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Covariant action for a string in *doubled yet gauged* spacetime

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Abstract

The section condition in double field theory has been shown to imply that a physical point should be one-to-one identified with a gauge orbit in the doubled coordinate space. Here we show the converse is also true, and continue to explore the idea of *spacetime being doubled yet gauged*. Introducing an appropriate gauge connection, we construct a string action, with an arbitrary generalized metric, which is completely covariant with respect to the coordinate gauge symmetry, generalized diffeomorphisms, world-sheet diffeomorphisms, world-sheet Weyl symmetry and $\mathbf{O}(D,D)$ T-duality. A topological term previously proposed in the literature naturally arises and a self-duality condition follows from the equations of motion. Further, the action may couple to a T-dual background where the Riemannian metric becomes everywhere singular.

Keywords: String; T-duality; Double Field Theory

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1. Introduction

In order to realize $\mathbf{O}(D,D)$ T-duality as a manifest symmetry [1–8], Double Field Theory (DFT) [9–14] doubles the spacetime dimension, from D to $D+D$, with doubled coordinates, $x^A=(\tilde{y}_\mu, y^\nu)$, of which the first and the last correspond to the 'winding' and the 'ordinary' coordinates respectively. However, DFT is not truly doubled since all the fields – including any local symmetry parameters – are subject to the section condition: the $\mathbf{O}(D,D)$ invariant d'Alembertian operator must be trivial, acting on arbitrary fields,

$$\partial_A\partial^A\Phi(x)=0, \tag{1.1}$$

as well as their products, or equivalently

$$\partial_A\Phi_1(x)\partial^A\Phi_2(x)=0. \tag{1.2}$$

While the section condition might appear somewhat odd or even invidious from the conventional Riemannian point of view, it is readily satisfied when all the fields are, up to $\mathbf{O}(D,D)$ rotations, independent of the dual winding coordinates, i.e. $\frac{\partial}{\partial\tilde{y}_\mu}\equiv 0$. This kind of explicit 'section-fixing' reduces DFT to *generalized geometry* [15–24] where the spacetime is not enlarged and the duality is less manifest.

Much progress has been made in recent years based on the notion of doubled spacetime subject to the section condition [25–59], including the state of the art reviews [52,54] and the construction of $\mathcal{N}=2$ $D=10$ maximally supersymmetric double field theory [46] as the unification of type IIA and IIB supergravities.[2] Analogous parallel developments on U-duality are also available [60–73] [3] which may all be incorporated into the *grand scheme of* E_{11} [74–77].

The $(D+D)$-dimensional doubled spacetime is far from being an ordinary Riemannian manifold, since it postulates the existence of a globally well-defined $\mathbf{O}(D,D)$ invariant constant metric,

$$\mathcal{J}_{AB}=\begin{pmatrix}0&1\\1&0\end{pmatrix}, \tag{1.3}$$

which serves to raise and lower the doubled spacetime indices.[4] Further, the diffeomorphism symmetry is generated not by the ordinary Lie derivative but by a generalized one,

$$\begin{aligned}\hat{\mathcal{L}}_V T_{A_1\cdots A_n} := {}& V^B\partial_B T_{A_1\cdots A_n}+\omega\partial_B V^B T_{A_1\cdots A_n}\\ &+\sum_{i=1}^{n}(\partial_{A_i}V_B-\partial_B V_{A_i})T_{A_1\cdots A_{i-1}}{}^{B}{}_{A_{i+1}\cdots A_n},\end{aligned} \tag{1.4}$$

where ω is the weight of the DFT-tensor, $T_{A_1\cdots A_n}(x)$, and $V^A(x)$ is an infinitesimal diffeomorphism parameter, as a DFT vector field which must also obey the section condition,

$$\partial_A\partial^A V^B(x)=0,\qquad \partial_A V^B(x)\partial^A\Phi(x)=0. \tag{1.5}$$

The generalized Lie derivative of the $\mathbf{O}(D,D)$ metric vanishes for consistency,

[2] Cf. http://strings2013.sogang.ac.kr//main/?skin=video_GS_2.htm.
[3] Cf. http://strings2013.sogang.ac.kr//main/?skin=video_27_5.htm.
[4] For example, $\partial^A=\mathcal{J}^{-1AB}\partial_B$ as done in (1.1) and (1.2).

$$\hat{\mathcal{L}}_V \mathcal{J}_{AB} = 0. \tag{1.6}$$

As pointed out in [51], the section condition implies that the coordinates in doubled spacetime do not represent the physical points in an injective manner. Rather, a physical point should be one-to-one identified with a 'gauge orbit' in the coordinate space, i.e. an *equivalence relation* holds for the doubled coordinates:

$$x^A \sim x^A + \phi \partial^A \varphi, \tag{1.7}$$

where ϕ and φ are two arbitrary functions in DFT. While we review the explicit realization of this equivalence below, cf. (2.3), it implies that *spacetime is doubled yet gauged*. The diffeomorphism symmetry then means an invariance under arbitrary reparametrizations of the gauge orbits. Henceforth, we call the equivalence relation on coordinates (1.7), 'coordinate gauge symmetry' so that the quotient/equivalence classes form a space diffeomorphic to the section. It follows that a similar equivalence relation holds for the infinitesimal diffeomorphism parameters,

$$V^A \sim V^A + \phi \partial^A \varphi, \tag{1.8}$$

and consequently there is more than one finite tensorial diffeomorphic transformation rule, while the simplest choice seems to be the one found in [44] which we shall recall in (3.1). In fact, it was the coordinate gauge symmetry that played a crucial role to resolve a puzzle left in [44] where the simple tensorial diffeomorphic transformation rule found therein did not coincide with the exponentiation of the generalized Lie derivative (1.4). Nevertheless, they are equivalent up to the coordinate gauge symmetry [51].

It is the purpose of this work to further explore the geometric significance of the coordinate gauge symmetry (1.7), in particular by considering a string which propagates in a doubled yet gauged spacetime. Compared to preceding works on sigma models in doubled space [1–3,7,8,37,45], the novelties of the action to be constructed in this paper, (4.2), are as follows. (i) As in DFT, *a priori* no isometry nor torus structures are assumed. For an arbitrary given background, our action is fully covariant under world-sheet diffeomorphism, world-sheet Weyl symmetry, target spacetime generalized diffeomorphisms, $\mathbf{O}(D, D)$ T-duality and the coordinate gauge symmetry. (ii) The full spacetime dimensions are doubled, yet they are gauged. (iii) The self-duality relation follows from the equations of motion of the auxiliary gauge fields, without breaking any symmetry. (iv) The action may also describe a string which propagates in a novel class of non-Riemannian geometries.

The rest of the paper is organized as follows.

- In Section 2, we prove that the coordinate gauge symmetry (1.7) implies the section condition, both (1.1) and (1.2). Hence, with the result of [51], they are in fact equivalent. This motivates us to propose to take the 'coordinate gauge symmetry' as the geometrical first principle for the doubled spacetime formalism.
- In Section 3, we explicitly introduce a gauge connection for the coordinate gauge symmetry and define *gauged differential one-forms for the doubled coordinates*, $Dx^M = \mathrm{d}x^M - \mathcal{A}^M$. We demonstrate their covariant properties under generalized diffeomorphisms and the coordinate gauge symmetry.
- In Section 4, in terms of the gauged differential one-forms, we construct an *action for a string in the doubled yet gauged spacetime*, which is completely covariant with respect to the coordinate gauge symmetry, generalized diffeomorphisms, $\mathbf{O}(D, D)$ T-duality, world-sheet diffeomorphisms and world-sheet Weyl symmetry.

- In Section 5, we discuss reductions to undoubled formalisms. We point out that there are generically two classes of reductions depending on the generalized metric. For the *non-degenerate* case, the generalized metric can as usual be parametrized by a Riemannian metric and a Kalb–Ramond B-field. The reduction then naturally recovers the doubled sigma model proposed in [8] including a 'topological term' [5] and a 'self-duality' relation [2,3]. On the other hand, the *degenerate* case deals with a background where the Riemannian metric, if interpreted in that way, is everywhere singular. As an example, we obtain such a singular background by T-dualizing a fundamental string geometry [78].
- We conclude with a summary and comments in Section 6.
- Appendices A and B contain a brief review of covariant derivatives in DFT and some useful formulae.

While the idea of spacetime being gauged might sound strange at first, since the spacetime coordinates are dynamic fields on world-sheet, the coordinate gauge symmetry will be realized as just one of the gauge symmetries of the constructed action along with others.

2. *Doubled yet gauged* spacetime

Coordinate gauge symmetry means an equivalence relation on the coordinates of the doubled spacetime which is generated by a *derivative-index-valued* shift [51],

$$x^A \sim x^A + \phi\partial^A\varphi, \tag{2.1}$$

where ϕ and φ are two arbitrary functions in DFT. The *coordinate gauge symmetry* is additive, being Abelian in nature,

$$x^A \sim x^A + \phi\partial^A\varphi, \qquad x^A \sim x^A + \phi'\partial^A\varphi' \quad \Longrightarrow \quad x^A \sim x^A + \phi\partial^A\varphi + \phi'\partial^A\varphi'. \tag{2.2}$$

All the functions of DFT, i.e. field variables, local symmetry parameters (including ϕ and φ in (2.1)) and their arbitrary derivative-descendants, are then by definition required to be invariant under the derivative-index-valued shift,

$$\partial_{A_1}\partial_{A_2}\cdots\partial_{A_n}\Phi(x+\Delta) = \partial_{A_1}\partial_{A_2}\cdots\partial_{A_n}\Phi(x), \qquad \Delta^A = \phi\partial^A\varphi, \tag{2.3}$$

where $n = 0, 1, 2, \ldots.$

For a generic (local) shift of the coordinates, with an arbitrary real parameter, $s \in \mathbb{R}$,

$$x^A \longrightarrow x^A + s\Delta^A(x), \tag{2.4}$$

a standard power expansion reads

$$T_{A_1\cdots A_m}(x+s\Delta) = T_{A_1\cdots A_m}(x) + \sum_{l=1}^{\infty}\frac{s^l}{l!}\Delta^{B_1}\Delta^{B_2}\cdots\Delta^{B_l}\partial_{B_1}\partial_{B_2}\cdots\partial_{B_l}T_{A_1\cdots A_m}(x). \tag{2.5}$$

From the consideration of e.g. putting $T_{A_1\cdots A_m} = \partial_{A_1}\cdots\partial_{A_m}\Phi_1$ and $\Delta^A = k^{B_1}\cdots k^{B_n}\partial_{B_1}\cdots\partial_{B_n}\partial^A\Phi_2$ with an arbitrary constant vector, k^B, it follows immediately that the coordinate gauge symmetry (2.1) implies the section condition which is quadratic in functions (1.2),

$$\big(\partial_{A_1}\partial_{A_2}\cdots\partial_{A_m}\partial^C\Phi_1\big)(\partial_{B_1}\partial_{B_2}\cdots\partial_{B_n}\partial_C\Phi_2) = 0, \qquad m, n \geqslant 0. \tag{2.6}$$

Further, as we show shortly, a particular case of this result ($\Phi_1 = \Phi_2$ and $m = n = 1$) leads to the other section condition, or the linear "weak" constraint (1.1). Hence, the coordinate gauge symmetry (2.1) implies the section condition, both (1.1) and (1.2). Since the converse is also true from (2.3) and (2.5) [51], we conclude the following:

The coordinate gauge symmetry (2.1) is equivalent to the section condition, both (1.1) and (1.2),

$$x^A \sim x^A + \phi\partial^A\varphi \quad \Longleftrightarrow \quad \partial_A\partial^A\Phi(x) = 0 \quad \text{and} \quad \partial_A\Phi_1(x)\partial^A\Phi_2(x) = 0,$$

and serves as a geometric first principle for the doubled spacetime formalism.

Theorem. *If* $\partial_A\partial_B\Phi\partial^A\partial_C\Phi = 0$ *then* $\partial_A\partial^A\Phi = 0$.

Proof. The given assumption implies the nilpotent property of a $(D+D)\times(D+D)$ square matrix,

$$M_A{}^B = \partial_A\partial^B\Phi, \qquad M^2 = 0. \tag{2.7}$$

Hence, with an arbitrary real parameter, $s \in \mathbb{R}$, we have

$$\det(1+sM) = e^{\operatorname{tr}\ln(1+sM)} = e^{s\operatorname{tr}M}, \tag{2.8}$$

or

$$\det\left(\delta_A{}^B + s\partial_A\partial^B\Phi\right) = e^{s\partial_A\partial^A\Phi} = 1 + \sum_{n=1}^{\infty}\frac{s^n}{n!}\left(\partial_A\partial^A\Phi\right)^n. \tag{2.9}$$

Since the determinant is a finite polynomial in the variable, s, while the exponential has an *a priori* infinite power series expansion, it is clear that each higher order term of the latter must vanish, or

$$\partial_A\partial^A\Phi = 0. \tag{2.10}$$

In particular, the determinant is one, being s-independent. This completes our proof. □

Comments.

(i) An alternative proof may be established by considering the 'Jordan normal form' of the square matrix, M. The nilpotent property of the matrix implies that all the diagonal elements of its Jordan normal form are zero, and hence the matrix is traceless (2.10).

(ii) With the relation,

$$0 = \int_{\mathbb{R}^{D+D}} \partial_A\partial_B\Phi\partial^A\partial^B\Phi = \int_{\mathbb{R}^{D+D}} \partial_A\left(\partial_B\Phi\partial^A\partial^B\Phi - \partial^A\Phi\partial_B\partial^B\Phi\right) + \left(\partial_A\partial^A\Phi\right)^2, \tag{2.11}$$

if we assume "sufficiently fast fall off behavior at infinity" in order to ignore the total derivative or the surface integral, with the positive definite property of $(\partial_A\partial^A\Phi)^2$ we might argue for (2.10). However, this assumption appears too strong to be realized in double field theory where explicitly the fields do not depend on the dual winding coordinates. Thus, it is desirable to establish a direct *proof*, as presented above, which holds irrespective of the boundary conditions.

(iii) It is worthwhile to note that, unlike $\partial_A\partial_B\Phi\partial^A\partial_C\Phi = 0$, an alternative condition, $\partial_A\Phi\partial^A\Phi = 0$, does not necessarily imply the weak constraint, $\partial_A\partial^A\Phi = 0$, except in dimension $D = 1$, since a counterexample exists for $D \geqslant 2$,

$$\Phi'\left(x^1,\ldots,x^D,\tilde{x}_1,\ldots,\tilde{x}_D\right) = \exp\left[2\sqrt{x^1\tilde{x}_1} + \sum_{\mu=2}^{D}\frac{x^\mu - \tilde{x}_\mu}{\sqrt{D-1}}\right], \tag{2.12}$$

which satisfies

$$\partial_A \Phi' \partial^A \Phi' = 0, \qquad \partial_A \partial^A \Phi' = \frac{1}{2\sqrt{x^1 \tilde{x}_1}} \Phi' \neq 0. \tag{2.13}$$

3. Gauge connection for the coordinate gauge symmetry

We recall two finite local symmetries of double field theory.

- *Generalized diffeomorphism*, $x^M \to x'^M$, in the 'passive' form [44,51],

$$T_{A_1 A_2 \cdots A_n}(x) \longrightarrow T'_{A_1 A_2 \cdots A_n}(x') = (\det L)^{-\omega} \bar{F}_{A_1}{}^{B_1} \bar{F}_{A_2}{}^{B_2} \cdots \bar{F}_{A_n}{}^{B_n} T_{B_1 B_2 \cdots B_n}(x), \tag{3.1}$$

where

$$L_A{}^B = \partial_A x'^B, \qquad \bar{L} = \mathcal{J} L^t \mathcal{J}^{-1},$$
$$F = \frac{1}{2}\big(L\bar{L}^{-1} + \bar{L}^{-1}L\big), \qquad \bar{F} = \mathcal{J} F^t \mathcal{J}^{-1} = \frac{1}{2}\big(L^{-1}\bar{L} + \bar{L}L^{-1}\big). \tag{3.2}$$

In particular, F can be shown to be an $\mathbf{O}(D, D)$ element,

$$F\bar{F} = 1, \quad F \in \mathbf{O}(D, D), \tag{3.3}$$

which agrees with (1.6). It should be also noted that it is not the transformed coordinates, x'^M, but the difference, $x'^M - x^M$, that satisfies the section condition, (1.1), (1.2),[5] such that, e.g.

$$\partial^A \Phi (L - 1)_A{}^B = 0. \tag{3.4}$$

- *Coordinate gauge symmetry* (2.1) [51],

$$x^A \sim x_s^M = e^{s\mathcal{V}\cdot\partial} x^M = x^M + s\mathcal{V}^M(x), \qquad T_{A_1 A_2 \cdots A_n}(x) = T_{A_1 A_2 \cdots A_n}(x_s), \tag{3.5}$$

which is generated by a *derivative-index-valued* vector, $\mathcal{V}^A$,

$$\mathcal{V}^M = \phi \partial^M \varphi, \tag{3.6}$$

satisfying the section condition (1.2),

$$\mathcal{V}^M \partial_M \Phi = 0. \tag{3.7}$$

The main difference of the above two local symmetries is formally whether the spacetime indices of the tensors are supposed to be rotated or not.[6]

We now introduce a gauge connection,[7] $\mathcal{A}^M$, and define *gauged differential one-forms for the doubled coordinates*,

[5] For example, $\partial_A x^M \partial^A \Phi = \partial^M \Phi \neq 0$.

[6] Of course, the transformation, $x^M \to x_s^M = x^M + s\mathcal{V}^M(x)$, can be taken as a sort of generalized diffeomorphism which can be then shown to reduce to the B-field gauge symmetry only, without involving any Riemannian diffeomorphism [51].

[7] A closely related earlier work is [4] where 'gauging' chiral currents on a world-sheet was discussed, see also [5,8].

$$Dx^M := \mathrm{d}x^M - \mathcal{A}^M. \tag{3.8}$$

We require the connection to be a *derivative-index-valued* one-form, satisfying

$$\mathcal{A}^M \partial_M \Phi = 0, \qquad \mathcal{A}^M (L-1)_M{}^N = 0. \tag{3.9}$$

Under the finite generalized diffeomorphism (3.1), the gauge connection must transform as

$$\mathcal{A}^M \longrightarrow \mathcal{A}'^M = \mathcal{A}^N F_N{}^M + \mathrm{d}x^N (L-F)_N{}^M, \tag{3.10}$$

such that the gauged differential one-forms are *covariant*,

$$Dx^M \longrightarrow D'x'^M = Dx^N F_N{}^M. \tag{3.11}$$

Furthermore, thanks to the following identities which derive from the section condition [51] (see also [44]),

$$\begin{aligned}
&\big(1-\bar{L}^{-1}\big)L = 1-\bar{L}^{-1},\\
&L-F = \frac{1}{2}(L+1)\big(1-\bar{L}^{-1}\big)L = \frac{1}{2}(L+1)\big(1-\bar{L}^{-1}\big),\\
&F\bar{L}-1 = \frac{1}{2}(L-1)(1+\bar{L}),\\
&\partial_A \Phi = L_A{}^B \partial'_B \Phi = F_A{}^B \partial'_B \Phi,\\
&(1-\bar{L})_A{}^B \partial_B \Phi = \big(1-\bar{L}^{-1}\big)_A{}^B \partial_B \Phi = 0,
\end{aligned} \tag{3.12}$$

the gauge connection remains still *derivative-index-valued* after the transformation (3.10). In other words, the derivative-index-valuedness of the connection is preserved under generalized diffeomorphisms,

$$\mathcal{A}'^M \partial'_M \Phi = \mathcal{A}^M \partial_M \Phi = 0. \tag{3.13}$$

With the (passive) tensorial transformation rule (3.1),

$$\mathcal{H}_{MN} \longrightarrow \mathcal{H}'_{MN} = \bar{F}_M{}^K \bar{F}_N{}^L \mathcal{H}_{KL}, \tag{3.14}$$

we can now ensure the invariance of the generalized metric under generalized diffeomorphisms,

$$\mathcal{H}_{MN} Dx^M \otimes Dx^N = \mathcal{H}'_{MN} D'x'^M \otimes D'x'^N. \tag{3.15}$$

On the other hand, under the coordinate gauge symmetry (3.5), the gauge connection transforms precisely the same way as the one-form, $\mathrm{d}x^M$,

$$\begin{aligned}
&\mathrm{d}x^M \longrightarrow \mathrm{d}x_s^M = \mathrm{d}x^M + s\mathrm{d}\big(\phi\partial^M\varphi\big),\\
&\mathcal{A}^M \longrightarrow \mathcal{A}_s^M = \mathcal{A}^M + s\mathrm{d}\big(\phi\partial^M\varphi\big),
\end{aligned} \tag{3.16}$$

such that it preserves the derivative-index-valuedness. The gauged differential one-forms are then simply *invariant*,

$$Dx^M = D_s x_s^M. \tag{3.17}$$

4. Completely covariant string action

We pull back the gauged differential one-forms (3.8) to a string world-sheet with coordinates, σ^i, $i = 0, 1$, and promote the doubled target spacetime coordinates and the gauge connection to the world-sheet fields, $X^M(\sigma)$ and $\mathcal{A}_i^M(\sigma)$. That is, σ^i denotes a coordinate on the world-sheet, Σ, and X^M are the components of $X : \Sigma \to \mathbf{R}^{D+D}$ so that

$$DX^M = \mathrm{d}\sigma^i D_i X^M = \mathrm{d}\sigma^i \big(\partial_i X^M - \mathcal{A}_i^M\big). \tag{4.1}$$

The string action we construct in this work is then

$$\begin{aligned} \mathcal{S} &= \frac{1}{4\pi\alpha'} \int_\Sigma \mathrm{d}^2\sigma\, \mathcal{L}, \\ \mathcal{L} &= -\frac{1}{2}\sqrt{-h}h^{ij} D_i X^M D_j X^N \mathcal{H}_{MN}(X) - \epsilon^{ij} D_i X^M \mathcal{A}_{jM}. \end{aligned} \tag{4.2}$$

Here h_{ij} corresponds to the usual auxiliary world-sheet metric which can raise or lower the world-sheet coordinate indices, i, j. The action describes a string propagating in a doubled yet gauged spacetime with an arbitrarily given generalized metric, $\mathcal{H}_{AB}(X)$. Apart from the section condition, the generalized metric only needs to satisfy the two defining properties,

$$\mathcal{H}_{AB} = \mathcal{H}_{BA}, \qquad \mathcal{H}_A{}^B \mathcal{H}_B{}^C = \delta_A{}^C. \tag{4.3}$$

Otherwise it is quite arbitrary. The string tension in the doubled spacetime is halved, i.e. $(4\pi\alpha')^{-1}$ instead of $(2\pi\alpha')^{-1}$ as stressed in [8]. It may recover the standard value, $(2\pi\alpha')^{-1}$, after reduction to an undoubled formalism, cf. (5.6). In fact, as we see in the next section, cf. (5.8), at least for “non-degenerate” cases of the generalized metric, the above doubled string action precisely reduces to the standard undoubled string action with the right number of degrees of freedom. While we reserve Section 5 for the exposition of the reductions to undoubled formalisms, in the remaining of the current section we focus on the covariant properties of the doubled action.

The Lagrangian is manifestly symmetric with respect to the $\mathbf{O}(D, D)$ T-duality and the world-sheet diffeomorphisms. Furthermore, from (3.9), (3.12) and (3.17), up to total derivatives it is invariant under the coordinate gauge symmetry as

$$\begin{aligned} \epsilon^{ij} D_{si} X_s^M \mathcal{A}_{sjM} &= \epsilon^{ij} D_i X^M \big[\mathcal{A}_{jM} + s\partial_j X^N \partial_N(\phi\partial_M\varphi)\big] \\ &= \epsilon^{ij} D_i X^M \mathcal{A}_{jM} + s\epsilon^{ij}\partial_i X^M \partial_j X^N \partial_N(\phi\partial_M\varphi) \\ &= \epsilon^{ij} D_i X^M \mathcal{A}_{jM} + \partial_j\big[s\epsilon^{ij}\partial_i X^M(\phi\partial_M\varphi)\big] \\ &= \epsilon^{ij} D_i X^M \mathcal{A}_{jM} - \partial_i\big(s\epsilon^{ij}\phi\partial_j\varphi\big), \end{aligned} \tag{4.4}$$

and also invariant under the generalized diffeomorphisms as

$$\begin{aligned} \epsilon^{ij} D'_i X'^M \mathcal{A}'_{jM} &= \epsilon^{ij} D_i X^L F_L{}^M \big[\bar{F}_M{}^N \mathcal{A}_{jN} + (\bar{L} - \bar{F})_M{}^N \partial_j X_N\big] \\ &= \epsilon^{ij} D_i X^M \mathcal{A}_{jM} + \epsilon^{ij} D_i X^M (F\bar{L} - 1)_M{}^N \partial_j X_N \\ &= \epsilon^{ij} D_i X^M \mathcal{A}_{jM} + \frac{1}{2}\epsilon^{ij}\partial_i X^M \big[(L-1)(1+\bar{L})\big]_M{}^N \partial_j X_N \\ &= \epsilon^{ij} D_i X^M \mathcal{A}_{jM} + \frac{1}{2}\epsilon^{ij}\partial_i X^M \partial_j X^N \big(L\mathcal{J} - \mathcal{J}L^t + L\mathcal{J}L^t - \mathcal{J}\big)_{[MN]} \\ &= \epsilon^{ij} D_i X^M \mathcal{A}_{jM} + \frac{1}{2}\epsilon^{ij}\partial_i X^M \partial_j X^N \big(L\mathcal{J} - \mathcal{J}L^t\big)_{[MN]} \end{aligned}$$

$$
\begin{aligned}
&= \epsilon^{ij} D_i X^M \mathcal{A}_{jM} + \epsilon^{ij} \partial_i X^M L_M{}^N \partial_j X_N \\
&= \epsilon^{ij} D_i X^M \mathcal{A}_{jM} + \epsilon^{ij} \partial_i X'^N \partial_j X_N \\
&= \epsilon^{ij} D_i X^M \mathcal{A}_{jM} + \partial_i \big(\epsilon^{ij} X'^N \partial_j X_N \big).
\end{aligned} \tag{4.5}
$$

Generically, under arbitrary variations of all the fields, the Lagrangian transforms as

$$
\begin{aligned}
\delta \mathcal{L} = &-\frac{1}{2}\sqrt{-h}\,\delta h^{ij}\left(D_i X^M D_j X^N - \frac{1}{2} h_{ij} D_k X^M D^k X^N \right)\mathcal{H}_{MN} \\
&+ \delta X^L \left[\partial_i \big(\sqrt{-h} D^i X^M \mathcal{H}_{ML} - \epsilon^{ij} D_j X_L \big) - \frac{1}{2}\sqrt{-h} D_i X^M D^i X^N \partial_L \mathcal{H}_{MN} \right] \\
&+ \delta \mathcal{A}_{iL} \big[\sqrt{-h}\mathcal{H}^L{}_M D^i X^M + \epsilon^{ij} \partial_j X^L \big] \\
&- \partial_i \big[\delta X^L \big(\sqrt{-h} D^i X^M \mathcal{H}_{LM} + \epsilon^{ij} \mathcal{A}_{jL} \big) \big],
\end{aligned} \tag{4.6}
$$

where the second line can be rewritten in an alternative manner,

$$
\begin{aligned}
&\delta X^L \left[\partial_i \big(\sqrt{-h} D^i X^M \mathcal{H}_{ML} - \epsilon^{ij} D_j X_L \big) - \frac{1}{2}\sqrt{-h} D_i X^M D^i X^N \partial_L \mathcal{H}_{MN} \right] \\
&\quad = \delta X^L \bigg[\mathcal{H}_{LM} \partial_i \big(\sqrt{-h} D^i X^M \big) + \sqrt{-h} D_i X^M D^i X^N \\
&\qquad \times \left(\partial_{(M} \mathcal{H}_{N)L} - \frac{1}{2} \partial_L \mathcal{H}_{MN} \right) + \epsilon^{ij} \partial_i \mathcal{A}_{jL} \bigg].
\end{aligned} \tag{4.7}
$$

Every line in (4.6) then corresponds to *the equation of motion for each field* as follows.

- For the world-sheet metric, h_{ij}, we have the Virasoro constraints,

$$
\left(D_i X^M D_j X^N - \frac{1}{2} h_{ij} D_k X^M D^k X^N \right) \mathcal{H}_{MN} = 0. \tag{4.8}
$$

- For the gauge connection, $\mathcal{A}_{iM}$, since it is not arbitrary but derivative-index-valued, from

$$
\mathcal{A}_i^M \delta \mathcal{A}_{jM} = 0, \qquad \delta \mathcal{A}_i^M \partial_M \Phi = 0, \tag{4.9}
$$

  the equation of motion amounts to

$$
\delta \mathcal{A}_{iM} \left(\mathcal{H}^M{}_N D^i X^N + \frac{1}{\sqrt{-h}} \epsilon^{ij} D_j X^M \right) = 0. \tag{4.10}
$$

  That is to say, the quantity inside the bracket should be derivative-index-valued too.
- For the dynamical field, X^L, from (B.1),

$$
\begin{aligned}
&\frac{1}{\sqrt{-h}} \partial_i \big(\sqrt{-h} D^i X^M \mathcal{H}_{ML} \big) - 2\Gamma_{LMN} (P D_i X)^M \big(\bar{P} D^i X \big)^N \\
&\quad + \frac{1}{\sqrt{-h}} \epsilon^{ij} \partial_i \mathcal{A}_{jL} = 0,
\end{aligned} \tag{4.11}
$$

  or equivalently from (4.7), (B.3),

$$
\begin{aligned}
&\frac{1}{\sqrt{-h}} \partial_i \big(\sqrt{-h} D^i X^N \big) + 2\Gamma_{KLM} \big[(P D_i X)^K \bar{P}^{LN} + (\bar{P} D_i X)^K P^{LN} \big] D^i X^M \\
&\quad + \frac{1}{\sqrt{-h}} \epsilon^{ij} \partial_i \mathcal{A}_{jM} \mathcal{H}^{MN} = 0.
\end{aligned} \tag{4.12}
$$

Here, with a pair of projectors,

$$P_{AB} = \frac{1}{2}(\mathcal{J} + \mathcal{H})_{AB}, \qquad \bar{P}_{AB} = \frac{1}{2}(\mathcal{J} - \mathcal{H})_{AB}, \tag{4.13}$$

we set

$$(PD_iX)^M = P^M{}_N D_i X^N, \qquad (\bar{P}D_iX)^M = \bar{P}^M{}_N D_i X^N, \tag{4.14}$$

and Γ_{LMN} is the DFT analogy of the Christoffel connection [28],[8]

$$\begin{aligned}\Gamma_{CAB} = {}& 2(P\partial_C P\bar{P})_{[AB]} + 2\big(\bar{P}_{[A}{}^D\bar{P}_{B]}{}^E - P_{[A}{}^D P_{B]}{}^E\big)\partial_D P_{EC} \\ & - \frac{4}{D-1}\big(\bar{P}_{C[A}\bar{P}_{B]}{}^D + P_{C[A}P_{B]}{}^D\big)\big(\partial_D d + \big(P\partial^E P\bar{P}\big)_{[ED]}\big).\end{aligned} \tag{4.15}$$

5. Reductions to undoubled formalisms

In this section, we consider reductions of the doubled formalism to undoubled formalisms by solving the section condition (1.1), (1.2) explicitly: we require all the target spacetime fields, including the generalized metric, to be independent of the dual coordinates,

$$\frac{\partial\Phi(\tilde{y}, y)}{\partial\tilde{y}_\mu} = 0. \tag{5.1}$$

As a consequence, the latter half of the components of the derivative-index-valued gauge connection is trivial,[9]

$$\mathcal{A}^M = A_\lambda\partial^M y^\lambda = (A_\mu, 0), \tag{5.2}$$

and the gauged differential one-forms are explicitly (cf. [5,8]),

$$D_i X^M = \big(\partial_i\tilde{Y}_\mu - A_{i\mu}, \partial_i Y^\nu\big). \tag{5.3}$$

Now we turn to the parametrization of the generalized metric which must satisfy the defining properties (4.3),

$$\mathcal{H}_{AB} = \begin{pmatrix} U^{\mu\nu} & N^\mu{}_\lambda \\ (N^t)_\rho{}^\nu & S_{\rho\lambda}\end{pmatrix}, \qquad \mathcal{H}_{AB} = \mathcal{H}_{BA}, \qquad \mathcal{H}_A{}^B\mathcal{H}_B{}^C = \delta_A{}^C. \tag{5.4}$$

With the fixing of the section by (5.1), (5.2) and (5.3), depending on whether the upper left $D \times D$ block, i.e. $U^{\mu\nu}$, is degenerate or not, there are two classes of parametrizations. Each of them requires separate analysis.

5.1. Non-degenerate case: Reduction to standard form

As emphasized in [25] and also recently discussed in [57], when the upper left $D \times D$ block of the generalized metric, $U^{\mu\nu}$, is non-degenerate, the remaining parts are completely determined by the non-degenerate symmetric matrix and one free anti-symmetric matrix which we may identify with the usual Riemannian metric and the Kalb–Ramond B-field respectively,

---

8 See Appendix A for a concise review of the covariant derivatives in DFT.

9 Note the ordering of the ordinary and the dual coordinates in our convention, $x^M = (\tilde{y}_\mu, y^\nu)$, $\partial^M = \mathcal{J}^{MN}\partial_N = (\partial_\mu, \tilde{\partial}^\nu)$.

$$\begin{aligned}&\mathcal{H}_{AB}=\begin{pmatrix}G^{-1} & -G^{-1}B\\ BG^{-1} & G-BG^{-1}B\end{pmatrix},\qquad G_{\mu\nu}=G_{\nu\mu},\\ &\det(G_{\mu\nu})\neq 0,\qquad B_{\mu\nu}=-B_{\nu\mu}.\end{aligned}\tag{5.5}$$

With this 'standard' parametrization of the generalized metric, the doubled yet gauged sigma model (4.2) reduces to an expression which is multiplied by the 'correct' value of the string tension,

$$\frac{1}{4\pi\alpha'}\mathcal{L}\equiv\frac{1}{2\pi\alpha'}\mathcal{L}',\tag{5.6}$$

$$\begin{aligned}\mathcal{L}'=&-\frac{1}{2}\sqrt{-h}h^{ij}\partial_i Y^\mu\partial_j Y^\nu G_{\mu\nu}(Y)+\frac{1}{2}\epsilon^{ij}\partial_i Y^\mu\partial_j Y^\nu B_{\mu\nu}(Y)+\frac{1}{2}\epsilon^{ij}\partial_i\tilde{Y}_\mu\partial_j Y^\mu\\ &-\frac{1}{4}\sqrt{-h}h^{ij}(\mathcal{C}_{i\mu}-A_{i\mu})(\mathcal{C}_{j\nu}-A_{j\nu})G^{\mu\nu}(Y).\end{aligned}\tag{5.7}$$

Here $\mathcal{C}_{i\mu}$ denotes the on-shell value of the connection,

$$\mathcal{C}_{i\mu}=\partial_i\tilde{Y}_\mu+\partial_i Y^\lambda B_{\lambda\mu}+\frac{1}{\sqrt{-h}}\epsilon_i{}^j\partial_j Y^\lambda G_{\lambda\mu}.\tag{5.8}$$

It is also useful to note

$$\begin{aligned}D_iX^MD_jX^N\mathcal{H}_{MN}=&\ \partial_iY^\mu\partial_jY^\nu G_{\mu\nu}\\ &+\big(D_i\tilde{Y}_\mu+\partial_iY^\lambda B_{\lambda\mu}\big)\big(D_j\tilde{Y}_\nu+\partial_jY^\rho B_{\rho\nu}\big)G^{\mu\nu}.\end{aligned}\tag{5.9}$$

Integrating out the connection, we may ignore the second line in (5.7). The remaining first line then consists of the standard undoubled string Lagrangian without the dual coordinate fields, $\tilde{Y}_\mu$, and a total derivative involving the dual fields,

$$\partial_i\left(\frac{1}{2}\epsilon^{ij}\tilde{Y}_\mu\partial_jY^\mu\right)=\partial_j\left(\frac{1}{2}\epsilon^{ij}Y^\mu\partial_i\tilde{Y}_\mu\right).\tag{5.10}$$

In fact, this total derivative is precisely the topological term introduced in [5,8] for the quantum equivalence of a doubled sigma model to the usual formalism for world-sheets of arbitrary genus. Our covariant action naturally reproduces it from the first principle of the 'coordinate gauge symmetry': The topological term shares the same geometric origin as the world-sheet pull-back of the B-field.

Furthermore, with $D_i\tilde{Y}_\mu=\partial_i\tilde{Y}_\mu-A_{i\mu}$ (5.3), the on-shell value of the connection (5.8) reads

$$D_i\tilde{Y}_\mu+\partial_iY^\lambda B_{\lambda\mu}+\frac{1}{\sqrt{-h}}\epsilon_i{}^j\partial_jY^\lambda G_{\lambda\mu}=0.\tag{5.11}$$

Since $G_{\mu\nu}$ is *non-degenerate*, this relation implies

$$G^{\mu\nu}D_i\tilde{Y}_\nu-\big(G^{-1}B\big)^\mu{}_\nu\partial_iY^\nu+\frac{1}{\sqrt{-h}}\epsilon_i{}^j\partial_jY^\mu=0,\tag{5.12}$$

and hence in particular,

$$\big(BG^{-1}\big)_\mu{}^\nu D_i\tilde{Y}_\nu-\big(BG^{-1}B\big)_{\mu\nu}\partial_iY^\nu+\frac{1}{\sqrt{-h}}\epsilon_i{}^jB_{\mu\nu}\partial_jY^\nu=0.\tag{5.13}$$

In addition to this result, Eq. (5.11) also implies, after contraction with $\frac{1}{\sqrt{-h}}\epsilon_k{}^i$ and using (B.6),

$$\frac{1}{\sqrt{-h}}\epsilon_k{}^i\big(D_i\tilde{Y}_\mu - B_{\mu\nu}\partial_i Y^\nu\big) + G_{\mu\nu}\partial_k Y^\nu = 0. \tag{5.14}$$

Finally, combining (5.13) and (5.14), we acquire

$$\big(BG^{-1}\big)_\mu{}^\nu D_i\tilde{Y}_\nu + \big(G - BG^{-1}B\big)_{\mu\nu}\partial_i Y^\nu + \frac{1}{\sqrt{-h}}\epsilon_i{}^j D_j\tilde{Y}_\mu = 0. \tag{5.15}$$

In terms of the doubled coordinates and the generalized metric, the two equations, (5.11) and (5.15), exactly amount to a self-duality relation in the doubled spacetime!, cf. (4.10),

$$\mathcal{H}^M{}_N D_i X^N + \frac{1}{\sqrt{-h}}\epsilon_i{}^j D_j X^M = 0. \tag{5.16}$$

Thus, for the *non-degenerate cases* the full set of self-duality relations (5.16) follows from the equation of motion (5.11) without being imposed by hand. On the other hand for *degenerate cases*, as we see below and also expected from the generic expression (4.10), this is not true in general: not all the components of the relation (5.16) are satisfied.

5.2. Degenerate case: Non-Riemannian geometry

We start with an exact solution of supergravity obtained in [78] which corresponds to a macroscopic fundamental string geometry with the $\mathbf{SO}(1,1)\times\mathbf{SO}(8)$ isometry. In string frame, the background reads

$$\begin{aligned}
&\mathrm{d}s^2 = f^{-1}\big(-\mathrm{d}t^2 + \big(\mathrm{d}x^1\big)^2\big) + \big(\mathrm{d}x^2\big)^2 + \cdots + \big(\mathrm{d}x^9\big)^2,\\
&B = \big(f^{-1}+c\big)\,\mathrm{d}t\wedge\mathrm{d}x^1,\\
&e^{-2\phi} = c' f,
\end{aligned} \tag{5.17}$$

where f is a harmonic function,

$$f = 1 + \frac{Q}{r^6}, \qquad r^2 = \sum_{a=2}^{9}\big(x^a\big)^2, \tag{5.18}$$

and c and c' are arbitrary constants. In particular, the former corresponds to a 'non-physical' constant shift of the B-field which we introduce. The corresponding generalized metric is then

$$\mathcal{H}_{MN} = \begin{pmatrix} f\eta^{\alpha\beta} & 0 & -(1+cf)\mathcal{E}^\alpha{}_\delta & 0\\ 0 & \delta^{ab} & 0 & 0\\ (1+cf)\mathcal{E}_\gamma{}^\beta & 0 & -c(2+cf)\eta_{\gamma\delta} & 0\\ 0 & 0 & 0 & \delta_{cd}\end{pmatrix}, \tag{5.19}$$

and the DFT-dilaton is given by

$$e^{-2d} = e^{-2\phi}\sqrt{-g} = c'. \tag{5.20}$$

Here the Greek letters, $\alpha,\beta,\gamma,\delta,\ldots$, denote the Minkowskian $\mathbf{SO}(1,1)$ vector indices subject to the flat metric, $\eta_{\alpha\beta}=\mathrm{diag}(-+)$, and the roman letters, $a,b,c,d,\ldots$, are for the Euclidean $\mathbf{SO}(8)$ vector indices with the Kronecker-delta flat metric. As seen in (5.19), the generalized metric then decomposes into sixteen blocks, $(2+8+2+8)\times(2+8+2+8)$. Further, with the 2×2 anti-symmetric Levi-Civita symbol, $\mathcal{E}_{\alpha\beta} = -\mathcal{E}_{\beta\alpha}$, $\mathcal{E}_{01}=+1$, we set

$$\mathcal{E}^{\alpha}{}_{\beta} = \eta^{\alpha\gamma}\mathcal{E}_{\gamma\beta} = -\mathcal{E}_{\beta}{}^{\alpha} = -\mathcal{E}_{\beta\delta}\eta^{\delta\alpha}, \tag{5.21}$$

which satisfies

$$\mathcal{E}^{\alpha}{}_{\beta}\mathcal{E}^{\beta}{}_{\gamma} = \delta^{\alpha}{}_{\gamma}. \tag{5.22}$$

Now we perform an $\mathbf{O}(D, D)$ rotation exchanging (t, x^1) and $(\tilde{t}, \tilde{x}_1)$ planes,[10]

$$\mathcal{H}_{AB} \longrightarrow \mathcal{O}_A{}^C\mathcal{O}_B{}^D\mathcal{H}_{CD}, \quad \mathcal{O}_A{}^B = \begin{pmatrix} 0 & 0 & \eta^{\mu\nu} & 0 \\ 0 & \delta^a{}_b & 0 & 0 \\ \eta_{\lambda\rho} & 0 & 0 & 0 \\ 0 & 0 & 0 & \delta_c{}^d \end{pmatrix}, \tag{5.23}$$

and obtain a T-dual background,

$$\mathcal{H}_{MN} = \begin{pmatrix} -c(2+cf)\eta^{\alpha\beta} & 0 & (1+cf)\mathcal{E}^{\alpha}{}_{\delta} & 0 \\ 0 & \delta^{ab} & 0 & 0 \\ -(1+cf)\mathcal{E}_{\gamma}{}^{\beta} & 0 & f\eta_{\delta\gamma} & 0 \\ 0 & 0 & 0 & \delta_{cd} \end{pmatrix}, \tag{5.24}$$

which corresponds to

$$\begin{aligned} ds^2 &= -\frac{1}{c(2+cf)}\left(-dt^2 + \left(dx^1\right)^2\right) + \left(dx^2\right)^2 + \cdots + \left(dx^9\right)^2, \\ B &= \frac{1+cf}{c(2+cf)}\,dt \wedge dx^1, \\ e^{-2\phi} &= c'c(2+cf). \end{aligned} \tag{5.25}$$

Clearly, as long as $c \neq 0$, the Riemannian metric is non-degenerate and well-defined. The corresponding doubled yet gauged sigma model then can be readily read off from the results above, (5.5), (5.7). On the other hand, when we take the limit, $c \to 0$, the Riemannian metric becomes everywhere singular.[11] But, the corresponding generalized metric is perfectly smooth and reduces in the limit to

$$\mathcal{H}_{MN} = \begin{pmatrix} 0 & 0 & \mathcal{E}^{\alpha}{}_{\delta} & 0 \\ 0 & \delta^{ab} & 0 & 0 \\ -\mathcal{E}_{\gamma}{}^{\beta} & 0 & f\eta_{\delta\gamma} & 0 \\ 0 & 0 & 0 & \delta_{cd} \end{pmatrix}. \tag{5.26}$$

A remarkable fact is then that, despite the "Riemannian" singularity, the doubled yet gauged sigma model (4.2) can describe a string propagating in such a background well, explicitly through

$$\frac{1}{4\pi\alpha'}\mathcal{L} \equiv \frac{1}{2\pi\alpha'}\mathcal{L}'', \tag{5.27}$$

$$\begin{aligned} \mathcal{L}'' = &-\frac{1}{4}\sqrt{-h}h^{ij}\partial_i Y^{\alpha}\partial_j Y^{\beta}\eta_{\alpha\beta}f(Y) - \frac{1}{2}\sqrt{-h}h^{ij}\partial_i\tilde{Y}_{\alpha}\partial_j Y^{\beta}\mathcal{E}^{\alpha}{}_{\beta} \\ &+ \frac{1}{2}\sqrt{-h}A_{i\alpha}\left(\mathcal{E}^{\alpha}{}_{\beta}h^{ij}\partial_j Y^{\beta} + \frac{1}{\sqrt{-h}}\epsilon^{ij}\partial_j Y^{\alpha}\right) \end{aligned}$$

[10] For discussion on T-duality along the temporal direction, see e.g. [79].

[11] Nevertheless we merely note that the combination, $e^{-2\phi}g_{\mu\nu}$, is finite. See also [68] for other examples of singular metrics.

$$
\begin{aligned}
&-\frac{1}{2}\sqrt{-h}h^{ij}\partial_i Y^a\partial_j Y_a+\frac{1}{2}\epsilon^{ij}\partial_i\tilde{Y}_a\partial_j Y^a\\
&-\frac{1}{4}\sqrt{-h}h^{ij}\Big(\partial_i\tilde{Y}_a+\frac{1}{\sqrt{-h}}\epsilon_i{}^k\partial_k Y_a-A_{ia}\Big)\Big(\partial_j\tilde{Y}^a+\frac{1}{\sqrt{-h}}\epsilon_j{}^l\partial_l Y^a-A_j{}^a\Big)\\
=&-\frac{1}{4}\sqrt{-h}h^{ij}\partial_i Y^\alpha\partial_j Y^\beta\eta_{\alpha\beta}f(Y)-\frac{1}{2}\sqrt{-h}h^{ij}\partial_i Y^a\partial_j Y_a+\frac{1}{2}\epsilon^{ij}\partial_i\tilde{Y}_\mu\partial_j Y^\mu\\
&+\frac{1}{2}\sqrt{-h}(A_{i\alpha}-\partial_i\tilde{Y}_\alpha)\Big(\mathcal{E}^\alpha{}_\beta h^{ij}\partial_j Y^\beta+\frac{1}{\sqrt{-h}}\epsilon^{ij}\partial_j Y^\alpha\Big)\\
&-\frac{1}{4}\sqrt{-h}h^{ij}\Big(\partial_i\tilde{Y}_a+\frac{1}{\sqrt{-h}}\epsilon_i{}^k\partial_k Y_a-A_{ia}\Big)\Big(\partial_j\tilde{Y}^a+\frac{1}{\sqrt{-h}}\epsilon_j{}^l\partial_l Y^a-A_j{}^a\Big).
\end{aligned}
\tag{5.28}
$$

We note that, while the $\mathbf{SO}(8)$ sector of $\{Y^a,\tilde{Y}_a,A_{ia}\}$ agrees with the non-degenerate result (5.7) having the flat Euclidean metric, δ_{ab}, and the vanishing B-field, the $\mathbf{SO}(1,1)$ sector of $\{Y^\alpha,\tilde{Y}_\alpha,A_{i\alpha}\}$ takes a novel exotic form. In particular, the gauge field components are quadratic for the non-degenerate $\mathbf{SO}(8)$ sector, whereas they are linear for the degenerate $\mathbf{SO}(1,1)$ sector.

Integrating out all the gauge fields, the doubled yet gauged sigma model reduces to

$$
\frac{1}{2\pi\alpha'}\Big[-\frac{1}{4}\sqrt{-h}h^{ij}\partial_i Y^\alpha\partial_j Y^\beta\eta_{\alpha\beta}f(Y)-\frac{1}{2}\sqrt{-h}h^{ij}\partial_i Y^a\partial_j Y_a+\frac{1}{2}\epsilon^{ij}\partial_i\tilde{Y}_\mu\partial_j Y^\mu\Big],\tag{5.29}
$$

where now the two of the 'ordinary' coordinate fields must satisfy a self-duality constraint,

$$
\partial_i Y^\alpha+\frac{1}{\sqrt{-h}}\epsilon_i{}^j\mathcal{E}^\alpha{}_\beta\partial_j Y^\beta=0.\tag{5.30}
$$

This is in contrast to the non-degenerate $\mathbf{SO}(8)$ sector of which the ordinary and the dual coordinate fields are, like (5.11), related by a different type of a self-duality relation,

$$
D_i\tilde{Y}_a+\frac{1}{\sqrt{-h}}\epsilon_i{}^j\partial_j Y_a=0\quad\Longleftrightarrow\quad\partial_i Y_a+\frac{1}{\sqrt{-h}}\epsilon_i{}^j D_j\tilde{Y}_a=0.\tag{5.31}
$$

To summarize, even for the degenerate sector for which the Riemannian metric is ill-defined, there exists a sigma model type Lagrangian description involving a self-duality constraint.

In order to illustrate this feature in a more general setup, let us consider an extreme case of the degeneracy where the upper left $D\times D$ block of a generalized metric vanishes itself,

$$
\mathcal{H}_{AB}=\begin{pmatrix}0 & N^\mu{}_\lambda\\ (N^t)_\rho{}^\nu & S_{\rho\lambda}\end{pmatrix},\qquad N^2=1,\qquad S=S^t,\qquad SN=-(SN)^t.\tag{5.32}
$$

In this background, the doubled yet gauged sigma model reduces to

$$
\begin{aligned}
\frac{1}{4\pi\alpha'}\mathcal{L}\equiv\frac{1}{2\pi\alpha'}\Big[&-\frac{1}{4}\sqrt{-h}h^{ij}\partial_i Y^\mu\partial_j Y^\nu S_{\mu\nu}(Y)+\frac{1}{2}\epsilon^{ij}\partial_i\tilde{Y}_\mu\partial_j Y^\mu\\
&+\frac{1}{2}\sqrt{-h}(A_{i\mu}-\partial_i\tilde{Y}_\mu)\Big(N^\mu{}_\nu h^{ij}\partial_j Y^\nu+\frac{1}{\sqrt{-h}}\epsilon^{ij}\partial_j Y^\mu\Big)\Big].
\end{aligned}
\tag{5.33}
$$

Integrating out the gauge field which is linear in the Lagrangian, we end up with the first line which consists of the topological term and a usual sigma model kinetic term for the ordinary coordinate fields with the halved tension. In addition, the ordinary coordinate fields must satisfy a self-dual relation,

$$\partial_i Y^\mu + \frac{1}{\sqrt{-h}}\epsilon_i{}^j N^\mu{}_\nu \partial_j Y^\nu = 0. \tag{5.34}$$

These agree with the $\mathbf{SO}(1,1)$ sector of the above example, (5.29), (5.30).

Especially for the trivial case of $S_{\rho\lambda} = 0$, the resulting reduced Lagrangian is purely topological,

$$\frac{1}{4\pi\alpha'}\mathcal{L} \equiv \frac{1}{4\pi\alpha'}\epsilon^{ij}\partial_i \tilde{Y}_\mu \partial_j Y^\mu. \tag{5.35}$$

6. Summary and comments

We summarize the main results with comments.

- The coordinate gauge symmetry (2.1) is equivalent to the section condition, both (1.1) and (1.2).
- The coordinate gauge symmetry implies that *spacetime is doubled yet gauged*.
- *Gauged differential one-forms for the doubled coordinates*, $Dx^M = \mathrm{d}x^M - \mathcal{A}^M$ (3.8) are fully covariant under the generalized diffeomorphisms and the coordinate gauge symmetry. In particular, the gauge connection, $\mathcal{A}^M$, is *derivative-index-valued* (3.9).
- The completely covariant Lagrangian description of a string propagating in the doubled yet gauged spacetime is given by (4.2), i.e.

$$\frac{1}{4\pi\alpha'}\mathcal{L} = \frac{1}{4\pi\alpha'}\left[-\frac{1}{2}\sqrt{-h}h^{ij}D_i X^M D_j X^N \mathcal{H}_{MN}(X) - \epsilon^{ij}D_i X^M \mathcal{A}_{jM}\right]. \tag{6.1}$$

- For non-degenerate cases, the covariant Lagrangian reduces to (5.7), i.e.

$$\begin{aligned}\frac{1}{2\pi\alpha'}\mathcal{L}' = \frac{1}{2\pi\alpha'}\Big[&-\frac{1}{2}\sqrt{-h}h^{ij}\partial_i Y^\mu \partial_j Y^\nu G_{\mu\nu}(Y)\\ &+\frac{1}{2}\epsilon^{ij}\partial_i Y^\mu \partial_j Y^\nu B_{\mu\nu}(Y) + \frac{1}{2}\epsilon^{ij}\partial_i \tilde{Y}_\mu \partial_j Y^\mu\\ &-\frac{1}{4}\sqrt{-h}h^{ij}(\mathcal{C}_{i\mu} - A_{i\mu})(\mathcal{C}_{j\nu} - A_{j\nu})G^{\mu\nu}(Y)\Big].\end{aligned} \tag{6.2}$$

  In particular, a self-duality condition (5.16) relating the ordinary and the dual coordinate fields follows from the equations of motion,

$$D_i X^M + \frac{1}{\sqrt{-h}}\epsilon_i{}^j D_j X^N \mathcal{H}_N{}^M = 0. \tag{6.3}$$

- Even for degenerate cases where the Riemannian metric becomes everywhere singular, it is still possible to have a Lagrangian description of a string propagating in such backgrounds (5.28), (5.33). Again a self-duality condition is implied by the equation of motion. However, unlike the non-degenerate cases, it is to be imposed only on the ordinary coordinate fields, (5.30), (5.34).
- In both the non-degenerate and the degenerate cases, (5.10), (5.29), (5.33), there appears a topological term bi-linear in ordinary and dual coordinates,

$$\frac{1}{4\pi\alpha'}\epsilon^{ij}\partial_i \tilde{Y}_\mu \partial_j Y^\mu, \tag{6.4}$$

which was previously introduced by hand in [5,8]. In our formalism, this term naturally arises and shares the same geometric origin, i.e. the second term in (6.1), as the world-sheet pull-back of the B-field in (6.2).

- Especially for open string, the topological term leads to a world-sheet boundary integral,

$$-\frac{1}{4\pi\alpha'}\oint_{\partial\Sigma} \mathrm{d}\sigma^i\, \tilde{Y}_\mu \partial_i Y^\mu, \tag{6.5}$$

which resembles the conventional coupling of the string to a (Born–Infeld) gauge potential, $\oint \mathrm{d}\sigma^i\, \hat{A}_\mu \partial_i Y^\mu$, and hints at an intriguing relation between the dual coordinate and the (Born–Infeld) gauge potential,

$$\tilde{Y}_\mu|_{\partial\Sigma} \iff -4\pi\alpha'\hat{A}_\mu. \tag{6.6}$$

In fact, for a derivative-index-valued vector, $\mathcal{V}^M = \phi\partial^M\varphi$ (3.6), which generates the coordinate gauge symmetry, the generalized Lie derivative (1.4) of the generalized metric, $\mathcal{H}_{AB}$ (5.5), implies (see (2.9) of [51]),

$$\begin{aligned} &\delta G_{\mu\nu} = 0, \qquad \delta B_{\mu\nu} = \partial_\mu(\phi\partial_\nu\varphi) - \partial_\nu(\phi\partial_\mu\varphi), \\ &\delta\tilde{Y}_\mu = \phi\partial_\mu\varphi, \qquad \delta(B_{\mu\nu} - 2\partial_{[\mu}\tilde{Y}_{\nu]}) = 0, \end{aligned} \tag{6.7}$$

of which the last would be, assuming the identification (6.6), consistent with the gauge invariant combination of the B-field and the field strength, $B + 4\pi\alpha'\hat{F}$.

- Our results are classical. Quantization remains as a future work, especially for the degenerate non-Riemannian backgrounds.
- The beta-functional world-sheet derivation of the DFT equations of motion, one-loop [30, 37,80] and beyond for the higher order α'-corrections [55,57], may be now worth while to revisit equipped with the full covariance.
- Thorough investigation of the degenerate geometry is desirable within the frameworks of both the doubled yet gauged sigma model and the maximally supersymmetric double field theory [46]. It seems natural to expect such a background to provide an alternative or enriched scheme for compactification.
- Understanding of the coordinate gauge symmetry from the Hamiltonian view point for a constrained system deserves a separate study [81].
- Supersymmetrization as well as generalization to a generic p-brane are also of interest, e.g. using the methods of [4,8,82].

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Appendix A. Covariant derivatives in DFT

Here we review the covariant derivatives in DFT.
With a pair of symmetric projectors,

$$P_{AB} = P_{BA} = \frac{1}{2}(\mathcal{J}+\mathcal{H})_{AB}, \qquad \bar{P}_{AB} = \bar{P}_{BA} = \frac{1}{2}(\mathcal{J}-\mathcal{H})_{AB}, \tag{A.1}$$

satisfying

$$P_A{}^B P_B{}^C = P_A{}^C, \qquad \bar{P}_A{}^B \bar{P}_B{}^C = \bar{P}_A{}^C, \qquad P_A{}^B \bar{P}_B{}^C = 0, \tag{A.2}$$

the DFT analogy of the Christoffel connection reads [28]

$$\begin{aligned}\Gamma_{CAB} = {} & 2(P\partial_C P\bar{P})_{[AB]} + 2\big(\bar{P}_{[A}{}^D \bar{P}_{B]}{}^E - P_{[A}{}^D P_{B]}{}^E\big)\partial_D P_{EC} \\ & - \frac{4}{D-1}\big(\bar{P}_{C[A}\bar{P}_{B]}{}^D + P_{C[A}P_{B]}{}^D\big)\big(\partial_D d + \big(P\partial^E P\bar{P}\big)_{[ED]}\big),\end{aligned} \tag{A.3}$$

which defines the *semi-covariant derivative* [25,28],

$$\begin{aligned}\nabla_C T_{A_1A_2\cdots A_n} := {} & \partial_C T_{\omega A_1A_2\cdots A_n} - \omega\Gamma^B{}_{BC} T_{A_1A_2\cdots A_n} \\ & + \sum_{i=1}^{n} \Gamma_{CA_i}{}^B T_{A_1\cdots A_{i-1}BA_{i+1}\cdots A_n}.\end{aligned} \tag{A.4}$$

The connection is the unique solution to the following requirements.

- The semi-covariant derivative is compatible with the $\mathbf{O}(D,D)$ metric,

$$\nabla_A \mathcal{J}_{BC} = 0 \quad \Longleftrightarrow \quad \Gamma_{CAB} + \Gamma_{CBA} = 0. \tag{A.5}$$

- The semi-covariant derivative annihilates the whole NS-NS sector, i.e. the DFT-dilaton[12] and the pair of projectors (A.1),

$$\nabla_A d = 0, \qquad \nabla_A P_{BC} = 0, \qquad \nabla_A \bar{P}_{BC} = 0. \tag{A.6}$$

- The cyclic sum of the connection vanishes,

$$\Gamma_{ABC} + \Gamma_{CAB} + \Gamma_{BCA} = 0. \tag{A.7}$$

- Lastly, the connection belongs to the kernels of rank-six projectors,

$$\mathcal{P}_{CAB}{}^{DEF}\Gamma_{DEF} = 0, \qquad \bar{\mathcal{P}}_{CAB}{}^{DEF}\Gamma_{DEF} = 0, \tag{A.8}$$

where

$$\begin{aligned}\mathcal{P}_{CAB}{}^{DEF} &= P_C{}^D P_{[A}{}^{[E} P_{B]}{}^{F]} + \frac{2}{D-1} P_{C[A}P_{B]}{}^{[E}P^{F]D}, \\ \bar{\mathcal{P}}_{CAB}{}^{DEF} &= \bar{P}_C{}^D \bar{P}_{[A}{}^{[E} \bar{P}_{B]}{}^{F]} + \frac{2}{D-1} \bar{P}_{C[A}\bar{P}_{B]}{}^{[E}\bar{P}^{F]D}.\end{aligned} \tag{A.9}$$

12 Since e^{-2d} is a scalar density with weight one, $\nabla_A d = -\frac{1}{2}e^{2d}\nabla_A e^{-2d} = \partial_A d + \frac{1}{2}\Gamma^B{}_{BA}$.

In particular, the two symmetric properties, (A.5) and (A.7), enable us to replace the ordinary derivatives in the definition of the generalized Lie derivative (1.4) by the semi-covariant derivatives (A.4),

$$\begin{aligned}\hat{\mathcal{L}}_V T_{A_1\cdots A_n} = {} & V^B\nabla_B T_{A_1\cdots A_n} + \omega\nabla_B V^B T_{A_1\cdots A_n} \\ & + \sum_{i=1}^{n}(\nabla_{A_i}V_B - \nabla_B V_{A_i})T_{A_1\cdots A_{i-1}}{}^{B}{}_{A_{i+1}\cdots A_n}.\end{aligned} \tag{A.10}$$

The rank-six projectors satisfy the projection property,

$$\mathcal{P}_{CAB}{}^{DEF}\mathcal{P}_{DEF}{}^{GHI} = \mathcal{P}_{CAB}{}^{GHI}, \qquad \bar{\mathcal{P}}_{CAB}{}^{DEF}\bar{\mathcal{P}}_{DEF}{}^{GHI} = \bar{\mathcal{P}}_{CAB}{}^{GHI}. \tag{A.11}$$

They are also symmetric and traceless,

$$\begin{aligned}&\mathcal{P}_{CABDEF} = \mathcal{P}_{DEFCAB} = \mathcal{P}_{C[AB]D[EF]}, \qquad \bar{\mathcal{P}}_{CABDEF} = \bar{\mathcal{P}}_{DEFCAB} = \bar{\mathcal{P}}_{C[AB]D[EF]}, \\ &\mathcal{P}^{A}{}_{ABDEF} = 0, \qquad P^{AB}\mathcal{P}_{ABCDEF} = 0, \\ &\bar{\mathcal{P}}^{A}{}_{ABDEF} = 0, \qquad \bar{P}^{AB}\bar{\mathcal{P}}_{ABCDEF} = 0.\end{aligned} \tag{A.12}$$

Now, under the infinitesimal DFT-coordinate transformation set by the generalized Lie derivative, the semi-covariant derivative transforms as

$$\begin{aligned}\delta(\nabla_C T_{A_1\cdots A_n}) = {} & \hat{\mathcal{L}}_V(\nabla_C T_{A_1\cdots A_n}) \\ & + \sum_{i=1}^{n} 2(\mathcal{P}+\bar{\mathcal{P}})_{CA_i}{}^{BDEF}\partial_D\partial_{[E}V_{F]}T_{A_1\cdots A_{i-1}BA_{i+1}\cdots A_n}.\end{aligned} \tag{A.13}$$

The sum on the right hand side corresponds to a potentially anomalous part against the full covariance. Hence, in general, the semi-covariant derivative is not necessarily covariant.[13] However, since the anomalous terms are projected by the rank-six projectors which satisfy the properties in (A.12), it is in fact possible to eliminate them. Combined with the projectors, the semi-covariant derivative – as the name indicates – can be converted into various fully covariant derivatives [28]:

$$\begin{aligned}&P_C{}^D\bar{P}_{A_1}{}^{B_1}\bar{P}_{A_2}{}^{B_2}\cdots\bar{P}_{A_n}{}^{B_n}\nabla_D T_{B_1B_2\cdots B_n}, \\ &\bar{P}_C{}^D P_{A_1}{}^{B_1}P_{A_2}{}^{B_2}\cdots P_{A_n}{}^{B_n}\nabla_D T_{B_1B_2\cdots B_n}, \\ &P^{AB}\bar{P}_{C_1}{}^{D_1}\bar{P}_{C_2}{}^{D_2}\cdots\bar{P}_{C_n}{}^{D_n}\nabla_A T_{BD_1D_2\cdots D_n}, \\ &\bar{P}^{AB}P_{C_1}{}^{D_1}P_{C_2}{}^{D_2}\cdots P_{C_n}{}^{D_n}\nabla_A T_{BD_1D_2\cdots D_n}, \\ &P^{AB}\bar{P}_{C_1}{}^{D_1}\bar{P}_{C_2}{}^{D_2}\cdots\bar{P}_{C_n}{}^{D_n}\nabla_A\nabla_B T_{D_1D_2\cdots D_n}, \\ &\bar{P}^{AB}P_{C_1}{}^{D_1}P_{C_2}{}^{D_2}\cdots P_{C_n}{}^{D_n}\nabla_A\nabla_B T_{D_1D_2\cdots D_n}.\end{aligned} \tag{A.14}$$

Appendix B. Useful formulae

For the DFT generalized diffeomorphism connection (A.3), we have

[13] However, (A.5) and (A.6) are exceptions as the anomalous terms vanish identically, thanks to (A.12).

$$\begin{aligned}
&\Gamma_{LJK} P^J{}_M \bar{P}^K{}_N = (P\partial_L P\bar{P})_{MN}, \\
&\Gamma_{LJK} \bar{P}^J{}_M P^K{}_N = -(\bar{P}\partial_L P P)_{MN}, \\
&\partial_L \mathcal{H}_{MN} = 4\Gamma_{LJK} P^J{}_{(M} \bar{P}^K{}_{N)}, \\
&\mathcal{H}_L{}^K \partial_K \mathcal{H}_{MN} = 4\big(\Gamma_{IJK} P^I{}_L P^J{}_{(M} \bar{P}^K{}_{N)} + \Gamma_{IJK} \bar{P}^I{}_L \bar{P}^J{}_{(M} P^K{}_{N)}\big),
\end{aligned} \tag{B.1}$$

such that

$$\begin{aligned}
\mathcal{H}_M{}^K \partial_N \mathcal{H}_{KL} + \mathcal{H}_N{}^K \partial_M \mathcal{H}_{KL} &= -\mathcal{H}_{LK} \partial_N \mathcal{H}^K{}_M - \mathcal{H}_{LK} \partial_M \mathcal{H}^K{}_N \\
&= 4\big(\Gamma_{MJK} P^J{}_{[N} \bar{P}^K{}_{L]} + \Gamma_{NJK} P^J{}_{[M} \bar{P}^K{}_{L]}\big),
\end{aligned} \tag{B.2}$$

and

$$\begin{aligned}
&\mathcal{H}_L{}^K \partial_K \mathcal{H}_{MN} + \mathcal{H}_M{}^K \partial_N \mathcal{H}_{KL} + \mathcal{H}_N{}^K \partial_M \mathcal{H}_{KL} \\
&\quad = -4\big(\Gamma_{JK(M} P^J{}_{N)} \bar{P}^K{}_L + \Gamma_{JK(M} \bar{P}^J{}_{N)} P^K{}_L\big).
\end{aligned} \tag{B.3}$$

It is also worth while to note

$$\begin{aligned}
&(PU)^M \nabla_M (\bar{P}V)^N + (\bar{P}U)^M \nabla_M (PV)^N \\
&\quad = \frac{1}{2} U^M \big[\partial_M V^N + 2P_M{}^K \Gamma_K{}^N{}_L (\bar{P}V)^L + 2\bar{P}_M{}^K \Gamma_K{}^N{}_L (PV)^L\big] \\
&\qquad - \frac{1}{2} (\mathcal{H}U)^M \partial_M (\mathcal{H}V)^N.
\end{aligned} \tag{B.4}$$

On the string world-sheet we have

$$\epsilon_i{}^j \epsilon_k{}^l = (-h)\big(\delta_i{}^l \delta_k{}^j - h_{ik} h^{jl}\big), \qquad \epsilon_{ij} \epsilon^{kl} = (-h)\big(\delta_i{}^l \delta_j{}^k - \delta_i{}^k \delta_j{}^l\big). \tag{B.5}$$

In particular,

$$\left(\frac{1}{\sqrt{-h}} \epsilon_i{}^j\right)\left(\frac{1}{\sqrt{-h}} \epsilon_j{}^k\right) = \delta_i{}^k. \tag{B.6}$$

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