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Phase transition between the BTZ black hole and AdS space

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Abstract

In three dimensions, a phase transition occurs between the non-rotating BTZ black hole and the massless BTZ black hole. Further, introducing the mass of a conical singularity, we show that a transition between the non-rotating BTZ black hole and thermal AdS space is also possible.

1. Introduction

Hawking's semiclassical analysis for the black hole radiation suggests that most of information in initial states is shield behind the event horizon and is never back to the asymptotic region far from the evaporating black hole [1]. This means that the unitarity is violated by an evaporating black hole. However, this conclusion has been debated ever since [2,3]. It is closely related to the information loss paradox which states the question of whether the formation and subsequent evaporation of a black hole is unitary. One of the most urgent problems in the black hole physics is to resolve the unitarity issue.

Maldacena proposed that the unitarity can be restored if one takes into account the topological diversity of gravitational instantons with the same AdS boundary in three-dimensional gravity [4]. Actually, three-dimensional gravity [5] is not directly related to the information loss problem because there is no physically propagating degrees of freedom [6]. If this gravity is part of string theory [7], the AdS/CFT correspondence [8] means that the black hole formation and evaporating process should be unitary because its boundary can be described by a unitary CFT. Recently, Hawking has withdrawn his argument on information loss and suggested that the unitarity can be preserved by extending Maldacena's proposal to four-dimensional gravity system [9].

We remark an interesting phenomenon in the AdS black hole thermodynamics. There exists the Hawking–Page transition between AdS–Schwarzschild black hole and thermal AdS space in four dimensions [10]. Some authors have proposed that this transition is also possible in three-dimensional spacetimes: Transition between the non-rotating BTZ black hole and thermal AdS space [11,12]. Recently the author has shown that there is no the first-order Hawking–Page transition between the non-rotating BTZ black hole and thermal AdS space [13], by comparing it with the phase transition between AdS–Schwarzschild black hole and thermal AdS space.

In this Letter, we show that a phase transition occurs between the non-rotating BTZ black hole and the massless BTZ black hole. If one introduces the mass of a conical singularity, a transition between the non-rotating BTZ black hole and thermal AdS space is also possible. We use the off-shell β-function which measures the mass of a conical singularity at the event horizon, and the off-shell free energy which is used to study the growth of the off-shell black hole.

We start with the non-rotating ($J=0$) BTZ black hole described by the line element

$$ds^2_{\mathrm{NBTZ}} = -\left[\frac{r^2}{l^2} - \mu\right]dt^2 + \frac{dr^2}{\frac{r^2}{l^2} - \mu} + r^2\,d\theta^2, \tag{1}$$

which possesses a continuous mass spectrum from $M = \frac{\mu}{8G_3}$ to the massless AdS black holes ($M = 0$) with different topology:

$$ds^2_{\mathrm{MADS}} = -\frac{r^2}{l^2}dt^2 + \frac{l^2}{r^2}dr^2 + r^2\,d\theta^2, \tag{2}$$

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where we find a degenerate event horizon at the origin of the coordinate ($r = 0$). Also the AdS spacetime is allowed by the line element

$$ds^2_{\text{TADS}} = -\left[1 + \frac{r^2}{l^2}\right]dt^2 + \frac{dr^2}{1 + \frac{r^2}{l^2}} + r^2\, d\theta^2. \tag{3}$$

In this work we consider three interesting cases [14,15]. (i) The non-rotating BTZ black hole (NBTZ) is given by $M = r_+^2/8G_3l^2$ and $T_H = r_+/2\pi l^2$ with the horizon radius $r_+ = l\sqrt{\mu}$. (ii) The massless BTZ black hole (MBTZ) with $M = T_H = 0$ is called the spacetime picture of the RR vacuum state. (iii) The thermal AdS spacetime (TADS) is determined by $M = -1/8G_3$ and $T_H = 0$. This case corresponds to the spacetime picture of the NS–NS vacuum state [16]. Although the thermodynamic properties of TADS and MBTZ are nearly the same, their Euclidean topologies are quite different: TADS (MBTZ) are topologically non-trivial (trivial). The TADS has a non-contractible S^1 at $r = 0$, while the MBTZ is contractible but it has a conical singularity at the event horizon ($r = 0$).

In $d \geqslant 4$ case, the Hawking–Page phase transition occurs between the Schwarzschild–AdS black hole and thermal AdS space. In this case, there exists a minimum temperature at $r_+ = r_0$. We have two solutions: for $r_+ < r_0$, the unstable black hole with the negative heat capacity; for $r_+ > r_0$, the stable black hole with the positive heat capacity. Even though the unstable solution is thermally unstable, it is important as the mediator of phase transition from thermal AdS to AdS black hole.

2. Transition between MBTZ and NBTZ

However, for Chern–Simons black holes (NBTZ case), the situation is quite different from the case of the Schwarzschild–AdS black hole [17,18]. The NBTZ could be thermally equilibrium with the heat reservoir at any temperature T. To show this, we introduce the on-shell free energy (energy) and heat capacity (entropy) as

$$F^{\text{on}}_{\text{NBTZ}} = -E_{\text{NBTZ}} = -\frac{r_+^2}{8G_3l^2},$$
$$C_{\text{NBTZ}} = S_{\text{NBTZ}} = \frac{\pi r_+}{2G_3}. \tag{4}$$

A condition for the thermal equilibrium is given by $T = T_H$. Then we always have a stable NBTZ at $r_s = 2\pi l^2 T$ without the minimum temperature. A positive heat capacity ($C_{\text{NBTZ}} > 0$) means that the NBTZ is a thermally stable system, irrespective of any size of the black hole. This point contrasts to the case of the Schwarzschild–AdS black hole. It is obvious that the NBTZ with $T_H = 0$ leads to the MBTZ case

$$F_{\text{MBTZ}} = E_{\text{MBTZ}} = C_{\text{MBTZ}} = S_{\text{MBTZ}} = 0. \tag{5}$$

On the other hand, thermodynamic quantities for thermal AdS space are given by

$$F_{\text{TADS}} = E_{\text{TADS}} = -\frac{1}{8G_3},$$
$$C_{\text{TADS}} = S_{\text{TADS}} = 0. \tag{6}$$

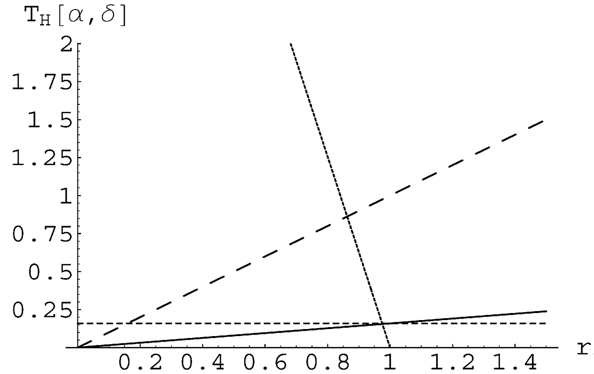

Fig. 1. The temperature picture of a cool (off-shell) black hole growth in a hotter heat bath at $T = T_c = 1/2\pi$ (small dashed line). Solid line shows a plot of the increasing temperature T_H of a cool black hole with $l = 1$. Large dashed line indicates the off-shell parameter $\alpha(r_+, T_c)$. Dotted line denotes the deficit angle $\delta(r_+, T_c)$. In this case we have a saddle point (stable NBTZ) at $r_s = 1$ ($\alpha = 1, \delta = 0, T_H = T_c$).

In order to study the phase transition clearly, we have to introduce the generalized (off-shell) free energy

$$F^{\text{off}}_{\text{NBTZ}} = E_{\text{NBTZ}} - T \cdot S_{\text{NBTZ}}. \tag{7}$$

Also the off-shell parameter α and the deficit angle δ take the forms

$$\alpha(r_+, T) = \frac{T_H}{T}, \qquad \delta(r_+, T) = 2\pi(1 - \alpha). \tag{8}$$

As is shown in Fig. 1, α is zero at $r_+ = 0$ and it is one at $r_+ = r_s$ with $l = 1$. On the other hand, δ has the maximum of 2π at $r_+ = 0$ and it is zero at $r_+ = r_s$. This means that the near horizon geometry at $r_+ = 0$ is the narrowest cone with the shape ($\prec$), while its geometry at $r_+ = r_s$ is a contractible manifold ($\subset$). In this sense $r_+ = 0$ is not a saddle point. We have $0 < \delta < 2\pi$ between $r_+ = 0$ and $r_+ = r_s$ and thus we have a cone singularity at the event horizon ($<$). Using α, we can rewrite the off-shell free energy as

$$F^{\text{off}}_{\text{NBTZ}}(r_+, T) = -F^{\text{on}}_{\text{NBTZ}}\left[1 - \frac{2}{\alpha}\right] \tag{9}$$

with the corresponding Euclidean action[1] $I^{\text{off}}_{\text{NBTZ}} = F^{\text{off}}_{\text{NBTZ}}/T$. At $\alpha = 1$ ($r_+ = r_s$), we recovers $F^{\text{off}}_{\text{NBTZ}} = F^{\text{on}}_{\text{NBTZ}}$. We confirm this from the operation

$$\frac{\partial F^{\text{off}}_{\text{NBTZ}}}{\partial r_+} = 0 \to T = T_H \to F^{\text{off}}_{\text{NBTZ}} = F^{\text{on}}_{\text{NBTZ}}. \tag{10}$$

In this sense the off-shell (off-equilibrium) free energy becomes the on-shell free energy at the saddle point of $r_+ = r_s =$

[1] In Appendix D of Ref. [12], there is a slightly different approach to this free energy. The on-shell action is given by $I^{\text{on}}_{\text{NBTZ}} = F^{\text{on}}_{\text{NBTZ}}/T_H = -\frac{\pi r_+}{4G_3}$ and a contribution from the conical singularity is $I_{\text{cs}} = -\frac{r_+\delta}{4G_3}$. The total gravitational action is then: $I_g = I^{\text{on}}_{\text{NBTZ}} + I_{\text{cs}} = \frac{\pi r_+}{4G_3}\alpha - \frac{\pi r_+}{2G_3} = -I^{\text{on}}_{\text{NBTZ}}\alpha[1 - \frac{2}{\alpha}] = F^{\text{off}}_{\text{NBTZ}}/T$. Here we prove $I_g = I^{\text{off}}_{\text{NBTZ}}$. If this conical deficit is created by a Euclidean point particle of mass M_{pp}, we include its action $I_{\text{pp}}(= -I_{\text{cs}}) = -\pi^2 l^2 T\alpha(\alpha - 1)/G_3$ as a counter term. Then the total action leads to the on-shell action: $I_{\text{tot}} = I^{\text{on}}_{\text{NBTZ}} + I_{\text{cs}} + I_{\text{pp}} = I^{\text{on}}_{\text{NBTZ}}$.

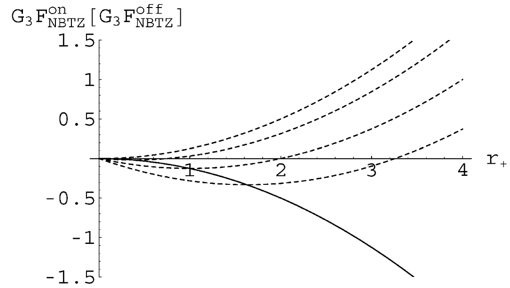

Fig. 2. The on-shell versus off-shell free energy. The solid line represents the on-free energy $F^{\rm on}_{\rm NBTZ}(r_+)$ in the units of G_3 and $l=1$, while the dashed line denotes the off-shell free energy $F^{\rm off}_{\rm NBTZ}(r_+,T)$ for four different temperatures: from the top down, $T=0, 0.059, T_c(=0.159), 0.259$. At each saddle point $r_+=r_s$, we have $F^{\rm out}_{\rm NBTZ}=F^{\rm on}_{\rm NBTZ}$.

$2\pi l^2 T>0$. Further, we obtain the β-function from the definition

$$\beta_{\rm NBTZ}(r_+,T)\propto\frac{\partial I^{\rm off}_{\rm NBTZ}}{\partial r_+}=-\frac{\mathcal{C}_{\rm NBTZ}}{6l}\delta(r_+,T), \tag{11}$$

where the $\mathcal{C}_{\rm NBTZ}$-function is related to the central charge on the boundary CFT. In this case, it is just a constant $\mathcal{C}_{\rm NBTZ}=3l/2G_3=c$. Further Eq. (11) means that the β-function measures the deficit angle δ mainly.

At this stage, we introduce an assumed picture of the phase transition in three dimensions. A phase transition may occur at $T=T_c=1/2\pi l(r_+=l)$ between NBTZ and MBTZ [13]. As is shown in Fig. 2, at $T=0$, the MBTZ is a saddle point as the ground state. For $T>0$, we have $F^{\rm off}_{\rm NBTZ}(r_+)<0$ at the saddle point $r_+=r_s$ so that a stable NBTZ is more probable than the MBTZ. Thus it is possible to flow from the MBTZ to the NBTZ along the path provided by the off-shell black hole configurations. At $T=T_c$, the situation is the same. This case is depicted in Fig. 3. The off-shell black holes can be modeled by the metric Eq. (1) with fixed T and varying $0<r_+<r_s$, and a conical singularity at the event horizon. This differs from the Hawking–Page transition where the unstable black hole plays an important role of the mediator from thermal AdS to AdS black hole. Here is no such a mediator. Hence there is no the Hawking–Page like transition in three dimensions. This states the censorship for the Hawking–Page transition in three dimensions. Since, in the canonical approach, the free energy corresponds to the effective potential, the transition between the MBTZ with and NBTZ may be regarded as the tunneling process.

On the AdS side, we check whether or not the Cardy–Velinde formula is satisfied with this picture. To obtain this formula of $S_{\rm NBTZ}=\frac{2\pi l}{d-2}\sqrt{E_c(E_{\rm NBTZ}-E_c)}$ [19], we have to define the Casimir energy $E_c=2E_{\rm NBTZ}-T_H S_{\rm NBTZ}$. However, it turns out

$$E_c=0. \tag{12}$$

Also considering the boundary topology of S^1 leads to $E_c=0$ because it is locally flat. Thus we no longer use this formula to show a relation between entropy and energy in three dimensions.

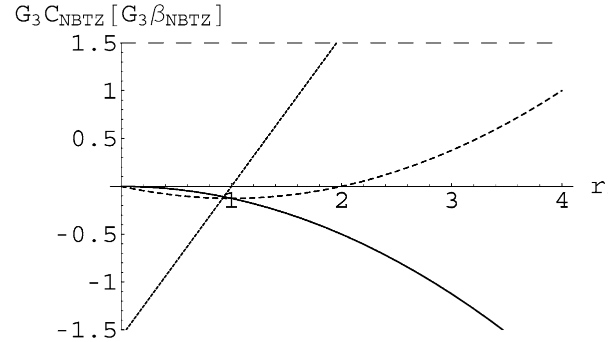

Fig. 3. The large dashed line denotes the $\mathcal{C}_{\rm NBTZ}=3l/2G_3=c$ as the central charge on the CFT boundary. The dotted line represents the off-shell β-function $\beta_{\rm NBTZ}(r_+,T_c)$, which measures the mass of a conical singularity. The solid line denotes the on-shell free energy $F^{\rm on}_{\rm NBTZ}(r_+)$, while the small dashed line shows the off-shell free energy $F^{\rm off}_{\rm NBTZ}(r_+,T_c)$. At the junction point of $r_+=r_s=1$, one has $F^{\rm on}_{\rm NBTZ}=F^{\rm off}_{\rm NBTZ}$, $\beta_{\rm NBTZ}=0$. This point is a stable NBTZ which comes out from the off-shell approach.

On the CFT side, we introduce the well-known Cardy formula in two dimensions

$$S_{\rm CFT}=2\pi\sqrt{\frac{c}{6}L_0}+2\pi\sqrt{\frac{\bar c}{6}\bar L_0} \tag{13}$$

with $c=\bar c=3l/2G_3$ and $L_0=\bar L_0=M_{\rm NBTZ}l/2$. Here the radius of S^1 is set to be $\rho=1$. Then we establish the AdS/CFT correspondence for the entropy: $S_{\rm CFT}=S_{\rm NBTZ}$ [15]. Finally we note that this model does not satisfies a higher-dimensional relation of $E_c\propto c$ because of $E_c=0$.

3. Transition between TADS and NBTZ

In three dimensions, one has a mass gap between MBTZ with $M_{\rm MBTZ}=0$ and TADS with $M_{\rm TADS}=-1/8G_3$. A conical singularity interpreted as a point mass source would be introduced to explain this. For this purpose, we use the relation of $I_{\rm cs}=-\frac{r_+\delta}{4G_3}\equiv 4\pi r_+M_{\rm cs}$. Then the mass of a conical singularity at the event horizon is given by [12]

$$M_{\rm cs}(r_+,T)=-\frac{1}{8G_3}\frac{\delta}{2\pi}=-\frac{1}{8G_3}(1-\alpha). \tag{14}$$

This is closely parallel to the point particle at the event horizon: $I_{\rm pp}=\frac{r_+\delta}{4G_3}\equiv 4\pi r_+M_{\rm pp}$ with $M_{\rm pp}=\frac{\delta}{16\pi G_3}=-M_{\rm cs}$. Here we obtain another relation $M_{\rm cs}=\beta_{\rm NBTZ}/4\pi$ between mass and β-function. The branch of $-1/8G_3\leqslant M_{\rm cs}<0$ is allowed only to a collection of off-shell black holes with a conical singularity for $0\leqslant r_+<r_s$. In this section we use the mass (energy) $M_{\rm cs}$ instead of the mass of black hole itself.

Furthermore, we introduce a new energy and free energy which are based on the Horowitz–Myers conjecture for the AdS soliton [20]. This implies that the soliton with a negative energy can be taken as the thermal background. We note that for a three-dimensional AdS space, the flat AdS black hole and spherical AdS black hole are the same because their horizons are one dimension. Thus, the three-dimensional AdS soliton is just the thermal AdS space [21]. Then we can calculate the new energy and free energy with respect to the soliton background

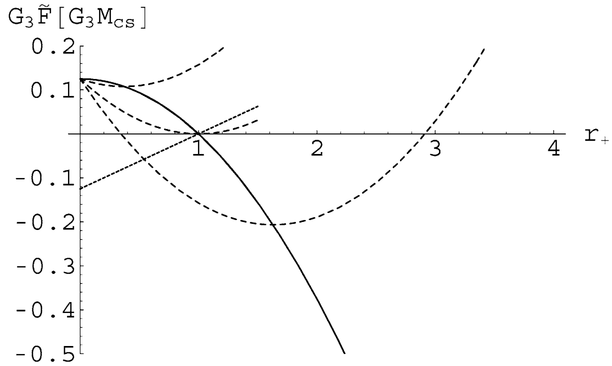

Fig. 4. The solid line denotes the on-shell free energy $\tilde{F}^{\text{on}}(r_+)$, while the dashed lines show the off-shell free energy $\tilde{F}^{\text{off}}(r_+, T)$ for three different temperatures: from the top down, $T = 0.059 (< T_c)$, $T_c = 0.159$, $0.259 (> T_c)$. These are shifted from $F^{\text{on}}_{\text{NBTZ}}$ and $F^{\text{off}}_{\text{NBTZ}}$ by $+1/8$. The dotted line represents the mass of a conical singularity $M_{\text{cs}}(r_+, T_c)$.

(TADS) using the standard regularization scheme:

$$\tilde{E}(r_+) = \frac{1}{8G_3}\left[1 + \frac{r_+^2}{l^2}\right],$$

$$\tilde{F}^{\text{on}}(r_+) = F^{\text{on}}_{\text{NBTZ}} - F_{\text{TADS}} = \frac{1}{8G_3}\left[1 - \frac{r_+^2}{l^2}\right]. \quad (15)$$

This leads to

$$\tilde{F}^{\text{off}}(r_+, T) = F^{\text{off}}_{\text{NBTZ}}(r_+, T) - F_{\text{TADS}}. \quad (16)$$

The new energy of $\tilde{E} = E_{\text{NBTZ}} - E_{\text{TADS}}$ is always positive with respect to the TADS. We have $\tilde{F}^{\text{on}} = \tilde{F}^{\text{off}} = 1/8G_3$ but M_{TADS} is found to be $M_{\text{cs}}(0, T_c) = -1/8G_3$ at $r_+ = 0$. On the other hand, at the saddle point $r_+ = r_s$, we have $\tilde{F}^{\text{on}}(r_+) = \tilde{F}^{\text{off}}(r_+, T_c) = M_{\text{cs}}(r_+, T_c) = 0$. This is depicted in Fig. 4. At $T = T_c$, the transition from the TADS to the NBTZ is possible. For $T < T_c$, the TADS dominates, while for $T > T_c$, the NBTZ dominates because of $\tilde{F}^{\text{off}}(r_+ = r_s) < 0$. There is a change of dominance at the critical temperature $T = T_c$.

Therefore, if one considers the mass of a conical singularity, we can connect the TADS with the NBTZ using the off-shell free energy approach. In this way, we could accommodate the TADS with a negative mass and free energy within our picture.

Alternatively, if one includes quantum fluctuations, there exits a possibility that the MBTZ is not the end of the Hawking evaporation and the end might be the TADS [22].

Consequently, the transition between the MBTZ and NBTZ is possible to occur. This does not belong to the first-order Hawking–Page transition because it is not a genuine process of a black hole nucleation mediated by an unstable black hole. Furthermore, if one introduces the mass of a conical singularity and the Horowitz–Myers conjecture, a transition between the TADS and NBTZ is also possible.

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