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Differently implicational universal triple I method of (1, 2, 2) type[☆]

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ABSTRACT

As a generalization of the triple I method, differently implicational universal triple I method of (1, 2, 2) type (universal triple I method for short) is investigated. First, the concepts of residual operators and strongly residual operators are given, and then related conclusions of residual pairs are provided. Second, the related universal triple I solutions (including FMP-solutions, FMT-solutions and so on) are strictly defined by the infimum, where such solutions are divided into two parts respectively corresponding to the minimum and infimum. Then, we put emphasis on the FMP-solutions, in which the unified forms of FMP-solutions w.r.t. strongly residual operators and a new idea for getting FMP-solutions w.r.t. infimum are achieved. Third, as a result of analyzing the logic basis of a sort of CRI (Compositional Rule of Inference) method, it is found that their CRI solutions can be regarded as special cases of FMP-solutions. Lastly, the response functions of fuzzy systems via universal triple I method are discussed, which demonstrates that the universal triple I method can provide bigger choosing space and get better fuzzy controllers by contrast with the triple I method and CRI method.

1. Introduction

Nowadays fuzzy reasoning plays an important role in fuzzy control, fuzzy data mining and artificial intelligence (see [1–4]). The basic problems of fuzzy reasoning are fuzzy modus ponens (FMP) and fuzzy modus tollens (FMT) as following:

$$\text{FMP: for a given rule } A \to B \text{ and input } A^*, \text{ to compute } B^* \text{ (output)}, \tag{1}$$

$$\text{FMT: for a given rule } A \to B \text{ and input } B^*, \text{ to compute } A^* \text{ (output)}, \tag{2}$$

where $A, A^* \in F(X)$ (the set of all fuzzy subsets on X) and $B, B^* \in F(Y)$ (the set of all fuzzy subsets on Y). To solve the FMP problem Zadeh put forward the famous CRI (Compositional Rule of Inference) method expressed as follows

$$B^*(y) = \sup_{x \in X}\{A^*(x) \wedge R(A(x), B(y))\}, \quad y \in Y \tag{3}$$

where R is an implication operator (defined as a mapping $[0, 1]^2 \to [0, 1]$); see [5–8]. $R(a, b)$ can also be written as $a \to b$. Later, Wang pointed out that there were some disadvantages in the CRI method in [9], and proposed the triple I method whose solution $B^*(y)$ was the minimum fuzzy set making

$$(A(x) \to B(y)) \to (A^*(x) \to B^*(y)) \tag{4}$$

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take its maximum. It is proved that the triple I method has many advantages embodied as its logic basis, reversibility, and the property of pointwise optimization (see [9,10]).

A whole fuzzy system is composed of a rule base, fuzzier, method of fuzzy reasoning (e.g. the triple I method) and defuzzier (see [11]). The rule base is usually given in advance. Some key capabilities (such as universal approximation, stability and practicability) can be effectively analyzed only if the last three parts in a fuzzy system are considered at the same time.

Currently, the fuzzy systems taking singleton fuzzier, centroid defuzzier and the triple I method or CRI method for fuzzy reasoning (respectively called the triple I systems or CRI systems in the sequel) [12,13], have been investigated by Li and Tang in [10,14–16]. It is found that, from the point of view of fuzzy systems, the effect of the triple I method is imperfect, which is embodied as inferior response ability and practicability of the triple I systems. However the CRI method has better capabilities than the triple I method and is basically acceptable (see Section 2.2 for the capabilities of triple I system and CRI system).

Although the triple I method has many acknowledged advantages mentioned above, it is imperfect from the point of view of fuzzy systems, which will hold back its broad application to a large extent. Moreover, the triple I method was proposed as the improvement on the CRI method, however it is inferior to the CRI method in this case (see [10,14–16]) leading to that the meaning of improvement is weakened to a certain extent. It is such shortcomings of the triple I method that make us take a strong interest in further investigation on it with high necessity.

Then a natural problem arise: How can we improve the triple I method? It should begin at its characteristic that three implication operators must be consistent in formula (4), which may be the root of existent hidden troubles. And Li pointed out in [10] the fact that the CRI method is a special case of the triple I method only if three implication operators in (4) are different. In detail, the CRI method (expressed as formula (3)) can be regarded as the triple I method where the second and third operators take the Mamdani operator R_M. Enlightened by it, we can let the latter two operators be same and the first one unlimited, that is, generalize (4) to:

$$(A(x) \to_1 B(y)) \to_2 (A^*(x) \to_2 B^*(y)). \tag{5}$$

The triple I method derived from (5) is called differently implicational universal triple I method of (1, 2, 2) type (universal triple I method for short) here. It is shown that the universal triple I method has better capabilities by our analysis (see Section 2.3 and the latter part), and we will focus on the universal triple I method in the present paper.

The rest of this paper is organized as follows. In Section 2, the state-of-the-art and analysis related to universal triple I method are introduced. In Section 3, a new definition of residual operator is provided, and based on it related results of residual pairs are proved. In Section 4, the FMP-solutions are strictly defined, and then correlative outcomes are achieved from three kinds of solutions, in which the unified forms of solutions w.r.t. a kind of operators and a new solving idea for part solutions are presented. In Section 5, the logic basis of a generalized CRI method is studied, and it is found that it is a special case of the universal triple I method. In Section 6, the response functions of fuzzy systems via the universal triple I method are analyzed; the significance of universal triple I method is further discussed. In Sections 7 and 8, the related results of FMT-universal triple I method, α-universal triple I method are respectively given. Section 9 is the conclusion.

2. State-of-the-art and related analysis

2.1. Triple I method

Since the triple I method was proposed, it has attracted rapidly growing interests. Wang et al. systemically investigated it and the α-triple I method (derived from $(A(x) \to B(y)) \to (A^*(x) \to B^*(y)) \geq \alpha$), together with their corresponding theories of sustaining degrees and reversibility (see [9,17,18]). Wang and Pei et al. constructed regular implication operators from left-continuous t-norms, and then gave the unified forms of triple I method (see [19–21]). Following Song et al. discussed similar forms, i.e. $(A(x) \to B(y)) \to (A^*(x) \to B^*(y)) \leq \alpha$, $(A^*(x) \to B^*(y)) \to (A(x) \to B(y)) \geq \alpha$ and $(A^*(x) \to B^*(y)) \to (A(x) \to B(y)) \leq \alpha$, they respectively established the restriction theory of triple I method, the reverse triple I method and the restriction theory of reverse triple I method (see [22–24]), which were also paid attention by Wang and Li in [25,26].

2.2. Capabilities of triple I system

Currently, the triple I systems and CRI systems have been investigated by Li and Tang. Li discussed the response ability of the triple I systems constructed by 51 implication operators, and found that only 2 systems can be used in such 51 systems (see [14]). However in the CRI systems constructed by 23 implication operators there are 12 usable systems in such 23 CRI systems (see [15]). Therefore there are very few usable triple I systems, demonstrating that the triple I method has inferior results. Later, Li contrasted the triple I systems and CRI systems, and found the triple I systems are basically not as good as the latter in general (see [10]). In [16], we achieved the fact that 2 systems are practicable in 11 triple I systems while 4 systems are usable in 11 CRI systems constructed by the same 11 implication operators. So, from the point of view of fuzzy systems, the effect of the triple I method is inferior, while the CRI method has better capabilities and is basically acceptable.

To analyze the reasons to bring out such shortcomings of the triple I method we find that there are two main reasons. First, the key reason is the triple I method itself. Second, chosen fuzzier and defuzzier are inappropriate, but in fact such fuzzier and defuzzier are familiar with wide application background while the CRI systems have acceptable effect. Therefore the fact that the capabilities of triple I systems are inferior, is mainly derived from the triple I method itself.

2.3. Universal triple I method

The universal triple I method is obviously a generalization of the triple I method, which holds many characteristics of the triple I method (such as solving process and solutions' forms in the sequel) and has contact with the CRI method. So the research of the universal triple I method will help to not only avoid shortcomings of the triple I method, but also comprehend the essence of the triple I method and CRI method, which is vital to the fundamental theory of fuzzy reasoning, and fuzzy controllers' study and so on. Ref. [10] does not definitely propose formula (5) and the universal triple I method, and has no further study. To the knowledge of the authors, there is no finding other literature to study universal triple I method. Consequently, we will investigate the universal triple I method.

3. Preliminaries

Recall that $R(a, b)$ is also written as $a \to b$, so $\to_i$ and R_i can be regarded as the same one for convenience ($i = 1, 2$). Currently there are many literatures investigating the theme of residual pairs (see e.g. [27–29]), in which Wang and Pei study the residual pair from left-continuous t-norms for the triple I method. Now we shall propose a new method for constructing the residual pair.

Definition 3.1. An implication operator R is called a residual operator if the following three conditions are satisfied:

(C1) $R(a, b)$ is nondecreasing w.r.t. b ($a, b \in [0, 1]$).
(C2) $R(a, b)$ is right-continuous w.r.t. b ($a \in [0, 1], b \in [0, 1)$).
(C3) $\{y \in [0, 1] \mid a \to y = 1\} \neq \varnothing$ ($a \in [0, 1]$).

Especially, if R also satisfies

(C4) $a \leq b$ iff $R(a, b) = 1$ ($a, b \in [0, 1]$, iff denotes "if and only if"),

then R is said to be a strongly residual operator.

In the present paper, 16 familiar implication operators are mainly considered. They are Lukasiewicz operator R_L, Gödel operator R_G, Goguen operator R_{Go}, Gaines–Rescher operator R_{GR}, R_0 operator, Zadeh operator R_Z, Yager operator R_Y, Reichenbach operator R_R, Dubois–Prade operator R_{DP}, Larsen operator R_{La}, Kleene–Dienes operator R_{KD}, Mamdani operator R_M, revised Reichenbach operator R_{13} (see [30]), R_{14} (see [15]), R_{15} and R_{16} (see [14]) as follows.

$$R_L(a, b) = \begin{cases} 1, & a \leq b \\ a' + b, & a > b, \end{cases} \qquad R_G(a, b) = \begin{cases} 1, & a \leq b \\ b, & a > b, \end{cases}$$

$$R_{Go}(a, b) = \begin{cases} 1, & a = 0 \\ (b/a) \wedge 1, & a \neq 0, \end{cases} \qquad R_{GR}(a, b) = \begin{cases} 1, & a \leq b \\ 0, & a > b, \end{cases}$$

$$R_0(a, b) = \begin{cases} 1, & a \leq b \\ a' \vee b, & a > b, \end{cases} \qquad R_Z(a, b) = a' \vee (a \wedge b),$$

$$R_Y(a, b) = b^a \quad (R_Y(0, 0) = 1), \qquad R_R(a, b) = a' + a \times b,$$

$$R_{DP}(a, b) = \begin{cases} 1, & a' \wedge b \neq 0 \\ a' \vee b, & a' \wedge b = 0 \end{cases} = \begin{cases} b, & a = 1 \\ a', & b = 0 \\ 1, & \text{else}, \end{cases} \qquad R_{La}(a, b) = a \times b,$$

$$R_{KD}(a, b) = a' \vee b, \qquad R_M(a, b) = a \wedge b,$$

$$R_{13}(a, b) = \begin{cases} 1, & a \leq b \\ a' + ab, & a > b, \end{cases} \qquad R_{14}(a, b) = \begin{cases} 0, & b = 0 \\ a, & b > 0, \end{cases}$$

$$R_{15}(a, b) = \begin{cases} 1, & a \leq b \\ a'/b', & a > b, \end{cases} \qquad R_{16}(a, b) = \begin{cases} 1, & a \leq b \\ a', & a > b \end{cases}$$

where x' denotes $1 - x$, $a, b \in [0, 1]$, $\vee = \max$ and $\wedge = \min$.

Definition 3.2. Let $\to$ and $\otimes$ be two $[0, 1]^2 \to [0, 1]$ mappings, $(\to, \otimes)$ is said to be a residual pair or, $\to$ and $\otimes$ are residual to each other, if

$$a \otimes b \leq c \quad \text{iff} \quad b \leq a \to c, \; a, b, c \in [0, 1].$$

Lemma 3.1. (i) *Let* $(\to_1, \otimes)$ *and* $(\to_2, \otimes)$ *be residual pairs, then* $\to_1 = \to_2$; (ii) *let* $(\to, \otimes_1)$ *and* $(\to, \otimes_2)$ *be residual pairs, then* $\otimes_1 = \otimes_2$.

Proof. (i) Note that, if one can prove that $x \le a$ iff $x \le b$, then $a = b$. Since $(\to_1, \otimes)$ and $(\to_2, \otimes)$ are residual pairs, we have $z \le a \to_1 b$ iff $a \otimes z \le b$ iff $z \le a \to_2 b$ $(a, b, z \in [0, 1])$, and then $a \to_1 b = a \to_2 b$ (i.e. $\to_1 = \to_2$) holds. (ii) It is similar to get the conclusion. □

Theorem 3.1. *Let* $\to: [0, 1]^2 \to [0, 1]$ *be a residual operator, and define* $\otimes_\to : [0, 1]^2 \to [0, 1]$ *as follows*

$$a \otimes_\to b = \wedge\{y \in [0, 1] \mid b \le a \to y\}, \quad a, b \in [0, 1] \tag{6}$$

then $(\to, \otimes_\to)$ *is a residual pair, and*

$$a \to b = \vee\{y \in [0, 1] \mid a \otimes_\to y \le b\}. \tag{7}$$

Proof. (i) Let $\Gamma = \{y \in [0, 1] \mid b \le a \to y\}$ and $d = \wedge\Gamma = a \otimes_\to b$. If $b \le a \to c$ $(a, b, c \in [0, 1])$, then $c \in \Gamma$ and hence $a \otimes_\to b \le c$ (noting that $a \otimes_\to b = \wedge\Gamma$).

Conversely, we shall prove that $d = a \otimes_\to b \le c$ implies $b \le a \to c$ $(a, b, c \in [0, 1])$. Since $\to$ satisfies (C3) (i.e. $\{y \in [0, 1] \mid a \to y = 1\} \ne \varnothing$), it follows that there exists $y \in [0, 1]$ such that $a \to y = 1$ for any $a \in [0, 1]$, and then $\{y \in [0, 1] \mid b \le a \to y\} \ne \varnothing$. We have two cases to be considered: $d < c$ or $d = c$.

(a) Suppose $d < c$, it follows from the definition of infimum that there exists $y_0 \in \Gamma$ such that $y_0 < d + \varepsilon < c$ for any $\varepsilon \in (0, c - d)$. This means $b \le a \to y_0 \le a \to c$ (noting that $\to$ satisfies (C1)).

(b) Suppose $d = c$, we show $c \in \Gamma$ and then $b \le a \to c$. Indeed, if $c \notin \Gamma$, then, in Γ, there exist $c_0 > c_1 > \cdots > c_n > \cdots$ such that $\lim_{i\to\infty} c_i = c$ and $c_i > c$, so c is the right limit of $\{c_i\}$. Notice that $\to$ satisfies (C2) and $b \le a \to c_i$ $(i = 0, 1, \ldots)$, by taking limit at both sides we achieve $b \le \lim_{i\to\infty}(a \to c_i) = a \to c$ (i.e. $c \in \Gamma$), which is a contradiction.

Summarizing the above, we have $a \otimes_\to b \le c$ iff $b \le a \to c$, i.e., $(\to, \otimes_\to)$ is a residual pair.

(ii) Since $(\to, \otimes_\to)$ is a residual pair, we get $a \otimes_\to y \le b$ iff $y \le a \to b$. Consequently, $\vee\{y \in [0, 1] \mid a \otimes_\to y \le b\} = \vee\{y \in [0, 1] \mid y \le a \to b\} = a \to b$. □

Proposition 3.1. (i) $R_G, R_L, R_0, R_{Go}, R_{GR}, R_{KD}, R_R, R_Y, R_{13}, R_{15}, R_{16}$ *satisfy* (C1), (C2) *and* (C3), *so they are residual operators;* $R_G, R_L, R_0, R_{Go}, R_{GR}, R_{13}, R_{15}, R_{16}$ *also satisfy* (C4), *and hence they are strongly residual operators.*

(ii) R_Z, R_M, R_{La} *satisfy* (C1), (C2)*but do not satisfy* (C3).

(iii) R_{DP} *satisfies* (C1), (C3) *and does not satisfy* (C2). R_{14} *satisfies* (C1) *and does not satisfy* (C2) *and* (C3). *But* R_{DP}, R_{14} *both are right-continuous w.r.t. second component on* (0, 1).

(iv) $R_G, R_L, R_0, R_{Go}, R_{GR}, R_{DP}, R_{KD}, R_R, R_Y, R_{13}, R_{15}, R_{16}$ *satisfy*

(C5) $a \to 1 = 1$ $(a \in [0, 1])$;

and R_M, R_{La}, R_{14} *satisfy*

(C6) $a \to 1 = a$ *and* $a \to 0 = 0$ $(a \in [0, 1])$.

Proof. (i) For implication operators listed above we only prove the case of R_0 as an example, the remainders can be proved similarly. We use x instead of b to analysis. If $0 \le x < a$, then $R_0(a, x) = (1 - a) \vee x < 1$; if $a \le x \le 1$, then $R_0(a, x) = 1$. These imply that R_0 obviously satisfies (C1) and (C4). If $x = a$, then $\lim_{x\to a+} R_0(a, x) = 1 = R_0(a, a)$, i.e., $R_0(a, x)$ is right-continuous w.r.t. x for the case of $x = a$. And $R_0(a, x)$ is evidently right-continuous on $[0, a) \cup (a, 1)$. Thus R_0 satisfies (C2). Since $R_0(a, 1) = 1$, we achieve $\{y \in [0, 1] \mid a \to y = 1\} \ne \varnothing$, and hence (C3) holds for R_0.

(ii) We only prove the case of R_Z as an example. If $a \le 1/2$, then $R_Z(a, x) = a'$. If $a > 1/2$, we have three cases to be considered: (a) If $x < a'$, then $R_Z(a, x) = a'$. (b) If $a' \le x \le a$, then $R_Z(a, x) = x$ (noting that $a' \le R_Z(a, x) \le a$). (c) If $x > a$, then $R_Z(a, x) = a$. Thus, R_Z satisfies (C1). It is evident that R_Z satisfies (C2). For $0 < a < 1, R_Z(a, x) = a' \vee (a \wedge x) \le a' \vee a < 1$, i.e. $\{x \in [0, 1] \mid a \to x = 1\} = \varnothing$, so R_Z does not satisfy (C3).

(iii) We only prove the case of R_{DP} as an example. It is similar to prove that R_{DP} satisfies (C1), (C3). We shall show that R_{DP} does not satisfy (C2). If $a = 1$, then $R_{DP}(1, x) = x$. If $a = 0$, then $R_{DP}(0, x) \equiv 1$. If $0 < a < 1$, it is evident that $R_{DP}(a, x)$ is right-continuous w.r.t. x on (0, 1). But $\lim_{x\to 0+} R_{DP}(a, x) = 1 \ne R_{DP}(a, 0) = a'$, it follows that $R_{DP}(a, x)$ is not right-continuous w.r.t. x for the case of $x = 0$. As a result, $R_{DP}(a, x)$ does not satisfy (C2). Inspecting the above proof, we can readily obtain that R_{DP} is right-continuous w.r.t. second component on (0, 1).

(iv) It is easy to prove it. □

Lemma 3.2. *Let* $A, B \subset [0, 1]$, *then* $\wedge(A \cup B) = (\wedge A) \wedge (\wedge B)$.

Proof. If A or B is empty, then it is easy to get the conclusion (noting that $\wedge\varnothing = 1$). Suppose A, B are nonempty. By the fact that $A \subset A \cup B$ and $B \subset A \cup B$, it follows that $\wedge A \ge \wedge(A \cup B)$ and $\wedge B \ge \wedge(A \cup B)$. Thus $(\wedge A) \wedge (\wedge B) \ge \wedge(A \cup B)$.

Further, we shall show $\wedge(A \cup B) \ge (\wedge A) \wedge (\wedge B)$. Notice that, if one can prove that $x \le b$ if $x \le a$, then $a \le b$. If $x \le (\wedge A) \wedge (\wedge B)$, then $x \le \wedge A$ and $x \le \wedge B$, i.e., $x \le x_1$ and $x \le x_2$ for $\forall x_1 \in A, \forall x_2 \in B$. Thus, $x \le x_3$ for $\forall x_3 \in A \cup B$ and then $x \le \wedge(A \cup B)$. These imply $\wedge(A \cup B) \ge (\wedge A) \wedge (\wedge B)$.

Summarizing the above, we achieve $\wedge(A \cup B) = (\wedge A) \wedge (\wedge B)$. □

Proposition 3.2. *The operations corresponding to $R_G, R_{Go}, R_{GR}, R_L, R_{KD}, R_0, R_Y, R_R, R_{13}, R_{15}, R_{16}$ in residual pairs are as follows, respectively.*

$$a \otimes_G b = a \wedge b, \qquad a \otimes_{Go} b = a \times b, \qquad a \otimes_{GR} b = \begin{cases} a, & b > 0 \\ 0, & b = 0, \end{cases}$$

$$a \otimes_L b = \begin{cases} a+b-1, & a+b > 1 \\ 0, & a+b \le 1, \end{cases} \qquad a \otimes_{KD} b = \begin{cases} b, & a+b > 1 \\ 0, & a+b \le 1, \end{cases}$$

$$a \otimes_0 b = \begin{cases} a \wedge b, & a+b > 1 \\ 0, & a+b \le 1, \end{cases} \qquad a \otimes_Y b = \begin{cases} \sqrt[a]{b}, & a > 0 \\ 0, & a = 0, \end{cases}$$

$$a \otimes_R b = \begin{cases} (a+b-1)/a, & a > 0 \\ 0, & a = 0, \end{cases}$$

$$a \otimes_{13} b = \begin{cases} [(a+b-1)/a] \wedge a, & a+b > 1 \\ 0, & a+b \le 1, \end{cases} \qquad a \otimes_{15} b = \begin{cases} (a+b-1)/b, & a+b > 1 \\ 0, & a+b \le 1, \end{cases}$$

$$a \otimes_{16} b = \begin{cases} a, & a+b > 1 \\ 0, & a+b \le 1. \end{cases}$$

Proof. It follows from Proposition 3.1 that such 11 implication operators are residual operators. We only prove R_{13} as an example. By (6) and Lemma 3.2, we achieve

$$\begin{aligned} a \otimes_{13} b &= \wedge\{y \in [0,1] \mid b \le R_{13}(a,y)\} \\ &= \wedge(\{y \in [0,1], a \le y \mid b \le R_{13}(a,y)\} \cup \{y \in [0,1], a > y \mid b \le R_{13}(a,y)\}) \\ &= [\wedge\{y \in [0,1], a \le y \mid b \le 1\}] \wedge [\wedge\{y \in [0,1], a > y \mid b \le 1 - a + ay\}] \\ &= a \wedge [\wedge\{y \in [0,1] \mid a+b-1 \le ay, y < a\}]. \end{aligned} \tag{8}$$

If $a + b > 1$, then $a > 0$ and we have two cases to be considered: (a) Suppose $(a+b-1)/a < a$, then it follows from (8) that $a \otimes_{13} b = a \wedge [(a+b-1)/a]$. (b) Suppose $(a+b-1)/a \ge a$, then it follows from (8) that $a \otimes_{13} b = a \wedge [\wedge\varnothing] = a \wedge 1 = a = a \wedge [(a+b-1)/a]$. (Notice that it is in poset $([0,1], \le)$ and $\le$ is less or equal relation, thus $\wedge\varnothing = 1$.)

If $a + b \le 1$, then $a \otimes_{13} b = a \wedge [\wedge\varnothing] = a \wedge 1 = a = 0$ for $a = 0$, or $a + b - 1 \le 0 \le ay$ holds and hence $a \otimes_{13} b = a \wedge 0 = 0$ for $a > 0$ by virtue of formula (8).

Thus, it follows that $a \otimes_{13} b = \begin{cases} [(a+b-1)/a] \wedge a, & a+b > 1 \\ 0, & a+b \le 1 \end{cases}$. □

Proposition 3.3. *If $\to$ satisfies*

(C7) $\wedge\{a \to x_i \mid i \in I\} = a \to \wedge\{x_i \mid i \in I\}$ $(a, x_i \in [0,1]; I \ne \varnothing)$,

then $\to$ satisfies (C1) *and* (C2).

Proof. Let $x_1, x_2 \in [0,1]$ and $x_1 \le x_2$. Since $\to$ satisfies (C7), we have $a \to x_1 = a \to (x_1 \wedge x_2) = (a \to x_1) \wedge (a \to x_2) \le a \to x_2$. Thus $\to$ satisfies (C1).

Further, we shall show that $\to$ satisfies (C2). On the one hand, since $\to$ satisfies (C7), it follows that $\wedge\{a \to x \mid x > b\} = a \to \wedge\{x \mid x > b\} = a \to b$ for $b \in [0,1)$, $x \in [0,1]$ (Noting that $\{x \mid x > b\} \ne \varnothing$.). Thus $a \to x \ge \wedge\{a \to x \mid x > b\}$ holds for $\forall x > b$, which implies $\lim_{x \to b+}(a \to x) \ge \wedge\{a \to x \mid x > b\} = a \to b$. On the other hand, it follows from the definition of infimum that for $\forall \varepsilon > 0$, there exists $x_0 > b$ such that $a \to x_0 < \wedge\{a \to x \mid x > b\} + \varepsilon$. Considering that $\to$ satisfies (C1), we obtain

$$\lim_{x \to b+}(a \to x) = \lim_{\substack{x \to b \\ x_0 > x > b}}(a \to x) < \wedge\{a \to x \mid x > b\} + \varepsilon,$$

and hence $\lim_{x \to b+}(a \to x) \le \wedge\{a \to x \mid x > b\} = a \to b$. Together we achieve $\lim_{x \to b+}(a \to x) = a \to b$ $(b \in [0,1)$, $x \in [0,1])$, i.e. $\to$ satisfies (C2). □

Lemma 3.3. *$\to$ satisfies* (C3), *iff $\to$ satisfies*

(C8) $\{y \in [0,1] \mid b \le a \to y\} \ne \varnothing$, $a, b \in [0,1]$.

Proof. If $\to$ satisfies (C8), then, by taking $b = 1$, we get $\{y \in [0,1] \mid a \to y = 1\} \ne \varnothing$ for $\forall a \in [0,1]$, i.e. $\to$ satisfies (C3). On the other hand, if $\to$ satisfies (C3), then there exists $y \in [0,1]$ such that $a \to y = 1$ for $\forall a, b \in [0,1]$, and hence $\{y \in [0,1] \mid b \le a \to y\} \ne \varnothing$, i.e. $\to$ satisfies (C8). □

By Proposition 3.3, Lemma 3.3, we can get Lemma 3.4.

Lemma 3.4. *If $\to$ satisfies* (C7) *and* (C8), *then $\to$ is a residual operator.*

In [31], Liu also gave a definition of residual pair (see Definition 2.5 in [31]). In order to differentiate, such a kind of residual pair is defined as symmetrical residual pair, that is:

Definition 3.3. Let $\to$ and $\otimes$ be two $[0,1]^2 \to [0,1]$ mappings, $(\to, \otimes)$ is said to be a symmetrical residual pair if $a \otimes b \le c$ iff $a \le b \to c$, $a, b, c \in [0,1]$.

Liu pointed out in Theorem 2.1 of [31] that if $\to$ satisfies (C7) and (C8) then $\otimes$ can be achieved by $a \otimes b = \wedge\{y \in [0,1] \mid a \le b \to y\}$ such that $(\to, \otimes)$ is a symmetrical residual pair. It is not difficult to get Proposition 3.4.

Proposition 3.4. *If $\to$ satisfies* (C7) *and* (C8)*, $\otimes_\to$ is generated by $\to$ according to* (6)*, $\otimes$ is generated by $\to$ from Theorem* 2.1 *in* [31]*, then $(\to, \otimes_\to)$ is a residual pair, and $(\to, \otimes)$ a symmetrical residual pair while $a \otimes b = b \otimes_\to a$ holds.*

4. FMP-universal triple I method and its solutions

4.1. FMP-universal triple I method

Definition 4.1. Let Z be any nonempty set and $F(Z)$ the set of all fuzzy subsets on Z, define partial order relation $\le_F$ on $F(Z)$ (according to pointwise order) as: $A(z) \le_F B(z)$ iff $A(z_0) \le B(z_0)$ for $\forall z_0 \in Z$, where $A(z), B(z) \in F(Z)$.

Lemma 4.1 (*Wang [32]*). *$\langle F(Z), \le_F \rangle$ is a complete lattice.*

Definition 4.2. Suppose that, $A, A^* \in F(X)$, $B \in F(Y)$, nonempty set $\mathbb{E}$ is the set of $B^*(y)$ which makes (5) get its maximum for any $x \in X, y \in Y$ in $\langle F(Y), \le_F \rangle$, and $D^*(y)$ is the infimum of $\mathbb{E}$. If $D^*(y)$ is the minimum of $\mathbb{E}$, then $D^*(y)$ is called a MinP-solution. If $D^*(y)$ is not the minimum of $\mathbb{E}$, then $D^*(y)$ is called an InfP-quasi-solution; in $\mathbb{E}$, we pick out a fuzzy set $D^{**}(y)$ as small as possible, and call $D^{**}(y)$ an InfP-solution.

Let FMP-universal triple I solution (FMP-solution for short) be a general designation of MinP-solution and InfP-solution.

Remark 4.1. The strict definition of FMP-solution is given by the infimum in Definition 4.2. FMP-solutions are divided into two parts respectively corresponding to the minimum and infimum, i.e. the MinP-solutions and InfP-solutions. It is similar to the triple I method that A, A^*, B should be unchangeable and B^* changeable in Definition 4.2. When there is not the minimum in $\mathbb{E}$, its infimum $D^*(y)$ commonly exists (see Theorem 4.1 in what follows) and $D^*(y)$ is not a strict universal triple I solution (since it cannot make (5) get its maximum). Obviously, $D^{**}(y)$ is not unique but $D^*(y)$ is. Specially, as a special case of universal triple I solution, the triple I solution has the similar case.

Theorem 4.1. *There exists a unique fuzzy set $D^*(y)$ such that*

(i) *$D^*(y) \le_F D(y)$ for $\forall D(y) \in \mathbb{E}$, and*
(ii) *there is $C(y) \in \mathbb{E}$ satisfying $C(y_0) < D^*(y_0) + \varepsilon$ for $\forall y_0 \in Y$ and $\forall \varepsilon > 0$;*

then $D^*(y)$ is the infimum of $\mathbb{E}$. Specially, if $D^*(y) \in \mathbb{E}$ also holds, then $D^*(y)$ is a MinP-solution.

Proof. It follows from Lemma 4.1 that $\langle F(Y), \le_F \rangle$ is a complete lattice. Thus $D^*(y) = \inf \mathbb{E}$ uniquely exists since nonempty set $\mathbb{E} \subset F(Y)$.

We shall construct the fuzzy set $D^*(y)$ that we need. Notice that $\langle [0,1], \le \rangle$ is a complete lattice, and $\{D(y_0) \mid D \in \mathbb{E}\} \subset [0,1]$ holds for any constant $y_0 \in Y$, hence $\inf\{D(y_0) \mid D \in \mathbb{E}\} \triangleq D^*(y_0)$ uniquely exists. By the definition of infimum, we have $D^*(y_0) \le D(y_0)$ for $\forall D \in \mathbb{E}$ and that there exists $C \in \mathbb{E}$ such that $C(y_0) < D^*(y_0) + \varepsilon$ for any $\varepsilon > 0$. Let y_0 respectively takes every element in Y, and then we obtain $D^*(y_0)$ for any $y_0 \in Y$, thus there is a fuzzy set $D^*(y)$ such that $D^*(y)|_{y=y_0} = D^*(y_0)$. $D^*(y)$ obviously satisfies (i).

Further, we shall show that there exists $C^*(y) \in \mathbb{E}$ such that $C^*(y_0) < D^*(y_0) + \varepsilon$ for any $\varepsilon > 0$ and $y_0 \in Y$. Notice that we already know that there is $C \in \mathbb{E}$ satisfying $C(y_0) < D^*(y_0) + \varepsilon$ for any constant $y_0 \in Y$ and $\varepsilon > 0$. Let y_0 respectively takes every element in Y, and then there is a fuzzy set $C^*(y)$ such that $C^*(y)|_{y=y_0} = C(y_0)$, which is evidently what we need. Thus, $D^*(y)$ satisfies (ii).

Since partial order relation is according to pointwise order, it is easy to get $D^*(y) = \inf \mathbb{E}$. Specially, if $D^*(y) \in \mathbb{E}$, then $D^*(y)$ is the minimum of $\mathbb{E}$, thus it follows from Definition 4.2 that $D^*(y)$ is a MinP-solution. □

Proposition 4.1. *If $\to_2$ satisfies* (C1) *and* (C2)*, then the FMP-solution $B^*(y)$ is the MinP-solution, and the maximum of* (5) *is $M(x,y) = (A(x) \to_1 B(y)) \to_2 (A^*(x) \to_2 1)$.*

Proof. At first, we shall show that the maximum of (5) is $M(x,y)$. Since $\to_2$ satisfies (C1), then for (5), we have: $(A(x) \to_1 B(y)) \to_2 (A^*(x) \to_2 B^*(y)) \le (A(x) \to_1 B(y)) \to_2 (A^*(x) \to_2 1) = M(x,y)$. This means that $M(x,y)$ is the maximum of (5) (noting that A, B, A^* are unchangeable in (5) by Remark 4.1).

Further, to prove that FMP-solution $B^*(y)$ is the MinP-solution, it is enough to prove that $B^*(y)$ is the minimum of $\mathbb{E}$. Note that $B^*(y) = \inf \mathbb{E}$ and $\mathbb{E} = \{D^*(y) \in F(Y) \mid (A(x) \to_1 B(y)) \to_2 (A^*(x) \to_2 D^*(y)) = M(x,y), x \in X, y \in Y\}$. If $B^*(y) \notin \mathbb{E}$, then there exist fuzzy sets $B_1, B_2, \dots$ in $\mathbb{E}$ such that

$$\lim_{n\to\infty} B_n(y) = B^*(y), \quad y \in Y. \tag{9}$$

From the fact that $B_1, B_2, \dots \in \mathbb{E}$, we get ($n = 1, 2, \dots; x \in X; y \in Y$)

$$(A(x) \to_1 B(y)) \to_2 (A^*(x) \to_2 B_n(y)) = M(x, y).$$

Since $B^* = \inf \mathbb{E}$, we achieve $B_n(y) \geq B^*(y)$ ($y \in Y$), and then it follows from (9) that $B^*(y)$ is the right limit of $\{B_n(y) \mid n = 1, 2, \dots\}$ ($y \in Y$), thus we have (noting that $\to_2$ satisfies (C1), (C2))

$$\begin{aligned} M(x, y) &= \lim_{n\to\infty} \{(A(x) \to_1 B(y)) \to_2 (A^*(x) \to_2 B_n(y))\} \\ &= (A(x) \to_1 B(y)) \to_2 (A^*(x) \to_2 B^*(y)), \quad x \in X,\ y \in Y. \end{aligned}$$

Thus $B^*(y) \in \mathbb{E}$, a contradiction. Consequently, we obtain $B^*(y) \in \mathbb{E}$, and hence $B^*(y)$ is the minimum of $\mathbb{E}$. □

Corollary 4.1. *If $\to_2$ is a residual operator, then FMP-solution $B^*(y)$ is the MinP-solution, and the maximum of (5) is $M(x, y)$. Especially, if $\to_2$ is also a strongly residual operator, then $M(x, y) = 1$.*

Remark 4.2. The fact that $\to_2$ satisfies (C1) and (C2), is a sufficient condition to ensure that FMP-solution $B^*(y)$ is the MinP-solution, but is not a necessary condition. For example, take $a \to_2 b = a(1 - b)$ (see [14]), then formula (5) is equal to $R_1(A(x), B(y)) \times [1 - (A^*(x) \times (1 - B^*(y)))]$ and the FMP-solution is $B^*(y) = \begin{cases} 1, & y \in E \\ 0, & y \in Y - E \end{cases}$ where $E = \{y \in Y \mid \sup_{x\in X}\{R_1(A(x), B(y)) \times A^*(x)\} > 0\}$ (the solving process is similar to Proposition 4.4 in what follows). It is easy to find that $\to_2$ does not satisfy (C1), but this FMP-solution is a MinP-solution.

4.2. MinP-solutions corresponding to strongly residual operators

Theorem 4.2. *Let $\to_2$ be a strongly residual operator and $\otimes$ the residual operation w.r.t. $\to_2$, then the MinP-solution can be expressed as follows:*

$$B^*(y) = \sup_{x\in X}\{A^*(x) \otimes R_1(A(x), B(y))\}, \quad y \in Y. \tag{10}$$

Proof. It follows from Corollary 4.1 that FMP-solution $B^*(y)$ is the MinP-solution, and the maximum of formula (5) is 1. First, we shall prove:

$$(A(x) \to_1 B(y)) \to_2 (A^*(x) \to_2 B^*(y)) = 1, \quad x \in X,\ y \in Y. \tag{11}$$

Indeed, it follows from the expression of $B^*(y)$ that $A^*(x) \otimes R_1(A(x), B(y)) \leq B^*(y), x \in X, y \in Y$. Since $(\to_2, \otimes)$ is a residual pair, we obtain: $R_1(A(x), B(y)) \leq A^*(x) \to_2 B^*(y), x \in X, y \in Y$. Thus formula (11) holds (noting that $\to_2$ satisfies (C4)).

Second, we shall show that $B^*(y)$ is the minimum. Let $D(y) \in \mathbb{E}$, then $(A(x) \to_1 B(y)) \to_2 (A^*(x) \to_2 D(y)) = M(x, y) = 1$ ($x \in X, y \in Y$). This implies $A(x) \to_1 B(y) \leq A^*(x) \to_2 D(y), x \in X, y \in Y$ by virtue of the conditions that $\to_2$ satisfies. And, considering $(\to_2, \otimes)$ is a residual pair, then $A^*(x) \otimes R_1(A(x), B(y)) \leq D(y), x \in X, y \in Y$. Thus $D(y)$ is an upper bound of

$$\{A^*(x) \otimes R_1(A(x), B(y)) \mid x \in X\}, \quad y \in Y.$$

Hence it follows from (10) that $B^*(y) \leq_F D(y)$. These imply that $B^*(y)$ is the minimum of $\mathbb{E}$, then $B^*(y) = \inf \mathbb{E}$.

Thus we obtain that $B^*(y)$ is the MinP-solution by Definition 4.2. □

From Proposition 3.2, Theorem 4.2 and Proposition 3.1, we can get Theorem 4.3.

Theorem 4.3. *Suppose that $R_2 \in \{R_G, R_L, R_0, R_{Go}, R_{GR}, R_{13}, R_{15}, R_{16}\}$, and $\otimes$ is the residual operation w.r.t. R_2, then the MinP-solution can be expressed as $B^*(y) = \sup_{x\in X}\{A^*(x) \otimes R_1(A(x), B(y))\},\ y \in Y$. Specially:*

(i) *If $R_2 = R_G$, then the MinP-solution is: $B^*(y) = \sup_{x\in X}\{A^*(x) \wedge R_1(A(x), B(y))\},\ y \in Y$. If $R_2 = R_{Go}$, then $B^*(y) = \sup_{x\in X}\{A^*(x) \times R_1(A(x), B(y))\},\ y \in Y$.*

(ii) *If $R_2 = R_{GR}$, then $B^*(y) = \sup_{x\in E_y}\{A^*(x)\}$ where $E_y = \{x \in X \mid R_1(A(x), B(y)) > 0\}$.*

(iii) *If $R_2 \in \{R_L, R_{13}, R_{15}, R_{16}, R_0\}$, then $E_y = \{x \in X \mid (A^*(x))' < R_1(A(x), B(y))\}$, and for the case of $R_2 = R_L$, $B^*(y) = \sup_{x\in E_y}\{A^*(x) + R_1(A(x), B(y)) - 1\}$; for the case of $R_2 = R_{13}$, $B^*(y) = \sup_{x\in E_y}\{[(A^*(x) + R_1(A(x), B(y)) - 1)/A^*(x)] \wedge A^*(x)\}$; for the case of $R_2 = R_{15}$, $B^*(y) = \sup_{x\in E_y}\{[A^*(x) + R_1(A(x), B(y)) - 1]/R_1(A(x), B(y))\}$; for the case of $R_2 = R_{16}$, $B^*(y) = \sup_{x\in E_y}\{A^*(x)\}$; for the case of $R_2 = R_0$, $B^*(y) = \sup_{x\in E_y}\{A^*(x) \wedge R_1(A(x), B(y))\}$.*

From Lemma 3.4, Theorem 4.2, we can obtain Proposition 4.2.

Proposition 4.2. *If R_2 satisfies* (C4), (C7) *and* (C8), *and $\otimes$ is its residual mapping, then the MinP-solution can be expressed as $B^*(y) = \sup_{x\in X}\{A^*(x) \otimes R_1(A(x), B(y))\}$, $y \in Y$.*

If $R_1 = R_2$, then the MinP-solution degenerates into the triple I solution for the FMP problem, and we have Corollaries 4.2 and 4.3.

Corollary 4.2. *If R_2 is a strongly residual operator, and $\otimes$ its residual mapping, and take $R_1 = R_2 \triangleq R$, then the MinP-solution (i.e. the triple* I *solution of FMP) can be expressed as follows:*

$$B^*(y) = \sup_{x\in X}\{A^*(x) \otimes R(A(x), B(y))\}, \quad y \in Y. \tag{12}$$

Corollary 4.3. *If R_2 satisfies* (C4), (C7) *and* (C8), *and $\otimes$ is its residual mapping, and take $R_1 = R_2 \triangleq R$, then the MinP-solution (i.e. the triple* I *method of FMP) is the same as Corollary 4.2.*

Remark 4.3. Wang gave the unified forms of triple I method derived from regular implication operators (see Theorem 1 of Ref. [19]), where got regular implication operators (from left-continuous t-norms) and then obtained the same expression as formula (12) in the present paper.

By Proposition 1 in [19], it is easy to obtain Lemma 4.2. And then we get Proposition 4.3 by Lemma 3.4.

Lemma 4.2. *Regular implication operators satisfy* (C4), (C7) *and* (C8).

Proposition 4.3. *Regular implication operators are all strongly residual operators.*

Remark 4.4. In Theorem 1 of Ref. [19], the triple I solution (that is (12)) is only suitable for regular implication operators. Proposition 4.3 demonstrates that regular implication operators are strongly residual operators. These imply that Theorem 1 of Ref. [19] is a special case of Corollary 4.2 in the present paper. Moreover, strongly residual operators $R_{GR}, R_{13}, R_{15}, R_{16}$ are not regular implication operators, but they can be used to (12). In Corollary 3.1 in Ref. [31], Liu gave the fact that if $(R, \otimes_2)$ is a symmetrical residual pair and R satisfies (C4), (C7) and (C8), then the triple I method of FMP can be expressed as $B^*(y) = \sup_{x\in X}\{R(A(x), B(y)) \otimes_2 A^*(x)\}$. By Proposition 3.4, we can achieve $R(A(x), B(y)) \otimes_2 A^*(x) = A^*(x) \otimes R(A(x), B(y))$, which means that Corollary 3.1 in [31] is the same as Corollary 4.3 in the present paper.

4.3. Other MinP-solutions

Proposition 4.4. *If $\rightarrow_2 \in \{R_Y, R_{La}, R_R, R_{KD}\}$, then the MinP-solution $B^*(y) = \begin{cases} 1, & y \in E \\ 0, & y \in Y - E \end{cases}$ where $E = \{y \in Y \mid \sup_{x\in X}\{R_1(A(x), B(y)) \times A^*(x)\} > 0\}$.*

Proof. It is obvious that $\rightarrow_2$ satisfies (C1) and (C2), then it follows from Proposition 4.1 that $B^*(y)$ is the MinP-solution. We only prove R_Y and R_{KD} as examples, the remainders can be proved similarly.

First, take $\rightarrow_2 = R_Y$. Then formula (5) is equal to $((B^*(y))^{A^*(x)})^{R_1(A(x),B(y))} = (B^*(y))^{R_1(A(x),B(y))\times A^*(x)}$ $(y \in Y)$. If $y \in E$, then (5) takes the maximum 1 iff $B^*(y) = 1$. If $y \in Y - E$ (i.e. $R_1(A(x), B(y)) \times A^*(x) \equiv 0, x \in X$), then it is independent of $B^*(y)$ that (5) takes its maximum, thus we should take $B^*(y) = 0$. Together we get that the conclusion is correct by Definition 4.2.

Second, take $\rightarrow_2 = R_{KD}$. Then (5) is equal to $[R_1(A(x), B(y))]' \vee (A^*(x))' \vee B^*(y)$ $(y \in Y)$. If $y \in E$, then (5) takes the maximum 1 iff $B^*(y) = 1$. If $y \in Y - E$ (i.e. $R_1(A(x), B(y)) \times A^*(x) \equiv 0, x \in X$), then $[R_1(A(x), B(y))]' \vee (A^*(x))' \equiv 1$ $(x \in X)$, and it is independent of $B^*(y)$ that (5) takes its maximum, thus we should take $B^*(y) = 0$. Together we get that the conclusion is correct by Definition 4.2. □

Proposition 4.5. *If $\rightarrow_2$ is R_M, then the MinP-solution $B^*(y) = \sup_{x\in X}\{A^*(x) \wedge R_1(A(x), B(y))\}$.*

Proof. Note that R_M satisfies (C1) and (C2), then it follows from Proposition 4.1 that $B^*(y)$ is MinP-solution. Formula (5) is equal to $R_1(A(x), B(y)) \wedge A^*(x) \wedge B^*(y)$, and it takes the maximum iff $B^*(y) \geq R_1(A(x), B(y)) \wedge A^*(x)$ $(y \in Y)$. By Definition 4.2, we know that the conclusion is correct. □

Similar to the induction process of previous propositions in the present paper and Algorithm 4.4.3 (the triple I method based on R_Z) in [17], we can prove Proposition 4.6.

Proposition 4.6. *If $\rightarrow_2$ is R_Z, then the MinP-solution $B^*(y) = \sup_{x\in E_y}\{A^*(x) \wedge R_1(A(x), B(y))\}$ where $E_y = \{x \in X \mid (A^*(x))' < R_1(A(x), B(y))\} \cap \{x \in X \mid R_1(A(x), B(y)) > 1/2\}$.*

Remark 4.5. By previous propositions, we can get triple I solutions where $R_1 = R_2 \in \{R_Y, R_{La}, R_R, R_{KD}, R_M, R_Z\}$. Hou and Li et al. gave triple I solutions w.r.t. R_{La}, R_R, R_{KD} in [14], which were all $B^*(y) = 1$. Subsequently, in [33], Li pointed out that the triple I solution w.r.t. R_{La} was $B^*(y) = \begin{cases} 1, & y \in E \\ 0, & y \in Y - E \end{cases}$ where $E = \{y \in Y \mid \sup_{x\in X}\{R(A(x), B(y)) \times A^*(x)\} > 0\}$ (let $R = R_1 = R_2$), which was different from related conclusion in [14]. As for this problem, by Definition 4.2 and correlative definitions and conclusions from other literatures, we find that the latter is correct. Further, it is found that there are similar cases in triple I solutions w.r.t. R_R, R_{KD}, R_Y, thus triple I solutions from Proposition 4.4 revise the related conclusions in [14].

4.4. InfP-solutions

Proposition 4.7. *If $\rightarrow_2$ is R_{DP}, then the InfP-quasi-solution is* $B^*(y) = \begin{cases} 1, & y \in E \\ 0, & y \in Y - E \end{cases}$ *where* $E = \{y \in Y \mid (\exists x)(A^*(x) \wedge R_1(A(x), B(y)) = 1)\}$.

Proof. We shall prove that $B^*(y)$ satisfies (i) and (ii) in Theorem 4.1.

(i) Suppose that $C(y)$ is any fuzzy set in $\mathbb{E}$ and that y_0 any element of Y. Then $C(y_0)$ makes formula (5), i.e.,

$$(A(x) \rightarrow_1 B(y_0)) \rightarrow_{DP} (A^*(x) \rightarrow_{DP} C(y_0)) \tag{13}$$

take the maximum for $\forall x \in X$. If $y_0 \in Y - E$, then $B^*(y_0) = 0 \leq C(y_0)$. If $y_0 \in E$, then there exists $x_0 \in X$ such that $A^*(x_0) \wedge R_1(A(x_0), B(y_0)) = 1$ (i.e. $A^*(x_0) = R_1(A(x_0), B(y_0)) = 1$), which implies that (13) is equal to $1 \rightarrow_{DP} (1 \rightarrow_{DP} C(y_0)) = C(y_0)$, thus we should take $C(y_0) = 1$ in order to make (5) get its maximum. Hence $B^*(y_0) \leq C(y_0)$ holds for $\forall y_0 \in Y$, and then we have $B^*(y) \leq_F C(y)$ for $\forall C(y) \in \mathbb{E}$, i.e. $B^*(y)$ satisfies (i) in Theorem 4.1.

(ii) Let $D(y) = \begin{cases} 1, & y \in E \\ \varepsilon/2, & y \in Y - E \end{cases}$ for $\forall y_0 \in Y$ and $\forall \varepsilon > 0$, thus $D(y_0) < B^*(y_0) + \varepsilon$. We shall show that $D(y_0)$ makes formula (5), i.e.,

$$(A(x) \rightarrow_1 B(y_0)) \rightarrow_{DP} (A^*(x) \rightarrow_{DP} D(y_0)) \tag{14}$$

take its maximum for $\forall x \in X$. If $y_0 \in E$, then $D(y_0) \in \mathbb{E}$ and (14) gets its maximum 1. If $y_0 \in Y - E$, then $D(y_0) = \varepsilon/2$ and $A^*(x_0) \wedge R_1(A(x_0), B(y_0)) < 1$ for $\forall x_0 \in X$, and it is easy to validate that (14) gets its maximum 1. Together we get $D(y) \in \mathbb{E}$, thus $B^*(y)$ satisfies (ii) in Theorem 4.1.

It follows from Theorem 4.1 that $B^*(y)$ is the infimum of $\mathbb{E}$. We show that $B^*(y)$ is not the minimum, thus it is the InfP-quasi-solution. Indeed, if there exists y_0 such that $R_1(A(x), B(y_0)) = 1$ and $0 < A^*(x) < 1$ $(x \in X)$, then $y_0 \in Y - E$, thus $B^*(y_0) = 0$ and formula (5) is equal to $1 \rightarrow_{DP} (A^*(x) \rightarrow_{DP} 0) = (A^*(x))' < 1$. These imply that $B^*(y)$ cannot ensure that it makes (5) get its maximum 1 for $\forall x \in X$, $\forall y \in Y$. Thus $B^*(y) \notin \mathbb{E}$, and $B^*(y)$ is not the minimum of $\mathbb{E}$. $\square$

Proposition 4.8. *If $\rightarrow_2$ is R_{DP}, then the InfP-solution is* $B^*(y) = \begin{cases} 1, & y \in E_1 \\ 0, & y \in E_2 \\ \varepsilon(y), & else \end{cases}$ *where* $E_1 = \{y \in Y \mid (\exists x)(A^*(x) \wedge R_1(A(x), B(y)) = 1)\}$, $E_2 = \{y \in Y \mid R_1(A(x), B(y)) \vee A^*(x) < 1, x \in X\}$, *and* $\varepsilon(y) \in (0, 1)$ *is a very small positive number* $(y \in E)$.

Proof. If $y \in E_1$, then $B^*(y) = 1$ and $B^*(y)$ obviously makes (5), i.e.,

$$R_1(A(x_0), B(y)) \rightarrow_{DP} (A^*(x_0) \rightarrow_{DP} B^*(y)) \tag{15}$$

take its maximum for $\forall x_0 \in X$. If $y \in E_2$, then $B^*(y) = 0$ and $R_1(A(x_0), B(y)) \vee A^*(x_0) < 1$ for $\forall x_0 \in X$, hence $A^*(x_0) \rightarrow_{DP} B^*(y) \geq (A^*(x_0))' > 0$ and (15) takes the maximum 1. If $y \in Y - E_1 - E_2$ (i.e., $A^*(x) \wedge R_1(A(x), B(y)) < 1$ for $\forall x \in X$ and there exists $x \in X$ such that $R_1(A(x), B(y)) \vee A^*(x) = 1$), then it is easy to validate that (15) takes the maximum 1. Thus we have $B^*(y) \in \mathbb{E}$.

Further, we show that it cannot take $B^*(y) = 0$ when $y \in Y - E_1 - E_2$. In fact, if we take $B^*(y) = 0$ when $y \in Y - E_1 - E_2$, then $B^*(y)$ cannot ensure that it makes (15) get its maximum 1 because there ordinarily exists $x_0 \in X$ such that $R_1(A(x_0), B(y)) \vee A^*(x_0) = 1$ and $0 < A^*(x_0) \wedge R_1(A(x_0), B(y)) < 1$. It follows from Definition 4.2 that we know the conclusion is correct. $\square$

Remark 4.6. Note that $\{y \in Y \mid R_1(A(x), B(y)) = 0, A^*(x) = 1, x \in X\} \cup \{y \in Y \mid R_1(A(x), B(y)) = 1, A^*(x) = 0, x \in X\} \subset Y - E_1 - E_2$, thus $B^*(y) = 0$ and it also makes (5) take its maximum. But we can leave such extreme case out of account.

Similar to Propositions 4.7 and 4.8, we can prove Propositions 4.9 and 4.10.

Proposition 4.9. *If $\rightarrow_2$ is R_{14}, then the InfP-quasi-solution is $B^*(y) = 0$.*

Proposition 4.10. *If $\rightarrow_2$ is R_{14}, then the InfP-solution is* $B^*(y) = \begin{cases} \varepsilon(y), & y \in E \\ 0, & else \end{cases}$ *where* $E = \{y \mid \sup_{x \in X}\{R_1(A(x), B(y)) \wedge A^*(x)\} > 0\}$, *and* $\varepsilon(y) \in (0, 1)$ *is a very small positive number* $(y \in E)$.

Remark 4.7. As for the InfP-solution, a new solving idea is given in previous propositions. First, we get the unique InfP-quasi-solution. Second, it follows from Theorem 4.1 that we achieve the InfP-solution by making slight adjustment to the InfP-quasi-solution. Obviously, such a solving idea is different from the one of MinP-solutions or triple I solutions.

5. Logic basis of CRI method

Definition 5.1. If a mapping $\otimes : [0, 1]^2 \rightarrow [0, 1]$ is associative and commutative, and satisfies the conditions $1 \otimes a = a$ and that $a \leq b$ implies $a \otimes c \leq b \otimes c$ $(a, b, c \in [0, 1])$, then $\otimes$ is defined as a t-norm. If a t-norm $\otimes$ satisfies $a \otimes \vee\{x_i \mid i \in I\} = \vee\{a \otimes x_i \mid i \in I\}$ where $a, x_i \in [0, 1]$ and $I \neq \varnothing$ $(i \in I)$, then it is left-continuous.

Lemma 5.1 (*Wang [19]*). *Let $\otimes$ be a left-continuous t-norm, define $a \to b = \vee\{y \in [0,1] \mid a \otimes y \leq b\}$, then $(\to, \otimes)$ is a symmetrical residual pair, and* (i) $a \to b = 1$ *iff* $a \leq b$; (ii) $a \leq b \to c$ *iff* $b \leq a \to c$; (iii) $a \to (b \to c) = b \to (a \to c)$; (iv) $1 \to a = a$; (v) $\wedge\{a \to x_i \mid i \in I\} = a \to \wedge\{x_i \mid i \in I\}$; (vi) $\vee\{a_i \mid i \in I\} \to b = \wedge\{a_i \to b \mid i \in I\}$; (vii) *$a \to b$ is increasing w.r.t. b and decreasing w.r.t. a.*

Remark 5.1. In Lemma 5.1, we have $a \otimes b = b \otimes a$ (noting that $\otimes$ is a left-continuous t-norm). Thus, the condition of symmetrical residual pairs (i.e. $a \otimes b \leq c$ iff $a \leq b \to c$), is equivalent to the one of residual pairs (i.e. $a \otimes b \leq c$ iff $b \leq a \to c$). This means that $(\to, \otimes)$ in Lemma 5.1 is also a residual pair.

Definition 5.2. If $\to: [0,1]^2 \to [0,1]$ is a strongly residual operator and its residual mapping $\otimes$ is a left-continuous t-norm, then $\to$ is said to be a t-strongly residual operator.

Proposition 5.1. *The implication operator $\to$ gotten in Lemma 5.1 is a t-strongly residual operator.*

Proof. Since $\to$ satisfies (i), (v) and (vii) in Lemma 5.1, we have that $\to$ obviously satisfies (C1), (C4) and (C7). Thus it is easy to get that $\to$ satisfies (C2) and (C3) by Proposition 3.3, hence $\to$ is a strongly residual operator. It follows from Lemma 3.1 that $\otimes_{\to}$ gotten by (6) in Theorem 3.1 is the same as $\otimes$ in Lemma 5.1. Thus we achieve that $\to$ is a t-strongly residual operator by Definition 5.2. □

It is easy to prove Proposition 5.2 by Lemma 3.1.

Proposition 5.2. *Suppose that, $(\to_1, \otimes_1)$ is a residual pair gotten by Lemma 5.1 where $\otimes_1$ is a left-continuous t-norm, and $(\to_2, \otimes_2)$ is a residual pair gotten by Theorem 3.1 where $\to_2$ is a t-strongly residual operator, then $\to_1 = \to_2$ iff $\otimes_1 = \otimes_2$.*

The CRI method is subsequently generalized to the following form

$$B^*(y) = \sup_{x \in X}\{A^*(x) \otimes R_1(A(x), B(y))\}, \quad y \in Y \tag{16}$$

where $\otimes$ is a t-norm (see [9,34]). Wang pointed out the fact that the CRI method has no reasonable interpretation in theory for a long time (see [34]). Aiming at this problem, from the point of view of universal triple I method, we shall give a interpretation for the CRI method for the case that $\otimes$ is a left-continuous t-norm. In Proposition 3.2, it is easy to verify that $\otimes_G, \otimes_{Go}, \otimes_L, \otimes_0$ are t-norms and that others are not.

Lemma 5.2 (*Wang [19]*). *$\otimes_G, \otimes_{Go}, \otimes_L, \otimes_0$ are all left-continuous t-norms, and their residual implication operations are respectively R_G, R_{Go}, R_L, R_0.*

Theorem 5.1. *Suppose that, the CRI solution is expressed as formula (16) where $\otimes$ is a left-continuous t-norm, and R_1 is the implication operation residual to $\otimes$, then the CRI solution is the MinP-solution where $R_2 = R_1$.*

Proof. Let R_2 be R_1, then it follows from Proposition 5.1 that R_2 is a strongly residual operator. Since $(R_2, \otimes)$ is a residual pair, by Theorem 4.2, we have that the MinP-solution is $B^*(y) = \sup_{x \in X}\{A^*(x) \otimes R_1(A(x), B(y))\}$ $(y \in Y)$, which is the same as the CRI solution. □

From Theorems 5.1 and 4.3 and Proposition 4.5, we can obtain Corollary 5.1.

Corollary 5.1. (i) *If $\otimes = \otimes_G$ in (16), then the CRI solution is $B^*(y) = \sup_{x \in X}\{A^*(x) \wedge R_1(A(x), B(y))\}$, and it is the MinP-solution where $R_2 \in \{R_G, R_M\}$.*
(ii) *If $\otimes = \otimes_{Go}$ in (16), then the CRI solution is $B^*(y) = \sup_{x \in X}\{A^*(x) \times R_1(A(x), B(y))\}$, and it is the MinP-solution where $R_2 = R_{Go}$.*
(iii) *If $\otimes = \otimes_L$ in (16), then the CRI solution is $B^*(y) = \sup_{x \in E_y}\{A^*(x) + R_1(A(x), B(y)) - 1\}$ where $E_y = \{x \in X \mid (A^*(x))' < R_1(A(x), B(y))\}$, and it is the MinP-solution where $R_2 = R_L$.*
(iv) *If $\otimes = \otimes_0$ in (16), then the CRI solution is $B^*(y) = \sup_{x \in E_y}\{A^*(x) \wedge R_1(A(x), B(y))\}$ where $E_y = \{x \in X \mid (A^*(x))' < R_1(A(x), B(y))\}$, and it is the MinP-solution where $R_2 = R_0$.*

In [11], Wang mentioned several t-norms, such as the t-norm of Dubois–Prade defined as $a \otimes_{dp-\beta} b = ab/\max(a, b, \beta)$ $(\beta \in [0,1])$, the t-norm of Yager defined as $a \otimes_{y-\omega} b = 1 - \min[1, ((1-a)^\omega + (1-b)^\omega)^{1/\omega}]$ $(\omega \in (0, \infty))$, and Einstein product defined as $a \otimes_{ep} b = ab/[2-(a+b-ab)]$. Obviously, $\otimes_{dp-\beta} = \otimes_{Go}$ where $\beta = 1$; and $\otimes_{dp-\beta} = \otimes_G$ where $\beta = 0$. For $\otimes_{y-\omega}$, if $\omega = 1$, then $\otimes_{y-\omega} = \otimes_L$; if $\omega = 0.5$, then $a \otimes_{y-0.5} b = \begin{cases} 1 - (g(a,b))^2, & g(a,b) \leq 1 \\ 0, & g(a,b) > 1 \end{cases}$ where $g(a,b) = \sqrt{1-a} + \sqrt{1-b}$.

Proposition 5.3. *$\otimes_{dp-\beta}, \otimes_{ep}, \otimes_{y-0.5}$ are all left-continuous t-norms, and their residual implication operations are respectively $R_{dp-\beta}, R_{ep}, R_{y-0.5}$ where $R_{dp-1} = R_{Go}$, $R_{dp-0} = R_G$, $R_{dp-\beta}(a,b) = \begin{cases} 1, & a \leq b \\ b\beta/a, & \beta \geq a > b \\ b, & a > b, a > \beta \end{cases}$ $(\beta \in (0,1))$, $R_{ep}(a,b) = \begin{cases} 1, & a \leq b \\ (2b-ab)/(a+b-ab), & a > b \end{cases}$, $R_{y-0.5}(a,b) = \begin{cases} 1, & a \leq b \\ 1 - (\sqrt{1-b} - \sqrt{1-a})^2, & a > b \end{cases}$.*

Proof. We only prove the case of $\otimes_{dp-\beta}$ as an example. If $\beta = 0$, then $\otimes_{dp-0} = \otimes_G$. If $\beta = 1$, then $\otimes_{dp-1} = \otimes_{Go}$. Thus it follows from Lemma 5.2 that $\otimes_{dp-\beta}$ is left-continuous where $\beta \in \{0, 1\}$.

If $\beta \in (0, 1)$, then we have three cases to be considered:

(a) Suppose $a > \beta$. Thus $a \otimes_{dp-\beta} b = a \wedge b$, and hence

$$a \otimes_{dp-\beta} \vee\{x_i \mid i \in I\} = a \wedge (\vee\{x_i \mid i \in I\}) = \vee\{a \wedge x_i \mid i \in I\} = \vee\{a \otimes_{dp-\beta} x_i \mid i \in I\}.$$

(b) Suppose $a \le \beta$ and $x_i \le \beta$ for $\forall i \in I$. Thus

$$a \otimes_{dp-\beta} \vee\{x_i \mid i \in I\} = a \times (\vee\{x_i \mid i \in I\})/\beta = \vee\{a \times x_i/\beta \mid i \in I\} = \vee\{a \otimes_{dp-\beta} x_i \mid i \in I\}.$$

(c) Suppose $a \le \beta$ and $(\exists i \in I)(x_i > \beta)$. Let $J = \{j \in I \mid x_j > \beta\}$, and hence $\vee\{x_i \mid i \in I\} = \vee\{x_j \mid j \in J\}$ holds. Considering $\otimes_{dp-\beta}$ is a t-norm, we obtain $\vee\{a \otimes_{dp-\beta} x_j \mid j \in J\} = \vee\{a \otimes_{dp-\beta} x_i \mid i \in I\}$, thus

$$\begin{aligned} a \otimes_{dp-\beta} \vee\{x_i \mid i \in I\} &= a \otimes_{dp-\beta} \vee\{x_j \mid j \in J\} = a \wedge (\vee\{x_j \mid j \in J\}) = \vee\{a \wedge x_j \mid j \in J\} \\ &= \vee\{a \otimes_{dp-\beta} x_j \mid j \in J\} = \vee\{a \otimes_{dp-\beta} x_i \mid i \in I\}. \end{aligned}$$

Thus it follows from Definition 5.1 that $\otimes_{dp-\beta}$ is a left-continuous t-norm.

Further, we shall prove that the implication operator residual to $\otimes_{dp-\beta}$ is $R_{dp-\beta}$. Since $a \otimes_{dp-0} b = a \wedge b = a \otimes_G b$ and $a \otimes_{dp-1} b = ab = a \otimes_{Go} b$, then the implication operators residual to $\otimes_{dp-0}, \otimes_{dp-1}$ are respectively $R_G \triangleq R_{dp-0}$, $R_{Go} \triangleq R_{dp-1}$ by Propositions 3.2 and 5.2.

If $\beta \in (0, 1)$, then $a \otimes_{dp-\beta} b = ab/\max(a, b, \beta) = \begin{cases} ab/\beta, & a \vee b \le \beta \\ a, & a \le b,\ a \vee b > \beta \\ b, & a > b,\ a \vee b > \beta \end{cases}$. Thus it follows from Lemma 5.1 that

$$\begin{aligned} R_{dp-\beta}(a, b) &= \vee\{y \in [0, 1] \mid a \otimes_{dp-\beta} y \le b\} \\ &= \vee(\{y \in [0, 1], a \vee y \le \beta \mid a \otimes_{dp-\beta} y \le b\} \cup \{y \in [0, 1], a > y, a \vee y > \beta \mid a \otimes_{dp-\beta} y \le b\} \\ &\quad \cup \{y \in [0, 1], a \le y, a \vee y > \beta \mid a \otimes_{dp-\beta} y \le b\}) \\ &= (\vee\{y \in [0, 1] \mid a \vee y \le \beta, ay/\beta \le b\}) \vee (\vee\{y \in [0, 1] \mid a > y, a \vee y > \beta, y \le b\}) \\ &\quad \vee (\vee\{y \in [0, 1] \mid a \le y, a \vee y > \beta, a \le b\}). \end{aligned}$$

If $a \le b$, then $\vee\{y \in [0, 1] \mid a \le y, a \vee y > \beta, a \le b\} = 1$, and hence $R_{dp-\beta}(a, b) = 1$. If $\beta \ge a > b$, then $R_{dp-\beta}(a, b) = (\vee\{y \in [0, 1] \mid y \le b\beta/a, y \le \beta\}) \vee (\vee\varnothing) \vee (\vee\varnothing) = b\beta/a$. If $a > b, a > \beta$, then $R_{dp-\beta}(a, b) = (\vee\varnothing) \vee (\vee\{y \in [0, 1] \mid y \le b\}) \vee (\vee\varnothing) = b$. Together we get that the implication operator residual to $\otimes_{dp-\beta}$ is $R_{dp-\beta}$. □

Thus the implication operators considered in the present paper are extended from 16 kinds to 19 kinds. Theorem 4.2 is also suitable for $R_{ep}, R_{dp-\beta}, R_{y-0.5}$. Obviously, $R_{ep}, R_{dp-\beta}, R_{y-0.5}$ are all regular implication operators. It follows from Theorem 5.1 and Proposition 5.3 that we can obtain Proposition 5.4.

Proposition 5.4. (i) *If $\otimes = \otimes_{ep}$ in (16), then the CRI solution is $B^*(y) = \sup_{x \in X}\{A^*(x) \otimes_{ep} R_1(A(x), B(y))\}$ $(y \in Y)$, and it is the MinP-solution where $R_2 = R_{ep}$.*

(ii) *If $\otimes = \otimes_{dp-\beta}$ in (16), then the CRI solution is $B^*(y) = \sup_{x \in X}\{A^*(x) \otimes_{dp-\beta} R_1(A(x), B(y))\}$ $(y \in Y)$, and it is the MinP-solution where $R_2 = R_{dp-\beta}$.*

(iii) *If $\otimes = \otimes_{y-0.5}$ in (16), then the CRI solution is $B^*(y) = \sup_{x \in X}\{A^*(x) \otimes_{y-0.5} R_1(A(x), B(y))\}$ $(y \in Y)$, and it is the MinP-solution where $R_2 = R_{y-0.5}$.*

6. Fuzzy systems constructed by FMP-universal triple I method and their response functions

It is evident that formula (1) is suitable for the case of one rule in FMP problem. If there are n rules, then (1) should be changed into:

FMP: for n given rules $A_i \to B_i$ and input A^*, to compute B^* (output).

The inference relation of rule $A_i \to B_i$ can be regarded as a fuzzy relation from X to Y $(i = 1, \ldots, n)$, denoting by $A_i(x) \to_1 B_i(y)$ where implication operator $\to_1$ is previously chosen, and the whole reference rule should be $R_1(x, y) \triangleq \vee_{i=1}^{n}(A_i(x) \to_1 B_i(y))$ (see [14–16]). Thus formula (5), i.e. $R_1(A(x), B(y)) \to_2 (A^*(x) \to_2 B^*(y))$, should be changed into:

$$R_1(x, y) \to_2 (A^*(x) \to_2 B^*(y)). \tag{17}$$

Proposition 6.1. *Suppose that $\to_2 \in \{R_G, R_{Go}, R_M, R_0, R_L, R_Z, R_{La}, R_R, R_Y, R_{KD}, R_{DP}, R_{GR}, R_{13}, R_{14}, R_{15}, R_{16}, R_{ep}, R_{dp-\beta}, R_{y-0.5}\}$, and the FMP-solution derived from formula (5) is $\varphi(R_1(A(x), B(y)))$, then the FMP-solution derived from formula (17) is $\varphi(R_1(x, y))$.*

Proof. Let $\to_2 \in \{R_G, R_{Go}, R_M, R_0, R_L, R_Z, R_{La}, R_R, R_Y, R_{KD}, R_{DP}, R_{GR}, R_{13}, R_{14}, R_{15}, R_{16}, R_{ep}, R_{dp-\beta}, R_{y-0.5}\}$. By foregoing proving process and conclusions of FMP-solutions, we have that the process of getting solutions derived from (5) regards $R_1(A(x), B(y))$ (or written as $A(x) \to_1 B(y)$) as a single whole. It is obvious that there is $R_1(x, y)$ instead of $R_1(A(x), B(y))$

in (17) comparing with (5), which is the unique difference. If we replace every $R_1(A(x), B(y))$ with $R_1(x, y)$ in the solving process derived from (5), then we achieve the one derived from (17), thus it is easy to get that the conclusion is correct. □

The process of fuzzy reasoning from the fuzzy set $A^*(x)$ to $B^*(y)$ has been given in the present paper. However, from the point of view of a whole fuzzy system, fuzzier and defuzzier should also be considered. The methods in common use are the singleton fuzzier and centroid defuzzier (see [10,14–16]). First, transform x^* into a singleton $A^*(x) = \begin{cases} 1, & x = x^* \\ 0, & x \neq x^* \end{cases}$. Second, carry through fuzzy reasoning via the FMP-universal triple I method to get fuzzy set $B^*(y)$. Lastly, use centroid defuzzier to achieve:

$$y^* = \int_Y yB^*(y)\mathrm{d}y \Big/ \int_Y B^*(y)\mathrm{d}y.$$

Thus, there is output $y^* = F(x^*)$ for each input x^*. Then a whole single-input single-output (SISO) fuzzy system is constructed where $y = F(x)$ is said to be the response function of this fuzzy system.

But the centroid method makes no sense when $B^*(y) \equiv 0$. In [11], Wang made mention of several defuzziers including centroid defuzzier, center average defuzzier and defuzzier of average from the maximum. And the last one (i.e. defuzzier of average from the maximum), which takes

$$y^* = \int_{hgt(Y)} y\mathrm{d}y \Big/ \int_{hgt(Y)} \mathrm{d}y$$

where $hgt(B^*) = \{y \in Y \mid B^*(y) = \sup_{y\in Y} B^*(y)\}$, is partly similar to the centroid defuzzier. Thus, we mainly adopts the centroid defuzzier, but utilizes the defuzzier of average from the maximum only if $B^*(y) \equiv 0$ (notice here $hgt(Y) = Y$) in the SISO fuzzy system in the present paper. Such a method (which uses two defuzziers) has already been proved to be effective in [16].

Definition 6.1. Let Z be any nonempty set and $\mathbb{C} = \{C_i\}_{(1\leq i\leq n)}$ a family of normal fuzzy sets on Z where the peak-point of C_i is z_i (i.e. the unique point satisfying $C_i(z_i) = 1$ in Z). $\mathbb{C}$ is called a fuzzy partition of Z if $(\forall z \in Z)\ \left(\sum_{i=1}^{n} C_i(z) = 1\right)$ holds, and C_i is defined as a base element in $\mathbb{C}$. Thus $\mathbb{C}$ is also said to be a group of base elements of Z.

Remark 6.1. Definition 6.1 obviously implies $(\forall i, j)\ (i \neq j \Rightarrow z_i \neq z_j)$ and that $\mathbb{C}$ has Kronecker property, i.e. $C_i(z_j) = \delta_{ij}$ where $\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$.

To analyze response functions of fuzzy systems, suppose that $\mathbb{A} = \{A_i\}_{(1\leq i\leq n)}$ and $\mathbb{B} = \{B_i\}_{(1\leq i\leq n)}$ are respectively fuzzy partitions of X and Y where A_i, B_i are integrable functions. We assume that X and Y are all real number intervals, e.g., $X = [a, b]$ and $Y = [c, d]$ in which $a < x_1 < x_2 < \cdots < x_n < b, c < y_1 < y_2 < \cdots < y_n < d$ where x_i, y_i are respectively peak-points of A_i, B_i. It is obvious that $\mathbb{A}$ and $\mathbb{B}$ have Kronecker property: $A_i(x_j) = \delta_{ij} = B_i(y_j)$ since they are all fuzzy partitions. Take $h_1 = y_1 - c, h_i = y_i - y_{i-1}$ $(i = 2, 3, \ldots, n)$ and $h = \max_{1\leq i\leq n}\{h_i\}$. From the definition of definite integral, for the centroid defuzzier, we obtain:

$$y^* = \int_Y yB^*(y)\mathrm{d}y \Big/ \int_Y B^*(y)\mathrm{d}y \approx \left[\sum_{i=1}^{n} y_i B^*(y_i) h_i\right] \Big/ \left[\sum_{i=1}^{n} B^*(y_i) h_i\right]. \tag{18}$$

Similarly, for the defuzzier of average from the maximum, we get:

$$y^* = \int_{hgt(Y)} y\mathrm{d}y \Big/ \int_{hgt(Y)} \mathrm{d}y \approx \sum_{i=1}^{n} y_i h_i \Big/ \sum_{i=1}^{n} h_i \triangleq c_0.$$

The response functions of some fuzzy systems constructed by the FMP-universal triple I method will be given in what follows.

Theorem 6.1. *Let* $\rightarrow_2 \in \{R_0, R_L, R_Z, R_{13}\}$.

(i) *If* $\rightarrow_1$ *satisfies* (C5), *then the SISO fuzzy system constructed by the FMP-universal triple* I *method (FMP-universal triple* I *system for short) is approximately a step response function (i.e.* $F(x) = c_0$*). Especially, if* $\rightarrow_1 \in \{R_L, R_G, R_{Go}, R_0, R_{GR}, R_{DP}, R_{KD}, R_R, R_Y, R_{13}, R_{15}, R_{16}, R_{ep}, R_{dp-\beta}, R_{y-0.5}\}$, *then we have the same conclusion.*

(ii) *If* $\rightarrow_1$ *satisfies* (C6), *then there are two cases to be considered:* (a) *Suppose* $x^* \in E_y$. *There exists a group of base functions* $\mathbb{A}^* = \{A_i^*\}_{(1\leq i\leq n)}$ *such that the FMP-universal triple* I *system is approximately a univariate piecewise interpolation function regarding* A_i^* *as its base functions (i.e.* $F(x) = \sum_{i=1}^{n} A_i^*(x) y_i$*), and* $\mathbb{A}^*$ *is a fuzzy partition on* X. *Especially, if* $\{y_i\}_{(1\leq i\leq n)}$ *is an equidistant partition, then* $\mathbb{A}^*$ *degenerates into* $\mathbb{A}$ *(i.e.* $F(x) = \sum_{i=1}^{n} A_i(x) y_i$*).* (b) *Suppose* $x^* \in X - E_y$. *The FMP-universal triple* I *system is approximately a step response function (i.e.* $F(x) = c_0$*).*

Especially, if $\rightarrow_1 \in \{R_M, R_{La}, R_{14}\}$, *then we have the same conclusion.*

(iii) *If* $\rightarrow_1$ *is* R_Z, *then there are two cases to be considered:* (a) *Suppose* $x^* \in E_y$. *There exists a group of base functions* $\mathbb{A}^* = \{A_i^*\}_{(1\leq i\leq n)}$ *such that the FMP-universal triple* I *system is approximately a univariate piecewise fitted function regarding* A_i^* *as its base functions (i.e.* $F(x) = \sum_{i=1}^{n} A_i^*(x) y_i$*).* (b) *Suppose* $x^* \in X - E_y$. *The FMP-universal triple* I *system is approximately a step response function (i.e.* $F(x) = c_0$*).*

Proof. We only prove the case of $\to_2 = R_0$ as an example. It follows from Theorem 4.3 and Proposition 6.1 that the MinP-solution can be expressed as $B^*(y) = \sup_{x\in E_y}\{A^*(x) \wedge R_1(x, y)\}$ where $E_y = \{x \in X \mid (A^*(x))' < R_1(x, y)\}$ and $R_1(x, y) = \bigvee_{i=1}^{n} R_1(A_i(x), B_i(y))$. As for input x^*, we get a singleton $A^*_{x^*} \triangleq A^*(x) = \begin{cases} 1, & x = x^* \\ 0, & x \neq x^* \end{cases}$.

(i) If $\to_1$ satisfies (C5), then we have two cases to be considered.

(a) Suppose $x^* \in E_y$. By the structure of E_y, we have $E_y = \{x^*\}$ and then the MinP-solution can be expressed as $B^*(y) = R_1(x^*, y) = \bigvee_{i=1}^{n} R_1(A_i(x^*), B_i(y))$. Since $B_k(y_i) = \delta_{ki}$, it follows from (18) that

$$y^* \approx \frac{\sum_{i=1}^{n} y_i B^*(y_i) h_i}{\sum_{i=1}^{n} B^*(y_i) h_i} = \frac{\sum_{i=1}^{n} h_i [\bigvee_{k=1}^{n} R_1(A_k(x^*), B_k(y_i))] y_i}{\sum_{i=1}^{n} h_i [\bigvee_{k=1}^{n} R_1(A_k(x^*), B_k(y_i))]} = \frac{\sum_{i=1}^{n} h_i y_i}{\sum_{i=1}^{n} h_i} = c_0.$$

(b) Suppose $x^* \notin E_y$. We have $B^*(y) = 0$ and then the centroid method makes no sense, so we utilize the defuzzier of average from the maximum. Thus we obtain $y^* \approx c_0$ and the response function can be expressed as $F(x) = c_0$.

It follows from Propositions 3.1 and 5.3 and Lemma 5.1 that $R_L, R_G, R_{Go}, R_0, R_{GR}, R_{DP}, R_{KD}, R_R, R_Y, R_{13}, R_{15}, R_{16}, R_{ep}, R_{dp-\beta}$, $R_{y-0.5}$ satisfy (C5), thus we have the same conclusion.

(ii) If $\to_1$ satisfies (C6), then we have two cases to be considered.

(a) Suppose $x^* \in E_y$. It is similar to get $E_y = \{x^*\}$ and $B^*(y) = \bigvee_{i=1}^{n} R_1(A_i(x^*), B_i(y))$. Since $B_k(y_i) = \delta_{ki}$, it follows from (18) that

$$y^* \approx \frac{\sum_{i=1}^{n} y_i B^*(y_i) h_i}{\sum_{i=1}^{n} B^*(y_i) h_i} = \frac{\sum_{i=1}^{n} h_i [\bigvee_{k=1}^{n} R_1(A_k(x^*), B_k(y_i))] y_i}{\sum_{i=1}^{n} h_i [\bigvee_{k=1}^{n} R_1(A_k(x^*), B_k(y_i))]} = \frac{\sum_{i=1}^{n} h_i A_i(x^*) y_i}{\sum_{i=1}^{n} h_i A_i(x^*)}.$$

Denote $A_i^*(x^*) \triangleq h_i A_i(x^*) / \left[\sum_{i=1}^{n} h_i A_i(x^*)\right]$, then we have $y^* = \sum_{i=1}^{n} A_i^*(x^*) y_i$. Let $\mathbb{A}^* \triangleq \{A_i^*\}_{(1\le i\le n)}$, $F(x) \triangleq \sum_{i=1}^{n} A_i^*(x) y_i$. Considering $A_k(x_i) = \delta_{ki}$ $(i, k = 1, \ldots, n)$, it follows

$$F(x_i) = \sum_{k=1}^{n} A_k^*(x_i) y_k = \sum_{k=1}^{n} h_k A_k(x_i) y_k \Big/ \sum_{k=1}^{n} h_k A_k(x_i) = y_i \quad (i = 1, \ldots, n),$$

then $F(x)$ is a univariate piecewise interpolation function which regards A_i^* as its base functions. Furthermore, $\sum_{i=1}^{n} A_i^*(x) = \sum_{i=1}^{n} \left[h_i A_i(x) / \left(\sum_{i=1}^{n} h_i A_i(x)\right)\right] = 1$ holds for $\forall x \in X$, thus $\mathbb{A}^*$ is a fuzzy partition on X. At last, if $\{y_i\}_{(1\le i\le n)}$ is an equidistant partition (i.e. $(\forall i)(h_i = h)$), then it is evident that $A_i^* = A_i$, $\mathbb{A}^* = \mathbb{A}$, and hence $F(x) = \sum_{i=1}^{n} A_i(x) y_i$.

(b) Suppose $x^* \in X - E_y$. We can similarly obtain $y^* \approx c_0$. Consequently, the conclusion is correct.

It follows from Proposition 3.1 that R_M, R_{La}, R_{14} satisfy (C6), thus we have the same conclusion.

(iii) If $\to_1 = R_Z$, then we have two cases to be considered.

(a) Suppose $x^* \in E_y$. Similarly we have $E_y = \{x^*\}$ and $B^*(y) = R_1(x^*, y) = \bigvee_{i=1}^{n} R_Z(A_i(x^*), B_i(y))$. Since $B_k(y_i) = \delta_{ki}$, it follows from (18) that

$$y^* \approx \frac{\sum_{i=1}^{n} y_i B^*(y_i) h_i}{\sum_{i=1}^{n} B^*(y_i) h_i} = \frac{\sum_{i=1}^{n} h_i [\bigvee_{k=1}^{n} ((1 - A_k(x^*)) \bigvee (A_k(x^*) \wedge B_k(y_i)))] y_i}{\sum_{i=1}^{n} h_i [\bigvee_{k=1}^{n} ((1 - A_k(x^*)) \bigvee (A_k(x^*) \wedge B_k(y_i)))]}$$

$$= \frac{\sum_{i=1}^{n} h_i [A_i(x^*) \bigvee (\bigvee_{k=1}^{n} (1 - A_k(x^*)))] y_i}{\sum_{i=1}^{n} h_i [A_i(x^*) \bigvee (\bigvee_{k=1}^{n} (1 - A_k(x^*)))]}.$$

Denote $C_i(x^*) \triangleq A_i(x^*) \vee (\bigvee_{k=1}^{n} (1 - A_k(x^*)))$ and $A_i^*(x^*) \triangleq h_i C_i(x^*) / \sum_{i=1}^{n} h_i C_i(x^*)$, then $y^* \approx \sum_{i=1}^{n} A_i^*(x^*) y_i$. Let $\mathbb{A}^* \triangleq \{A_i^*\}_{(1\le i\le n)}$ and $F(x) \triangleq \sum_{i=1}^{n} A_i^*(x) y_i$. Since $A_k(x_i) = \delta_{ki}$, we obtain:

$$F(x_i) = \frac{\sum_{j=1}^{n} h_j [A_j(x_i) \vee (\bigvee_{k=1}^{n} (1 - A_k(x_i)))] y_j}{\sum_{j=1}^{n} h_j [A_j(x_i) \vee (\bigvee_{k=1}^{n} (1 - A_k(x_i)))]} = \frac{\sum_{j=1}^{n} h_j y_j}{\sum_{j=1}^{n} h_j} = c_0 \quad (i = 1, \ldots, n).$$

Obviously, it cannot make $F(x_i) = y_i$ hold for any i, thus $F(x)$ is a univariate piecewise fitted function which regards A_i^* as its base functions.

(b) Suppose $x^* \in X - E_y$. We can similarly obtain $y^* \approx c_0$, thus the conclusion is correct. □

Remark 6.2. From Theorem 6.1, if $\rightarrow_1 = \rightarrow_2 \in \{R_0, R_L, R_{13}\}$ (which is the case of the triple I method), then response function is a step response function, which can hardly be used in practical fuzzy system. However, when we let $\rightarrow_1$ be other implication operator (e.g. $\rightarrow_1 \in \{R_M, R_{La}, R_{14}, R_Z\}$), the corresponding fuzzy systems can be used and then their capabilities are greatly improved. Thus the FMP-universal triple I method provides some ideal choices which cannot be offered by the triple I method of FMP. This implies that the FMP-universal triple I method is more excellent.

Theorem 6.2. *Let $\rightarrow_2 \in \{R_G, R_{Go}, R_M\}$.*

(i) *If $\rightarrow_1$ satisfies* (C5)*, then the conclusion is the same as Theorem* 6.1(i).

(ii) *If $\rightarrow_1$ satisfies* (C6)*, then there exists a group of base functions $\mathbb{A}^* = \{A_i^*\}_{(1\le i\le n)}$ such that the FMP-universal triple* I *system is approximately a univariate piecewise interpolation function taking A_i^* as its base functions (i.e. $F(x) = \sum_{i=1}^n A_i^*(x)y_i$), and $\mathbb{A}^*$ is a fuzzy partition on X; especially, if $\{y_i\}_{(1\le i\le n)}$ is an equidistant partition, then $\mathbb{A}^*$ degenerates into $\mathbb{A}$ (i.e. $F(x) = \sum_{i=1}^n A_i(x)y_i$). Especially, if $\rightarrow_1 \in \{R_M, R_{La}, R_{14}\}$, then we have the same conclusion.*

(iii) *If $\rightarrow_1$ is R_Z, then there exists a group of base functions $\mathbb{A}^* = \{A_i^*\}_{(1\le i\le n)}$ such that the FMP-universal triple* I *system is approximately a univariate piecewise fitted function taking A_i^* as its base functions (i.e. $F(x) = \sum_{i=1}^n A_i^*(x)y_i$).*

Proof. We only prove the case of $\rightarrow_2 = R_G$ as an example. It follows from Theorem 4.3 and Proposition 6.1 that the MinP-solution can be expressed as $B^*(y) = \sup_{x\in X}\{A^*(x) \wedge R_1(x, y)\}$ where $R_1(x, y) = \bigvee_{i=1}^n R_1(A_i(x), B_i(y))$. As for input x^*, we get a singleton $A^*_{x^*}$. Thus $B^*(y) = R_1(x^*, y) = \bigvee_{i=1}^n R_1(A_i(x^*), B_i(y))$.

(i) Suppose that $\rightarrow_1$ satisfies (C5). Since $B_k(y_i) = \delta_{ki}$, it follows from (18) that

$$y^* \approx \frac{\sum_{i=1}^n y_i B^*(y_i)h_i}{\sum_{i=1}^n B^*(y_i)h_i} = \frac{\sum_{i=1}^n h_i[\bigvee_{k=1}^n R_1(A_k(x^*), B_k(y_i))]y_i}{\sum_{i=1}^n h_i[\bigvee_{k=1}^n R_1(A_k(x^*), B_k(y_i))]} = \frac{\sum_{i=1}^n h_i y_i}{\sum_{i=1}^n h_i} = c_0.$$

Therefore the response function can be expressed as $F(x) = c_0$. Similar to Theorem 6.1(i), we can get the same conclusion for the case of $\rightarrow_1 \in \{R_L, R_G, R_{Go}, R_0, R_{GR}, R_{DP}, R_{KD}, R_R, R_Y, R_{13}, R_{15}, R_{16}, R_{ep}, R_{dp-\beta}, R_{y-0.5}\}$.

(ii) Suppose that $\rightarrow_1$ satisfies (C6). It is similar to Theorem 6.1(ii) (when $x^* \in E_y$) that we can prove it. Moreover, it follows from Proposition 3.1 that R_M, R_{La}, R_{14} satisfy (C6), thus we have the same conclusion.

(iii) Suppose that $\rightarrow_1 = R_Z$. It is similar to Theorem 6.1(iii) (when $x^* \in E_y$) that we can prove it. □

Theorem 6.3. *Let $\rightarrow_2 \in \{R_{La}, R_Y, R_R, R_{KD}, R_{GR}, R_{15}, R_{16}\}$, then the FMP-universal triple* I *system is approximately a step response function (i.e. $F(x) = c_0$).*

Proof. Suppose that $\rightarrow_2 \in \{R_{La}, R_Y, R_R, R_{KD}\}$. We only prove the case of $\rightarrow_2 = R_{La}$ as an example. It follows from Propositions 4.4 and 6.1 that the MinP-solution can be expressed as $B^*(y) = \begin{cases} 1, & y \in E \\ 0, & y \in Y - E \end{cases}$ where $E = \{y \in Y \mid \sup_{x\in X}\{R_1(x, y) \times A^*(x)\} > 0\}$ and $R_1(x, y) = \bigvee_{i=1}^n R_1(A_i(x), B_i(y))$. As for input x^*, we get a singleton $A^*_{x^*}$. We have two cases to be considered.

(a) Suppose $R_1(x^*, y) > 0$. Then we get $y \in E$ and $B^*(y) = 1$, therefore

$$y^* \approx \frac{\sum_{i=1}^n y_i B^*(y_i)h_i}{\sum_{i=1}^n B^*(y_i)h_i} = \frac{\sum_{i=1}^n y_i h_i}{\sum_{i=1}^n h_i} = c_0.$$

(b) Suppose $R_1(x^*, y) = 0$. Then we get $y \in Y - E$ and $B^*(y) = 0$, thus the centroid method makes no sense and we utilize the defuzzier of average from the maximum, then we obtain $y^* \approx c_0$. Therefore the response function can be expressed as $F(x) = c_0$.

Suppose that $\rightarrow_2 \in \{R_{GR}, R_{15}, R_{16}\}$. We only prove the case of $\rightarrow_2 = R_{16}$ as an example. The MinP-solution can be expressed as $B^*(y) = \sup_{x\in E_y}\{A^*(x)\}$ where $E_y = \{x \in X \mid (A^*(x))' < R_1(x, y)\}$. For the case of $x^* \in E_y$, we have $E_y = \{x^*\}$ and $B^*(y) = 1$. For the case of $x^* \notin E_y$, we have $B^*(y) = 0$. It is similar to Theorem 6.1(i) that we can obtain the response function which is expressed as $F(x) = c_0$. □

When $\rightarrow_2 \in \{R_G, R_M, R_{Go}, R_L, R_0, R_{ep}, R_{dp-\beta}, R_{y-0.5}\}$, the FMP-universal triple I method degenerates into the CRI method (by Corollary 5.1 and Proposition 5.4). Thus, in Theorems 6.1 and 6.2, we have already given the conclusions of the CRI method corresponding to $\rightarrow_2 \in \{R_G, R_M, R_{Go}, R_L, R_0\}$. When $\rightarrow_1 = \rightarrow_2$, the FMP-universal triple I method degenerates into the triple I method for the FMP problem, and we can easily achieve the following corollary.

Corollary 6.1. *Let $\rightarrow_1 = \rightarrow_2 \triangleq \rightarrow$ (i.e., aiming at the case of the triple* I *method).*

(i) *Suppose that $\rightarrow \in \{R_0, R_L, R_{13}, R_G, R_{Go}, R_{La}, R_R, R_Y, R_{KD}, R_{GR}, R_{15}, R_{16}\}$, then the FMP-universal triple* I *system is approximately a step response function.*

(ii) *Suppose that* $\rightarrow$ *is* R_M*, then there exists a group of base functions* $\mathbb{A}^* = \{A_i^*\}_{(1\le i\le n)}$ *such that the FMP-universal triple* I *system is approximately a univariate piecewise interpolation function which takes* A_i^* *as its base functions.*

(iii) *Suppose that* $\rightarrow$ *is* R_Z*, then there are two cases to be considered:* (a) *Suppose* $x^* \in E_y$*, there exists* $\mathbb{A}^* = \{A_i^*\}_{(1\le i\le n)}$ *such that the FMP-universal triple* I *system is approximately a univariate piecewise fitted function which regards* A_i^* *as its base functions.* (b) *Suppose* $x^* \in X - E_y$*, the FMP-universal triple* I *system is approximately a step response function.*

Remark 6.3. In [14], Li provided response functions of the triple I methods based on some implication operators, which uniformly used the centroid method, and took complementary definition when $B^*(y) \equiv 0$. For example, let $\rightarrow$ be R_Z. When $x^* \notin E_y$, $B^*(y) \equiv 0$ and Li took $B^*(y) = \sup\{A^*(x) \wedge R_Z(x, y)\}$, and then got y^* by the centroid method in [14]. It is obvious that such treating method violates the fact that $B^*(y)$ is a constant. However, in the present paper, we utilize the defuzzier of average from the maximum and achieve $y^* \equiv c_0$, thus it is evident that this treating method in the present paper is more reasonable.

The response function where $\rightarrow_2 \in \{R_0, R_L, R_Z, R_{13}, R_G, R_{Go}, R_M, R_{La}, R_R, R_Y, R_{KD}, R_{GR}, R_{15}, R_{16}\}$ has been given. But it becomes complicated for the case of $\rightarrow_2 \in \{R_{DP}, R_{14}, R_{dp-\beta}, R_{ep}, R_{y-0.5}\}$, so we do not investigate it here. Especially, when $\rightarrow_2 \in \{R_{DP}, R_{14}\}$, it is corresponding to the InfP-solution whose properties (including response ability) are also interesting research topics, which will be analyzed in other paper in the future.

We shall summarize the FMP-universal triple I method.

By previous theorems, such FMP-universal triple I systems can be divided into 3 kinds. (i) The FMP-universal triple I system is approximately an interpolation function. Thus it can be universal approximator and then usable in practice, such as the FMP-universal triple I system where $(\rightarrow_1, \rightarrow_2) \in \{R_M, R_{La}, R_{14}\} \times \{R_0, R_L, R_Z, R_{13}, R_G, R_{Go}, R_M\}$ (which may demand $x^* \in E_y$). (ii) The FMP-universal triple I system is approximately a fitted function. Hence it may be usable, such as the FMP-universal triple I system where $(\rightarrow_1, \rightarrow_2) \in \{R_Z\} \times \{R_0, R_L, R_Z, R_{13}, R_G, R_{Go}, R_M\}$ (which may demand $x^* \in E_y$). (iii) The FMP-universal triple I system is approximately a step response function. Thus it only has step response ability, therefore it can hardly be used in practice, such as the FMP-universal triple I system where $\rightarrow_2 \in \{R_{La}, R_R, R_Y, R_{KD}, R_{GR}, R_{15}, R_{16}\}$ or $(\rightarrow_1, \rightarrow_2) \in \{R_L, R_G, R_{Go}, R_0, R_{GR}, R_{DP}, R_{KD}, R_R, R_Y, R_{13}, R_{15}, R_{16}, R_{ep}, R_{dp-\beta}, R_{y-0.5}\} \times \{R_0, R_L, R_Z, R_{13}, R_G, R_{Go}, R_M\}$.

Considering related CRI method and triple I method, it is similar to get corresponding conclusions. The CRI system (i.e. the SISO fuzzy system constructed by the CRI method) where $(\rightarrow_1, \rightarrow_2) \in \{R_M, R_{La}, R_{14}, R_Z\} \times \{R_G, R_{Go}, R_M, R_0, R_L\}$ can be practicable. Meanwhile, the triple I system (i.e. the SISO fuzzy system constructed by the triple I method) where $\rightarrow \in \{R_M, R_Z\}$ can be practicable, however the triple I system where $\rightarrow \in \{R_0, R_L, R_{13}, R_G, R_{Go}, R_{La}, R_R, R_Y, R_{KD}, R_{GR}, R_{15}, R_{16}\}$ can hardly be used.

It is readily seen that the FMP-universal triple I method provides some effective choices which cannot be offered by the CRI method and triple I method of FMP. Therefore, the FMP-universal triple I method has bigger choosing space, and there are more useful FMP-universal triple I systems than the CRI systems and triple I systems. In practical application, such as design of fuzzy controllers, we can get better and more usable fuzzy controllers (e.g. having universal approximation, stability) and fuzzy reasoning strategies. Consequently, from such point of view, the FMP-universal triple I method is superior.

Meanwhile, the FMP-universal triple I method has close relationship with the triple I method and CRI method. Thus it provides an important idea to investigate the latter two methods, which is analyzed from the characteristic and relationship of $\rightarrow_1$ and $\rightarrow_2$. So the research of universal triple I method will help to obtain the essence of triple I method and CRI method, and to achieve the contacts and differences between them.

Further, we shall analyze the significance of generalization from the triple I method to the universal triple I method. The triple I method is proposed for formula (4) (i.e. $(A(x) \rightarrow B(y)) \rightarrow (A^*(x) \rightarrow B^*(y))$). But it follows from Theorem 6.3 that the FMP-solution and its response function are totally determined by the second and third implication operator in formula (4) (corresponding to $\rightarrow_2$ in (5)) if $\rightarrow_2 \in \{R_{La}, R_Y, R_R, R_{KD}, R_{GR}, R_{15}, R_{16}\}$. And, by Theorems 6.1 and 6.2, if $\rightarrow_2 \in \{R_0, R_L, R_Z, R_{13}, R_G, R_{Go}, R_M\}$, then the response function is unitedly determined by $\rightarrow_1$ and $\rightarrow_2$. So it is reasonable to let the first implication operator take $\rightarrow_1$, and the second and third operators take $\rightarrow_2$. Thus, it has clear basis to generalize (4) to (5) (i.e. $(A(x) \rightarrow_1 B(y)) \rightarrow_2 (A^*(x) \rightarrow_2 B^*(y))$). Moreover, from the previous conclusions, in the basically same range, there are only two usable fuzzy systems via the triple I method, and meanwhile there are 28 usable fuzzy systems via the universal triple I method. As a result, the overladen request to keep $\rightarrow_1 = \rightarrow_2$ holds back the development of the triple I method to a certain extent. Summarizing the above, it has clear theoretical value and practical meaning to generalize the triple I method to the universal triple I method.

7. FMT-universal triple I method

In this section, we shall focus on the FMT problem expressed as (2).

Definition 7.1. Suppose that $A \in F(X)$, $B, B^* \in F(Y)$, nonempty set $\mathbb{F}$ is the set of $A^*(x)$ which makes (5) get its maximum for any $x \in X$ and $y \in Y$ in $\langle F(X), \le_F\rangle$, $C^*(x)$ is the supremum of $\mathbb{F}$. If $C^*(x)$ is the maximum of $\mathbb{F}$, then $C^*(x)$ is called a MaxT-solution. If $C^*(x)$ is not the maximum of $\mathbb{F}$, then $C^*(x)$ is called a SupT-quasi-solution; in $\mathbb{F}$, pick out a fuzzy set $C^{**}(x)$ as big as possible, and call $C^{**}(x)$ a SupT-solution.

Let FMT-universal triple I solution (FMT-solution for short) be a general designation of MaxT-solution and SupT-solution.

It is similar to Theorem 4.1 that we can prove Theorem 7.1.

Theorem 7.1. *There exists a unique fuzzy set $C^*(x)$ such that*

(i) $C(x) \leq_F C^*(x)$ *for* $\forall C(x) \in \mathbb{F}$, *and*
(ii) *there is* $D(x) \in \mathbb{F}$ *satisfying* $D(x_0) > C^*(x_0) - \varepsilon$ *for* $\forall x_0 \in X$ *and* $\forall \varepsilon > 0$;

then $C^*(x)$ is the supremum of $\mathbb{F}$. Specially, if $C^*(x) \in \mathbb{F}$ also holds, then $C^*(x)$ is a MaxT-solution.

Definition 7.2. A residual operator R is called a FMT-residual operator if it satisfies the following conditions:

(C9) $R(a, b)$ is decreasing w.r.t. a $(a, b \in [0, 1])$;
(C10) $R(a, b)$ is left-continuous w.r.t. a $(a \in (0, 1], b \in [0, 1])$.

Especially, if R also satisfies (C4), then R is said to be a FMT strongly residual operator.

It is similar to Proposition 4.1 that we can prove Proposition 7.1.

Proposition 7.1. *If R_2 is a FMT-residual operator, then the FMT-solution $A^*(x)$ is the MaxT-solution, and the maximum of (5) is $N(x, y) = (A(x) \to_1 B(y)) \to_2 (0 \to_2 B^*(y))$.*

Corollary 7.1. *If R_2 is a FMT strongly residual operator, then the FMT-solution $A^*(x)$ is the MaxT-solution, and the maximum of (5) is $N(x, y) = 1$.*

Definition 7.3. We say that an implication operator R has contrapositive symmetry if it satisfies

(C11) $R(a, b) = R(b', a')$ $(a, b \in [0, 1])$.

If a residual operator R has contrapositive symmetry, then define R as a symmetrical residual operator. If a symmetrical residual operator R is also a strongly residual operator, then call R a strongly symmetrical residual operator.

Similar to Theorem 4.2, we can prove Theorem 7.2.

Theorem 7.2. *Suppose that, R_2 is a strongly symmetrical residual operator, and $\otimes$ is its residual mapping, then the MaxT-solution can be expressed as $A^*(x) = \inf_{y \in Y}\{B^*(y) \oplus (R_1(A(x), B(y)))'\}$ $(x \in X)$ where $a \oplus b = (a' \otimes b')'$ $(a, b \in [0, 1])$.*

Lemma 7.1. *$R_L, R_0, R_{GR}, R_{DP}, R_{KD}, R_R, R_{13}$ satisfy* (C11) *in the considered* 19 *implication operators.*

Corollary 7.2. *Suppose that, $R_2 \in \{R_L, R_0, R_{GR}, R_{13}\}$, and $\otimes$ is the residual operation w.r.t. R_2, then the MaxT-solution can be expressed as $A^*(x) = \inf_{y \in Y}\{B^*(y) \oplus (R_1(A(x), B(y)))'\}$, $x \in X$.*

Theorem 7.3. *Suppose that $\to_2$ satisfies* (C4) *and*

(C12) $a \leq R(b, c)$ *iff* $b \leq R(a, c)$ $(a, b, c \in [0, 1])$,

then MaxT-solution can be expressed as $A^*(x) = \inf_{y \in Y}\{R_1(A(x), B(y)) \to_2 B^*(y)\}$, $x \in X$.

Proof. First, we shall prove:

$$(A(x) \to_1 B(y)) \to_2 (A^*(x) \to_2 B^*(y)) = 1, \quad x \in X, y \in Y. \tag{19}$$

In fact, it follows from the expression of $A^*(x)$ that $A^*(x) \leq R_1(A(x), B(y)) \to_2 B^*(y)$, $x \in X$, $y \in Y$. Since $\to_2$ satisfies (C12), we have $R_1(A(x), B(y)) \leq A^*(x) \to_2 B^*(y)$. Thus formula (19) holds (noting that $\to_2$ satisfies (C4)).

Second, we shall show that $A^*(x)$ is the maximum. Let $C(x) \in \mathbb{F}$, then $(A(x) \to_1 B(y)) \to_2 (C(x) \to_2 B^*(y)) = N(x, y) = 1$ $(x \in X, y \in Y)$ since $\to_2$ satisfies (C4) and (C12). And we get $R_1(A(x), B(y)) \leq C(x) \to_2 B^*(y)$ and then $C(x) \leq R_1(A(x), B(y)) \to_2 B^*(y)$. Therefore $C(x)$ is a lower bound of

$$\{R_1(A(x), B(y)) \to_2 B^*(y) \mid y \in Y\}, \quad x \in X.$$

Thus, it follows from the expression of $A^*(x)$ that $C(x) \leq A^*(x)$ $(x \in X)$.

Together we obtain that $A^*(x)$ is the MaxT-solution by Definition 7.1. □

Lemma 7.2. *$R_L, R_G, R_{Go}, R_0, R_{dp-\beta}, R_{ep}, R_{y-0.5}$ satisfy* (C12) *in the considered* 19 *implication operators.*

Corollary 7.3. *If $\to_2 \in \{R_L, R_G, R_{Go}, R_0, R_{dp-\beta}, R_{ep}, R_{y-0.5}\}$, then MaxT-solution can be expressed as $A^*(x) = \inf_{y \in Y}\{R_1(A(x), B(y)) \to_2 B^*(y)\}$, $x \in X$.*

Definition 7.4. If a regular implication operator R satisfies (C11), then R is said to be normal.

From Proposition 4.3, we can get Lemma 7.3. If $R_1 = R_2$, then the MaxT-solution degenerates into the triple I solution of FMT, and we have Corollary 7.4 from Theorem 7.2.

Lemma 7.3. *Normal implication operators are all strongly symmetrical residual operators.*

Corollary 7.4. *Suppose that, R_2 is a strongly symmetrical residual operator, and $\otimes$ its residual mapping, and take $R_1 = R_2 \triangleq R$, then the MaxT-solution (i.e. the triple* I *solution of FMT) can be expressed as $A^*(x) = \inf_{y\in Y}\{B^*(y) \oplus (R(A(x), B(y)))'\}$ where $a \oplus b = (a' \otimes b')'$ $(a, b \in [0, 1], x \in X)$.*

By Lemma 7.3, it follows from Corollary 7.4 that we can get Corollary 7.5. From Theorem 7.3, we can achieve Corollary 7.6.

Corollary 7.5. *Suppose that, R_2 is a normal implication operator, and $\otimes$ its residual mapping, and take $R_1 = R_2 \triangleq R$, then the MaxT-solution (i.e. the triple* I *solution of FMT) can be expressed as $A^*(x) = \inf_{y\in Y}\{B^*(y) \oplus (R(A(x), B(y)))'\}$ where $a \oplus b = (a' \otimes b')'$ $(a, b \in [0, 1], x \in X)$.*

Corollary 7.6. *Suppose that, R_2 satisfies* (C4) *and* (C12)*, and take $R_1 = R_2 \triangleq R$, then the MaxT-solution (i.e. the triple* I *solution of FMT) can be expressed as $A^*(x) = \inf_{y\in Y}\{R(R(A(x), B(y)), B^*(y))\}, x \in X$.*

Remark 7.1. Corollary 7.5 is the same as Theorem 2 in Ref. [19], while Corollary 7.5 is a special case of Corollary 7.4 which can be applicable for more implication operators. The form of triple I solution in Corollary 7.6 is the same as Corollary 3.3 in Ref. [31], however the latter needs the conditions of (C4), (C12), (C7) and (C8). Thus Corollary 3.3 in Ref. [31] is a special case of Corollary 7.6 in the present paper.

8. α-universal triple I method

It is similar to the idea of universal triple I method, the α-triple I method can be generalized to the α-universal triple I method. We define the α-universal triple I method as the α-triple I method derived from

$$(A(x) \rightarrow_1 B(y)) \rightarrow_2 (A^*(x) \rightarrow_2 B^*(y)) \geq \alpha. \tag{20}$$

If the maximum of (5) is constantly 1 (e.g., when $\rightarrow_2$ is a strongly residual operator), then the universal triple I method can be regarded as the α-universal triple I method where $\alpha = 1$, so it is a special case of the α-universal triple I method. But, if the maximum of (5) is not constantly a number (e.g., when $\rightarrow_2 \in \{R_Z, R_M\}$), then the universal triple I method has no direct relationship with the α-universal triple I method.

The α-FMP-solution will be defined in what follows, and the definition of α-FMT-solution (including α-MaxT-solution, α-SupT-quasi solution, α-SupT-solution and $\mathbb{F}_\alpha$) can be achieved similarly. For convenience, assume that $\mathbb{E}_\alpha$ and $\mathbb{F}_\alpha$ are nonempty sets.

Definition 8.1. Suppose that $A, A^* \in F(X), B \in F(Y)$, nonempty set $\mathbb{E}_\alpha$ is the set of $B^*(y)$ which makes (20) hold for any $x \in X$ and $y \in Y$ in $\langle F(Y), \leq_F\rangle$, $D^*(y)$ is the infimum of $\mathbb{E}_\alpha$. If $D^*(y)$ is the minimum of $\mathbb{E}_\alpha$, then $D^*(y)$ is called an α-MinP-solution. If $D^*(y)$ is not the minimum of $\mathbb{E}_\alpha$, then $D^*(y)$ is called an α-InfP-quasi solution; in $\mathbb{E}_\alpha$, pick out a fuzzy set $D^{**}(y)$ as small as possible, and call $D^{**}(y)$ an α-InfP-solution.

Let α-FMP-universal triple I solution (α-FMP-solution for short) be a general designation of α-MinP-solution and α-InfP-solution.

It is similar to Theorem 4.1 that we can prove Theorems 8.1 and 8.2.

Theorem 8.1. *There exists a unique fuzzy set $D^*(y)$ such that*

(i) *$D^*(y) \leq_F D(y)$ for $\forall D(y) \in \mathbb{E}_\alpha$, and*
(ii) *there is $C(y) \in \mathbb{E}_\alpha$ satisfying $C(y_0) < D^*(y_0) + \varepsilon$ for $\forall y_0 \in Y$ and $\forall \varepsilon > 0$;*

then $D^*(y)$ is the infimum of $\mathbb{E}_\alpha$. Specially, if $D^*(y) \in \mathbb{E}_\alpha$ also holds, then $D^*(y)$ is an α-MinP-solution.

Theorem 8.2. *There exists a unique fuzzy set $C^*(x)$ such that*

(i) *$C(x) \leq_F C^*(x)$ for $\forall C(x) \in \mathbb{F}_\alpha$, and*
(ii) *there is $D(x) \in \mathbb{F}_\alpha$ satisfying $D(x_0) > C^*(x_0) - \varepsilon$ for $\forall x_0 \in X$ and $\forall \varepsilon > 0$;*

then $C^*(x)$ is the supremum of $\mathbb{F}_\alpha$. Specially, if $C^*(x) \in \mathbb{F}_\alpha$ also holds, then $C^*(x)$ is an α-MaxT-solution.

Theorem 8.3. *Suppose that, $\rightarrow_2$ is a residual operator, and $\otimes$ its residual mapping, then the α-MinP-solution can be expressed as*

$$B^*(y) = \sup_{x\in X}\{A^*(x) \otimes [R_1(A(x), B(y)) \otimes \alpha]\}, \quad y \in Y. \tag{21}$$

Specially, if $\rightarrow_2 \in \{R_G, R_L, R_0, R_{Go}, R_{GR}, R_{KD}, R_R, R_Y, R_{13}, R_{15}, R_{16}, R_{dp-\beta}, R_{ep}, R_{y-0.5}\}$, then we have the same conclusion.

Proof. First, we shall prove:

$$(A(x) \rightarrow_1 B(y)) \rightarrow_2 (A^*(x) \rightarrow_2 B^*(y)) \geq \alpha, \quad x \in X, \ y \in Y. \tag{22}$$

In fact, it follows from the expression of $B^*(y)$ that $A^*(x) \otimes [R_1(A(x), B(y)) \otimes \alpha] \leq B^*(y), x \in X, y \in Y$. Since $(\rightarrow_2, \otimes)$ is a residual pair, we obtain: $R_1(A(x), B(y)) \otimes \alpha \leq A^*(x) \rightarrow_2 B^*(y)$ and then $\alpha \leq R_1(A(x), B(y)) \rightarrow_2 (A^*(x) \rightarrow_2 B^*(y))$, i.e. (22) holds. Thus $B^*(y) \in \mathbb{E}_\alpha$.

Second, we shall show that $B^*(y)$ is the minimum. Let $D(y) \in \mathbb{E}_\alpha$, then $(A(x) \to_1 B(y)) \to_2 (A^*(x) \to_2 D(y)) \geq \alpha$ $(x \in X, y \in Y)$. Since $(\to_2, \otimes)$ is a residual pair, we obtain: $R_1(A(x), B(y)) \otimes \alpha \leq A^*(x) \to_2 D(y)$ and then $A^*(x) \otimes [R_1(A(x), B(y)) \otimes \alpha] \leq D(y)$ $(x \in X, y \in Y)$. Hence, $D(y)$ is an upper bound of

$$\{A^*(x) \otimes [R_1(A(x), B(y)) \otimes \alpha] \mid x \in X\}, \quad y \in Y.$$

Thus it follows from (21) that $B^*(y) \leq_F D(y)$. These imply that $B^*(y)$ is the minimum of $\mathbb{E}_\alpha$.

Together we get that $B^*(y)$ is the α-MinP-solution by Definition 8.1. Especially, if $\to_2 \in \{R_G, R_L, R_0, R_{Go}, R_{GR}, R_{KD}, R_R, R_Y, R_{13}, R_{15}, R_{16}, R_{dp-\beta}, R_{ep}, R_{y-0.5}\}$, then $\to_2$ is a residual operator, thus we have the same conclusion. $\square$

If $R_1 = R_2$, then the α-MinP-solution degenerates into the α-triple I solution of FMP, and we have Corollary 8.1 from Theorem 8.3.

Corollary 8.1. *Suppose that, R_2 is a residual operator, and $\otimes$ its residual mapping, and take $R_1 = R_2 \triangleq R$, then the α-MinP-solution (i.e. the α-triple* I *solution of FMP) can be expressed as $B^*(y) = \sup_{x \in X}\{A^*(x) \otimes [R(A(x), B(y)) \otimes \alpha]\}$, $y \in Y$.*

By Proposition 4.3 and Lemma 3.4, it follows from Corollary 8.1 that we can get Corollaries 8.2 and 8.3.

Corollary 8.2. *Suppose that, R_2 is a regular implication operator, and $\otimes$ its residual mapping, and take $R_1 = R_2 \triangleq R$, then the α-MinP-solution (i.e. the α-triple* I *solution of FMP) can be expressed as $B^*(y) = \sup_{x \in X}\{A^*(x) \otimes R(A(x), B(y)) \otimes \alpha\}$, $y \in Y$.*

Corollary 8.3. *Suppose that, R_2 satisfies* (C7) *and* (C8), *and $\otimes$ its residual mapping, and take $R_1 = R_2 \triangleq R$, then the α-MinP-solution (i.e. the α-triple* I *solution of FMP) can be expressed as $B^*(y) = \sup_{x \in X}\{A^*(x) \otimes [R(A(x), B(y)) \otimes \alpha]\}$, $y \in Y$.*

Remark 8.1. Corollary 8.2 is the same as Theorem 3 in Ref. [19] (noting that $\otimes$ is associative, commutative). However Corollary 8.2 is a special case of Corollary 8.1 which can be applicable for more implication operators.

Remark 8.2. In [31], Liu investigated the triple I method based on pointwise sustaining degrees (i.e. the triple I method from $(A(x) \to B(y)) \to (A^*(x) \to B^*(y)) \geq \alpha(x, y)$). By Theorem 3.1 of [31], he pointed out the fact that the $\alpha(x, y)$-triple I solution of FMP can be expressed as $B^*(y) = \sup_{x \in X}\{[\alpha(x, y) \otimes_2 R(A(x), B(y))] \otimes_2 A^*(x)\}$ where $(\to, \otimes_2)$ is a symmetrical residual pair if R satisfies (C7) and (C8). When $\alpha(x, y)$ degenerates into α, the α-triple I solution of FMP is $B^*(y) = \sup_{x \in X}\{[\alpha \otimes_2 R(A(x), B(y))] \otimes_2 A^*(x)\}$. By Proposition 3.4 in the present paper, it follows that $B^*(y) = \sup_{x \in X}\{A^*(x) \otimes [R(A(x), B(y)) \otimes \alpha]\}$ where $(\to, \otimes)$ is a residual pair. Thus the α-triple I solution of FMP from Theorem 3.1 of [31] is equivalent to Corollary 8.3 in the present paper.

It is similar to Theorem 8.3 that we can prove Theorem 8.4.

Theorem 8.4. *Suppose that, $\to_2$ is a symmetrical residual operator, and $\otimes$ its residual mapping, then the α-MaxT-solution can be expressed as $A^*(x) = \inf_{y \in Y}\{[(B^*(y))' \otimes (R_1(A(x), B(y)) \otimes \alpha)]'\}$, $x \in X$. Specially, if $\to_2 \in \{R_L, R_0, R_{GR}, R_{KD}, R_R, R_{13}\}$, then we have the same conclusion.*

Theorem 8.5. *Suppose that, $\to_2$ is a residual operator satisfying* (C12), *and $\otimes$ its residual mapping, then the α-MaxT-solution can be expressed as $A^*(x) = \inf_{y \in Y}\{[R_1(A(x), B(y)) \otimes \alpha] \to_2 B^*(y)\}$, $x \in X$. Specially, if $\to_2 \in \{R_L, R_G, R_{Go}, R_0, R_{dp-\beta}, R_{ep}, R_{y-0.5}\}$, then we have the same conclusion.*

Proof. First, we shall prove:

$$(A(x) \to_1 B(y)) \to_2 (A^*(x) \to_2 B^*(y)) \geq \alpha, \quad x \in X,\ y \in Y. \tag{23}$$

In fact, it follows from the expression of $A^*(x)$ that $A^*(x) \leq [R_1(A(x), B(y)) \otimes \alpha] \to_2 B^*(y)$, $x \in X, y \in Y$. Since $\to_2$ satisfies (C12) and $(\to_2, \otimes)$ is a residual pair, we have $R_1(A(x), B(y)) \otimes \alpha \leq A^*(x) \to_2 B^*(y)$ and (23) holds. Thus $A^*(x) \in \mathbb{F}_\alpha$.

Second, we shall show that $A^*(x)$ is the maximum. Let $C(x) \in \mathbb{F}_\alpha$, then $(A(x) \to_1 B(y)) \to_2 (C(x) \to_2 B^*(y)) \geq \alpha$ $(x \in X, y \in Y)$. Since $(\to_2, \otimes)$ is a residual pair and $\to_2$ satisfies (C12), we have $R_1(A(x), B(y)) \otimes \alpha \leq C(x) \to_2 B^*(y)$ and then $C(x) \leq [R_1(A(x), B(y)) \otimes \alpha] \to_2 B^*(y)$. Therefore $C(x)$ is a lower bound of

$$\{[R_1(A(x), B(y)) \otimes \alpha] \to_2 B^*(y) \mid y \in Y\}, \quad x \in X.$$

Thus it follows from the expression of $A^*(x)$ that $C(x) \leq_F A^*(x)$. These imply that $A^*(x)$ is the maximum of $\mathbb{F}_\alpha$.

Therefore $A^*(x)$ is the α-MaxT-solution. Especially, if $\to_2 \in \{R_L, R_G, R_{Go}, R_0, R_{dp-\beta}, R_{ep}, R_{y-0.5}\}$, then it is easy to know that $\to_2$ is a residual operator satisfying (C12), thus we have the same conclusion. $\square$

If $R_1 = R_2$, then the α-MaxT-solution degenerates into the α-triple I solution of FMT. It follows from Lemma 7.3, Theorem 8.4, Lemma 3.4 and Theorem 8.5 that we can get Corollaries 8.4 and 8.5.

Corollary 8.4. *Suppose that, R_2 is a normal implication operator, and $\otimes$ its residual mapping, and take $R_1 = R_2 \triangleq R$, then the α-MaxT-solution (i.e. the α-triple* I *solution of FMT) can be expressed as $A^*(x) = \inf_{y \in Y}\{[(B^*(y))' \otimes (R(A(x), B(y)) \otimes \alpha)]'\}$, $x \in X$.*

Corollary 8.5. *Suppose that, R_2 satisfies* (C7), (C8) *and* (C12), *and $\otimes$ is its residual mapping, and take $R_1 = R_2 \triangleq R$, then the α-MaxT-solution (i.e. the α-triple* I *solution of FMT) can be expressed as $A^*(x) = \inf_{y\in Y}\{R[R(A(x), B(y)) \otimes \alpha, B^*(y)]\}, x \in X$.*

Remark 8.3. In Theorem 4 of [19], Wang got the fact that if R is a normal implication operator, then the α-triple I solution of FMT is $A^*(x) = \inf_{y\in Y}\{\alpha' \oplus B^*(y) \oplus (R(A(x), B(y)))'\} = \inf_{y\in Y}\{[(B^*(y))' \otimes (R(A(x), B(y)) \otimes \alpha)]'\}$, which is the same as Corollary 8.4 in the present paper. By the way, we can get the similar result from Corollary 3.2 in [31].

Remark 8.4. In Theorem 3.2 of [31], Liu pointed out the fact that the $\alpha(x, y)$-triple I solution of FMT can be expressed as $A^*(x) = \inf_{y\in Y}\{R[\alpha(x, y) \otimes_2 R(A(x), B(y)), B^*(y)]\}$ where $(R, \otimes_2)$ is a symmetrical residual pair if R satisfies (C7), (C8) and (C12). Similar to Remark 8.2, we can achieve that the α-triple I solution of FMT from Theorem 3.2 of [31] is equivalent to Corollary 8.5 in the present paper.

9. Conclusions

In the present paper, the triple I method is generalized to the differently implicational universal triple I method of (1, 2, 2) type. The main contributions and conclusions are as follows.

(i) A new definition of residual operator is given, and then the definition and related results of residual pairs are provided.

(ii) The universal triple I method is investigated. The related universal triple I solutions (including FMP-solution, FMT-solution, α-FMP-solution, α-FMT-solution) are strictly defined by the infimum, where such solutions are divided into two parts respectively corresponding to the minimum and infimum. Moreover, we put emphasis on the FMP-solutions, where the unified forms of solutions w.r.t. strongly residual operators are achieved, and a new idea for getting InfP-solutions is put forward.

(iii) The logic basis of a sort of CRI method is studied. It is found that the CRI method is a special case of the universal triple I method where $\otimes$ is a left-continuous t-norm in (16). This makes progress in relationships among the universal triple I method, triple I method and CRI method.

(iv) We discuss the response functions of SISO fuzzy systems respectively constructed by the universal triple I method, triple I method and CRI method. It is found that the universal triple I method can provide bigger choosing space and get better fuzzy controllers. Further, we analyze the significance of generalization from the triple I method to the universal triple I method.

If formula (5) is further generalized, that is, three implication operators are chosen without any limitation, then formula (4) is changed into the following form

$$(A(x) \to_1 B(y)) \to_2 (A^*(x) \to_3 B^*(y)). \tag{24}$$

The triple I method derived from formula (24) is called differently implicational universal triple I method. It is obvious that the differently implicational universal triple I method of (1, 2, 2) type is its special case. The work concerning differently implicational universal triple I method will be a research emphasis in the future.

What is more, the problems related to the differently implicational universal triple I method of (1, 2, 2) type (more widely, the differently implicational universal triple I method), such as its reversibility, continuity, universal approximation, stability, constructing and design of reasonable fuzzy systems, will be involved step by step in the further research.

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