent partial slip for viscous flow over solid surfaces occurs in many important applications. In all cases the bulk or core fluid experiences partial slip due to surface conditions. These include flow of rarefied gasses,[1] flow past rough or porous surfaces,[2] chemically treated hydrophobic surfaces and micro-hydrophobic surfaces,[3] dense particulate fluids such as emulsions, suspensions, foam and polymer solutions,[4] including blood. The partial slip condition, originally proposed by Navier,[5] states that the slip velocity u'_s is proportional to the local shear stress

$$u'_s = N\tau'_{ns}, \tag{1}$$

where N is a constant. Equation (1) is experimentally established for rarefied gasses and can be proven theoretically for the apparent slip of a steady bulk fluid bounded by minute grooved surfaces[2] and also easily proven for lubricated surfaces using two fluid layers. Navier's condition is thus firmly established for steady partial slip problems.

Recently, Naviers slip condition was also used by some authors for unsteady viscous flows.[6–12] But can Navier slip condition be applied to unsteady flows?

In order to address this question, we turn to a counter example of oscillatory viscous flow in a surface coated channel. Owing to the lowered viscosity of the coating, the bulk fluid appears to experience apparent partial slip. We shall use a two-fluid model which has successfully predicted apparent partial slip of particulate solutions,[4] especially blood.[13]

Figure 1 shows a channel of width $2L$ with axes x', y' placed on the center plane of the channel. The core fluid (of thickness $2\lambda L$) has higher viscosity than the fluid layers along the walls. Let subscript 1 denote the core fluid and subscript 2 denote the boundary fluid. For long channels, the unsteady Navier-Stokes equations are reduced to

$$u_1{}'_{t'} = -p'_{x'}/\rho_1 + \nu_1 u_1{}'_{y'y'}, \tag{2}$$

$$u_2{}'_{t'} = -p'_{x'}/\rho_2 + \nu_2 u_2{}'_{y'y'}. \tag{3}$$

Here is u' the axial velocity, t' is the time, ρ is the density and ν is the kinematic viscosity. The oscillatory pressure gradient is given as

$$-p'_{x'} = G\cos(\omega t'), \tag{4}$$

where G is a constant and ω is the frequency. Normalize all lengths by L, the velocities by $GL^2/(\rho_1\nu_1)$ and drop primes. Equations (2) and (3) become

$$s^2 u_{1t} = \mathrm{e}^{\mathrm{i}t} + u_{1yy}, \quad |y| \le \lambda, \tag{5}$$

$$\beta^2 s^2 u_{2t} = \gamma\beta^2 \mathrm{e}^{\mathrm{i}t} + u_{2yy}, \quad \lambda \le |y| \le 1. \tag{6}$$

Here $s^2 = \omega L^2/\nu_1$ is a non-dimensional parameter representing unsteadiness, $\beta^2 = \nu_1/\nu_2$, $\rho = \rho_1/\rho_2$, $\mathrm{i} = \sqrt{-1}$ and only the real part of any physical quantity is relevant. The boundary conditions are that u_2 is zero on the walls and that the velocities and shear stresses match between the two fluids

$$u_1(\lambda,t) = u_2(\lambda,t), \quad \gamma\beta^2 u_{1y}(\lambda,t) = u_{2y}(\lambda,t). \tag{7}$$

Let

$$u_1 = \mathrm{e}^{\mathrm{i}t} f_1(y), \quad u_2 = \mathrm{e}^{\mathrm{i}t} f_2(y). \tag{8}$$

The solution to Eq. (5) which is even in y is

$$f_1 = -\mathrm{i}/s^2 + C_1\cosh(\sqrt{\mathrm{i}}sy). \tag{9}$$

The solution to Eq. (6) which is zero on the wall is

$$f_2 = -\mathrm{i}\gamma(1 - e^{\sqrt{\mathrm{i}}s\beta(y-1)})/s^2 + C_2\sinh[\sqrt{\mathrm{i}}s\beta(y-1)]. \tag{10}$$

The complex constants C_1, C_2 are determined by Eqs. (7) and(8). We find

$$C_1 = \mathrm{i}\{\gamma - (\gamma-1)\cosh[\beta(\lambda-1)s\sqrt{\mathrm{i}}]\}\cdot \Big\{s^2\{\cosh[\beta(\lambda-1)s\sqrt{\mathrm{i}}]\cosh(\lambda s\sqrt{\mathrm{i}}) - \beta\gamma\sinh[\beta(\lambda-1)s\sqrt{\mathrm{i}}]\sinh(\lambda s\sqrt{\mathrm{i}})\}\Big\}^{-1}, \tag{11}$$

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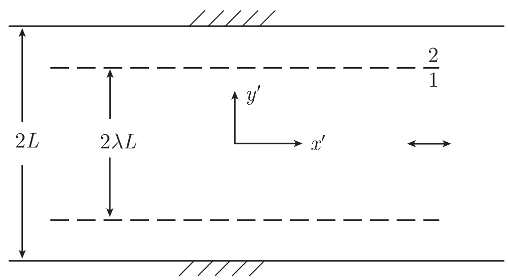

Fig. 1. Coated channel

$$C_2 = \left\{-\mathrm{i}\gamma\{\mathrm{e}^{\beta(\lambda-1)s\sqrt{\mathrm{i}}}\cosh(\lambda s\sqrt{\mathrm{i}})\} - \beta[1-(1-\mathrm{e}^{\beta(\lambda-1)s\sqrt{\mathrm{i}}})\gamma]\sinh(\lambda s\sqrt{\mathrm{i}})\right\}\cdot \left\{s^2\{\cosh[\beta(\lambda-1)s\sqrt{\mathrm{i}}]\cosh(\lambda s\sqrt{\mathrm{i}}) - \beta\gamma\sinh[\beta(\lambda-1)s\sqrt{\mathrm{i}}]\sinh(\lambda s\sqrt{\mathrm{i}})\}\right\}^{-1}. \quad (12)$$

After the velocity is obtained, the instantaneous flow rate, normalized by $GL^3/(\rho_1\nu_1)$, is

$$Q = \int_0^\lambda u_1\mathrm{d}y + \int_\lambda^1 u_2\mathrm{d}y = \mathrm{e}^{\mathrm{i}t}\left\{\frac{-\mathrm{i}\lambda}{s^2} + C_1\frac{\sinh(\lambda s\sqrt{\mathrm{i}})}{s\sqrt{\mathrm{i}}} + C_2\frac{1-\cosh[\beta(\lambda-1)s\sqrt{\mathrm{i}}]}{\beta s\sqrt{\mathrm{i}}} - \frac{\mathrm{i}\gamma}{s_2}\left[1-\lambda-\frac{1-\mathrm{e}^{\beta(\lambda-1)s\sqrt{\mathrm{i}}}}{\beta s\sqrt{\mathrm{i}}}\right]\right\}. \quad (13)$$

The velocity magnitudes and the flow rate decrease rapidly as the frequency parameters increases.

Owing to the coated walls, apparent partial slip is experienced by the core fluid. The normalized slip velocity and the local shear stress of the core on the core boundary are

$$V = u_1(\lambda, t), \quad \tau = u_{1y}(\lambda, t). \quad (14)$$

Figure 2 shows typical results for a complete period of time. The slip velocity and the shear stress are almost in phase for low frequencies but they are quite out of phase for higher frequencies. Even more revealing is the ratio V/τ which should be a constant if the Naviers condition were valid. Figure 3 shows that the ratio is somewhat constant for $s = 1$ but varies dramatically for $s = 10$. In fact only in the steady flow limit ($s < 0.01$) does the ratio remain truly constant. Thus with the surface dynamics (in our case, Region 2 viscous layer) ignored, Naviers condition cannot be applied to a core fluid in unsteady slip flow problems. Our analysis can be applied to oscillatory or pulsatile flows in microchannels.[3,10]

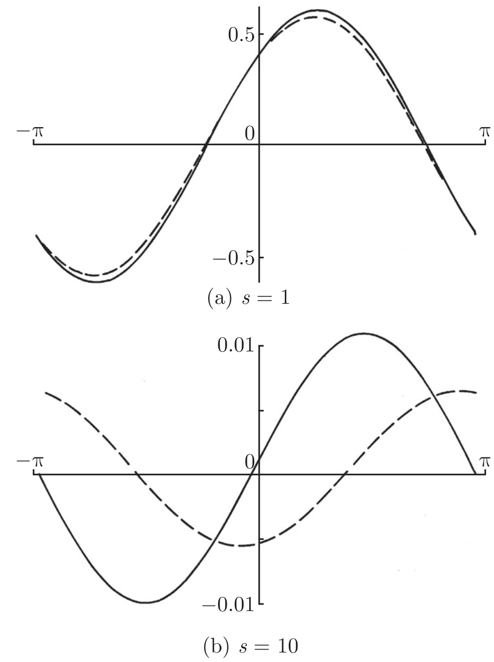

Fig. 2. Normalized slip velocity (solid line) and local shear stress (dashed line) of the core fluid.

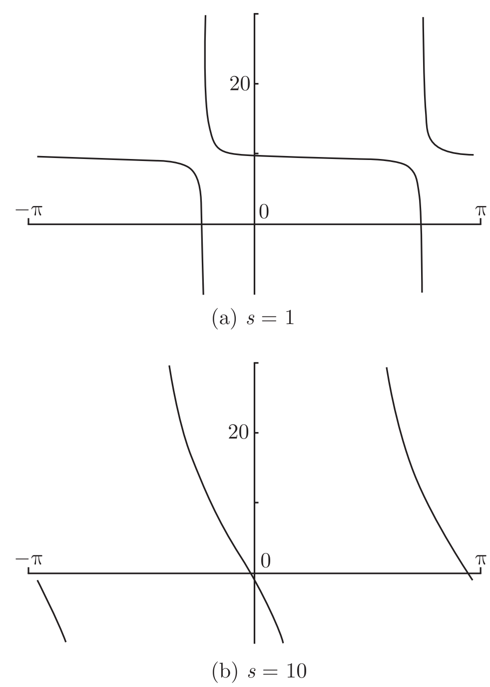

Fig. 3. The ratio of slip velocity to local shear stress of the core fluid.

1. F. Sharipov, and V. Seleznev. J. Phys. Chem. Ref. Data **27**, 657 (1998).
2. C. Y. Wang, Phys. Fluids **15**, 1114 (2003).
3. D. C. Tretheway, and C. D. Meinhart, Phys. Fluids, **14**, L9 (2002).
4. A. Yoshimura, and R. K. Prudhomme, J. Rheol. **32**, 53 (1988).
5. C. L. M. Navier, C. R. Acad. Sci. Fr. **6**, 389 (1827).
6. A. R. A. Khaled, and K. Vafai, Int. J. Nonlinear Mech. **39**, 795 (2004).

7. J. Zhang, M.J. Miksis, and S.G. Bankoff, Phys. Fluids **18**, 072106 (2006).
8. M. R. Mohyuddin, J. Mech. Eng. Sci. Tech. **21**, 829 (2007).
9. S. Asghar, K. Hanif, and T. Hayat, Meccanica **42**, 141 (2007).
10. C. I. Chen, C. K. Chen, and H. J. Lin, J. Appl. Mech. **75**, 014506 (2008).
11. S. Mukhopadhyay, and H. I. Andersson. Heat Mass Trans. **45**, 1447 (2009).
12. Z. Abbas, Y. Wang, T. Hayat, and M. Oberlack. Int. J. Num. Meth. Fluids **59**, 443 (2009).
13. C. Y. Wang, and J.B. Bassingthwaighte, J. Biomech **125**, 910 (2003).