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Large sets of extended directed triple systems with even orders ☆

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ABSTRACT

For three types of triples: unordered, cyclic and transitive, the corresponding extended triple, extended triple system and their large sets are introduced. The existence of *LESTS*(v) and *LEMTS*(v) were completely solved. In this paper, we shall discuss the existence problem of *LEDTS*(v) and give the following conclusion: there exists an *LEDTS*(v) for any even v except $v = 4$. The existence of *LEDTS*(v) with odd order v will be discussed in another paper, we are working at it.

1. Introduction

Let x, y, z be distinct elements in a finite set X. A *triple* $\{x, y, z\}$ (or *cyclic triple* $\langle x, y, z\rangle$, or *transitive triple* (x, y, z)) on X is a set of three unordered pairs $\{x, y\}$, $\{y, z\}$, $\{z, x\}$ (or ordered pairs (x, y), (y, z), (z, x), or ordered pairs (x, y), (y, z), (x, z)) of X. For these (classical) triples, the elements in each pair and triple must be distinct. When this restriction is broken, we have the so-called *extended* unordered pair (or ordered pair) and *extended* triple (or extended cyclic triple, or extended transitive triple), which were firstly introduced by Johnson and Mendelsohn in 1972, see [5].

An *extended* Steiner (or Mendelsohn, or directed) triple system *ESTS*(v) (or *EMTS*(v), or *EDTS*(v)) is a pair $(X, \mathcal{A})$, where X is a v-set and $\mathcal{A}$ is a collection of extended triples (or cyclic triples, or transitive triples) on X, called *blocks*, such that every extended unordered (or ordered) pair of X belongs to exactly one block of $\mathcal{A}$. A *large set* of *ESTS*(v) (or *EMTS*(v), or *EDTS*(v)), denoted by *LESTS*(v) (or *LEMTS*(v), or *LEDTS*(v)), is a collection $\{(X, \mathcal{A}_k)\}_k$, where X is a v-set, each $(X, \mathcal{A}_k)$ is an *ESTS*(v) (or *EMTS*(v), or *EDTS*(v)) and these $\mathcal{A}_k$ form a partition of all extended triples (or cyclic triples, or transitive triples) on X. The types of extended triples (or cyclic triples, or transitive triples) and the extended pairs contained in them are listed in the following table.

System	Forms of triple	Pairs covered by triple	Number of triples in a v-set	Number of systems in a large set
ESTS	$S_1 : \{x, x, x\}$	$\{x, x\}$	v	v
	$S_2 : \{x, x, y\}$	$\{x, x\}, \{x, y\}$	$v(v-1)$	
	$S_3 : \{x, y, z\}$	$\{x, y\}, \{y, z\}, \{z, x\}$	$v(v-1)(v-2)/6$	
EMTS	$M_1 : \langle x, x, x\rangle$	(x, x)	v	v
	$M_2 : \langle x, x, y\rangle$	$(x, y), (y, x), (x, x)$	$v(v-1)$	
	$M_3 : \langle x, y, z\rangle$	$(x, y), (y, z), (z, x)$	$v(v-1)(v-2)/3$	

(*continued on next page*)

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System	Forms of triple	Pairs covered by triple	Number of triples in a v-set	Number of systems in a large set
	$D_1 : (x, x, x)$	(x, x)	v	
	$D_2 : (x, x, y)$	(x, x), (x, y)	$v(v-1)$	
EDTS	$D_3 : (x, y, y)$	(x, y), (y, y)	$v(v-1)$	$3v-2$
	$D_4 : (x, y, x)$	(x, y), (y, x), (x, x)	$v(v-1)$	
	$D_5 : (x, y, z)$	(x, y), (y, z), (x, z)	$v(v-1)(v-2)$	

The existence problem of extended Steiner triple system and extended Mendelsohn triple system have been solved in [1,2,5]. The existence problem of extended directed triple system with some additional conditions has also been discussed in [3,4]. In this paper, we will discuss the existence problems for the large sets of *ESTS*, *EMTS* and *EDTS*. For the last designs, i.e., *LEDTS*(v), our conclusion is: there exists an *LEDTS*(v) for any even v except $v = 4$. The existence of *LEDTS*(v) with odd order v will be discussed in another paper, we are working at it.

Theorem 1.1. *There exists an LESTS*(v) *for any integer* $v \geq 1$.

Proof. For $v \equiv 1, 2 \bmod 3$, the collection $\{(Z_v, \mathcal{A}_x) : x \in Z_v\}$ forms an *LESTS*(v), where

$$\mathcal{A}_0 = \{\{i, j, k\} : i + j + k \equiv 0 \bmod v\}, \qquad \mathcal{A}_x = \mathcal{A}_0 + x, \quad x \in Z_v.$$

For $v \equiv 0 \bmod 3$, the collection $\{(Z_v, \mathcal{A}_{s,x}) : x \in Z_{v/3}, 0 \leq s \leq 2\}$ forms an *LESTS*(v), where

$$\mathcal{A}_{s,0} = \{\{i, j, k\} : i + j + k \equiv s \bmod v\}, \quad 0 \leq s \leq 2,$$

$\mathcal{A}_{s,x} = \{B + x : B \in \mathcal{A}_{s,0}\}$, where $(i, j, k) + x = (i + x, j + x, k + x)$ for $i, j, k \in Z_v$, the addition is taken modulo v, $x \in Z_{v/3}$, $0 \leq s \leq 2$. ■

In [9], Wang gave the existence spectrum for *LEMTS*(v). Here, we give a simpler proof.

Theorem 1.2. *There exists an LEMTS*(v) *for any integer* $v \geq 1$.

Proof. Let $\{(Z_v, \mathcal{A}_x) : x \in Z_v\}$ be an *LESTS*(v). Replace each (S_3 type's) extended triple $\{x, y, z\}$ in $\mathcal{A}_x$ by (M_3 type's) extended cyclic triples $\langle x, y, z\rangle$ and $\langle z, y, x\rangle$. As well, by replacing each (S_1 and S_2 type's) extended triples $\{x, x, x\}$ and $\{x, x, y\}$ by (M_1 and M_2 type's) extended cyclic triples $\langle x, x, x\rangle$ and $\langle x, x, y\rangle$, the triple system $\{(Z_v, \mathcal{A}_x) : x \in Z_v\}$ will become an *LEMTS*(v). ■

In this paper, we shall focus on the existence of *LEDTS*(v) with even orders v. Let k, g, n be positive integers. A k-*GDD*(g^n) is a triple $(\mathcal{V}, \mathcal{G}, \mathcal{B})$, where $\mathcal{V}$ is a gn-set, $\mathcal{G}$ is a partition of $\mathcal{V}$, which consists of n subsets (called *groups*) with size g, and $\mathcal{B}$ is a family of some subsets (called *blocks*) of $\mathcal{V}$ such that if $B \in \mathcal{B}$, then $|B| = k$ and every pair of distinct elements of $\mathcal{V}$ occurs in exactly one block or one group but not both.

Let K be a set of positive integers, $t, v, g_1, \ldots, g_r, n_1, \ldots, n_r$ be positive integers, s be a non-negative integer and $\sum_{i=1}^{r} n_i g_i = v - s$. A *candelabra t-system* (t, K)-*CS*$(v : s)$ or (t, K)-*CS*$(g_1^{n_1} g_2^{n_2} \cdots g_r^{n_r} : s)$, see [7], is a quadruple $(X, S, \mathcal{G}, \mathcal{A})$ that satisfying the following conditions:

(1) X is a v-set (called *points*), S is its s-subset (called *a stem*);
(2) $\mathcal{G}$ is a partition of $X \setminus S$, which consists of n_i subsets with size g_i (called *groups*);
(3) $\mathcal{A}$ is a family of some subsets of X, each member (called *block*) has the size from K;
(4) Every t-subset T of X is contained in exactly one block if $|T \cap (S \cup G)| < t$, $\forall G \in \mathcal{G}$, or in no block if $T \subseteq S \cup G$ for some $G \in \mathcal{G}$.

Especially, a (t, K)-*CS*$(1^v : 0)$ is just a t-wise balanced design $S(t, K, v)$, briefly denoted by t-*BD*, and a (t, k)-*CS*$(1^v : 0)$ is just a t-design $S(t, k, v)$.

$F(3, 3, g^n)$ is a triple $(X, \mathcal{G}, \mathcal{A})$ where X is a gn-set of points, $\mathcal{G}$ is a collection of n non-empty subsets (called *groups*) of size g of X which partition X, $\mathcal{A}$ is a collection of all triples satisfying each triple intersects any given group in at most one point and $\mathcal{A}$ can be partitioned into $gn\mathcal{A}_x$, $x \in G \in \mathcal{G}$ such that each $(X \setminus G, \mathcal{G} \setminus \{G\}, \mathcal{A}_x)$ is a 3-*GDD*(g^{n-1}).

Let v be a positive integer, X be a v-set, $\mathcal{G}$ be a partition of X, and $K_1, \ldots, K_s, K_{\mathcal{T}}$ be sets of positive integers. Suppose that $\mathcal{B}_1, \ldots, \mathcal{B}_s$ and $\mathcal{T}$ are collections of some subsets of X with size from $K_1, \ldots, K_s$ and $K_{\mathcal{T}}$ respectively. An *s-fan design* s-*FG*$(3, (K_1, K_2, \ldots, K_{\mathcal{T}}), v)$ is an $(s + 3)$-tuple $(X, \mathcal{G}, \mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_s, \mathcal{T})$, where $(X, \mathcal{G})$ is a 1-*BD*, $(X, \mathcal{G} \cup \mathcal{B}_i)$ is a 2-*BD* for each $1 \leq i \leq s$, and $(X, \mathcal{G} \cup (\cup_{i=1}^{s} \mathcal{B}_i) \cup \mathcal{T})$ is a 3-*BD*.

Below, I_n is an n-set, Z_n is a residual ring module n and F_q is a finite field of order q. Denote $Z_n^* = Z_n \setminus \{0\}$ and $F_q^* = F_q \setminus \{0\}$. Denote extended transitive triple by (a, b, c) or abc. For a family of extended transitive triples $\mathcal{A}$ on Z_n (or F_q) and $x, m \in Z_n$ (or F_q), denote

$$\mathcal{A} + x = \{(a + x, b + x, c + x) : (a, b, c) \in \mathcal{A}\}, \qquad m\mathcal{A} = \{(ma, mb, mc) : (a, b, c) \in \mathcal{A}\},$$
$$-\mathcal{A} = \{(-a, -b, -c) : (a, b, c) \in \mathcal{A}\} \quad \text{and} \quad \mathcal{A}^{-1} = \{(c, b, a) : (a, b, c) \in \mathcal{A}\}.$$

Definition 1.1. For positive integers n_i and g_i, $1 \le i \le r$, a *directed group divisible triple system DGDD*$(g_1^{n_1} \cdots g_r^{n_r})$ is a trio $(X, \mathcal{G}, \mathcal{A})$ satisfying the following conditions:
(1) X is a set containing $\sum_{i=1}^{r} n_i g_i$ *points*;
(2) $\mathcal{G}$ is a partition of X, which consists of n_i subsets of size g_i (called *groups*);
(3) $\mathcal{A}$ is a family of some transitive triples of X (called *blocks*) such that $|A \cap G| \le 1$, $\forall A \in \mathcal{A}$, $G \in \mathcal{G}$;
(4) Each ordered pair on X from distinct (or same) groups is contained in exactly one (or no) block.

Definition 1.2. For positive integers n, g, s and $s \ge 2$, a *PDGDD*$(g^n : s)$ is a trio $(X, \mathcal{G}, \mathcal{A})$ satisfying the following conditions:
(1) X is a set containing $ng + s$ *points*;
(2) $\mathcal{G} = \{G_0, G_1, \ldots, G_n\}$ forms a partition of X, where $G_i = \{a_{i,j} : j \in I_g\}$ is called *group*, $i \in Z_n$. $|G_0| = s$ and other $|G_i| = g$;
(3) $\mathcal{A}$ consists of all transitive triples on X, intersecting each group in at most one points. And, $\mathcal{A}$ can be partitioned into $\{\mathcal{B}_{i,j}^r : i \in I_n, j \in I_g, r \in I_3\} \cup \{\mathcal{C}_k : 1 \le k \le 3(s-2)\}$, where each $\mathcal{B}_{i,j}^r$ forms a *DGDD*$(g^{n-1}(s+1)^1)$ on $X \setminus (G_i \setminus \{a_{i,j}\})$ with the group set $(\mathcal{G} \setminus \{G_0, G_i\}) \cup \{G_0 \cup \{a_{i,j}\}\}$, and each $\mathcal{C}_k$ forms a *DGDD*(g^n) on $X \setminus G_0$ with the group set $\mathcal{G} \setminus \{G_0\}$.

Definition 1.3. For positive integers n, g and s, an *EDGDD*$(g^n s^1)$ (*extended directed group divisible triple system*) is a trio $(X, \mathcal{G}, \mathcal{A})$ satisfying the following conditions:
(1) X is a set containing $ng + s$ *points*;
(2) $\mathcal{G} = \{G_0, G_1, \ldots, G_n\}$ forms a partition of X, where G_i $(i \in Z_n)$ is called *group*. $|G_0| = s$ and other $|G_i| = g$;
(3) $\mathcal{A}$ is a family of extended transitive triples of X (called *blocks*) such that $A \not\subseteq G \cup S$ for any $A \in \mathcal{A}$ and $G \in \mathcal{G}$;
(4) Each ordered 2-subset (x, y) of X is contained in exactly one (or no) block of $\mathcal{A}$ if x, y in distinct (or same) groups;
(5) Each pair (x, x) is contained in exactly one (or no) block of $\mathcal{A}$ if $x \notin G_0$ (or $x \in G_0$).

Especially, an *EDGDD*$(1^{n-s}s^1) = (X, \mathcal{G}, \mathcal{A})$ is named as *EDTS*$(n, s) = (X, Y, \mathcal{A})$, where the long group $G_0 = Y$ with size s is called *hole*.

Definition 1.4. For positive integers $w < v$, let X be a v-set, Y be its w-subset. An *LEDTS*(v, w) is a collection $\{(X, Y, \mathcal{A}_i) : 1 \le i \le 3v - 2\}$ such that all extended transitive triples from X, not belonging to Y, are partitioned into $\mathcal{A}_i$, $1 \le i \le 3v - 2$, where each $(X, Y, \mathcal{A}_i)$ is an *EDTS*(v, w) for $1 \le i \le 3w - 2$ or an *EDTS*(v) for $3w - 1 \le i \le 3v - 2$. Obviously, *LEDTS*$(v, w) \cup$ *LEDTS*$(w) =$ *LEDTS*(v).

Definition 1.5. For positive integers n, g and s, a *PECS*$(g^n : s)$ is a quadruple $(X, S, \mathcal{G}, \mathcal{A})$ satisfying the following conditions:
(1) X is an $(ng + s)$-set, S is its s-subset (called *stem*);
(2) $\mathcal{G} = \{G_1, \ldots, G_n\}$ partition $X \setminus S$, where each G_i is a g-subset;
(3) $\mathcal{A}$ consists of all extended transitive triples from X, not belonging $S \cup G$, $\forall G \in \mathcal{G}$. $\mathcal{A}$ can be partitioned into $\{\mathcal{B}_{i,j}^r : i \in I_n, j \in I_g, r \in I_3\} \cup \{\mathcal{C}_k : 1 \le k \le 3s - 2\}$, where each $\mathcal{B}_{i,j}^r$ forms an *EDGDD*$(1^{g(n-1)}(g+s)^1)$ on X with the long group $G_i \cup S$, each $\mathcal{C}_k$ forms a *DGDD*(g^n) on $X \setminus S$ with the groups $\mathcal{G}$.

Definition 1.6. For positive integers n, g and non-negative integer s, a *PECS*$^*(g^n : s)$ is a quadruple $(X, S, \mathcal{G}, \mathcal{A})$ satisfying the following conditions:
(1) X is an $(ng + s)$-set, S is its s-subset (called *stem*);
(2) $\mathcal{G} = \{G_1, \ldots, G_n\}$ partition $X \setminus S$, where each $G_i = \{a_{i,j} : j \in I_g\}$ is a g-subset, $i \in I_n$;
(3) $\mathcal{A}$ consists of all transitive directed triples (called *blocks*), not belonging $S \cup G$, $\forall G \in \mathcal{G}$. $\mathcal{A}$ can be partitioned into $\{\mathcal{B}_{i,j}^r : i \in I_n, j \in I_g, r \in I_3\} \cup \{\mathcal{C}_k : 1 \le k \le 3s + 4\}$, where each $\mathcal{B}_{i,j}^r$ forms an *EDGDD*$(1^{g(n-1)}(g+s-1)^1)$ on $X \setminus \{a_{i,j}\}$ with the long group $(G_i \cup S) \setminus \{a_{i,j}\}$, and each $\mathcal{C}_k$ forms a *DGDD*(g^n) on $X \setminus S$ with the groups $\mathcal{G}$.

Definition 1.7. For positive integers n and g, a *DF*(g^n) is a trio $(X, \mathcal{G}, \mathcal{A})$ where X is a gn-set of *points*, $\mathcal{G}$ is a partition of X into n subsets (called *groups*) with size g, $\mathcal{A}$ is a collection of all transitive triples intersecting any given group in at most one point, and $\mathcal{A}$ can be partitioned into $3gn$ $\mathcal{A}_x^j$ such that each $(X \setminus G, \mathcal{G} \setminus \{G\}, \mathcal{A}_x^j)$ is a *DGDD*(g^{n-1}), where $x \in G \in \mathcal{G}$ and $j \in I_3$.

Lemma 1.1. *There exists a DF*(g^n) *for positive integers* g, n *satisfying the following conditions:*

(1) $n \equiv 1, 2 \bmod 3$; (2) $6|n$ and $3|g$; (3) $n \equiv 3 \bmod 6$, $n > 3$ and $6|g$.

Proof. By [8], there exists an *OLDTS*(n) if and only if $n \equiv 0, 1 \bmod 3$, and if there exists an *OLDTS*(n) then there exists a *DF*(g^{n+1}). So we can get the conclusion (1).

From [6], there exists an $F(3, 3, g^n) = (X, \mathcal{G}, \mathcal{A})$ for $2|gn$, $3|g(n-1)(n-2)$ and $n > 3$, $n \ne 5$. By the definition, $\mathcal{A}$ can be partitioned into $gn\mathcal{A}_x$, $x \in G \in \mathcal{G}$, such that each $(X \setminus G, \mathcal{G} \setminus \{G\}, \mathcal{A}_x)$ is a 3-*GDD*(g^{n-1}). For $x \in G \in \mathcal{G}$, define

$$\mathcal{A}_x^1 = \{(a, b, c), (c, b, a) : (a, b, c) \in \mathcal{A}_x\},$$
$$\mathcal{A}_x^2 = \{(a, c, b), (b, c, a) : (a, b, c) \in \mathcal{A}_x\},$$
$$\mathcal{A}_x^3 = \{(b, a, c), (c, a, b) : (a, b, c) \in \mathcal{A}_x\}.$$

It is easy to see that each $(X \setminus G, \mathcal{G} \setminus \{G\}, \mathcal{A}_x^j)$ is a *DGDD*(g^{n-1}) and these $\mathcal{A}_x^j$, $x \in G \in \mathcal{G}$, $j \in I_3$, form a *DF*(g^n) on X with the groups $\mathcal{G}$. Thus, we can get the conclusion (2) and (3) for the case $3|n$. ■

2. Recursive construction

Theorem 2.1. *If there exist a PECS*$(g^n : s)$, *an LEDTS*$(g+s, s)$ *and an LEDTS*$(g+s)$, *then there exists an LEDTS*$(gn+s)$.

Proof. Let $PECS(g^n : s) = (X, S, \mathcal{G}, \mathcal{A})$, where $|X| = gn+s$, $|S| = s$, $\mathcal{G} = \{G_i : i \in I_n\}$ and $|G_i| = g$. $\mathcal{A}$ consists of all extended transitive triples from X, not belonging any $S \cup G_i$. $\mathcal{A}$ can be partitioned into $\{\mathcal{B}^r_{i,j} : i \in I_n, j \in I_g, r \in I_3\} \cup \{\mathcal{B}_k : 1 \le k \le 3s-2\}$, where each $\mathcal{B}^r_{i,j}$ forms an $EDGDD(1^{g(n-1)}(g+s)^1)$ on X with the long group $G_i \cup S$, each $\mathcal{B}_k$ forms a $DGDD(g^n)$ on $X \setminus S$ with the groups $\mathcal{G}$.

By the assumption, there exists an $LEDTS(g+s, s)$ on $G_i \cup S$ for each $i \in I_n \setminus \{1\}$, which contains

$$3g \text{ disjoint } EDTS(g+s) = (G_i \cup S, \mathcal{C}^r_{i,j}), \quad j \in I_g, r \in I_3;$$
$$3s-2 \text{ disjoint } EDTS(g+s, s) = (G_i \cup S, \mathcal{D}_{i,k}), \quad 1 \le k \le 3s-2.$$

And, there exists an $LEDTS(g+s)$ on $G_1 \cup S$ which contains

$$3g \text{ disjoint } EDTS(g+s) = (G_1 \cup S, \mathcal{C}^r_{1,j}), \quad j \in I_g, r \in I_3;$$
$$3s-2 \text{ disjoint } EDTS(g+s) = (G_1 \cup S, \mathcal{E}_k), \quad 1 \le k \le 3s-2.$$

Now, define

$$\Gamma^r_{i,j} = \mathcal{B}^r_{i,j} \cup \mathcal{C}^r_{i,j}, \quad i \in I_n, j \in I_g, r \in I_3;$$
$$\Lambda_k = \left(\bigcup_{i=2}^{n} \mathcal{D}_{i,k}\right) \bigcup \mathcal{B}_k \bigcup \mathcal{E}_k, \quad 1 \le k \le 3s-2.$$

Then each $\Gamma^r_{i,j}(x)$ or Λ_k forms an $EDTS(gn+s)$ on $X \cup S$, and they form an $LEDTS(gn+s)$. ■

Theorem 2.2. *If there exist e-FG*$(3, (K_0, K_1, \ldots, K_{e-1}, K_{\mathcal{T}}), g^n)$, *PECS*$(m^k : r)\ \forall k \in K_1$, *DF*$(m^k)\ \forall k \in K_{\mathcal{T}}$, *and DF*$(m^{k_j+1})\ \forall k_j \in K_j,\ 2 \le j \le e$, *then there exists a PECS*$((mg)^n : (e-1)m + r)$.

Construction. Let $e\text{-}FG(3, (K_0, K_1, \ldots, K_{e-1}, K_{\mathcal{T}}), g^n) = (X, \mathcal{G}, \mathcal{A}_0, \mathcal{A}_1, \ldots, \mathcal{A}_{e-1}, \mathcal{T})$, where $\mathcal{G}$ is a partition of the gn-set X into n groups with size g. Denote $\mathcal{G}_A = \{\{x\} \times I_m : x \in A\}$ and $A' = A \times I_m$, where $A \subseteq X$. Let $S_0, S_1, \ldots, S_{e-1}$ and $X \times I_m$ be pairwise disjoint sets, where $S_0 = \{\infty\} \times Z_r$, $S_t = \{(\infty, r+(t-1)m), \ldots, (\infty, r+tm-1)\}$, $t \in Z_e^*$. Denote $S = \bigcup_{t \in Z_e} S_t$, $X' = (X \times I_m) \cup S$, $G' = G \times I_m$, $G \in \mathcal{G}$. By assumption, we can give the following designs (1)–(3):

(1) $PECS(m^{|A|} : r) = (A' \cup S_0, S_0, \mathcal{G}_A, \mathcal{B}_A)$ for each $A \in \mathcal{A}_0$, where $\mathcal{B}_A$ can be partitioned into $3m|A|$ disjoint $\mathcal{B}^j_{x,i}(A)$ and $3r-2$ disjoint $\mathcal{B}_k(A)$, $x \in A$, $i \in I_m$, $j \in I_3$, $1 \le k \le 3r-2$, such that each $\mathcal{B}^j_{x,i}(A)$ forms an $EDGDD(1^{m(|A|-1)}(m+r)^1)$ on $A' \cup S_0$ with the long group $(\{x\} \times I_m) \cup S_0$, and each $\mathcal{B}_k(A)$ forms a $DGDD(m^{|A|})$ on A' with the groups $\mathcal{G}_A$.

(2) $DF(m^{|A|+1}) = (A' \cup S_t, \mathcal{G}_A \cup S_t, \mathcal{C}_A)$ for each $A \in \mathcal{A}_t$, $t \in Z_e^*$, where $\mathcal{C}_A$ can be partitioned into $3m|A|$ disjoint $\mathcal{C}^j_{x,i}(t, A)$ and $3m$ disjoint $\mathcal{C}^j_i(t, A)$, $x \in A$, $i \in I_m$, $j \in I_3$, such that each $\mathcal{C}^j_{x,i}(t, A)$ forms a $DGDD(m^{|A|})$ on $((A \setminus \{x\}) \times I_m) \cup S_t$ with the groups $\mathcal{G}_{A\setminus\{x\}} \cup \{S_t\}$, and each $\mathcal{C}^j_i(t, A)$ forms a $DGDD(m^{|A|})$ on A' with the groups $\mathcal{G}_A$.

(3) $DF(m^{|A|}) = (A', \mathcal{G}_A, \mathcal{D}_A)$ for each $A \in \mathcal{T}$, where $\mathcal{D}_A$ can be partitioned into $3m|A|$ disjoint $\mathcal{D}^j_{x,i}(A)$, $x \in A, i \in I_m, j \in I_3$, such that each $\mathcal{D}^j_{x,i}(A)$ forms a $DGDD(m^{|A|-1})$ on $(A \setminus \{x\}) \times I_m$ with the groups $\mathcal{G}_{A\setminus\{x\}}$.

Now, for $x \in X$, $i \in I_m$, $j \in I_3$, $1 \le k \le 3r-2$ and $t \in Z_e^*$, define

$$\mathcal{F}^j_{x,i} = \left(\bigcup_{x \in A \in \mathcal{A}_0} \mathcal{B}^j_{x,i}(A)\right) \bigcup \left(\bigcup_{x \in A \in \mathcal{A}_t, t \in Z_e^*} \mathcal{C}^j_{x,i}(t, A)\right) \bigcup \left(\bigcup_{x \in A \in \mathcal{T}} \mathcal{D}^j_{x,i}(A)\right);$$
$$\mathcal{F}_k = \bigcup_{A \in \mathcal{A}_0} \mathcal{B}_k(A);$$
$$\mathcal{F}^j_{i,t} = \bigcup_{A \in \mathcal{A}_t} \mathcal{C}^j_i(t, A).$$

Then, $\mathcal{F} = \{\mathcal{F}^j_{x,i}, x \in X, i \in I_m, j \in I_3\} \cup \{\mathcal{F}_k, 1 \le k \le 3r-2\} \cup \{\mathcal{F}^j_{i,t}, i \in I_m, j \in I_3, t \in Z_e^*\}$ forms a desired $PECS((mg)^n : (e-1)m + r)$ on X' with the groups $\{G' : G \in \mathcal{G}\}$ and the stem S.

Proof. (1) Each $\mathcal{F}^j_{x,i}$ ($x \in X, i \in I_m, j \in I_3$) forms an $EDGDD(1^{mg(n-1)}(mg+s)^1)$ on X' with the long group $G' \cup S$, where $x \in G \in \mathcal{G}$. In fact, any extended ordered pair $P = \{(\alpha, a), (\beta, b)\} \not\subset G' \cup S$ occurs exactly one block of $\mathcal{F}^j_{x,i}$:

$*$ Case $\infty \in \{\alpha, \beta\}$. If $\alpha = \infty$ ($\beta = \infty$ is similar). Then $(\alpha, a) \in S$ and $\beta \notin G$.

When $(\alpha, a) \in S_0$, there exists the unique block A in $\mathcal{A}_0$ containing x and β, since $\mathcal{A}_0$ forms a $GDD(g^n)$ on X. Then, there exists the unique block in $\mathcal{B}^j_{x,i}(A)$ containing P, since $\mathcal{B}^j_{x,i}(A)$ forms an $EDGDD(1^{m(|A|-1)}(m+r)^1)$ on $A' \cup S_0$ with the long group $(\{x\} \times I_m) \cup S_0$. Further, let us show the uniqueness for the block containing P. Suppose that there exists another block $C \in \mathcal{F}_{x,i}^j$ containing P. Since $(\alpha, a) \in S_0$, C must belong $\bigcup_{x \in A \in \mathcal{A}_0} \mathcal{B}_{x,i}^j(A)$. Then, there must be some $A_1 \in \mathcal{A}_0$ such that $C \in \mathcal{B}_{x,i}^j(A_1)$ and $\{x, \beta\} \subset A_1$. Since $\mathcal{A}_0$ forms a $GDD(g^n)$ on X and $\{x, \beta\} \subset A$, we have $A_1 = A$, i.e., $C \in \mathcal{B}_{x,i}^j(A)$. However, in $\mathcal{B}_{x,i}^j(A)$, the block containing P is unique.

When $(\alpha, a) \in S_t$ $(t \in Z_e^*)$, there exists the unique block $A \in \mathcal{A}_t$ containing x and β, since $\mathcal{A}_t$ $(t \in Z_e^*)$ forms a $GDD(g^n)$ on X. Then, there exists the unique block in $\mathcal{C}_{x,i}^j(t, A)$ containing P, since $\mathcal{C}_{x,i}^j(t, A)$ forms a $DGDD(m^{|A|})$ on $((A \setminus \{x\}) \times I_m) \cup S_t$ with the groups $\mathcal{G}_{A \setminus \{x\}} \cup \{S_t\}$. Similarly, we can show the uniqueness for the block containing P.

* Case $\infty \notin \{\alpha, \beta\}$. If $\alpha = \beta$ or $\alpha = x$ ($\beta = x$ is similar), then $\beta \notin G$. Since there exists the unique block $A \in \mathcal{A}_0$ containing x and β, there exists the unique block in $\mathcal{B}_{x,i}^j(A)$ containing P. If $\alpha \neq \beta$ and $x \notin \{\alpha, \beta\}$, then $\{x, \alpha, \beta\}$ is contained in the unique block $A \in (\bigcup_{t \in Z_e} \mathcal{A}_t) \bigcup \mathcal{T}$. Then,

$A \in \mathcal{A}_0 \longrightarrow$ there exists the unique block in $\mathcal{B}_{x,i}^j(A)$ containing P.

$A \in \mathcal{A}_t$ $(t \in Z_e^*) \longrightarrow$ there exists a unique block in $\mathcal{C}_{x,i}^j(t, A)$ containing P.

$A \in \mathcal{T} \longrightarrow$ there exists the unique block in $\mathcal{D}_{x,i}^j(A)$ containing P, since $((A \setminus \{x\}) \times Z_m, \mathcal{G}_{A \setminus \{x\}}, \mathcal{D}_{x,i}^j(A))$ is a $DGDD(m^{|A|-1})$.

The uniqueness for the block containing P can be similarly shown.

(2) Each $\mathcal{F}_{i,t}^j$ or $\mathcal{F}_k$ $(i \in I_m, j \in I_3, t \in Z_e^*, 1 \le k \le 3r - 2)$ forms a $DGDD((mg)^n)$ on $X \times I_m$. In fact, for any ordered pair $P = \{(\alpha, a), (\beta, b)\}$ from distinct groups,

* There exists the unique block $A \in \mathcal{A}_t$ containing α, β. And, by the construction, $\mathcal{C}_i^j(t, A)$ forms a $DGDD(m^{|A|})$ on A' with the groups $\mathcal{G}_A$. So, there exists the unique block in $\mathcal{C}_i^j(t, A) \subset \mathcal{F}_{i,t}^j$ containing P.

* There exists the unique block $A \in \mathcal{A}_0$ containing α, β. And, by the construction, $\mathcal{B}_k(A)$ forms a $DGDD(m^{|A|})$ on A' with the groups $\mathcal{G}_A$. So, there exists the unique block in $\mathcal{B}_k(A) \subset \mathcal{F}_k$ containing P.

(3) Any extended transitive triple $T = \{(\alpha, a), (\beta, b), (\gamma, c)\} \not\subset G' \cup S$, $\forall G \in \mathcal{G}$, belongs $\mathcal{F}$. In fact,

* $\alpha = \infty$ (or $\infty \in \{\beta, \gamma\}$). Then $(\alpha, a) \in S$ and β, γ are in distinct groups. When $(\alpha, a) \in S_0$, there exists the unique block $A \in \mathcal{A}_0$ containing β and γ. And, by the construction, $\mathcal{B}_A$ forms a $PECS(m^{|A|} : r)$ on $(A \times I_m) \cup S_0$, so $T \in \mathcal{B}_A \subset \mathcal{F}$. When $(\alpha, a) \in S_t$ $(t \in Z_e^*)$, there exists the unique block $A \in \mathcal{A}_t$ containing β and γ. And, by the construction, $\mathcal{C}_A$ forms a $DF(m^{|A|+1})$ with group set $\mathcal{G}_A \cup S_t$, so $T \in \mathcal{C}_A \subset \mathcal{F}$.

* $\infty \notin \{\alpha, \beta, \gamma\}$. By the definition of e-$FG(3, (K_0, K_1, \ldots, K_{e-1}, K_{\mathcal{T}}), g^n)$, there exists $A \in (\bigcup_{t \in Z_e} \mathcal{A}_t) \bigcup \mathcal{T}$ such that $\{\alpha, \beta, \gamma\} \subseteq A$. Therefore, $T \in \mathcal{B}_A \cup \mathcal{C}_A \cup \mathcal{D}_A \subset \mathcal{F}$. ■

Theorem 2.3. *If there exist* 2- $FG(3, (K_{\mathcal{B}}, K_{\mathcal{C}}, K_{\mathcal{D}}), g^n)$, $PECS^*(m^k : r)$ $\forall k \in K_{\mathcal{B}}$, $PDGDD(m^k : s)$ $\forall k \in K_{\mathcal{C}}$ *and* $DF(m^k)$ $\forall k \in K_{\mathcal{D}}$, *then there exists a* $PECS((mg)^n : r + s)$.

Proof. Let 2-$FG(3, (K_{\mathcal{B}}, K_{\mathcal{C}}, K_{\mathcal{D}}), g^n) = (X, \mathcal{G}, \mathcal{B}, \mathcal{C}, \mathcal{D})$, where $\mathcal{G}$ is a partition of the gn-set X into n groups with size g. Denote $\mathcal{G}_A = \{\{x\} \times I_m : x \in A\}$ where $A \subseteq X$. Let R, S and $X \times I_m$ are pairwise disjoint sets where $|R| = r$, $|S| = s$. By assumption, we can give the following designs (1)–(3):

(1) $PECS^*(m^{|A|} : r) = ((A \times I_m) \cup R, R, \mathcal{G}_A, \mathcal{B}_A)$ for each $A \in \mathcal{B}$, where $\mathcal{B}_A$ can be partitioned into $3m|A|$ disjoint $\mathcal{B}_{x,i}^j(A)$ and $3r + 4$ disjoint $\mathcal{B}_k(A)$, $x \in A$, $i \in I_m$, $j \in I_3$, $1 \le k \le 3r + 4$, such that each $\mathcal{B}_{x,i}^j(A)$ forms an $EDGDD(1^{m(|A|-1)}(m + r - 1)^1)$ on $((A \times I_m) \cup R) \setminus \{x_i\}$ with the long group $((\{x\} \times (I_m \setminus \{i\}))) \cup R$, and each $\mathcal{B}_k(A)$ forms a $DGDD(m^{|A|})$ on $A \times I_m$ with the groups $\mathcal{G}_A$.

(2) $PDGDD(m^k : s) = ((A \times I_m) \cup S, \mathcal{G}_A, \mathcal{C}_A)$ for each $A \in \mathcal{C}$, where $\mathcal{C}_A$ can be partitioned into $3m|A|$ disjoint $\mathcal{C}_{x,i}^j(A)$ and $3(s - 2)$ disjoint $\mathcal{C}_k(A)$, $x \in A$, $i \in I_m$, $j \in I_3$, $1 \le k \le 3(s - 2)$, such that each $\mathcal{C}_{x,i}^j(A)$ forms a $DGDD(m^{|A|-1}(s + 1)^1)$ on $((A \setminus \{x\}) \times I_m) \cup S \cup \{x_i\}$ with a $(s + 1)$-group $S \cup \{x_i\}$ and $|A| - 1m$-groups $\{y\} \times I_m$, $y \in A \setminus \{x\}$, and each $\mathcal{C}_k(A)$ forms a $DGDD(m^{|A|})$ on $A \times I_m$ with the groups $\mathcal{G}_A$.

(3) $DF(m^{|A|}) = (A \times I_m, \mathcal{G}_A, \mathcal{D}_A)$ for each $A \in \mathcal{D}$, where $\mathcal{D}_A$ can be partitioned into $3m|A|$ disjoint $\mathcal{D}_{x,i}^j(A)$, $x \in A, i \in I_m, j \in I_3$, such that each $\mathcal{D}_{x,i}^j(A)$ forms a $DGDD(m^{|A|-1})$ on $(A \setminus \{x\}) \times I_m$ with the groups $\mathcal{G}_{A \setminus \{x\}}$.

Now, define

$$\mathcal{F}_{x,i}^j = \left(\bigcup_{x \in A \in \mathcal{B}} \mathcal{B}_{x,i}^j(A) \right) \bigcup \left(\bigcup_{x \in A \in \mathcal{C}} \mathcal{C}_{x,i}^j(A) \right) \bigcup \left(\bigcup_{x \in A \in \mathcal{D}} \mathcal{D}_{x,i}^j(A) \right), \quad x \in X, i \in I_m, j \in I_3;$$

$$\mathcal{F}_k = \begin{cases} \bigcup_{A \in \mathcal{B}} \mathcal{B}_k(A) & 1 \le k \le 3r + 4 \\ \bigcup_{A \in \mathcal{C}} \mathcal{C}_{k-3r-4}(A) & 3r + 5 \le k \le 3(r + s) - 2. \end{cases}$$

Then, the collection $\{\mathcal{F}_{x,i}^j, x \in X, i \in I_m, j \in I_3\} \cup \{\mathcal{F}_k, 1 \le k \le 3(r + s) - 2\}$ forms a $PECS((mg)^n : r + s)$ on $(X \times I_m) \cup (R \cup S)$ with the groups $\{G \times I_m : G \in \mathcal{G}\}$ and the stem $R \cup S$. ■

Theorem 2.4 ([9]). *If there exists an* $LEDTS(v)$ *then there exist an* $LEDTS(3v)$ *and an* $LEDTS(3v, 3)$ *for* $v \ge 3$ *and* $v \neq 6$.

3. Structure equations and orbits

For a given order v, an $EDTS(v)$ may contain distinct amount of triples, and an $LEDTS(v)$ may consist of $EDTS(v)$ with distinct structure. In order to construct a large set of disjoint $EDTS(v)$, or to show its non-existence, we have to consider the structure of possible $EDTS(v)$ and $LEDTS(v)$. For example,

(1) How many D_i-triples may be contained in an $EDTS(v)$ for $1 \le i \le 5$?

(2) What structure each $EDTS(v)$ in an $LEDTS(v)$ has?

By the enumeration of the pairs (x, y) for $x = y$ and $x \ne y$, we have two equations:

$$|D_1| + |D_2| + |D_3| + |D_4| = v, \qquad |D_2| + |D_3| + 2|D_4| + 3|D_5| = v(v-1).$$

Let $x = |D_1|, y = |D_2| + |D_3|, z = |D_4| + |D_5|$. Adding the two equations, we obtain $x + 2y + 3z = v^2$ and $x + y \le v$, for $v \ge 3$. As well, in [3], Huang gave the further necessary conditions to exist an $EDTS(v)$:

$$|D_2| + |D_3| \ne 1 \quad \text{and} \quad |D_4| \equiv \begin{cases} |D_2| + |D_3| \bmod 3 & (\text{if } v \equiv 0, 1 \bmod 3) \\ |D_2| + |D_3| + 1 \bmod 3 & (\text{if } v \equiv 2 \bmod 3). \end{cases}$$

Structure equation for $EDTS(v)$: $x + 2y + 3z = v^2$, where $x + y \le v$ and $y \ne 1$.

Suppose it has m non-negative integer solutions (x_i, y_i, z_i), $1 \le i \le m$. Each solution (x_i, y_i, z_i) will give a possible $EDTS(v)$, which consists of x_iD_1-triples, y_iD_2- or D_3-triples and z_iD_4- or D_5-triples. The $EDTS(v)$ is called (x_i, y_i, z_i)-type's. Suppose an $LEDTS(v)$ consists of w_i (x_i, y_i, z_i)-type's $EDTS(v)$s, $1 \le i \le m$. Of course, $\sum_{i=1}^{m} w_i = 3v - 2$. These parameters w_i will be determined by

Structure equation system for $LEDTS(v)$:

$$\begin{pmatrix} x_1 & x_2 & \cdots & x_m \\ y_1 & y_2 & \cdots & y_m \\ z_1 & z_2 & \cdots & z_m \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{pmatrix} = \begin{pmatrix} v \\ 2v(v-1) \\ v(v-1)^2 \end{pmatrix}.$$

Take Z_v as the point set. Under the action of the automorphic group Z_v, all ordered pairs from Z_v can be partitioned into v *differences*:

$$\langle d \rangle = \{(x, x+d) : x \in Z_v\}, \quad d \in Z_v,$$

where $\langle 0 \rangle = \{(x, x) : x \in Z_v\}$ is a special difference only for extended triple systems. Under the action of the automorphic group Z_v, all extended transitive triples can be partitioned into *orbits*:

$$O(d, d') = \{(x-d, x, x+d') : x \in Z_v\}, \quad d, d' \in Z_v,$$

which covers three differences $\langle d \rangle$, $\langle d' \rangle$ and $\langle d + d' \rangle$ (one may equal to another), so the orbit $O(d, d')$ is denoted by $[d, d', d + d']$ sometimes. Among these orbits, there are

one D_1-orbit $O(0, 0) = \{(x, x, x) : x \in Z_v\}$;
$v - 1D_2$-orbits $O(0, d') = \{(x, x, x + d') : x \in Z_v\}$, $\quad d' \in Z_v^*$;
$v - 1D_3$-orbits $O(d, 0) = \{(x - d, x, x) : x \in Z_v\}$, $\quad d \in Z_v^*$;
$v - 1D_4$-orbits $O(d, -d) = \{(x - d, x, x - d) : x \in Z_v\}$, $\quad d \in Z_v^*$;
$(v-1)(v-2)D_5$-orbits $O(d, d') = \{(x - d, x, x + d') : x \in Z_v\}$, $\quad d, d' \in Z_v^*, d' \ne -d$.

Each orbit covers one difference ($\langle 0 \rangle$ for D_1-orbit), or two differences ($\langle 0 \rangle$, $\langle d' \rangle$ for D_2-orbits, $\langle 0 \rangle$, $\langle d \rangle$ for D_3-orbits) or three differences ($\langle 0 \rangle$, $\langle d \rangle$, $\langle -d \rangle$ for D_4-orbits, $\langle 0 \rangle$, $\langle d \rangle$, $\langle d' \rangle$ for D_5-orbits).

Furthermore, if v is a prime power q, and g is a primitive element of F_q, the index set of all non-zero elements in F_q is denoted by Z_{q-1}. Under the action of the multiplicative group of F_q, all orbits on F_q can be partitioned into the following *orbit families*.

one D_1-orbit family : $\overline{\mathcal{O}}_1 = \{O(0, 0)\}$, $\qquad$ one D_2-orbit family : $\overline{\mathcal{O}}_2 = \{O(0, g^i) : i \in Z_{q-1}\}$,
one D_3-orbit family : $\overline{\mathcal{O}}_3 = \{O(g^i, 0) : i \in Z_{q-1}\}$, $\qquad$ one D_4-orbit family : $\overline{\mathcal{O}}_4 = \{O(g^i, -g^i) : i \in Z_{q-1}\}$,

$$q - 2D_5\text{-orbit families} : \overline{\mathcal{O}}_5(k) = \{g^i \cdot O(1, g^k) : i \in Z_{q-1}\}, \quad k \in \begin{cases} Z_{q-1} \setminus \left\{ \dfrac{q-1}{2} \right\} & \text{for odd } q \\ Z_{q-1}^* & \text{for even } q. \end{cases}$$

4. *LEDTS*(v) of small orders

Lemma 4.1. *There exists no LEDTS*(4).

Proof. The structure equation for *EDTS*(4)

$$x + 2y + 3z = 16, \quad (x + y \le 4 \text{ and } y \ne 1)$$

has four non-negative integer solutions $(x, y, z) = (0, 2, 4), (1, 0, 5), (1, 3, 3), (4, 0, 4)$. But, the structure equation system for *LEDTS*(4)

$$\begin{pmatrix} 0 & 1 & 1 & 4 \\ 2 & 0 & 3 & 0 \\ 4 & 5 & 3 & 4 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix} = \begin{pmatrix} 4 \\ 24 \\ 36 \end{pmatrix}$$

has unique solution $(w_1, w_2, w_3, w_4) = (6, 0, 4, 0)$. Let the unique possible *LEDTS*(4) be $\{(Z_4, \mathcal{A}_k) : 1 \le k \le 6\} \bigcup \{(Z_4, \mathcal{B}_k) : 1 \le k \le 4\}$, where

$$|D_1| = 0, \quad |D_2 \cup D_3| = 2, \quad |D_4 \cup D_5| = 4 \quad \text{for each } \mathcal{A}_k,$$
$$|D_1| = 1, \quad |D_2 \cup D_3| = 3, \quad |D_4 \cup D_5| = 3 \quad \text{for each } \mathcal{B}_k.$$

Since $|\bigcup\{D_i : 1 \le i \le 4\}|$ must be 4. Consider these $\mathcal{A}_k$ only, it is easy to see that $|D_4| = |D_5| = 2$ in each $\mathcal{A}_k$. However, if an *EDTS*(4) contains two D_5-triples: (a, b, c) and (a', b', c'), there are two cases:

(1) $|\{a, b, c\} \cup \{a', b', c'\}| = 4$. Then, among the remaining arcs in K_4^* (the complete symmetric directed graph of order 4), there is only one pair of opposite arcs (x, y) and (y, x). The *EDTS*(4) cannot contain two D_4-triples, since each D_4-triple covers a pair of opposite arcs.

(2) $|\{a, b, c\} \cup \{a', b', c'\}| = 3$, i.e., $\{a, b, c\} = \{a', b', c'\}$. Let the other vertex in K_4^* be d, then two D_4-triples in the *EDTS*(4) should be (x, d, x) and (y, d, y), where $x \ne y \in \{a, b, c\}$, i.e., they have the same middle element d. However, it is impossible to partition all $6 \times 2 = 12 D_4$-triples into six parts in this form, because, for any element $x \in Z_4$, there are just three D_4-triples with the same middle element. ■

Lemma 4.2. *There exist an LEDTS*(2) *and an LEDTS*(6).

Construction. $LEDTS(2) = \{(Z_2, \mathcal{A}_i) : 0 \le i \le 3\}$, where

$\mathcal{A}_0$: 000 101; $\mathcal{A}_1$: 111 010; $\mathcal{A}_2$: 110 001; $\mathcal{A}_3$: 011 100.

$LEDTS(6) = \{(Z_6, \mathcal{A}_x) : x \in Z_6\} \cup \{(Z_6, \mathcal{B}_x) : x \in Z_6\} \cup \{(Z_6, \mathcal{C}_j) : 1 \le j \le 4\}$, where $\mathcal{A}_x = \mathcal{A}_0 + x$, $\mathcal{B}_x = \mathcal{B}_0 + x$, $x \in Z_6$, and

$\mathcal{A}_0$: 000 112 221 544 455 303 150 051 523 325 134 431 024 420;
$\mathcal{B}_0$: 003 330 115 551 422 244 012 210 504 405 352 253 413 314;

(The first two triples of $\mathcal{B}_0 + 3$, $\mathcal{B}_0 + 4$ and $\mathcal{B}_0 + 5$ need to be replaced by their inverse.)

$\mathcal{C}_1$: 010 121 202 313 424 505 235 532 340 043 451 154;
$\mathcal{C}_2$: 020 101 242 323 404 545 125 521 341 143 503 305;
$\mathcal{C}_3$: 040 151 232 343 454 535 502 205 013 310 124 421;
$\mathcal{C}_4$: 050 131 212 353 434 515 014 410 230 032 452 254.

Proof. The correctness for *LEDTS*(2) is obvious. Next, checking the appearance of each ordered pair, we can show that each A_0, B_0 and C_j forms an *EDTS*(6). Further, checking the appearance of each extended transitive triple (or each block orbit for A_0 and B_0), we can prove that all A_x, B_x and C_j, $x \in Z_6$, $1 \le j \le 4$, forms an *LEDTS*(6). ■

Lemma 4.3. *There exist an LEDTS*(8) *and an LEDTS*(8, 2).

Construction. Let g be a primitive element of the finite field F_8, and $g^3 = 1 + g$. Construct three families of extended transitive triples on F_8 as follows, where $F_8 = R \cup S$, $R = \{0, 1, g, g^3\}$, $S = \{g^2, g^4, g^5, g^6\}$.

$\mathcal{A}_0$: $(0, 0, 0), (g^j, 0, g^j), (g^{j+3}, g^{j+1}, g^{j+4}), (g^{j+2}, g^{j+6}, g^{j+1})$, $j \in Z_7$.
$\mathcal{A}_1$: $(0, 0, g^5) + x, x \in F_8$; $(1, 0, g^3) + x, (g^2, 0, g^6) + x$ for $x \in R$;
$(g^3, 0, g^4) + x, (g^2, 0, g) + x$ for $x \in S$.
$\mathcal{A}_2$: $(g^5, 0, 0) + x, x \in F_8$; $(1, 0, g^3) + x, (g^2, 0, g^6) + x$ for $x \in S$;
$(g^3, 0, g^4) + x, (g^2, 0, g) + x$ for $x \in R$.

Let $\mathcal{B}_x = \mathcal{A}_0 + x$, $\mathcal{C}_k = g^k\mathcal{A}_1$, $\mathcal{D}_k = g^k\mathcal{A}_2$, where $x \in F_8$ and $k \in Z_7$. Then, $\{(F_8, \mathcal{B}_x) : x \in F_8\} \cup \{(F_8, \mathcal{C}_k) : k \in Z_7\} \cup \{(F_8, \mathcal{D}_k) : k \in Z_7\}$ forms an *LEDTS*(8). Furthermore, define

$$\mathcal{B}'_0 = \mathcal{B}_0 \setminus \{(0,0,0), (g^5, 0, g^5)\}, \qquad \mathcal{B}'_{g^5} = \mathcal{B}_{g^5} \setminus \{(g^5, g^5, g^5), (0, g^5, 0)\} \quad \text{and} \quad \mathcal{B}'_x = \mathcal{B}_x \quad \text{for other } x \in F_8;$$

$$\mathcal{C}'_0 = \mathcal{C}_0 \setminus \{(0,0,g^5), (g^5, g^5, 0)\} \quad \text{and} \quad \mathcal{C}'_k = \mathcal{C}_k \quad \text{for } k \in Z_7^*;$$

$$\mathcal{D}'_0 = \mathcal{D}_0 \setminus \{(0, g^5, g^5), (g^5, 0, 0)\} \quad \text{and} \quad \mathcal{D}'_k = \mathcal{D}_k \quad \text{for } k \in Z_7^*.$$

Then, $\{(F_8, \mathcal{B}'_x) : x \in F_8\} \cup \{(F_8, \mathcal{C}'_k) : k \in Z_7\} \cup \{(F_8, \mathcal{D}'_k) : k \in Z_7\}$ forms an *LEDTS*(8, 2).

Proof. (1) $\mathcal{A}_0$ forms an *EDTS*(8) on F_8. In fact, it is easy to see that each of the ordered pairs (x, x), $(0, g^j)$, $(g^j, 0)$ and (g^j, g^{j+k}), $x \in F_8,\ j \in Z_7,\ k \in Z_7^*$, appears once in $\mathcal{A}_0$. Furthermore, each $\mathcal{B}_x$ or $\mathcal{B}'_y$ $x \in F_8,\ y \in F_8^* \setminus \{g^5\}$, is also an *EDTS*(8) on F_8. And, $\mathcal{B}'_0$ (and $\mathcal{B}'_{g^5}$) is an *EDTS*(8, 2) on F_8 with the hole $\{0, g^5\}$.

(2) $\mathcal{A}_1$ forms an *EDTS*(8) on F_8 (similarly, for $\mathcal{A}_2$). In fact, by the additive table

$+$	0	g^0	g^1	g^2	g^3	g^4	g^5	g^6
0	0	g^0	g^1	g^2	g^3	g^4	g^5	g^6
g^0	g^0	0	g^3	g^6	g^1	g^5	g^4	g^2
g^1	g^1	g^3	0	g^4	g^0	g^2	g^6	g^5
g^2	g^2	g^6	g^4	0	g^5	g^1	g^3	g^0
g^3	g^3	g^1	g^0	g^5	0	g^6	g^2	g^4
g^4	g^4	g^5	g^2	g^1	g^6	0	g^0	g^3
g^5	g^5	g^4	g^6	g^3	g^2	g^0	0	g^1
g^6	g^6	g^2	g^5	g^0	g^4	g^3	g^1	0

,

we can know that $R + R = R = S + S,\ R + S = S = S + R$ and

$$(0,0) \in \langle 0\rangle; \qquad (1,0), (g^2, g^6) \in \langle g^0\rangle; \qquad (1, g^3), (0, g) \in \langle g^1\rangle; \qquad (g^2, 0) \in \langle g^2\rangle;$$
$$(0, g^3), (g^3, 0) \in \langle g^3\rangle; \qquad (0, g^4), (g^2, g) \in \langle g^4\rangle; \qquad (0, g^5) \in \langle g^5\rangle; \qquad (0, g^6), (g^3, g^4) \in \langle g^6\rangle.$$

Obviously, the pairs in the orbits $\langle 0\rangle$ and $\langle g^5\rangle$ are filled. For the other orbits, we have

$$\left.\begin{cases} x \in R & (1,0) \in (R,R) \longrightarrow (1+x, x) \in (R,R) \\ x \in R & (g^2, g^6) \in (S,S) \longrightarrow (g^2+x, g^6+x) \in (S,S) \end{cases}\right\} \quad \langle g^0\rangle;$$

$$\left.\begin{cases} x \in R & (1,g^3) \in (R,R) \longrightarrow (1+x, g^3+x) \in (R,R) \\ x \in S & (0, g) \in (R,R) \longrightarrow (x, g+x) \in (S,S) \end{cases}\right\} \quad \langle g^1\rangle;$$

$$\left.\begin{cases} x \in R & (g^2,0) \in (S,R) \longrightarrow (g^2+x, x) \in (S,R) \\ x \in S & (g^2, 0) \in (S,R) \longrightarrow (g^2+x, x) \in (R,S) \end{cases}\right\} \quad \langle g^2\rangle;$$

$$\left.\begin{cases} x \in R & (0,g^3) \in (R,R) \longrightarrow (x, g^3+x) \in (R,R) \\ x \in S & (g^3, 0) \in (R,R) \longrightarrow (g^3+x, x) \in (S,S) \end{cases}\right\} \quad \langle g^3\rangle;$$

$$\left.\begin{cases} x \in S & (0,g^4) \in (R,S) \longrightarrow (x, g^4+x) \in (S,R) \\ x \in S & (g^2, g) \in (S,R) \longrightarrow (g^2+x, g+x) \in (R,S) \end{cases}\right\} \quad \langle g^4\rangle;$$

$$\left.\begin{cases} x \in R & (0,g^6) \in (R,S) \longrightarrow (x, g^6+x) \in (R,S) \\ x \in S & (g^3, g^4) \in (R,S) \longrightarrow (g^3+x, g^4+x) \in (S,R) \end{cases}\right\} \quad \langle g^6\rangle.$$

Therefore, the system $\mathcal{A}_1$ forms an *EDTS*(8) on F_8 indeed. Furthermore, each $\mathcal{C}_k$, $\mathcal{D}_k$ or $\mathcal{C}'_r$, $\mathcal{D}'_r$, $k \in Z_7,\ r \in Z_7^*$, is also an *EDTS*(8) on F_8. And, $\mathcal{C}'_0$ (and $\mathcal{D}'_0$) is an *EDTS*(8, 2) on F_8 with the hole $\{0, g^5\}$.

(3) $\{(F_8, \mathcal{B}_x) : x \in F_8\} \cup \{(F_8, \mathcal{C}_k) : k \in Z_7\} \cup \{(F_8, \mathcal{D}_k) : k \in Z_7\}$ forms an *LEDTS*(8). In fact,

$$(g^{j+3}, g^{j+1}, g^{j+4}) = g^j(g-1, g, g+g^2) \in g^j \cdot O(1, g^2) \in \overline{\mathcal{O}}_5(2),$$
$$(g^{j+2}, g^{j+6}, g^{j+1}) = g^j(g^6-1, g^6, g^6+g^5) \in g^j \cdot O(1, g^5) \in \overline{\mathcal{O}}_5(5),$$
$$(1+x, x, g^3+x) \in O(1, g^3) \in \overline{\mathcal{O}}_5(3), \qquad (g^2+x, x, g^6+x) \in g^2 \cdot O(1, g^4) \in \overline{\mathcal{O}}_5(4),$$
$$(g^3+x, x, g^4+x) \in g^3 \cdot O(1, g) \in \overline{\mathcal{O}}_5(1), \qquad (g^2+x, x, g+x) \in g^2 \cdot O(1, g^6) \in \overline{\mathcal{O}}_5(6).$$

Therefore, the D_5-triples in $\mathcal{A}_0$, $\mathcal{A}_1$ and $\mathcal{A}_2$ appear in all D_5-orbit families $\overline{\mathcal{O}}_5(k),\ k \in Z_7^*$. For $1 \le i \le 4$, the D_i-triples in $\mathcal{A}_0$, $\mathcal{A}_1$ and $\mathcal{A}_2$ appear in all D_i-orbit families $\overline{\mathcal{O}}_i$.

(4) $\{(F_8, \mathcal{B}'_x) : x \in F_8\} \cup \{(F_8, \mathcal{C}'_k) : k \in Z_7\} \cup \{(F_8, \mathcal{D}'_k) : k \in Z_7\}$ forms an $LEDTS(8, 2)$. In fact, the distinction between the collections (4) and (3) lies only in removing two blocks for each procedure

$$\mathcal{B}_0 \longrightarrow \mathcal{B}'_0, \qquad \mathcal{B}_{g^5} \longrightarrow \mathcal{B}'_{g^5}, \qquad \mathcal{C}_0 \longrightarrow \mathcal{C}'_0, \qquad \mathcal{D}_0 \longrightarrow \mathcal{D}'_0.$$

However, the removed eight blocks form just an $LEDTS(2)$ on the hole $\{0, g^5\}$. ■

Lemma 4.4. *There exists an* $LEDTS(10)$.

Construction. Construct an $LEDTS(10)$ on $X = Z_9 \cup \{u\}$ as follows, where $u \notin Z_9$ is a fixed element.

$\mathcal{A}_u$:	uuu	055	$0u4$	028	031(mod 9)								
$\mathcal{A}_0$:	$0uu$	$u00$	171	422	233	544	355	666	737	818			
	$78u$	$24u$	$36u$	$15u$	$u62$	$u41$	$u53$	$u87$	651	321	134	570	825
	380	160	405	843	012	568	746	063	286	647	752	207	048
$\mathcal{B}_0$:	$u1u$	070	112	226	335	484	551	663	767	828			
	$38u$	$74u$	$20u$	$65u$	$u45$	$u68$	$u03$	$u72$	785	564	247	580	713
	187	215	086	831	610	052	416	423	537	014	340	362	
$\mathcal{C}_0$:	$uu4$	$00u$	110	822	343	744	$5u5$	266	771	886			
	$7u8$	$1u3$	$3u6$	$6u2$	$4u1$	$2u0$	$8u7$	172	854	705	021	037	450
	156	427	657	325	763	046	614	830	381	518	523	608	248.

Define $\mathcal{A}_k = \mathcal{A}_0 + k$, $\mathcal{B}_k = \mathcal{B}_0 + k$ and $\mathcal{C}_k = \mathcal{C}_0 + k$, where $k \in Z_9$. Then, $\{(X, \mathcal{A}_k), (X, \mathcal{B}_k), (X, \mathcal{C}_k) : k \in Z_{10}\} \bigcup \{(X, \mathcal{A}_u)\}$ is an $LEDTS(10)$ desired.

Proof. First, it is not difficult to check that $\mathcal{A}_0$ (or $\mathcal{B}_0, \mathcal{C}_0, \mathcal{A}_u$) forms an $EDTS(10)$. Furthermore, in order to show the collection $\{(X, \mathcal{A}_k), (X, \mathcal{B}_k), (X, \mathcal{C}_k) : k \in Z_{10}\} \bigcup \{(X, \mathcal{A}_u)\}$ forms an $LEDTS(10)$ indeed, we list the following two tables. The first table shows the orbits of the triples containing u in every block set.

	$D_1 \sim D_4$	$(u, x, x+d)$	$(x, x+d, u)$	$(x, u, x+d)$
$\mathcal{A}_u$	(u, u, u)			$d = 4$
$\mathcal{A}_0$	$(*, u, u), (u, *, *)$	$d = 5, 6, 7, 8$	$d = 1, 2, 3, 4$	
$\mathcal{B}_0$	$(u, *, u)$	$d = 1, 2, 3, 4$	$d = 5, 6, 7, 8$	
$\mathcal{C}_0$	$(*, *, u), (u, u, *), (*, u, *)$			$d = 1, 2, 3, 5, 6, 7, 8$

The second table shows the orbits of the triples not containing u in every block set, where A_u (or $\mathcal{A}_0$, $\mathcal{B}_0$, $\mathcal{C}_0$) in the position (i, j) means that there exists some block in A_u (or $\mathcal{A}_0$, $\mathcal{B}_0$, $\mathcal{C}_0$) belonging to the orbit $O(i, j)$.

	0	1	2	3	4	5	6	7	8
0	$\mathcal{A}_0$	$\mathcal{B}_0$	$\mathcal{B}_0$	$\mathcal{C}_0$	$\mathcal{B}_0$	$\mathcal{B}_0$	$\mathcal{B}_0$	$\mathcal{C}_0$	$\mathcal{C}_0$
1	$\mathcal{A}_0$	$\mathcal{A}_0$	$\mathcal{A}_0$	$\mathcal{B}_0$	$\mathcal{C}_0$	$\mathcal{B}_0$	$\mathcal{B}_0$	$\mathcal{B}_0$	$\mathcal{C}_0$
2	$\mathcal{A}_0$	$\mathcal{A}_0$	$\mathcal{A}_0$	$\mathcal{B}_0$	$\mathcal{C}_0$	$\mathcal{C}_0$	$\mathcal{A}_u$	$\mathcal{A}_0$	$\mathcal{C}_0$
3	$\mathcal{C}_0$	$\mathcal{B}_0$	$\mathcal{B}_0$	$\mathcal{A}_0$	$\mathcal{C}_0$	$\mathcal{B}_0$	$\mathcal{B}_0$	$\mathcal{A}_u$	$\mathcal{C}_0$
4	$\mathcal{C}_0$	$\mathcal{C}_0$	$\mathcal{C}_0$	$\mathcal{C}_0$	$\mathcal{A}_0$	$\mathcal{B}_0$	$\mathcal{C}_0$	$\mathcal{B}_0$	$\mathcal{B}_0$
5	$\mathcal{A}_u$	$\mathcal{A}_0$	$\mathcal{C}_0$	$\mathcal{A}_0$	$\mathcal{A}_0$	$\mathcal{A}_0$	$\mathcal{B}_0$	$\mathcal{C}_0$	$\mathcal{A}_0$
6	$\mathcal{C}_0$	$\mathcal{C}_0$	$\mathcal{A}_0$	$\mathcal{A}_0$	$\mathcal{C}_0$	$\mathcal{B}_0$	$\mathcal{A}_0$	$\mathcal{A}_0$	$\mathcal{C}_0$
7	$\mathcal{A}_0$	$\mathcal{B}_0$	$\mathcal{B}_0$	$\mathcal{A}_0$	$\mathcal{B}_0$	$\mathcal{C}_0$	$\mathcal{A}_0$	$\mathcal{A}_0$	$\mathcal{B}_0$
8	$\mathcal{A}_0$	$\mathcal{B}_0$	$\mathcal{C}_0$	$\mathcal{C}_0$	$\mathcal{B}_0$	$\mathcal{A}_0$	$\mathcal{C}_0$	$\mathcal{B}_0$	$\mathcal{A}_0$

■

Lemma 4.5. *There exists an* $LEDTS(10, 4)$.

Proof. Suppose $\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4} \notin Z_6$ where $\bar{0}$ is only an auxiliary symbol. Let us construct an $LEDTS(10, 4)$ on $X = Z_6 \cup \{\bar{1}, \bar{2}, \bar{3}, \bar{4}\}$ with the hole $\{\bar{1}, \bar{2}, \bar{3}, \bar{4}\}$ as follows. Define

$$S_0 = \{(1, 2), (3, 4), (5, 0)\}, \qquad S_1 = \{(0, 4), (1, 5), (2, 3)\}, \qquad S_2 = \{(2, 0), (3, 1), (4, 5)\},$$
$$S_3 = \{(4, 2), (5, 3), (0, 1)\}, \qquad S_4 = \{(1, 4), (2, 5), (0, 3)\}.$$

For $i \in Z_5^*$ and $j \in Z_5$, denote

$$\bar{i}S_j = \{(\bar{i}, x, y), (y, x, \bar{i}) : (x, y) \in S_j\}, \qquad \bar{i}S'_j = \{(\bar{i}, y, x), (x, y, \bar{i}) : (x, y) \in S_j\},$$
$$\bar{0}S_j = \{(x, x, y), (y, y, x) : (x, y) \in S_j\}, \qquad \bar{0}S'_j = \{(x, y, y), (y, x, x) : (x, y) \in S_j\}.$$

Then, define ten families of extended transitive triples on X where the subscripts are taken in Z_5.

$$\mathcal{B}_k = \{\bar{i}S_{i+k} : i \in Z_5\}, \qquad \mathcal{B}'_k = \{\bar{i}S'_{i+k} : i \in Z_5\}, \quad k \in Z_5.$$

And, construct three families of extended transitive triples on X:

$$\begin{array}{llllllllllllll}
\mathcal{A}_0^0: & 1\bar{1}\bar{1} & 3\bar{2}\bar{2} & 0\bar{3}\bar{3} & 4\bar{4}\bar{4} & \bar{3}00 & \bar{2}11 & 212 & \bar{4}33 & \bar{1}44 & 535 & 0\bar{1}\bar{2} & 3\bar{1}\bar{3} \\
 & 5\bar{1}\bar{4} & 4\bar{2}\bar{3} & 1\bar{2}\bar{4} & 2\bar{3}\bar{4} & 4\bar{1}0 & 2\bar{2}0 & 5\bar{3}1 & 3\bar{4}0 & \bar{2}\bar{1}2 & \bar{3}\bar{1}5 & \bar{4}\bar{1}1 & \bar{3}\bar{2}3 \\
 & \bar{4}\bar{2}5 & \bar{4}\bar{3}2 & 2\bar{1}3 & 5\bar{2}4 & 1\bar{3}4 & 0\bar{4}4 & 013 & 245 & 342 & 150 & 052 & 431 \\
\mathcal{A}_0^1: & \bar{1}\bar{1}3 & \bar{2}\bar{2}1 & \bar{3}\bar{3}2 & \bar{4}\bar{4}4 & 030 & 11\bar{1} & 22\bar{3} & 33\bar{4} & 44\bar{2} & 515 & \bar{1}\bar{2}0 & \bar{1}\bar{3}4 \\
 & \bar{1}\bar{4}1 & \bar{2}\bar{3}3 & \bar{2}\bar{4}5 & \bar{3}\bar{4}0 & 4\bar{1}2 & 1\bar{2}2 & 4\bar{3}5 & 0\bar{4}2 & 3\bar{2}\bar{1} & 5\bar{3}\bar{1} & 2\bar{4}\bar{1} & 0\bar{3}\bar{2} \\
 & 5\bar{4}\bar{2} & 1\bar{4}\bar{3} & 0\bar{1}5 & 2\bar{2}4 & 3\bar{3}1 & 4\bar{4}3 & 014 & 250 & 354 & 213 & 410 & 532 \\
\mathcal{A}_0^2: & \bar{1}4\bar{1} & \bar{2}5\bar{2} & \bar{3}1\bar{3} & \bar{4}0\bar{4} & 0\bar{3}0 & 1\bar{1}1 & 2\bar{2}2 & 343 & 4\bar{4}4 & 555 & & \\
 & \bar{1}0\bar{2} & \bar{1}2\bar{3} & \bar{1}5\bar{4} & \bar{2}4\bar{3} & \bar{2}1\bar{4} & \bar{3}3\bar{4} & 0\bar{1}3 & 3\bar{2}0 & 3\bar{3}2 & 2\bar{4}3 & \bar{2}3\bar{1} & \bar{3}5\bar{1} \\
 & \bar{4}2\bar{1} & \bar{3}4\bar{2} & \bar{4}1\bar{2} & \bar{4}5\bar{3} & 012 & 135 & 531 & 054 & 140 & 241 & 520 & 425.
\end{array}$$

Let $\mathcal{A}_x^j = \mathcal{A}_0^j + x$ for $x \in Z_6$ and $j \in Z_3$. It is not difficult to check that each $\mathcal{A}_x^j$ forms an $EDTS(10)$ on X, and each $\mathcal{B}_k$ (or B'_k) forms an $EDTS(10, 4)$ on X with the holes $\{\bar{1}, \bar{2}, \bar{3}, \bar{4}\}$. So, the collection $\{(X, \mathcal{A}_x^j) : x \in Z_6, j \in Z_3\} \cup \{(X, \mathcal{B}_k) : k \in Z_5\} \cup \{(X, \mathcal{B}'_k) : k \in Z_5\}$ is an $LEDTS(10, 4)$ desired. ■

Lemma 4.6. *There exists an* $LEDTS(12)$.

Proof. We construct an $LEDTS(12)$ on $X = Z_{10} \cup \{u, v\}$.

$$\begin{array}{llllllllllllll}
\mathcal{A}_0^0: & u6u & v4v & 000 & 511 & 227 & 3v3 & 144 & 559 & 466 & 677 & 8u8 & 299 & \\
 & u1v & 07u & u57 & 2u4 & 9u2 & 3u0 & 1u9 & 4u3 & 238 & 805 & 583 & 372 & 490 \\
 & v5u & 60v & v87 & 8v9 & 9v1 & 5v2 & 2v0 & 7v6 & 781 & 695 & 341 & 170 & 745 \\
 & 963 & 864 & 098 & 621 & 256 & 012 & 135 & 036 & 482 & 739 & 168 & 947 & 504 \\
\mathcal{A}_0^1: & uu3 & vv6 & 020 & 114 & 252 & 331 & 445 & 55u & 668 & 77v & 887 & 909 & \\
 & uv2 & 34u & 80u & 69u & 71u & u40 & u85 & u19 & u76 & 046 & 158 & 073 & 423 \\
 & 2vu & 36v & 94v & 51v & 08v & v01 & v59 & v38 & v74 & 610 & 839 & 291 & 862 \\
 & 172 & 248 & 926 & 841 & 657 & 305 & 750 & 327 & 564 & 953 & 798 & 163 & 497 \\
\mathcal{A}_0^2: & 6uu & 9vv & u00 & 151 & 202 & 393 & 414 & 545 & 866 & 727 & v88 & 099 & \\
 & 3uv & 05u & 84u & 97u & 21u & u56 & u92 & u71 & u38 & 128 & 573 & 374 & 698 \\
 & vu4 & 78v & 46v & 52v & 10v & v91 & v03 & v62 & v75 & 430 & 580 & 429 & 823 \\
 & 631 & 904 & 081 & 067 & 965 & 136 & 264 & 179 & 859 & 325 & 760 & 487 &
\end{array}$$

$$\begin{array}{llllllll}
\mathcal{B}_0: & uuu & vuv & 006 & 0u4 & 0v5 & 013 & 310 \pmod{10}; \\
\mathcal{B}_1: & vvv & uvu & 044 & 0u5 & 0v6 & 023 & 320 \pmod{10}; \\
\mathcal{B}_2: & uuv & vvu & 055 & 0u6 & 0v4 & 029 & 031 \pmod{10}; \\
\mathcal{B}_3: & uvv & vuu & 007 & 0u1 & 0v3 & 082 & 095 \pmod{10}.
\end{array}$$

Let $\mathcal{A}_x^j = \mathcal{A}_0^j + x$ for $x \in Z_{10}$ and $j \in Z_3$. It is not difficult to check that each $\mathcal{A}_x^j$ (or $\mathcal{B}_k$, $k \in Z_4$) forms an $EDTS(12)$ on X and they are pairwise disjoint. Therefore, the collection $\{(X, \mathcal{A}_x^j) : x \in Z_{10}, j \in Z_3\} \bigcup \{(X, \mathcal{B}_k) : k \in Z_4\}$ is an $LEDTS(12)$ desired. ■

Lemma 4.7. *There exists an* $LEDTS(14)$.

Proof. We construct an $LEDTS(14)$ on $X = Z_{13} \cup \{u\}$, where 10, 11, 12 are written in $\bar{0}, \bar{1}, \bar{2}$.

$$\mathcal{A}_u: \quad uuu \quad 0u0 \quad 034 \quad 057 \quad 750 \quad 430 \quad \pmod{13}.$$

$$\begin{array}{lllllllllllll}
\mathcal{A}_0^0: \text{(i)} & u0u & 000 & 141 & 282 & 373 & 4\bar{1}4 & 565 & 686 & 747 & 838 & 979 & \bar{0}6\bar{0} \\
 & \bar{1}3\bar{1} & \bar{2}\bar{1}\bar{2} & & & & & & & & & & \\
\text{(ii)} & 3u4 & 2u5 & 9u1 & 8u\bar{0} & 7u\bar{1} & 6u\bar{2} & 036 & 9\bar{0}0 & \bar{2}59 & 18\bar{2} & 012 & 726 \\
 & 135 & 392 & 16\bar{1} & \bar{1}05 & \bar{2}3\bar{0} & 07\bar{2} & 4\bar{2}2 & 5\bar{0}4 & 694 & 857 & 048 & 9\bar{1}8 \\
 & \bar{0}17 & \bar{1}2\bar{0} & & & & & & & & & & \\
\text{(iii)} & \{cba : abc \in \text{(ii)}\} & & & & & & & & & & & \\
\mathcal{A}_0^1: \text{(i)} & uu0 & 00u & 11\bar{0} & 227 & 339 & 446 & 558 & 664 & 772 & 885 & 993 & \bar{0}\bar{0}1 \\
 & \bar{1}\bar{1}\bar{2} & \bar{2}\bar{2}\bar{1} & & & & & & & & & & \\
\text{(ii)} & 24u & 37u & 16u & \bar{2}5u & 8\bar{1}u & 9\bar{0}u & 480 & 78\bar{0} & 9\bar{2}8 & \bar{0}\bar{1}6 & 459 & 128 \\
 & 560 & 341 & 25\bar{0} & 67\bar{2} & 23\bar{2} & 47\bar{1} & 02\bar{1} & \bar{2}4\bar{0} & 571 & 683 & 790 & 692 \\
 & 03\bar{0} & 01\bar{2} & 9\bar{1}1 & 35\bar{1} & & & & & & & & \\
\text{(iii)} & \{cba : abc \in \text{(ii)}\} & & & & & & & & & & &
\end{array}$$

$\mathcal{A}_0^2 = (-\mathcal{A}_0^1)^{-1}$.

Let $\mathcal{A}_x^j = \mathcal{A}_0^j + x$ for $x \in Z_{13}$ and $j \in Z_3$. It is not difficult to check that each $\mathcal{A}_x^j$ (or $\mathcal{A}_u$) forms an *EDTS*(14) on X and they are pairwise disjoint. Therefore, the collection $\{(X, \mathcal{A}_x^j) : x \in Z_{13}, j \in Z_3\} \bigcup \{(X, \mathcal{A}_u)\}$ is an *LEDTS*(14) desired. ■

Lemma 4.8. *There exists an LEDTS*(16).

Proof. We construct an *LEDTS*(16) on $X = Z_{15} \cup \{u\}$, where 10, 11, 12, 13, 14 are written in $\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}$.

$\mathcal{A}_u : uuu \quad 055 \quad 0u\bar{0} \quad 034 \quad 068 \quad 430 \quad 860 \quad (\text{mod } 15).$

$\mathcal{A}_0^0$: (i) $uu7 \quad 0u0 \quad 411 \quad 622 \quad \bar{1}33 \quad 144 \quad 505 \quad 266 \quad \bar{2}77 \quad \bar{4}88 \quad 979 \quad \bar{0}0\bar{0}$
$3\bar{1}\bar{1} \quad \bar{2}\bar{2}u \quad \bar{3}\bar{2}\bar{3} \quad 8\bar{4}\bar{4} \quad 7u\bar{2}$

(ii) $1u5 \quad 3u9 \quad 4u6 \quad 8u\bar{1} \quad \bar{3}u\bar{4} \quad \bar{0}u2 \quad 56\bar{4} \quad 34\bar{3} \quad 29\bar{2} \quad 452 \quad 683 \quad 23\bar{4}$
$01\bar{4} \quad \bar{0}\bar{2}4 \quad \bar{1}\bar{3}6 \quad 9\bar{1}5 \quad 02\bar{3} \quad 47\bar{4} \quad 078 \quad 812 \quad \bar{3}19 \quad 03\bar{2} \quad 7\bar{1}2 \quad 04\bar{1}$
$069 \quad 6\bar{2}1 \quad \bar{3}57 \quad 713 \quad 894 \quad \bar{0}35 \quad \bar{2}58 \quad 1\bar{0}\bar{1} \quad \bar{4}9\bar{0} \quad \bar{3}8\bar{0} \quad \bar{0}67 \quad \bar{4}\bar{1}\bar{2}$

(iii) $\{cba : abc \in \text{(ii)}\}$.

$\mathcal{A}_0^1$: (i) $u\bar{4}u \quad 0\bar{1}0 \quad 141 \quad 2\bar{1}2 \quad 3\bar{1}3 \quad 4\bar{1}4 \quad 595 \quad 766 \quad 677 \quad 898 \quad 99\bar{1} \quad \bar{2}\bar{0}\bar{0}$
$\bar{1}\bar{1}9 \quad \bar{0}\bar{2}\bar{2} \quad \bar{3}4\bar{3} \quad \bar{4}\bar{4}\bar{4}$

(ii) $34u \quad 79u \quad 26u \quad 5\bar{1}u \quad \bar{0}\bar{3}u \quad 18u \quad 0\bar{2}u \quad \bar{2}25 \quad 7\bar{1}\bar{2} \quad 05\bar{0} \quad 17\bar{3} \quad 012$
$\bar{4}38 \quad 07\bar{4} \quad 8\bar{0}\bar{1} \quad \bar{4}56 \quad 68\bar{2} \quad 9\bar{2}1 \quad 457 \quad 239 \quad \bar{4}69 \quad 9\bar{3}0 \quad 37\bar{0} \quad 9\bar{0}\bar{4}$
$135 \quad 036 \quad \bar{2}\bar{3}3 \quad \bar{2}\bar{4}4 \quad 58\bar{3} \quad 278 \quad 048 \quad 16\bar{0} \quad \bar{1}\bar{4}1 \quad 6\bar{1}\bar{3} \quad \bar{3}\bar{4}2 \quad 24\bar{0}$

(iii) $\{cba : abc \in \text{(ii)}\}$.

$\mathcal{A}_0^2$: (i) $\bar{1}uu \quad 006 \quad 115 \quad 229 \quad 353 \quad 44\bar{4} \quad 551 \quad 660 \quad 77\bar{0} \quad 858 \quad 992 \quad \bar{0}\bar{0}7$
$u\bar{1}\bar{1} \quad \bar{2}\bar{2}\bar{3} \quad \bar{3}\bar{3}\bar{2} \quad \bar{4}\bar{4}4$

(ii) $21u \quad 73u \quad 50u \quad 49u \quad \bar{4}8u \quad \bar{3}6u \quad \bar{2}\bar{0}u \quad 09\bar{2} \quad 0\bar{3}\bar{4} \quad 6\bar{1}2 \quad 382 \quad 018$
$07\bar{1} \quad 14\bar{0} \quad 8\bar{2}7 \quad \bar{1}19 \quad 27\bar{4} \quad \bar{0}\bar{3}8 \quad 1\bar{2}\bar{4} \quad 9\bar{3}7 \quad 716 \quad 574 \quad \bar{0}59 \quad 3\bar{3}1$
$2\bar{0}0 \quad 5\bar{2}2 \quad 39\bar{4} \quad \bar{2}3\bar{1} \quad \bar{3}5\bar{1} \quad 6\bar{0}3 \quad 968 \quad 403 \quad 6\bar{4}5 \quad 4\bar{3}2 \quad \bar{1}48 \quad 6\bar{2}4$
$\bar{1}\bar{4}\bar{0}$

(iii) $\{cba : abc \in \text{(ii)}\}$.

Let $\mathcal{A}_x^j = \mathcal{A}_0^j + x$ for $x \in Z_{15}$ and $j \in Z_3$. It is not difficult to check that each $\mathcal{A}_x^j$ (or $\mathcal{A}_u$) forms an *EDTS*(16) on X and they are pairwise disjoint. Therefore, the collection $\{(X, \mathcal{A}_x^j) : x \in Z_{15}, j \in Z_3\} \bigcup \{(X, \mathcal{A}_u)\}$ is an *LEDTS*(16) desired. ■

Lemma 4.9. *There exists an LEDTS*(18).

Proof. Construct an *LEDTS*(18) on $X = Z_{16} \cup \{u, v\}$, where 10, 11, 12, 13, 14, 15 are written in $\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}$.

$\mathcal{A}_0^0$: (i) $u7u \quad v\bar{1}v \quad 00\bar{5} \quad 11\bar{0} \quad 224 \quad 3\bar{1}3 \quad 442 \quad 575 \quad 6\bar{2}6 \quad 7\bar{0}7 \quad 88\bar{3} \quad 9\bar{0}9$
$\bar{0}\bar{0}1 \quad \bar{1}\bar{1}4 \quad \bar{2}\bar{2}\bar{2} \quad \bar{3}\bar{3}8 \quad \bar{4}\bar{4}\bar{1} \quad \bar{5}\bar{5}0$

(ii) $45u \quad \bar{0}\bar{2}u \quad \bar{3}0u \quad 6\bar{1}u \quad 39u \quad \bar{4}2u \quad 18u \quad \bar{5}uv \quad 4\bar{3}\bar{5} \quad \bar{4}19 \quad \bar{4}5\bar{2} \quad 58\bar{5}$
$68v \quad 25v \quad \bar{3}\bar{4}v \quad 90v \quad \bar{2}1v \quad 4\bar{0}v \quad 37v \quad 05\bar{0} \quad 3\bar{0}\bar{3} \quad 672 \quad 012 \quad 07\bar{4}$
$23\bar{5} \quad 34\bar{4} \quad 56\bar{4} \quad \bar{5}1\bar{1} \quad 9\bar{1}4 \quad \bar{1}\bar{3}5 \quad 9\bar{2}5 \quad 135 \quad 036 \quad 048 \quad 0\bar{1}\bar{2} \quad 2\bar{2}\bar{3}$
$\bar{3}69 \quad \bar{4}8\bar{0} \quad 17\bar{3} \quad \bar{5}79 \quad 2\bar{0}\bar{1} \quad 461 \quad 892 \quad \bar{2}47 \quad 8\bar{2}3 \quad \bar{5}6\bar{0} \quad \bar{1}78$

(iii) $\{cba : abc \in \text{(ii)}\}$

$\mathcal{A}_0^1$: (i) $1uu \quad 0vv \quad u00 \quad v11 \quad 822 \quad 3v3 \quad 484 \quad 5u5 \quad \bar{3}66 \quad \bar{1}77 \quad 288 \quad \bar{4}99$
$\bar{0}6\bar{0} \quad 7\bar{1}\bar{1} \quad \bar{2}1\bar{2} \quad 6\bar{3}\bar{3} \quad 9\bar{4}\bar{4} \quad \bar{5}6\bar{5} \quad u1v \quad v8u \quad 0u8 \quad 8v0$

(ii) $2u3 \quad 6u9 \quad 7u\bar{2} \quad 4u\bar{1} \quad \bar{0}u\bar{4} \quad \bar{3}u\bar{5} \quad 03\bar{3} \quad 241 \quad 01\bar{5} \quad 02\bar{4} \quad 04\bar{2} \quad 05\bar{1}$
$7v\bar{0} \quad \bar{3}v2 \quad \bar{1}v\bar{2} \quad 4v6 \quad \bar{4}v5 \quad v9\bar{5} \quad 2\bar{1}\bar{5} \quad \bar{2}\bar{5}\bar{0} \quad 473 \quad 927 \quad 594 \quad \bar{0}09$
$8\bar{5}7 \quad 13\bar{0} \quad \bar{5}4\bar{4} \quad \bar{3}1\bar{1} \quad \bar{4}3\bar{2} \quad \bar{4}6\bar{1} \quad 615 \quad 836 \quad 7\bar{3}5 \quad 8\bar{2}5 \quad 5\bar{0}2 \quad 17\bar{4}$
$\bar{1}8\bar{0} \quad \bar{3}9\bar{2} \quad \bar{4}8\bar{3} \quad \bar{3}4\bar{0} \quad 706 \quad 918 \quad 5\bar{5}3 \quad \bar{1}39 \quad 26\bar{2}$

(iii) $\{cba : abc \in \text{(ii)}\}$

$\mathcal{A}_0^2$: (i) $uu7 \quad vv9 \quad 200 \quad 1\bar{4}1 \quad 022 \quad 313 \quad 4\bar{4}4 \quad 55\bar{1} \quad 656 \quad 77u \quad 818 \quad 99v$
$\bar{0}5\bar{0} \quad \bar{1}\bar{1}5 \quad \bar{5}\bar{2}\bar{2} \quad \bar{4}\bar{3}\bar{3} \quad \bar{3}\bar{4}\bar{4} \quad \bar{2}\bar{5}\bar{5}$

(ii) $u68 \quad u03 \quad u9\bar{3} \quad u\bar{5}4 \quad u\bar{2}2 \quad u\bar{4}5 \quad u\bar{0}\bar{1} \quad 1vu \quad 06\bar{1} \quad 17\bar{1} \quad 1\bar{3}\bar{5} \quad 0\bar{0}\bar{3}$
$v8\bar{1} \quad v\bar{0}\bar{2} \quad v34 \quad v05 \quad v7\bar{5} \quad v6\bar{3} \quad v\bar{4}2 \quad 014 \quad 489 \quad 0\bar{4}\bar{5} \quad 08\bar{2} \quad 259$
$9\bar{0}1 \quad \bar{1}\bar{5}2 \quad 58\bar{3} \quad 69\bar{5} \quad 8\bar{0}\bar{5} \quad \bar{1}\bar{3}3 \quad 126 \quad 079 \quad \bar{3}24 \quad \bar{5}35 \quad 6\bar{2}\bar{4} \quad 238$
$27\bar{0} \quad 46\bar{0} \quad 39\bar{2} \quad 7\bar{2}\bar{3} \quad 4\bar{1}\bar{2} \quad 745 \quad 367 \quad \bar{4}78 \quad 5\bar{2}1 \quad \bar{4}9\bar{1} \quad \bar{4}3\bar{0}$

(iii) $\{cba : abc \in \text{(ii)}\}$

$\mathcal{B}_0:\quad u\,u\,v\quad v\,v\,u\quad 0\,0\,8\quad 0\,u\,6\quad 0\,v\,\bar{0}\quad 0\,1\,3\quad 0\,4\,9\quad 3\,1\,0\quad 9\,4\,0\quad$ mod 16;
$\mathcal{B}_1:\quad u\,v\,v\quad v\,u\,u\quad 0\,8\,8\quad 0\,v\,6\quad 0\,u\,\bar{0}\quad 0\,2\,3\quad 0\,5\,9\quad 3\,2\,0\quad 9\,5\,0\quad$ mod 16;

$\mathcal{B}_2:$ (i) $u\,u\,u\quad 1\,9\,u\quad 2\,\bar{0}\,u\quad 3\,\bar{1}\,u\quad 4\,\bar{2}\,u\quad 5\,\bar{3}\,u\quad 6\,\bar{4}\,u\quad 7\,\bar{5}\,u\quad u\,8\,0$
$v\,u\,v\quad u\,9\,1\quad u\,\bar{0}\,2\quad u\,\bar{1}\,3\quad u\,\bar{2}\,4\quad u\,\bar{3}\,5\quad u\,\bar{4}\,6\quad u\,\bar{5}\,7\quad 0\,8\,u$
(ii) $0\,v\,4\quad 0\,0\,\bar{2}\quad 0\,1\,7\quad 0\,2\,5\quad 7\,1\,0\quad 5\,2\,0\quad$ mod 16;

$\mathcal{B}_3:$ (i) $v\,v\,v\quad 9\,1\,u\quad \bar{0}\,2\,u\quad \bar{1}\,3\,u\quad \bar{2}\,4\,u\quad \bar{3}\,5\,u\quad \bar{4}\,6\,u\quad \bar{5}\,7\,u\quad u\,0\,8$
$u\,v\,u\quad u\,1\,9\quad u\,2\,\bar{0}\quad u\,3\,\bar{1}\quad u\,4\,\bar{2}\quad u\,5\,\bar{3}\quad u\,6\,\bar{4}\quad u\,7\,\bar{5}\quad 8\,0\,u$
(ii) $0\,v\,\bar{2}\quad 0\,0\,4\quad 0\,6\,7\quad 0\,3\,5\quad 5\,3\,0\quad 7\,6\,0\quad$ mod 16.

Let $\mathcal{A}_x^j = \mathcal{A}_0^j + x$ for $x \in Z_{16}$ and $j \in Z_3$. It is not difficult to check that each $\mathcal{A}_x^j$ or each $\mathcal{B}_k$ ($k \in Z_4$) is the block set of an *EDTS*(18) on X and they are pairwise disjoint. Therefore, the collection $\{(X, \mathcal{A}_x^j) : x \in Z_{16}, j \in Z_3\} \bigcup \{(X, \mathcal{B}_k) : k \in Z_4\}$ is an *LEDTS*(18) desired. ■

5. Existence of *LEDTS*$(6t + 2)$

Lemma 5.1 ([10]). *There exist a PDGDD*$(3^3 : 2)$ *and a PDGDD*$(3^5 : 2)$.

Lemma 5.2. *There exists a PECS*$^*(3^3 : 0)$.

Proof. Let g be the primitive element of the field F_9, and $g^2 = 1 + 2g$. We will construct a *PECS*$^*(3^3 : 0)$, which consists of

(1) $27EDGDD(2^1 1^6)s$, denoted by $\mathcal{A}_x^j$, $x \in F_9, j \in Z_3$, where $\mathcal{A}_x^j = \mathcal{A}_0^j + x$;

(2) $4DGDD(3^3)s$, denoted by $\mathcal{B}_k$, $k \in I_4$.

Now, construct these $\mathcal{A}_0^j$ and $\mathcal{B}_k$ as follows.

(1) For each $\mathcal{A}_0^j$, the point set is $F_9 \setminus \{0\}$, the long group is $G_0 = \{g, g^5\}$, and the blocks are listed as follows, where the point g^a is briefly denoted by its index a.

$\mathcal{A}_0^0:$ 050 252 353 454 656 757 076 210 136 670 012
631 403 732 642 304 237 246 147 741

$\mathcal{A}_0^1:$ 003 422 733 440 266 776 146 613 341 712 307
435 362 201 064 025 247 170 523 560 754 657

$\mathcal{A}_0^2 = (\mathcal{A}_0^1)^{-1}$.

Clearly, each $\mathcal{A}_x^j$ will be on $F_9 \setminus \{x\}$ with the long group $G_0 + x$, $x \in F_9$.

(2) For each $\mathcal{B}_k$, the point set is F_9, the group set is $\{\{x, x+g, x+g^5\} : x = 0, 1, g^3\}$, and the blocks are listed as follows.

$\mathcal{B}_1 = \{(0, 1, g^3) + i, (0, g^4, g^7) + i, i \in F_9\}$; $\quad\mathcal{B}_2 = \{(0, 1, g^6) + i, (0, g^3, g^2) + i, i \in F_9\}$;
$\mathcal{B}_3 = \{(0, g^2, g^3) + i, (0, g^6, g^7) + i, i \in F_9\}$; $\quad\mathcal{B}_4 = \{(0, g^7, g^6) + i, (0, g^4, g^2) + i, i \in F_9\}$.

It is not difficult to verify that each $\mathcal{A}_0^j$ forms an *EDGDD*$(2^1 1^6)$ on $F_9 \setminus \{0\}$, each $\mathcal{B}_k$ forms a *DGDD*(3^3) on F_9, and all $\mathcal{A}_x^j$ and $\mathcal{B}_k$ ($x \in F_9, j \in Z_3, k \in I_4$) are mutually disjoint. Therefore, these designs form the desired *PECS*$^*(3^3 : 0)$ indeed. ■

Lemma 5.3. *There exists a PECS*$^*(3^5 : 0)$.

Proof. Take Z_{15} as the points. We will construct a *PECS*$^*(3^5 : 0)$, which consists of

(1) $45EDGDD(2^1 1^{12})s$, denoted by $\mathcal{A}_x^j$, $x \in Z_{15}, j \in Z_3$, where $\mathcal{A}_x^j = \mathcal{A}_0^j + x$;

(2) $4DGDD(3^5)s$, denoted by $\mathcal{B}_k$, $k \in I_4$.

Now, construct these $\mathcal{A}_0^j$ and $\mathcal{B}_k$ ($j \in Z_3, k \in I_4$) as follows.

(1) Each $\mathcal{A}_0^j$ is on $Z_{15} \setminus \{0\}$ with the long group $G_0 = \{5, 10\}$. The blocks in $\mathcal{A}_0^0$ and $\mathcal{A}_0^1$ are listed as follows, where 10, 11, 12, 13, 14 are written in $\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}$. And, $\mathcal{A}_0^2 = (-\mathcal{A}_0^1)^{-1}$.

$\mathcal{A}_0^0:$ (i) $1\,\bar{2}\,1\quad 2\,\bar{4}\,2\quad 3\,\bar{2}\,3\quad 4\,5\,4\quad 6\,\bar{4}\,6\quad 7\,5\,7\quad 8\,\bar{1}\,8\quad 9\,\bar{3}\,9\quad \bar{1}\,3\,\bar{1}\quad \bar{2}\,1\,\bar{2}\quad \bar{3}\,5\,\bar{3}\quad \bar{4}\,5\,\bar{4}$
(ii) $1\,2\,3\quad 2\,4\,6\quad 1\,4\,7\quad 1\,5\,9\quad 5\,\bar{1}\,2\quad \bar{2}\,\bar{3}\,2\quad 4\,\bar{1}\,3\quad \bar{0}\,\bar{1}\,1\quad \bar{1}\,\bar{4}\,7\quad 3\,7\,8\quad 1\,6\,8\quad \bar{4}\,3\,9\quad 4\,9\,\bar{0}$
$6\,9\,\bar{1}\quad \bar{3}\,\bar{4}\,1\quad \bar{3}\,4\,8\quad \bar{2}\,\bar{4}\,4\quad \bar{0}\,\bar{3}\,3\quad 3\,5\,6\quad 2\,8\,9\quad 6\,7\,\bar{3}\quad \bar{2}\,5\,8\quad 2\,7\,\bar{0}\quad 8\,\bar{0}\,\bar{4}\quad 7\,9\,\bar{2}\quad 6\,\bar{0}\,\bar{2}$
(iii) $\{cba : abc \in \text{(ii)}\}$

$$\begin{array}{lllllllllllllll}
\mathcal{A}_0^1: & \text{(i)} & \bar{2}\,1\,1 & \bar{4}\,2\,2 & \bar{1}\,3\,3 & 6\,4\,4 & 4\,6\,6 & \bar{3}\,7\,7 & 9\,8\,8 & 8\,9\,9 & 3\,\bar{1}\,\bar{1} & 1\,\bar{2}\,\bar{2} & 7\,\bar{3}\,\bar{3} & 2\,\bar{4}\,\bar{4} & \\
 & \text{(ii)} & 2\,6\,1 & 3\,8\,2 & 1\,4\,\bar{4} & \bar{1}\,1\,9 & 7\,\bar{2}\,4 & \bar{3}\,4\,9 & \bar{2}\,\bar{3}\,8 & \bar{1}\,2\,6 & \bar{0}\,\bar{1}\,7 & 7\,9\,2 & 7\,8\,6 & 1\,3\,\bar{3} & 2\,5\,\bar{2} \\
 & & 5\,9\,\bar{4} & 3\,7\,\bar{4} & \bar{1}\,\bar{4}\,8 & \bar{1}\,\bar{3}\,5 & \bar{2}\,\bar{4}\,\bar{0} & 4\,5\,8 & 6\,3\,5 & 3\,9\,\bar{2} & \bar{0}\,6\,9 & 8\,\bar{0}\,1 & 4\,\bar{1}\,2 & 6\,\bar{3}\,\bar{4} & \bar{0}\,3\,4 \\
 & & \bar{3}\,2\,\bar{0} & 7\,1\,5 & & & & & & & & & & &
\end{array}$$

(iii) $\{cba : abc \in \text{(ii)}\}$.

Clearly, each $\mathcal{A}_x^j$ will be on $Z_{15} \setminus \{x\}$ with the long group $G_0 + x,\ x \in Z_{15}$.

(2) For each $\mathcal{B}_k$, the point set is Z_{15}, the group set is $\{\{x, x+5, x+10\} : 0 \le x \le 4\}$, and the blocks are listed as follows.

$$\begin{array}{llllll}
\mathcal{B}_1: & 0\,3\,7 & 0\,1\,\bar{3} & 0\,6\,\bar{4} & 0\,2\,\bar{1} & \pmod{15}; \\
\mathcal{B}_2: & 0\,4\,7 & 0\,\bar{2}\,\bar{3} & 0\,8\,\bar{4} & 0\,9\,\bar{1} & \pmod{15}; \\
\mathcal{B}_3: & 0\,3\,2 & 0\,7\,1 & 0\,6\,4 & 0\,\bar{1}\,8 & \pmod{15}; \\
\mathcal{B}_4: & 0\,9\,1 & 0\,\bar{4}\,2 & 0\,\bar{3}\,4 & 0\,\bar{2}\,8 & \pmod{15}.
\end{array}$$

It is not difficult to verify that each $\mathcal{A}_0^j$ forms an $EDGDD(2^1 1^{12})$ on $Z_{15} \setminus \{0\}$, each $\mathcal{B}_k$ forms a $DGDD(3^5)$ on Z_{15}, and all $\mathcal{A}_x^j$ and $\mathcal{B}_k$ ($x \in Z_{15}, j \in Z_3, k \in I_4$) are mutually disjoint. Therefore, these designs form the desired $PECS^*(3^3 : 0)$ indeed. ■

Lemma 5.4. *There exists a $PECS(6^k : 2)$ for any integer $k \ge 3$.*

Proof. From [6], for $k \ge 3$, there exists a 2-$FG(3, (\{3, 5\}, \{3, 5\}, \{4, 6\}), 2^k)$. Furthermore, taking $m = 3, g = 2, r = 0, s = 2$ and using Theorem 2.3, since

$\exists\, PECS^*(3^k : 0)$ for $k \in \{3, 5\}$ by Lemmas 5.2 and 5.3,

$\exists\, PDGDD(3^k : 2)$ for $k \in \{3, 5\}$ by Lemma 5.1,

$\exists\, DF(3^k)$ for $k \in \{4, 6\}$ by Lemma 1.1,

we can get a $PECS(6^k : 2)$. ■

Theorem 5.1. *There exists an $LEDTS(6k + 2)$ for any integer $k \ge 0$.*

Proof. For $k = 0, 1, 2$, there exists an $LEDTS(6k + 2)$ by Lemmas 4.2, 4.3 and 4.7. For $k \ge 3$, there exist a $PECS(6^k : 2)$, an $LEDTS(8, 2)$ and an $LEDTS(8)$ by Lemmas 4.3 and 5.4. Therefore, there exists an $LEDTS(6k + 2)$ by Theorem 2.1. ■

6. Existence of $LEDTS(6t + 4)$

Lemma 6.1. *There exists a $PECS(3^3 : 1)$.*

Proof. Let g be the primitive element of the field F_9, and $g^2 = 1 + 2g$. Take $u \notin F_9$. We will construct a $PECS(3^3 : 1)$, which consists of

(1) $27 EDGDD(4^1 1^6)s$, denoted by $\mathcal{A}_x^j,\ x \in F_9, j \in Z_3$, where $\mathcal{A}_x^j = \mathcal{A}_0^j + x$;

(2) one $DGDD(3^3)$, denoted by $\mathcal{B}$.

Now, give the constructions for these $\mathcal{A}_0^j$ and $\mathcal{B}_k$ as follows.

(1) For each $\mathcal{A}_0^j$, the point set is $F_9 \cup \{u\}$, the long group is $G_0 = \{0, g, g^5, u\}$, and the blocks are listed as follows, where the point g^a is briefly denoted by its index a and the point 0 is denoted 8, but the point u is kept.

$$\begin{array}{lllllllllllll}
\mathcal{A}_0^0: & 0\,8\,0 & 2\,8\,2 & 3\,1\,3 & 4\,1\,4 & 6\,1\,6 & 7\,8\,7 & 0\,u\,6 & 2\,u\,3 & 3\,u\,2 & 4\,u\,7 & 6\,u\,0 & 7\,u\,4 \\
 & 8\,3\,4 & 7\,3\,6 & 6\,3\,8 & 5\,6\,4 & 0\,2\,4 & 2\,6\,7 & 3\,5\,7 & 4\,2\,5 & 5\,3\,0 & 6\,5\,2 & 0\,1\,7 & 1\,2\,0 \\
 & 7\,0\,5 & 4\,0\,3 & 7\,2\,1 & 4\,8\,6 & & & & & & & &
\end{array}$$

$$\begin{array}{lllllllllllll}
\mathcal{A}_0^1: & 0\,0\,3 & 2\,2\,4 & 3\,3\,0 & 4\,4\,2 & 6\,6\,7 & 7\,7\,6 & u\,6\,0 & u\,2\,3 & u\,7\,4 & 0\,6\,u & 3\,2\,u & 4\,7\,u \\
 & 8\,0\,4 & 8\,2\,6 & 8\,3\,7 & 4\,0\,8 & 6\,2\,8 & 7\,3\,8 & 4\,6\,1 & 0\,5\,7 & 1\,6\,3 & 4\,3\,5 & 2\,7\,1 & 6\,5\,4 \\
 & 5\,2\,0 & 3\,1\,4 & 7\,2\,5 & 1\,7\,0 & 5\,3\,6 & 0\,1\,2 & & & & & &
\end{array}$$

$$\begin{array}{lllllllllllll}
\mathcal{A}_0^2: & 4\,0\,0 & 6\,2\,2 & 2\,3\,3 & 7\,4\,4 & 0\,6\,6 & 3\,7\,7 & u\,3\,0 & u\,2\,4 & u\,6\,7 & 0\,3\,u & 4\,2\,u & 7\,6\,u \\
 & 8\,2\,0 & 8\,6\,3 & 3\,8\,4 & 0\,5\,2 & 6\,4\,5 & 5\,4\,3 & 4\,6\,8 & 7\,2\,8 & 0\,8\,7 & 3\,5\,6 & 5\,7\,0 & 2\,7\,5 \\
 & 2\,6\,1 & 4\,7\,1 & 1\,6\,0 & 1\,7\,3 & 0\,1\,4 & 3\,1\,2. & & & & & &
\end{array}$$

Clearly, each $\mathcal{A}_x^j$ will be on $F_9 \cup \{u\}$ with the long group $G_0 + x,\ x \in F_9$. Obviously, $G_0 + 0 = G_0 + g = G_0 + g^5,\ G_0 + 1 = G_0 + g^2 = G_0 + g^7$ and $G_0 + g^3 = G_0 + g^4 = G_0 + g^6$.

(2) For $\mathcal{B}$, the point set is F_9, the group set is $\{\{x, x + g, x + g^5\} : x = 0, 1, g^3\}$, and the blocks are

$$\mathcal{B} = \{(0, g^7, g^4) + i, (0, g^3, 1) + i : i \in F_9\}.$$

It is not difficult to verify that each $\mathcal{A}_0^j$ forms an $EDGDD(4^1 1^6)$ on $F_9 \cup \{u\}$, the $\mathcal{B}$ forms a $DGDD(3^3)$ on F_9, and all $\mathcal{A}_x^j$ and $\mathcal{B}$ ($x \in F_9, j \in Z_3$) are mutually disjoint. Therefore, these designs form the desired $PECS(3^3 : 1)$ indeed. ■

Lemma 6.2. *There exists a* $PECS(3^5 : 1)$.

Proof. Take $Z_{15} \cup \{u\}$ as the points, where $u \notin Z_{15}$. Denote $G_0 = \{0, 5, 10\}$ and $G_x = G_0 + x,\ 0 \le x \le 4$. We will construct a $PECS(3^5 : 1)$, which consists of

(1) $45EDGDD(4^1 1^{12})s$, denoted by $\mathcal{A}_x^j,\ x \in Z_{15}, j \in Z_3$, where $\mathcal{A}_x^j = \mathcal{A}_0^j + x$;

(2) one $DGDD(3^5)$, denoted by $\mathcal{B}$.

Now, construct these $\mathcal{A}_0^j$ ($j \in Z_3$) and $\mathcal{B}$ as follows.

(1) Each $\mathcal{A}_0^j$ is on $Z_{15} \cup \{u\}$ with the long group $G_0 \cup \{u\}$, and the blocks are listed as follows, where 10, 11, 12, 13, 14 are written in $\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}$.

$$\begin{array}{lllllllllllll}
\mathcal{A}_0^0 : \text{(i)} & 141 & 242 & 3\bar{0}3 & 4\bar{3}4 & 6\bar{4}6 & 7\bar{1}7 & 8\bar{4}8 & 9\bar{0}9 & \bar{1}8\bar{1} & \bar{2}8\bar{2} & \bar{3}1\bar{3} & \bar{4}3\bar{4} \\
\text{(ii)} & 6u9 & \bar{2}u\bar{3} & 3u7 & 2u8 & \bar{4}u1 & 4u\bar{1} & \bar{3}23 & \bar{3}09 & \bar{2}\bar{4}5 & \bar{0}\bar{1}2 & 389 & 257 \\
 & 168 & 79\bar{2} & 9\bar{1}1 & 58\bar{3} & \bar{1}\bar{2}0 & \bar{1}\bar{4}3 & 1\bar{0}\bar{2} & 4\bar{0}\bar{4} & 467 & 6\bar{0}\bar{3} & 048 & 9\bar{4}2 \\
 & 78\bar{0} & 26\bar{2} & 459 & \bar{2}34 & 012 & 135 & 036 & 17\bar{3} & 07\bar{4} & 56\bar{1} & & \\
\text{(iii)} & \multicolumn{12}{l}{\{cba : abc \in \text{(ii)}\}} \\
\mathcal{A}_0^1 : \text{(i)} & 117 & 229 & 33\bar{4} & 446 & 664 & 771 & 88\bar{1} & 992 & \bar{1}\bar{1}8 & \bar{2}\bar{2}\bar{3} & \bar{3}\bar{3}\bar{2} & \bar{4}\bar{4}3 \\
\text{(ii)} & 91u & \bar{4}2u & 37u & \bar{3}4u & 68u & \bar{1}\bar{2}u & 04\bar{1} & \bar{3}\bar{4}8 & \bar{1}\bar{4}\bar{0} & 894 & 381 & \bar{1}27 \\
 & 6\bar{1}3 & 24\bar{2} & 7\bar{2}6 & \bar{2}\bar{4}9 & 56\bar{4} & 352 & 47\bar{4} & 9\bar{3}3 & 4\bar{0}3 & \bar{1}59 & 451 & 7\bar{3}5 \\
 & 8\bar{2}5 & \bar{3}1\bar{1} & 8\bar{0}2 & 9\bar{0}7 & \bar{0}\bar{2}1 & \bar{0}\bar{3}6 & 069 & 078 & 261 & 01\bar{4} & 02\bar{3} & 03\bar{2} \\
\text{(iii)} & \multicolumn{12}{l}{\{cba : abc \in \text{(ii)}\}} \\
\mathcal{A}_0^2 : \text{(i)} & 711 & 922 & \bar{4}33 & 644 & 466 & 177 & \bar{1}88 & 299 & 8\bar{1}\bar{1} & \bar{3}\bar{2}\bar{2} & \bar{2}\bar{3}\bar{3} & 3\bar{4}\bar{4} \\
\text{(ii)} & 19u & 2\bar{4}u & 73u & 4\bar{3}u & 86u & \bar{2}\bar{1}u & \bar{4}8\bar{3} & \bar{4}9\bar{2} & 813 & 948 & \bar{0}79 & \bar{3}6\bar{0} \\
 & \bar{1}36 & 27\bar{1} & 4\bar{1}0 & 4\bar{2}2 & 1\bar{1}\bar{3} & \bar{4}\bar{0}\bar{1} & 523 & 612 & 7\bar{4}4 & \bar{3}39 & \bar{0}34 & 6\bar{4}5 \\
 & 59\bar{1} & \bar{2}1\bar{0} & \bar{2}58 & \bar{0}28 & \bar{3}57 & 514 & 690 & 780 & 1\bar{4}0 & 2\bar{3}0 & 3\bar{2}0 & \bar{2}67 \\
\text{(iii)} & \multicolumn{12}{l}{\{cba : abc \in \text{(ii)}\}.}
\end{array}$$

Clearly, each $\mathcal{A}_x^j$ will be on $Z_{15} \cup \{u\}$ with the long group $G_{\bar{x}},\ 0 \le \bar{x} \le 4,\ x \equiv \bar{x} \bmod 5$.

(2) For $\mathcal{B}$, the point set is Z_{15}, the group set is $\{G_0, G_1, G_2, G_3, G_4\}$, and the blocks are

$$\mathcal{B} = \{(0, 3, 4), (4, 3, 0), (0, 6, 8), (8, 6, 0) \bmod 15\}.$$

It is not difficult to verify that each $\mathcal{A}_0^j$ forms an $EDGDD(4^1 1^{12})$ on $Z_{15} \cup \{u\}$, the $\mathcal{B}$ forms a $DGDD(3^3)$ on Z_{15}, and all $\mathcal{A}_x^j$ and $\mathcal{B}$ ($x \in Z_{15}, j \in Z_3$) are mutually disjoint. Therefore, these designs form the desired $PECS(3^5 : 1)$ indeed. ■

Lemma 6.3. *There exists a* $PECS(6^k : 4)$ *for any integer* $k \ge 3$.

Proof. From [6], for $k \ge 3$, there exists a 2-$FG(3, (\{3, 5\}, \{3, 5\}, \{4, 6\}), 2^k)$. Furthermore, taking $m = 3, g = 2, r = 1$ and using Theorem 2.2, since

$\exists\, PECS(3^k : 1)$ for $k \in \{3, 5\}$ by Lemmas 6.1 and 6.2,

$\exists\, DF(3^{k+1})$ for $k \in \{3, 5\}$ and $\exists\, DF(3^k)$ for $k \in \{4, 6\}$ by Lemma 1.1,

we can get a $PECS(6^k : 4)$. ■

Theorem 6.1. *There exists an* $LEDTS(6k + 4)$ *if and only if* $k \ge 1$.

Proof. For $k = 0$, there does not exist $LEDTS(4)$ by Lemma 4.1. For $k = 1, 2$, there exists an $LEDTS(6k + 4)$ by Lemmas 4.4 and 4.8. For $k \ge 3$, there exist $PECS(6^k : 4)$, $LEDTS(10, 4)$ and $LEDTS(14)$ by Lemmas 4.5, 4.7 and 6.3. Then, there exists an $LEDTS(6k + 4)$ by Theorem 2.1. ■

7. Existence of *LEDTS*(6*t*)

Theorem 7.1. *There exists an* $LEDTS(6k)$ *for any integer* $k \ge 1$.

Proof. Let $6k = 3^t m$, where $t \ge 1,\ m \equiv 2, 4 \bmod 6$. By Theorems 5.1 and 6.1, there exists an $LEDTS(m)$ for any integer $m \ge 2$ and $m \ne 4$. Using Theorem 2.4, we can get an $LEDTS(3^t m)$ for $(t, m) \ne (1, 2), (1, 4), (2, 2)$. However, from Lemmas 4.2, 4.6 and 4.9, we can get

$$LEDTS(3^1 \cdot 2) = LEDTS(6), \qquad LEDTS(3^1 \cdot 4) = LEDTS(12) \quad \text{and} \quad LEDTS(3^2 \cdot 2) = LEDTS(18).$$

So, there exists an $LEDTS(6k)$ for any integer $k \ge 1$. ■

8. Conclusion

Theorem 8.1. *There exists an LEDTS(v) for any even v except $v = 4$.*

Proof. We can get the conclusion by Theorems 5.1, 6.1 and 7.1 and Lemma 4.1. ■

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