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# On the Reducibility of $\Pi_1^1$ Sets*

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## 1. INTRODUCTION

Although our purpose below is to study some rather technical, metarecursion-theoretic questions raised by G. Kreisel, it is possible to give a sample of our principal construction in the familiar language of descriptive set theory. Let $T$ be a member of $2^\omega$, and let $\mathscr{B}$ be a lightface $\Delta_1^1$ subset of $2^\omega$. We say *$T$ is generic with respect to $\mathscr{B}$* if there exists a hyperarithmetically encodable,[1] perfect closed subset $P$ of $2^\omega$ such that $T \in P$ and either $P \subseteq \mathscr{B}$ or $P \subseteq 2^\omega - \mathscr{B}$. Corollary 3.5 provides a $\Pi_1^1 T$ such that $T$ is generic with respect to every lightface $\Delta_1^1$ subset of $2^\omega$.

The proof of 3.5 combines the methods of forcing [1] and priority [2, 3] (cf. Hinman [4]). The priority method is an essential tool for the construction of nontrivial, recursively enumerable sets. Its use in 3.5 is appropriate for two reasons: $T$ is a $\Pi_1^1$ subset of $\omega$ if and only if $T$ is metarecursively enumerable; the forcing relation employed in it is metarecursively enumerable. Forcing with finite conditions will not suffice for the proof of this corollary. This last remark is made precise in 2.12, but for now consider the following intuitive argument. Suppose $T$ is Cohen-generic; i.e. $T$ is constructed by means of a forcing argument involving finite conditions and some ramified language $\mathscr{L}$ strong enough to define all the lightface $\Delta_1^1$ sets. Then $T$ will be generic with respect to all lightface $\Delta_1^1$ sets in the sense of generic defined above. $T$ will be infinite, since the collection of all finite sets is lightface $\Delta_1^1$ but contains no perfect closed subset. Suppose $T$ is $\Pi_1^1$; then $T$ must contain some infinite hyperarithmetic set $H$. The set of all $T$'s containing $H$ is lightface $\Delta_1^1$, so the language $\mathscr{L}$ must have the power to express the fact that $H$ is

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[1] A closed subset $P$ of $2^\omega$ is said to be hyperarithmetically encodable if the set of all finite initial segments of members of $P$ is hyperarithmetic.

a subset of $T$. Consequently, there must be some finite condition on $T$ which forces $H$ to be a subset of $T$. But no *finite* condition on $T$ can force a fixed *infinite* set to be a subset of $T$.

The second half of Section 2 describes the forcing relation needed in 3.5; the relation connects certain infinite, hyperarithmetic forcing conditions with sentences of recursive ordinal rank. The $\Pi_1^1$ set $A$ constructed in the proof of 3.5 is the range of a metarecursive function from recursive $\omega_1$ into $\omega$. Thus $A$ must be enumerated in a sequence of stages whose order-type is $\omega_1$. Since $A$ is a generic object, there must be an associated sequence of forcing conditions converging to $A$, whose order-type is recursive $\omega_1$ rather than the usual $\omega$. The fact that $A$ is built in $\omega_1$ stages rather than $\omega$ stages is the sole conceptual difficulty confronting the would-be reader of the proof of 3.5. This point is discussed further at the beginning of Section 3.

Many of the results of metarecursion theory, and perhaps the theory itself, have their origin in Kreisel's critique of a result of Spector [5]. Spector showed that all nonhyperarithmetic $\Pi_1^1$ sets occupied the same hyperdegree; he then interpreted his result as the solution of Post's problem for $\Pi_1^1$ sets. Kreisel didn't agree; he had an abundance of convincing arguments to support the view that $\Pi_1^1$, subsets of $\omega$ were analogous to recursively enumerable subsets of $\omega$, but he saw no good reason to think that hyperdegrees were analogous to Turing degrees. In particular, he did not accept the almost universally held contention of those early days that the hyperarithmetic subsets of $\omega$ were analogous to the recursive subsets of $\omega$; moreover, he had rational reasons for not accepting it. He conjectured that "incomparable" $\Pi_1^1$ sets did exist, and subsequent mathematical events proved him right.

It was shown in [6] that there exist two $\Pi_1^1$ subsets of $\omega$ such that neither is metarecursive in the other. It was announced in [6] and [7] that there exist two $\Pi_1^1$ sets such that neither is $\omega_1$-computable from the other; the proof is given in Section 3. In the light of 2.3, it appears that 3.15 is the strongest possible "incomparability" result for $\Pi_1^1$ sets.

Section 4 concludes the paper. It contains further results and open questions about $\Pi_1^1$ subsets of $\omega$ and countable admissible ordinals [9, 20].

## 2. Machinery

The first part of Section 2 is an attempt to make our paper as self-contained as possible without engaging in costly duplication. We repeat some but not all of the definitions of [8, 6, 7] and none of the proofs. We recommend [8] to the reader seeking motivation for the definitions, and [7] to his more pragmatic brother in need of intuitive descriptions of those basic constructions which have proved useful.

Let $\omega_1$ be the least nonrecursive ordinal. Let $\Phi$ be a function from $\omega_1$ into $\omega_1$; $\Phi$ is metarecursive if it can be computed according to the rules of Kripke's equation calculus [9]. His calculus is similar to Kleene's [10], but he adds an $\omega$-rule that makes infinite computations possible. For each $\alpha < \omega_1$, let $\bar{\alpha}$ be a numeral for $\alpha$. The primitive symbols of Kripke's calculus are: function letters $f, g, h, \ldots$; variables $x, y, z, \ldots$; numerals, successor ($'$), equality ($=$), and a bounded existential quantifier ($\mathrm{E}x<$). If $t, t_0, \ldots, t_{n-1}$ are terms, then variables, numerals, $t', f(t_0, \ldots, t_{n-1})$, and $(\mathrm{E}x < t_0)\, t_1(x)$ are terms. If $t_0$ and $t_1$ are terms, then $t_0 = t_1$ is an equation. The intended meaning of the equation

$$(\mathrm{E}x < t_0)\, t_1(x) = \bar{0}$$

is: There is an $x < t_0$ such that $t_1(x) = \bar{0}$. Similarly, the intended meaning of

$$(\mathrm{E}x < t_0)\, t_1(x) = \bar{1}$$

is: There is no $x < t_0$ such that $t_1(x) = \bar{0}$.

Kripke's first two computation rules are the standard substitution rules of Kleene [10]. His third rule has two parts:

(1) $(\mathrm{E}x < \bar{\beta})\, t(x) = \bar{0}$ is an immediate consequence of $t(\bar{\alpha}) = \bar{0}$, if $\alpha < \beta$;

(2) $(\mathrm{E}x < \bar{\beta})\, t(x) = \bar{1}$ is an immediate consequence of the set $\{t(\bar{\alpha}) = \bar{1} \mid \alpha < \beta\}$.

Let $E$ be a finite set of equations. Kripke defines $S^E$, the set of all equations metarecursively computable from $E$, by transfinite induction. $S_0^E = E$. If $\alpha$ is an arbitrary ordinal $>0$, then $S_\alpha^E$ is the set of all immediate consequences of members or subsets of $\bigcup \{S_\gamma^E \mid \gamma < \alpha\}$. $S^E = \bigcup \{S_\alpha^E \mid \alpha \geqslant 0\}$. Clearly, $S^E = \bigcup \{S_\alpha^E \mid \alpha \text{ countable}\}$, but more to the point, $S^E = \bigcup \{S_\alpha^E \mid \alpha \text{ recursive}\}$. This particular result of Kripke implies that each equation $c \in S^E$ can be computed via $E$ by means of a computation which, when put in standard tree-form, has recursive ordinal height. It is routine to encode such a computation as a set of natural numbers. If the encoding is done in a standard fashion, then each computation associated with $S^E$ will be encoded as a hyperarithmetic set, and then the set of all such computations can be regarded as a $\Pi_1^1$ set of indices of hyperarithmetic sets.

Let $\Phi$ be a function whose graph is a subset of $\omega_1^2$. We say $\Phi$ is partial metarecursive if there exists an $E$ with principal function letter $f$ such that

$$\Phi(\alpha) = \beta \leftrightarrow f(\bar{\alpha}) = \bar{\beta} \in S^E$$

for all $\alpha < \beta < \omega_1$. A set is *metarecursively enumerable* if it is the range of a partial metarecursive function. A function is *metarecursive* if it is partial metarecursive and its domain is all of $\omega_1$. A set $A$ is metarecursive if its characteristic function is metarecursive, or equivalently, if both $A$ and $\omega_1 - A$ are metarecursively enumerable. A set is *metafinite* if it is metarecursive and bounded (by some ordinal $< \omega_1$). There exists a metarecursive indexing of the metafinite sets. Namely, there exist metarecursive functions $j$ and $k$ such that for each metafinite $H$ there is a unique $\delta$ such that

$$H = \{\alpha \mid j(\delta, \alpha) = 0 \;\&\; \alpha < k(\delta)\};$$

if the above holds, then we write $H = K_\delta$.

The fundamental lemma of metarecursion theory is: If $f$ is a metarecursive function (hence total) and $K$ is a metafinite set, then the range of $f$ restricted to $K$ is a metafinite set. It follows that the union of "metafinitely many" metafinite sets is metafinite. More precisely, if $I$ is metafinite, then $\bigcup \{K_\delta \mid \delta \in I\}$ is metafinite.

PROPOSITION. 2.1 [8]. *Let $A \subseteq \omega$. $A$ is metafinite $\leftrightarrow$ $A$ is hyperarithmetic. $A$ is metarecursively enumerable $\leftrightarrow$ $A$* is $\Pi_1^1$.

Let $A$ and $B$ be sets of recursive ordinals. We consider several ways of reducing $A$ to $B$. $A$ is *weakly metarecursive in* $B$ ($A \leqslant_{\mathrm{w}} B$) if there are partial metarecursive functions $\Psi$ and $\Phi$ such that for all $\alpha$,

$$\alpha \in A \leftrightarrow (EH)(EK)[\Psi(H, K, \alpha) = 0 \;\&\; H \subseteq B \;\&\; K \subseteq \omega_1 - B],$$

$$\alpha \notin A \leftrightarrow (EH)(EK)[\Phi(H, K, \alpha) = 0 \;\&\; H \subseteq B \;\&\; K \subseteq \omega_1 - B],$$

where $H$ and $K$ are variables ranging over the metafinite sets. Driscoll [11] showed that $\leqslant_{\mathrm{w}}$ is not transitive!

Kripke's definition of $S^E$ can be relativized to $B \subseteq \omega_1$ as follows. Let $g$ be a function letter. Let

$$S_0^{E,B} = E \cup \{g(\bar{\delta}) = \bar{0} \mid \delta \in B\} \cup \{g(\bar{\delta}) = \bar{1} \mid \delta \notin B\};$$

if $\alpha$ is an arbitrary ordinal $> 0$, let $S_\alpha^{E,B}$ be the set of all immediate consequences of members or subsets of $\bigcup \{S_\gamma^{E,B} \mid \gamma < \alpha\}$. $S^{E,B} = \bigcup \{S_\alpha^{E,B} \mid \alpha \geqslant 0\}$. Clearly,

$$S^{E,B} = \bigcup \{S_\alpha^{E,B} \mid \alpha \text{ countable}\}.$$

More to the point, there is a least countable ordinal $\omega^B$ such that

$$S^{E,B} = \bigcup \{S_\alpha^{E,B} \mid \alpha < \omega^B\}.$$

If $B$ is metarecursive, then $\omega^B = \omega_1$. If $B \subseteq \omega$, then $\omega^B = \omega_1{}^B =$ the least ordinal not recursive in $B$. Thus if $B \subseteq \omega$ and $B$ is a nonhyperarithmetic $\Pi_1^1$ set, then $\omega^B > \omega_1$. On the other hand there does exist a nonmetarecursive, metarecursively enumerable $B$ such that $\omega^B = \omega_1$ [6].

Each equation $e \in S^{E,B}$ can be computed from $B$ via $E$ by means of a computation $c$ which, when put in standard tree-form, has ordinal height $< \omega^B$. If the standard encoding of $c$ as a subset of $\omega$ is hyperarithmetic, then we say $c$ is metafinite. If a computation is metafinite, then its ordinal height $< \omega_1$, but not conversely.

Let $S^{E,B,m}$ be the set of all members of $S^{E,B}$ computed from $B$ via $E$ by means of metafinite computations. $A$ is *metafinitely computable from* $B$ ($A \leqslant_{\mathrm{mc}} B$) if there exists an $E$ with principal function letter $f$ such that

$$\alpha \in A \leftrightarrow f(\bar{\alpha}) = \bar{0} \in S^{E,B,m},$$
$$\alpha \in A \leftrightarrow f(\bar{\alpha}) = \bar{1} \in S^{E,B,m},$$

for all $\alpha < \omega_1$.

THEOREM 2.2 [6]. $A \leqslant_{\mathrm{w}} B \leftrightarrow A \leqslant_{\mathrm{mc}} B$.

For each ordinal $\delta$, $A$ is *$\delta$-computable from $B$* ($A \leqslant_\delta B$) if there exists an $E$ with principal function letter $f$ such that

$$\alpha \in A \leftrightarrow f(\bar{\alpha}) = \bar{0} \in \bigcup \{S_\gamma^{E,B} \mid \gamma < \delta\},$$
$$\alpha \notin A \leftrightarrow f(\bar{\alpha}) = \bar{1} \in \bigcup \{S_\gamma^{E,B} \mid \gamma < \delta\},$$

for all $\alpha < \omega_1$. If $A \leqslant_{\mathrm{mc}} B$, then $A \leqslant_{\omega_1} B$, but not conversely. It can be shown that $\leqslant_{\omega_1}$ is not transitive.

PROPOSITION 2.3. *Let $A$ and $B$ be $\Pi_1^1$ subsets of $\omega$. If $B$ is not hyperarithmetic and $\delta > \omega_1$, then $A \leqslant_\delta B$.*

*Proof.* In the formalism of Kripke's equation calculus, rewrite Spector's proof [5] that any two nonhyperarithmetic $\Pi_1^1$ sets have the same hyperdegree. Utilize Kleene's result [12] that every $\Pi_1^1$ set is one-one reducible to $O$.

By 2.3 the strongest instance of "incomparability" possible for $\Pi_1^1$ subsets of $\omega$ is a pair of $\Pi_1^1$ sets such that neither is $\omega_1$-computable from the other. This possibility is realized in Section 3. One obvious source of difficulty in the study of $\omega_1$-computability is the presence of nonmetafinite computations of recursive ordinal height. The notion of $\omega_1$-subgenericity [6] enables us to avoid a direct confrontation with nonmetafinite computations. Let $B$ be a subset of $\omega_1$. $B$ is *$\omega_1$-subgeneric* if there exists a metarecursive function $t$ such that for every equation $e$ and every finite system of equations $E$,

$$e \in \bigcup \{S_\alpha^{E,B} \mid \alpha < \omega_1\} \leftrightarrow e \in \bigcup \{S_\alpha^{t(E),B,m} \mid \alpha < \omega_1\}.$$

PROPOSITION 2.4. *If $A \leqslant_{\omega_1} B$ and $B$ is $\omega_1$-subgeneric, then $A \leqslant_{\mathrm{w}} B$.*

THEOREM 2.5. *If $A \leqslant_{\omega_1} B$ and $B$ is $\omega_1$-subgeneric, then $A$ is $\omega_1$-subgeneric.*

2.4 is an immediate consequence of 2.2. Since we have no need of 2.5 in this paper, we save its proof for [13].

The remainder of Section 2 is an exposition of the forcing relation needed in Section 3 to construct an $\omega_1$-subgeneric, nonhyperarithmetic $\Pi_1^1$ subset of $\omega$.

We define a ramified analytic language $\mathscr{L}_{\omega_1}(\mathscr{T})$ similar to that employed by Feferman [14]. The primitive symbols of $\mathscr{L}_{\omega_1}(\mathscr{T})$ are: number variables $x, y, z, \ldots$; finite numerals $\bar{0}, \bar{1}, \bar{2}, \ldots$; unranked set variables $X, Y, Z, \ldots$; for each recursive ordinal $\alpha$, ranked set variables $X^\alpha, Y^\alpha, Z^\alpha, \ldots$; predicates $\epsilon$ and $=$; functions $+$, $\cdot$, and $'$ (successor); and a set constant $\mathscr{T}$. $\mathscr{F}$ is said to be a *ranked formula* if every set variable occurring in $\mathscr{F}$ is ranked; in that case the *ordinal rank* of $\mathscr{F}$ is the least ordinal $\gamma$ such that $\gamma > \beta$ for every quantifier $(\mathrm{E}X^\beta)$ occuring in $\mathscr{F}$ and such that $\gamma \geqslant \delta$ for every variable $X^\delta$ occurring freely in $\mathscr{F}$. A formula is existential if it is ranked or of it is of the form $(\mathrm{EX})\mathscr{F}$, where $\mathscr{F}$ is ranked.

Let $T$ be an arbitrary subset of $\omega$. We simultaneously inductively define the structure $\mathscr{M}_\alpha(T)$ and truth in the chain of structures $\bigcup\{\mathscr{M}_\beta(T) \mid \beta < \alpha\}$ for each recursive ordinal $\alpha$.

(A) Let $\mathscr{F}$ be a sentence of rank $\leqslant\alpha$. Then $\mathscr{F}$ is true in $\bigcup\{\mathscr{M}_\beta(T) \mid \beta < \alpha\}$ if $\mathscr{F}$ is true under the following interpretation: Each quantified set variable $X^\beta$ occurring in $\mathscr{F}$ is restricted to $\mathscr{M}_\beta(T)$; each number variable $x$ is restricted to $\omega$; $\mathscr{T}$ denotes $T$.

(B) Let $\mathscr{G}(x)$ be a formula of rank $\leqslant \alpha$ whose sole free variable is $x$. Let $G$ be the set of all $n$ such that $\mathscr{G}(\bar{n})$ is true in $\bigcup\{\mathscr{M}_\beta(T) \mid \beta < \alpha\}$. Then $\mathscr{M}_\alpha(T)$ is the set of all $G$'s thus obtained.

Let $\mathscr{M}(T) = \bigcup\{\mathscr{M}_\alpha(T) \mid \alpha < \omega_1\}$. Let $\mathscr{F}$ be an arbitrary sentence of $\mathscr{L}_{\omega_1}(\mathscr{T})$. Then $\mathscr{F}$ is true in $\mathscr{M}(T)$ ($\mathscr{M}(T) \models \mathscr{F}$) if $\mathscr{F}$ is true when each quantified, unranked set variable $X$ is restricted to $\mathscr{M}(T)$ and the remaining symbols of $\mathscr{F}$ are interpreted as in clause (A) above. It will be convenient to use an abstraction symbol $\hat{x}$ in the following standard fashion. Let $\mathscr{G}(x)$ be as in clause (B) above; then $\hat{x}\mathscr{G}(x)$ will denote the set $G$. If $t$ is a number theoretic term, then $t \in \hat{x}\mathscr{G}(x)$ is equivalent to $\mathscr{G}(t)$.

It is routine to assign recursive ordinals as Gödel numbers to the formulas of $\mathscr{L}_{\omega_1}(\mathscr{T})$ with the following consequences: the set of Gödel numbers of existential formulas is metarecursive; the predicate "$\alpha$ is the Gödel number of a formula of rank $\beta$" is metarecursive; for each $\beta$, the set of Gödel numbers of formulas of rank $<\beta$ is metafinite.

A coinfinite forcing condition $P$ is a pair $(P^+, P^-)$ of disjoint subsets of $\omega$ such that $\omega - (P^+ \cup P^-)$ is infinite. Coinfinite forcing conditions have been used by Silver [15] in the context of the set theory to define a new kind of generic set with interesting properties. $P$ is said to be metafinite if $P = (P^+, P^-)$ and $P^+$ and $P^-$ are metafinite. Metafinite $P$'s were employed in the proof of Theorem 2 of [6].

$P, Q, R, \ldots$ will denote metafinite, coinfinite forcing conditions.

$T \in P$ means that $P = (P^+, P^-)$, $P^+ \subseteq T$, and $P^- \subseteq \omega - T$.

$P \supseteq Q$ means $(T)(T \in Q \rightarrow T \in P)$.

We define a forcing relation, $P \Vdash \mathscr{F}$, where $\mathscr{F}$ is a sentence of $\mathscr{L}_{\omega_1}(\mathscr{T})$. The ranked sentences are treated as in [16]:

(1) $P \Vdash \mathscr{F}$ if $\mathscr{F}$ is ranked and $(T)[T \in P \rightarrow \mathscr{M}(T) \models \mathscr{F}]$.

(2) $P \Vdash (\mathrm{Ex})\, \mathscr{F}(x)$ if $\mathscr{F}(x)$ is unranked and $P \Vdash \mathscr{F}(\bar{n})$ for some $n$.

(3) $P \Vdash (EX^\alpha)\,\mathscr{F}(X^\alpha)$ if $\mathscr{F}(X^\alpha)$ is unranked and $P \Vdash \mathscr{F}(\hat{x}\mathscr{G}(x))$ for some $\mathscr{G}(x)$ of rank $\leqslant\alpha$ whose sole free variable is $x$.

(4) $P \Vdash (EX)\,\mathscr{F}(X)$ if $P \Vdash (EX^\alpha)\,\mathscr{F}(X^\alpha)$ for some $\alpha$.

(5) $P \Vdash \mathscr{F} \,\&\, \mathscr{G}$ if $\mathscr{F} \,\&\, \mathscr{G}$ is unranked, $P \Vdash \mathscr{F}$, and $P \Vdash \mathscr{G}$.

(6) $P \Vdash \sim\mathscr{F}$ if $\mathscr{F}$ is unranked and $(Q)_{P \supseteq Q} \sim [Q \Vdash \mathscr{F}]$.

LEMMA 2.6. *The relation $P \Vdash \mathscr{F}$, restricted to existential $\mathscr{F}$'s, is metarecursively enumerable.*

*Proof.* By clause (4) of the definition of $\Vdash$, it is sufficient to show $P \Vdash \mathscr{F}$, restricted to ranked $\mathscr{F}$'s, is metarecursively enumerable[2]. There exists a $\Pi_1^1$ set $I \subseteq \omega$ such that $\{P_i \mid i \in I\}$ is an enumeration of all $P$'s; in addition, there is an arithmetical formula $A(T, i)$ such that for all $T$ and all $i \in I$,

$$T \in P_i \leftrightarrow A(T, i).$$

Similarly, there exists a $\Pi_1^1$ set $J \subseteq \omega$ such that $\{\mathscr{F}_j \mid j \in J\}$ is an enumeration of all ranked $\mathscr{F}$'s; in addition, there is a $\Pi_1^1$ formula $K(T, j)$ such that for all $T$ and all $j \in J$,

$$[\mathscr{M}(T) \models \mathscr{F}_j] \leftrightarrow K(T, j).$$

The existence of $K$ follows from the fact that $\mathscr{M}(T) \models \mathscr{F}$ was defined for ranked $\mathscr{F}$ inductively by means of arithmetical closure conditions. Let $V(i, j)$ be the formula

$$(T)[A(T, i) \rightarrow K(T, j)].$$

Then for all $i \in I$ and $j \in J$, $P \Vdash \mathscr{F}_i$ if and only if $V(i, j)$. Since $V(i, j)$ is $\Pi_1^1$, the lemma follows from 2.1.

$P$ is said to weakly force $\mathscr{F}$ ($P \Vdash^* \mathscr{F}$) if $(Q)_{P \supseteq Q}(ER)_{Q \supseteq R}[R \Vdash \mathscr{F}]$.

LEMMA 2.7. *Let $\mathscr{F}$ be existential, let $P \Vdash^* \mathscr{F}$, and let $k$ be a finite subset of $\omega - (P^+ \cup P^-)$. Then there exists a $Q \subseteq P$ such that $Q \Vdash \mathscr{F}$ and $k$ is a finite subset of $\omega - (Q^+ \cup Q^-)$.*

*Proof.* For the sake of clarity assume $k = \{m\}$. Choose $Q_0$ so that $P \supseteq Q_0$, $m \in Q_0^+$, and $Q_0 \Vdash \mathscr{F}$. Clearly, $P \supseteq (Q_0^+ - \{m\}, Q_0^- \cup \{m\})$. Choose $Q_1 \subseteq (Q_0^+ - \{m\}, Q_0^- \cup \{m\})$ so that $Q_1 \Vdash \mathscr{F}$.

[2] Would a more intricate argument show that the relation $P \Vdash \mathscr{F}$, restricted to ranked $\mathscr{F}$'s, is metarecursive?

Let $Q^+ = Q_1^+$ and $Q^- = Q_1^- - \{m\}$. Clearly, $P \supseteq Q$, $m \notin Q^+ \cup Q^-$, and

$$T \in Q \rightarrow (T \in Q_0 \vee T \in Q_1)$$

for all $T$. Since $\mathscr{F}$ is existential, $Q \Vdash \mathscr{F}$. If $k$ has $n$ elements, the above construction is inflated so as to include definitions of $Q_i$ for all $i < 2^n$.

Lemma 2.8. *Let $\{\mathscr{F}_i \mid i < \omega\}$ be a metafinite sequence of existential sentences. Suppose*

$$(i)(Q)_{P \supseteq Q}(ER)_{Q \supseteq R}[R \Vdash \mathscr{F}_i].$$

*Then $(ER)_{P \supseteq R}(i)[R \Vdash \mathscr{F}_i]$.*

*Proof.* By 2.6 and 2.7 there exist partial metarecursive functions $R_n$ and $t(n)$ defined for all $n < \omega$ such that:

$$R_0 = P;$$
$$t(n) = \mu m[m \notin R_n^+ \cup R_n^- \ \&\ (i)_{i<n}(m > t(i))];$$
$$R_n \supseteq R_{n+1}, \qquad R_{n+1} \Vdash \mathscr{F}_n;$$
$$(i)_{i \leqslant n}[t(i) \notin R_{n+1}^+ \cup R_{n+1}^-].$$

Let $R^+ = \bigcup \{R_n^+ \mid n < \omega\}$ and $R^- = \bigcup \{R_n^- \mid n < \omega\}$. Then $R$ is a coinfinite forcing condition, since $(i)(t(i) \notin R^+ \cup R^-)$. And $R \Vdash \mathscr{F}_n$, since $R \subseteq R_{n+1}$ and $R_{n+1} \Vdash \mathscr{F}_n$.

Lemma 2.9. $(P)(\mathscr{F})(EQ)_{P \supseteq Q}[Q \Vdash \mathscr{F}$ *or* $Q \Vdash \sim\mathscr{F}]$.

*Proof.* It suffices to consider ranked $\mathscr{F}$'s. It is routine to define a concept of rank for a ranked formula $\mathscr{F}$ which takes into account both the ordinal rank of $\mathscr{F}$ and the number of logical symbols in $\mathscr{F}$, and which has the following natural property: if $(EX^\alpha)\,\mathscr{H}(X^\alpha)$ is a ranked sentence and if $\mathscr{G}(x)$ is a ranked formula of ordinal rank $\leqslant \alpha$, then $\mathscr{H}(\hat{x}\mathscr{G}(x))$ has lower rank than $(EX^\alpha)\,\mathscr{H}(X^\alpha)$. The lemma is proved by induction on the rank of $\mathscr{F}$. Let $\mathscr{F}$ be $(EX^\alpha)\,\mathscr{H}(X^\alpha)$. If there is a $Q \subseteq P$ and $\mathscr{G}(x)$ of ordinal rank $\leqslant \alpha$ such that $Q \Vdash \mathscr{H}(\hat{x}\mathscr{G}(x))$, then all is well. Suppose not! Let $\{\mathscr{G}_i(x) \mid i < \omega\}$ be a metafinite enumeration of all formulas of ordinal rank $\leqslant \alpha$ whose sole free variable is $x$. By the induction hypothesis,

$$(i)(Q)_{P \supseteq Q}(ER)_{Q \supseteq R}[R \Vdash \sim\mathscr{H}(\hat{x}\mathscr{G}_i(x))].$$

By 2.8 there is an $R \subseteq P$ such that $R \Vdash \sim \mathscr{H}(\hat{x}\mathscr{G}_i(x))$ for all $i < \omega$. But then $R \Vdash \sim(EX^\alpha)\,\mathscr{F}(X^\alpha)$.

Let $\mathscr{F}$ be a sentence of $\mathscr{L}_{\omega_1}(\mathscr{T})$. $T$ is said to be generic with respect to $\mathscr{F}$ if for some $P$: $T \in P$, and $P \Vdash \mathscr{F}$ or $P \Vdash \sim\mathscr{F}$. $T$ is *generic* if $T$ is generic with respect to every sentence $\mathscr{F}$. By 2.9, generic $T$'s exist. If $T$ is generic, then a routine argument shows: $\mathscr{M}(T) \models \mathscr{F}$ if and only if there is a $P$ such that $T \in P$ and $P \Vdash \mathscr{F}$. It can be shown with the help of 2.8 that if $T$ is generic, then $\omega_1{}^T = \omega_1$.

LEMMA 2.10. *Let $T$ be a subset of $\omega$. Then* (i) *is equivalent to* (ii), *and* (ii) *implies* (iii).

(i) *$T$ is $\omega_1$-subgeneric and not hyperarithmetic.*

(ii) *$T$ is generic with respect to all ranked sentences of $\mathscr{L}_{\omega_1}(\mathscr{T})$.*

(iii) *$T$ is generic with respect to all lightface $\Delta_1{}^1$ subsets of $2^\omega$.*

*Proof.* The equivalence of (i) and (ii) follows from the equivalence of two formalisms. The first formalism is concerned with computations of recursive ordinal height, the second with ranked sentences of $\mathscr{L}_{\omega_1}(\mathscr{T})$. Let $E$ be a finite system of equations in Kripke's equation calculus, let $B$ be an arbitrary subset of $\omega$, let $e$ be an equation, and let $\alpha$ be a recursive ordinal greater than every ordinal occurring in $E$ or $e$. Then there exists a ranked sentence $\mathscr{F}^E_{\alpha,e}$ such that

$$e \in S^{E,B}_\alpha \leftrightarrow \mathscr{M}(B) \models \mathscr{F}^E_{\alpha,e}$$

for *all* $B$; furthermore, $\mathscr{F}^E_{\alpha,e}$ is a metarecursive function of $E$, $\alpha$, and $e$. $\mathscr{F}^E_{\alpha,e}$ is of course defined by recursion on $\alpha$; the details of the recursion are similar to those encountered in Kleene's proof [17] of the equivalence of the hyperarithmetic sets and the ramified analytic sets of recursive ordinal rank. The ordinal rank of $\mathscr{F}^E_{\alpha,e}$ will be approximately $\alpha$.

Let $\mathscr{F}$ be a ranked sentence. Then there exist $E(\mathscr{F})$, $e(\mathscr{F})$, and $\alpha(\mathscr{F})$ such that

$$e(\mathscr{F}) \in S^{E(\mathscr{F}),B}_{\alpha(\mathscr{F})} \leftrightarrow \mathscr{M}(B) \models \mathscr{F}$$

for all $B$.

$E(\mathscr{F})$, $e(\mathscr{F})$, and $\alpha(\mathscr{F})$ are metarecursive functions of $\mathscr{F}$. It is natural to require that

$$e(\mathscr{F}) \in S^{E(\mathscr{F}),B}_{\gamma} \rightarrow e(\mathscr{F}) \in S^{E(\mathscr{F}),B}_{\alpha(\mathscr{F})}$$

for all recursive $\gamma > \alpha(\mathscr{F})$, since

$$\mathscr{M}_\gamma(B) \models \mathscr{F} \to \mathscr{M}(B) \models \mathscr{F}$$

for all $\gamma$ greater than the ordinal rank of $\mathscr{F}$.

Fix $T$ and suppose that $T$ is $\omega_1$-subgeneric but not hyperarithmetic, and that $\mathscr{M}(T) \models \mathscr{F}$, where $\mathscr{F}$ is a ranked sentence. We need a $P$ such that $T \in P$ and $P \Vdash \mathscr{F}$. We have

$$e(\mathscr{F}) \in S_{\alpha(\mathscr{F})}^{E(\mathscr{F}),T}.$$

Since $T$ is $\omega_1$-subgeneric, there is a metarecursive $t$ such that

$$e(\mathscr{F}) \in S_\gamma^{t(E(\mathscr{F})),T,m}$$

for some recursive $\gamma$. Then there is pair of metafinite subsets of $\omega$, $H$ and $K$, such that $H \subseteq T$, $K \subseteq \omega - T$, and

$$e(\mathscr{F}) \in S_\gamma^{t(E(\mathscr{F})),B,m}$$

for all $B$ such that $H \subseteq B$ and $K \subseteq \omega - B$. $P = (H, K)$ is a coinfinite condition, because $T \in P$ and $T$ is not hyperarithmetic. Clearly, $P \Vdash \mathscr{F}$.

Now suppose $T$ is generic with respect to the ranked sentences of $\mathscr{L}_{\omega_1}(\mathscr{T})$. First we check that $T$ is not hyperarithmetic. Suppose $T = H$, where $H$ is hyperarithmetic. There exists a ranked formula $\mathscr{H}(x)$ such that for all $B$,

$$B = H \leftrightarrow \mathscr{M}(B) \models \mathscr{T} = \hat{x}\mathscr{H}(x).^3$$

So there must be a $P$ such that $T \in P$ and $P \Vdash \mathscr{T} = \hat{x}\mathscr{H}(x)$. But then $(T)(T \in P \to T = H)$, an impossibility!

Fix $E$. We need a metarecursive function $t$ such that

$$e \in \bigcup \{S_\alpha^{E,T} \mid \alpha < \omega_1\} \leftrightarrow e \in \{S_\alpha^{t(E),T,m} \mid \alpha < \omega_1\}$$

for all $e$ and $E$. We have

$$e \in S_\alpha^{E,T} \leftrightarrow (EP)[T \in P \,\&\, P \Vdash \mathscr{F}_{\alpha,e}^E].$$

[3] The existence of $\mathscr{H}(x)$ is implicit in Kleene [17].

The relation $P \Vdash \mathscr{F}^{E}_{e,\alpha}$ is (by 2.6) metarecursively enumerable in $P$, $E$, $\alpha$ and $e$. $t(E)$ is a finite system of equations strong enough to enumerate the relation $P \Vdash \mathscr{F}^{E}_{e,\alpha}$. In addition, $t(E)$ has the property that $e$ is computable from $T$ via $t(E)$ if and only if there is a $P$ such that $T \in P$ and $P \Vdash \mathscr{F}^{E}_{\alpha,e}$.

Finally, we observe that (ii) $\rightarrow$ (iii). This follows immediately from the fact that if $\mathscr{B}$ is a lightface $\Delta_1{}^1$ subset of $2^{\omega}$, then there is a ranked sentence $\mathscr{F}$ of $\mathscr{L}_{\omega_1}(\mathscr{T})$ such that

$$T \in \mathscr{B} \leftrightarrow \mathscr{M}(T) \models \mathscr{F}$$

for *all* $T$.
This is proved, not by induction on the recursive ordinals, but by a direct argument involving Brouwer–Kleene trees (cf. Rogers [12, p. 454, 16–85]).

Let $A$ and $B$ be subsets of $\omega$. We say *$A$ is $\Sigma_1{}^1$ in $B$ over recursive $\omega_1$* if there exists an existential formula $\mathscr{G}(x)$ such that

$$n \in A \leftrightarrow \mathscr{M}(B) \models \mathscr{G}(\bar{n})$$

for all $n$. We say *$A$ is $\Delta_1{}^1$ in $B$ over recursive $\omega_1$* if both $A$ and $\omega - A$ are $\Sigma_1{}^1$ in $B$ over recursive $\omega_1$.

THEOREM 2.11. *Let $A$ and $B$ be subsets of $\omega$.*

(i) *If $A$ is $\Delta_1{}^1$ in $B$ over recursive $\omega_1$, then $A \leqslant_{\omega_1} B$.*

(ii) *If $B$ is $\omega_1$-subgeneric and $A \leqslant_{\omega_1} B$, then $A$ is $\Delta_1{}^1$ in $B$ over recursive $\omega_1$.*

*Proof.* The proof of (i) is similar to that of 2.10; we save it for [13], since it is not needed in Section 3. The proof of (ii) is obtained by adding one more fact to the proof of 2.10. Let $J$ be an arbitrary $\Pi_1{}^1$ subset of $\omega$. Then $J$ is $\Sigma_1{}^1$ over the hyperarithmetic sets [12]. Thus there is an existential formula $(EY)\,\mathscr{G}(x, Y)$ of $\mathscr{L}_{\omega_1}(\mathscr{T})$ (such that $\mathscr{G}(x, Y)$ contains no unranked bound variables) with the property that

$$n \in J \leftrightarrow \mathscr{M} \models (EY)\,\mathscr{G}(\bar{n}, Y)$$

for all $n$ ($\mathscr{M}$ = the set of all hyperarithmetic sets = $\mathscr{M}(T)$ for any hyperarithmetic $T$).

We claim that

$$n \in J \leftrightarrow \mathscr{M}(B) \models (EY)\,\mathscr{G}(\bar{n}, Y)$$

for all $\omega_1$-subgeneric $B$. If $B$ is hyperarithmetic, then $\mathscr{M}(B) = \mathscr{M}$. Suppose $B$ is not hyperarithmetic; then $B$ is generic with respect to all ranked sentences of $\mathscr{L}_{\omega_1}(\mathscr{T})$ by 2.10. If $n \in J$, then $\mathscr{M}(B) \models (EY)\,\mathscr{G}(\bar{n}, Y)$ because $\mathscr{M} \subseteq \mathscr{M}(B)$. Suppose $\mathscr{M}(B) \models (EY)\,\mathscr{G}(\bar{n}, Y)$. Then for some metafinite $P$: $B \in P$ and $P \Vdash (EY)\,\mathscr{G}(\bar{n}, Y)$. There exists a hyperarithmetic $T \in P$; $\mathscr{M}(T) = \mathscr{M}$, and

$$\mathscr{M}(T) \models (EY)\,\mathscr{G}(\bar{n}, Y),$$

so $n \in J$.

Let $A \leqslant_{\omega_1} B$. It suffices to show that $A$ is $\Sigma_1^1$ in $B$ over recursive $\omega_1$. Since $B$ is $\omega_1$-subgeneric, there is a finite system $E$ of equations such that $A$ is computable from $B$ via $E$ and metafinite computations only. Let $f$ be the principal function letter of $E$; let $e_n$ be the equation $f(\bar{n}) = \bar{0}$. Thus

$$n \in A \leftrightarrow (E\alpha)_{\alpha<\omega_1}[e_n \in S_\alpha^{E,B,m}].$$

Let $D$ be a typical metafinite computation that puts $e_n$ in $S_\alpha^{E,B,m}$ for some $\alpha < \omega_1$. Clearly $D$ draws only upon a metafinite set of membership facts about $B$. Thus there are metafinite sets $H^D$ and $K^D$ such that $e_n \in S_\alpha^{E,T,m}$ for *every* $T$ such that $H^D \subseteq T$ and $K^D \subseteq \omega - T$.

It is routine to use notations for recursive ordinals as Gödel numbers of metafinite computations so that the relation

$J(n, d)$ : $d$ is the Gödel number of a metafinite computation of $e_n$ via $E$

is $\Pi_1^1$. In addition, there are arithmetic predicates $\mathscr{H}(d, x)$ and $\mathscr{K}(d, x)$ such that

$$J(n, d) \rightarrow H^d = \hat{x}\mathscr{H}(d, x)\ \&\ K^d = \hat{x}\mathscr{K}(d, x),$$

for all $n$ and $d$ [12, p. 456, 16–98]. Let $(EY)\,\mathscr{G}(x_0, x_1, Y)$ be an existential formula of $\mathscr{L}(\mathscr{T})$ such that

$$J(n, d) \leftrightarrow \mathscr{M}(B) \models (EY)\,\mathscr{G}(\bar{n}, \bar{d}, Y)$$

for all $n$ and $d$. Let $Q(x_0)$ be

$$(\mathrm{E}x_1)[(EY)\,\mathscr{G}(x_0, x_1, Y)\ \&\ (x)(\mathscr{H}(x_1, x) \rightarrow x \in \mathscr{T})\ \&\ (x)(\mathscr{K}(x_1, x) \rightarrow x \notin \mathscr{T})].$$

Then $Q(x_0)$ is existential, and for each $n$,

$$n \in A \leftrightarrow \mathscr{M}(B) \models Q(x_0).$$

Feferman [14] studied Cohen-forcing in the context of $\mathscr{L}_{\omega_1}(\mathscr{T})$. Let $p, q, r \cdots$ be finite consistent conjunctions of formulas of the form: $\bar{m} \in \mathscr{T}$, $\bar{n} \notin \mathscr{T}$. Then $p \Vdash \mathscr{F}$ was defined by Feferman by induction on the recursive ordinals in a fashion analogous to Cohen's original definition of $\Vdash$. $T$ is Cohen-generic with respect to the sentence $\mathscr{F}$ of $\mathscr{L}_{\omega_1}(\mathscr{T})$ if there is a $p$ satisfied by $T$ such that $p \Vdash \mathscr{F}$ or $p \Vdash \sim\mathscr{F}$.

PROPOSITION 2.12. *If $T$ is Cohen-generic with respect to all ranked sentences of $\mathscr{L}_{\omega_1}(\mathscr{T})$, then $T$ is not $\Pi_1^1$.*

*Proof.* Let $T$ be $\Pi_1^1$ and Cohen-generic with respect to all ranked sentences. $T$ must be infinite, since no finite condition can force $T$ to be finite. But then $T$ must contain an infinite hyperarithmetic set, let us say, $H$. Let $\mathscr{H}(x)$ be a ranked formula such that

$$n \in H \leftrightarrow \mathscr{M}(T) \models \mathscr{H}(\bar{n})$$

for all $n$ and *all* $T$. There must be a $p$ such that $T$ satisfies $p$ and $p \Vdash (x)[\mathscr{H}(x) \to x \in \mathscr{T}]$. This last is impossible; simply choose an $n$ not mentioned in $p$ and belonging to $H$. Then $p \,\&\, n \notin \mathscr{T} \Vdash \mathscr{H}(\bar{n}) \,\&\, \bar{n} \notin \mathscr{T}$.

Proposition 2.12 should be compared with Theorem 3.4. The comparison highlights certain differences between various forcing relations on recursive $\omega_1$ that we plan to explore further in [13]. For example, we can show there is a $T$, Cohen-generic with respect to all ranked sentences of $\mathscr{L}_{\omega_1}(\mathscr{T})$, such that $T$ has the same hyperdegree as Kleene's $O$.

## 3. CONSTRUCTIONS

We begin with a preliminary construction in the hope of clarifying the principal technical device employed in the main construction. The preliminary construction provides a $\Pi_1^1 A \subseteq \omega$ such that $A$ is generic with respect to all ranked sentences of $\mathscr{L}_{\omega_1}(\mathscr{A})$. The main construction provides a pair of $\Pi_1^1$ sets which are $\Delta_1^1$-incomparable over recursive $\omega_1$. As we hinted in the introduction, a peculiar problem arises when one attempts to metarecursively enumerate exotic $\Pi_1^1$ sets by imitating certain exotic recursive enumerations. Since a $\Pi_1^1 A \subseteq \omega$ is metarecursively enumerated in $\omega_1$ many steps, there must be an intermediate stage $\sigma > \omega$ of the enumeration by which infinitely much of $A$ (call it $A^\sigma$) has been enumerated. If we are overly generous in putting numbers into $A$, then at some stage $\sigma < \omega_1$, $\omega - A^\sigma$ will be finite. So we must take precautions that guarantee that $\omega - A^\sigma$ is infinite for every $\sigma < \omega_1$; the $k$ function below is designed to retard the growth of $A$.

Let $f$ be a function from $\omega_1$ into $\omega_1$. We say $f$ *changes finitely often below* $\sigma$ $(\sigma \leqslant \omega_1)$ if

$$\{\delta \mid (E\varphi)[\delta < \varphi < \sigma \quad \& \quad f(\delta) \neq f(\varphi)]\}$$

is finite. We say $f$ changes finitely often if $f$ changes finitely often below $\omega_1$. Note that if $f$ is metarecursive and changes finitely often below $\sigma$ for every $\sigma > \omega_1$, then $f$ changes finitely often. If $f$ changes finitely often below $\sigma(\sigma \leqslant \omega_1)$, then

$$\lim\{f(\delta) \mid \delta < \sigma\} = f(\varphi),$$

where $\varphi$ is such that $f(\delta) = f(\varphi)$, whenever $\sigma > \delta > \varphi$.

Let $p$ be a one-one metarecursive function from $\omega_1$ into $\omega$. Let $\{\mathscr{G}_\delta \mid \delta < \omega_1\}$ be a metarecursive enumeration of all ranked sentences of $\mathscr{L}_{\omega_1}(\mathfrak{A})$. We say $\mathscr{F}_n$ is *well-defined* and equal to $\mathscr{G}_\delta$ at stage $\sigma$ if $(E\delta)(\delta < \sigma \,\&\, p(\delta) = n)$; $\mathscr{F}_n$ $(n < \omega)$ is called a partial metarecursive enumeration of ranked sentences. We define three metarecursive functions, $P(\sigma, n)$, $k(\sigma, n)$ and $A^\sigma$, for all $\sigma < \omega_1$ and all $n < \omega$ by recursion on $\sigma$. If $n$ is well-defined at stage $\sigma$, then our aim is to define a coinfinite forcing condition $P(\sigma, n) \neq \Phi$[4] and an $A^\sigma$ such that

$$P(\sigma, n) \Vdash \mathscr{F}_n \qquad \text{or} \qquad P(\sigma, n) \Vdash \sim\mathscr{F}_n$$

and $A^\sigma$ satisfies $P(\sigma, n)$. We also aim to define $k(\sigma, m)$ so that $\{k(\sigma, m) \mid m < \omega\} - (A^\sigma \cup P(\sigma, n)^-)$ is infinite for all $n$. Thus $k(\sigma, m)$ will be a witness to two important facts at the end of stage $\sigma$:

(1) $\omega - A^\sigma$ is infinite;

(2) $(A^\sigma, P(\sigma, n)^-)$ is a coinfinite forcing condition.

If (1) is false, then there is no point in continuing the enumeration of $A$. Suppose (2) were false. Then we might find ourselves in the following unfortunate position: $\mathscr{F}_n$ and $\mathscr{F}_{n+1}$ are well-defined at stage $\sigma$; $P(\sigma, n) \Vdash \mathscr{F}_n$; it is *not* the case that either $P(\sigma, n + 1) \Vdash \mathscr{F}_{n+1}$ or $P(\sigma, n + 1) \Vdash \sim\mathscr{F}_{n+1}$; $A^\sigma$ satisfies $P(\sigma, n)$; $A^\sigma \cup P(\sigma, n)^-$ is cofinite.

[4] $\Phi$ denotes both the trivial forcing condition satisfied by all $T$ and the empty set.

The natural thing to do at stage $\sigma + 1$ is to use Lemma 2.9 to define $P(\sigma + 1, n + 1)$ so that either

$$P(\sigma + 1, n + 1) \Vdash \mathscr{F}_{n+1} \qquad \text{or} \qquad P(\sigma + 1, n + 1) \Vdash \sim\mathscr{F}_{n+1},$$

and so that $P(\sigma + 1, n + 1)^+ \supseteq A^\sigma$. However, we also want to preserve the fact that $A^\sigma$ satisfies $P(\sigma, n)$, since $P(\sigma, n) \Vdash \mathscr{F}_n$ and since we intend to give $\mathscr{F}_n$ higher priority than $\mathscr{F}_{n+1}$. This can be done only if $P(\sigma + 1, n + 1)^+ \cap P(\sigma, n)^- = \Phi$. But then the natural use of Lemma 2.9 would be to provide a $P(=P(\sigma + 1, n + 1))$ such that either $P \Vdash \mathscr{F}_{n+1}$ or $P \Vdash \sim\mathscr{F}_{n+1}$ and such that $P^+ \supseteq A^\sigma$ and $P^- \supseteq P(\sigma, n)^-$; there is no hope of finding such a $P$, since $(A^\sigma, P(\sigma, n)^-)$ is not a cofinite forcing condition.

Stage $\sigma = 0$. $P(0, n) = \Phi$, $k(0, n) = n$, $A^0 = \Phi$.

Stage $\sigma = \lambda =$ limit ordinal. If $P(\delta, n)$ has changed finitely often below $\lambda$, then

$$P(\lambda, n) = \lim\{P(\delta, n) \mid \delta < \lambda\};$$

otherwise $P(\lambda, n) = P(0, n)$. Define $k(0, n)$ similarly. $A^\lambda = \bigcup \{A^\delta \mid \delta < \lambda\}$.

Stage $\sigma = \alpha + 1$. Let $n_\alpha$ be the least $n$ such that $\mathscr{F}_n$ is well-defined, $P(\alpha, n) = \Phi$, and

$$Q_\alpha{}^n = \left(A^\alpha, \bigcup \{\{k(\alpha, m)\} \cup P(\alpha, m)^- \mid m < n\}\right)$$

is a coinfinite condition. (If no such $n$ exists, then all values at stage $\sigma = \alpha + 1$ are the same as those at stage $\alpha$.) By 2.9 there is a $P \neq \Phi$ such that $Q_\alpha^{n_\alpha} \supseteq P$ and either $P \Vdash \mathscr{F}_{n_\alpha}$ or $P \Vdash \sim\mathscr{F}_{n_\alpha}$; let $P(n_\alpha, \sigma)$ be the first such $P$ occuring in the standard metarecursive enumeration of $\Vdash$ provided by 2.6. Let $A^\sigma = P(\sigma, n_\alpha)^+ \supseteq A^\alpha$.

$$P(\sigma, n) = \begin{cases} P(\alpha, n) & \text{if } n < n_\alpha, \\ \Phi & \text{if } n > n_\alpha. \end{cases}$$

Define $k(\sigma, n)$ so that

$$k(\sigma, m) = k(\alpha, m) \text{ for } m < n_\alpha,\ k(\sigma, i) < k(\sigma, m) < k(\sigma, m + 1) \text{ for } i < n_\alpha \leqslant m,$$

and

$$\{k(\sigma, m) \mid m \geqslant n_\alpha\} \subseteq \omega - (A^\sigma \cup P(\sigma, n_\alpha)^-).$$

LEMMA 3.1. *For each $n$, the functions $P(\sigma, n)$ and $k(\sigma, n)$ change only finitely often.*

*Proof.* Let $n^*$ be the least $n$ such that $P(\sigma, n)$ changes infinitely often. Let $\lambda^*$ be the least limit ordinal $\lambda < \omega_1$ such that $P(\sigma, n^*)$ changes infinitely often below $\lambda$. As $\sigma$ approaches $\lambda^*$, the value of $P(\sigma, n^*)$ alternates infinitely often between the trivial value $\Phi$ and various nontrivial values. There must be infinitely many $\alpha < \lambda^*$ such that $P(\alpha, n^*)$ is nontrivial and $P(\alpha + 1, n^*) = \Phi$; for every such $\alpha$, $n_\alpha < n^*$. But then $P(\sigma, m)$ changes infinitely often below $\lambda^*$ for some $m < n^*$. If $k(\sigma, n)$ changes infinitely often, then $P(\sigma, m)$ changes infinitely often for some $m \leqslant n$.

LEMMA 3.2. *$k(\sigma, i) \notin A^\sigma \cup P(\sigma, m)^-$ for all $i \geqslant m$ and all $\sigma$.*

*Proof.* By induction on $\sigma$. Suppose $\sigma = \alpha + 1$ and $n_\alpha$ is defined. Clearly, $k(\sigma, i) \notin A^\sigma \cup P(\sigma, m)^-$ for $i \geqslant m \geqslant n_\alpha$. $P(\sigma, m)^- \subseteq P(\sigma, n_\alpha)^-$ for $m < n_\alpha$, so $k(\sigma, i) \notin A^\sigma \cup P(\sigma, m)^-$ for $i \geqslant n_\alpha > m$. By the inductive hypothesis,

$$k(\sigma, i) \notin A^\alpha \cup P(\sigma, m)^-$$

for $n_\alpha > i \geqslant m$. And happily, $k(\sigma, i) \notin A^\sigma$ for $n_\alpha > i$.

Suppose $\sigma = \lambda =$ limit ordinal; apply 3.1.

LEMMA 3.3. *$(A^\sigma, \bigcup \{P(\sigma, m)^- \mid m < n\})$ is a coinfinite condition for all $\sigma$ and all $n$.*

*Proof.* First observe by induction on $\sigma$ that $A^\sigma \cap P(\sigma, m)^- = \Phi$ and $k(\sigma, m) < k(\sigma, m + 1)$ for all $m$; then apply 3.2.

THEOREM 3.4. *There exists a $\Pi_1^1 A$ generic with respect to all ranked sentences of $\mathscr{L}_{\omega_1}(\mathfrak{A})$.*

*Proof.* Let $A = \bigcup \{A^\sigma \mid \sigma < \omega_1\}$, where $A^\sigma$ is defined above. $A$ is $\Pi_1^1$, thanks to 2.1. Let $\mathscr{G}$ be a ranked sentence of $\mathscr{L}_{\omega_1}(\mathfrak{A})$. Let $n$ be such that $\mathscr{F}_n$ is well-defined and equal to $\mathscr{G}$ for all sufficiently large $\sigma$. By 3.1 $n_\alpha \geqslant n$ for all sufficiently large $\alpha$. Then by 3.2 and 3.3, $P(\sigma, n) \neq \Phi$ for arbitrarily large $\sigma$.

By 3.1

$$P(n) = \lim\{P(\sigma, n) \mid \sigma < \omega_1\}$$

exists. Since $P(n) \neq \Phi$, $P(n) \Vdash \mathscr{F}_n$ or $P(n) \Vdash \sim\mathscr{F}_n$ ; and $A \supseteq P(n)^+$. By 3.3, $\omega - A^\sigma \supseteq P(n)^-$ for all sufficiently large $\sigma$, so $A$ satisfies $P(n)$.

COROLLARY 3.5. *There exists a $\Pi_1^1$ subset of $\omega$ generic with respect to every lightface $\Delta_1^1$ subset of $2^\omega$.*

COROLLARY 3.6. *There exists a nonhyperarithmetic, $\omega_1$-subgeneric $\Pi_1^1$ subset of $\omega$.*

Corollaries 3.5 and 3.6 are consequences of 2.10. Now we undertake some modifications of the construction underlying 3.4 in the hope of obtaining two $\Pi_1^1$ subsets of $\omega$, each generic with respect to all ranked sentences of $\mathscr{L}_{\omega_1}(\mathscr{A})$, neither $\Delta_1^1$ in the other over recursive $\omega_1$. Let $\mathscr{F}_n^A$ $(n < \omega)$ be the partial metarecursive enumeration $\mathscr{F}_n (n < \omega)$ of ranked sentences of $\mathscr{L}_{\omega_1}(\mathscr{A})$ used in 3.4. Let $\mathscr{F}_n^B$ $(n < \omega)$ be an isomorphic enumeration of the ranked sentences of $\mathscr{L}_{\omega_1}(\mathscr{B})$. Let $\mathscr{G}_n^A(x)$ $(n < \omega)$ be a partial metarecursive enumeration of all $\Sigma_1^1$ formulas of $\mathscr{L}_{\omega_1}(\mathscr{A})$ whose sole free variable is $x$; thus for each $\Sigma_1^1$ formula of $\mathscr{L}_{\omega_1}(\mathscr{A})$ of the form $\mathscr{G}(x)$, there exists an $n < \omega$ such that $\mathscr{G}_n^A(x)$ is well-defined and equal to $\mathscr{G}(x)$ for all sufficiently large $\sigma$. Let $\mathscr{G}_n^B(x)$ $(n < \omega)$ be an isomorphic enumeration of $\Sigma_1^1$ formulas of $\mathscr{L}_{\omega_1}(\mathscr{B})$. ($\Sigma_1^1$ = existential.)

We define eight metarecursive functions, $P_{\mathscr{F}}^A(\sigma, n)$, $P_{\mathscr{G}}^A(\sigma, n)$, $k^A(\sigma, n)$, $A^\sigma$, $P_{\mathscr{F}}^B(\sigma, n)$, $P_{\mathscr{G}}^B(\sigma, n)$, $k^B(\sigma, n)$, and $B^\sigma$, for all $\sigma < \omega_1$ and all $n < \omega$ by recursion on $\sigma$. $P_{\mathscr{F}}^A$ and $k^A$ bear the same relation to $A^\sigma$ that $P$ and $k$ bear to $A^\sigma$ in 3.4; the same holds for $P_{\mathscr{F}}^B$, $k^B$, and $B^\sigma$.

Suppose $\mathscr{G}_n^A(x)$ is well-defined for all sufficiently large $\sigma$. Then we try to control the course of events so that $A$ satisfies $P_{\mathscr{G}}^A(\sigma, n)$,

$$k^B(\sigma, n) \in B^\sigma \qquad \text{and} \qquad P_{\mathscr{G}}^A(\sigma, n) \Vdash \mathscr{G}_n^A(k^B(\sigma, n)).$$

If we succeed, then $\omega - B$ will not be $\Sigma_1^1$ in $A$ over recursive $\omega_1$ via $\mathscr{G}_n^A$. If we fail, all is not lost since we also try to achieve

$$k^B(\sigma, n) \notin B^\sigma \qquad \text{and} \qquad \mathscr{M}(A) \models \sim\mathscr{G}_n^A(k^B(\sigma, n)).$$

The genericity of $A$ with respect to ranked sentences is essential, since it guarantees that either

$$(EP)_{A\in P}[P \Vdash \mathscr{G}_n^A(k^B(\sigma, n))] \qquad \text{or} \qquad \mathscr{M}(A) \models \sim\mathscr{G}_n^A(k^B(\sigma, n)).$$

In a symmetrical fashion we strive to make $\omega - A$ not $\Sigma_1^1$ in $B$ over recursive $\omega_1$. The usual Friedbergian conflict results from our simultaneous attempts to make $A$ and $B$ "incomparable", and is resolved by the usual priorities. But there is an additional conflict also resolved by priorities. Our attempts to make $A$ and $B$ generic will interfere with our attempts to make $A$ and $B$ "incomparable". In ordinary recursion theory this problem cannot occur, since the finiteness of ordinary computations compels every set to be generic.

*Stage* $\sigma = 0$. $P_{\mathscr{F}}{}^A(0, n) = P_{\mathscr{G}}{}^A(0, n) = P_{\mathscr{F}}{}^B(0, n) = P_{\mathscr{G}}{}^B(0, n) = \Phi$, $k^A(0, n) = k^B(0, n) = n$, $A^0 = B^0 = \Phi$.

*Stage* $\sigma = \lambda =$ limit ordinal. If $P_{\mathscr{F}}{}^A(\delta, n)$ has changed finitely often below $\lambda$, then

$$P_{\mathscr{F}}{}^A(\lambda, n) = \lim\{P_{\mathscr{F}}{}^A(\delta, n) \mid \delta < \lambda\};$$

otherwise $P_{\mathscr{F}}{}^A(\lambda, n) = P_{\mathscr{F}}{}^A(0, n)$. Define $P_{\mathscr{G}}{}^A(\lambda, n)$, $P_{\mathscr{F}}{}^B(\lambda, n)$, $P_{\mathscr{G}}{}^B(\lambda, n)$, $k^A(\lambda, n)$, and $k^B(\lambda, n)$ similarly. $A^\lambda = \bigcup\{A^\delta \mid \delta < \lambda\}$; $B^\lambda = \bigcup\{B^\delta \mid \delta < \lambda\}$.

*Stage* $\sigma = \alpha + 1$. If $\sigma \equiv i (\mathrm{mod}\ 4)$, go to case $i$.

*Case 0.* Let $n_\alpha$ be the least $n$ such that $\mathscr{F}_n{}^A$ is well-defined, $P_{\mathscr{F}}{}^A(\alpha, n) = \Phi$, and such that

$${}^A\underset{\sim}{Q}_\alpha{}^n = \left(A^\alpha, \bigcup\{\{k^A(\alpha, m)\} \cup P_{\mathscr{F}}{}^A(\alpha, m)^- \cup P_{\mathscr{G}}{}^A(\alpha, m)^- \mid m < n\}\right)$$

is a coinfinite condition. (If no such $n$ exists, then all values at stage $\sigma = \alpha + 1$ are the same as those at stage $\alpha$.) By 2.9 there is a $P \neq \Phi$ such that ${}^A\underset{\sim}{Q}_\alpha{}^n \supseteq P$, and either $P \Vdash \mathscr{F}^A_{n_\alpha}$ or $P \Vdash \sim\mathscr{F}^A_{n_\alpha}$. Let $P_{\mathscr{F}}{}^A(\sigma, n_\alpha)$ be the first such $P$ in the enumeration of $\Vdash$ provided by 2.6. The *only other changes* to be made in passing from stage $\alpha$ to stage $\sigma = \alpha + 1$ are

$$P_{\mathscr{F}}{}^A(\sigma, m + 1) = P_{\mathscr{G}}{}^A(\sigma, m) = \Phi \qquad \text{for} \quad m \geq n_\alpha\,;$$

$A^\sigma = P_{\mathscr{F}}{}^A(\sigma, n_\alpha)^+ \supseteq A^\alpha$; $k^A(\sigma, m)$ $(m \geq n_\alpha)$ is defined so that

$$k^A(\sigma, i) < k^A(\sigma, m) < k^A(\sigma, m + 1) \qquad \text{for} \quad i < n_\alpha \leq m,$$

and

$$\{k^A(\sigma, m) \mid m \geq n_\alpha\} \subseteq \omega - (A^\sigma \cup P_{\mathscr{F}}{}^A(\sigma, n_\alpha)^-).$$

*Case 1.* Let $r_\alpha^A$ be the least $n$ such that $\mathscr{G}_n^A(x)$ is well-defined, $P_{\mathscr{G}}^A(\alpha, n) = \Phi$, and for some $\gamma \leqslant \alpha$,

$$\begin{gathered} A^\alpha \text{ satisfies } P_\gamma,\ \omega - (A^\alpha \cup P_\gamma^-) \text{ is infinite,} \\ P_\gamma \neq \Phi, \quad P_\gamma \not\Vdash \mathscr{G}_n^A(k^B(\alpha, n)), \\ (m)_{m<n}[P_{\mathscr{G}}^A(\alpha, m) \supseteq P_\gamma], \\ (m)_{m\leqslant n}[P_{\mathscr{F}}^A(\alpha, m) \supseteq P_\gamma]. \end{gathered}$$

Let the first such $P_\gamma$ be $P_{\mathscr{G}}^A(\sigma, r_\alpha^A)$. (If $r_\alpha^A$ is not defined, then all values at stage $\sigma$ equal those at stage $\alpha$.) The only other changes are

$$\begin{aligned} B^\sigma &= B^\alpha \bigcup \{k^B(\alpha, r_\alpha^A)\}; \\ P_{\mathscr{F}}^B(\sigma, m) &= P_{\mathscr{G}}^B(\sigma, m) = P_{\mathscr{F}}^A(\sigma, m) = P_{\mathscr{G}}^A(\sigma, m) = \Phi \qquad \text{for} \quad m > r_\alpha^A; \\ k^B(\sigma, m) &= k^B(\alpha, m+1) \qquad \text{for} \quad m \geqslant r_\alpha^A; \end{aligned}$$

$A^\sigma = A^\alpha$; $k^A(\sigma, m)$ $(m \geqslant r_\alpha^A)$ is defined so that

$$k^A(\sigma, i) < k^A(\sigma, m) < k^A(\sigma, m+1) \qquad \text{for} \quad i < n_\alpha \leqslant m,$$

and

$$\{k^A(\sigma, m) \mid m \geqslant r_\alpha^A\} \subseteq \omega - (A^\sigma \cup P_{\mathscr{G}}^A(\sigma, r_\alpha^A)^-).$$

*Case 2.* Same as Case 0 with $A$ replaced by $B$ throughout.

*Case 3.* Same as Case 1 with $A$ and $B$ interchanged throughout save for one asymmetry. The equation $P_{\mathscr{G}}^B(\sigma, m) = \Phi$ $(m > r_\alpha^A)$ of Case 1 becomes

$$P_{\mathscr{G}}^A(\sigma, m) = \Phi \qquad \text{for} \quad m \geqslant r_\alpha^B.$$

LEMMA 3.7. *For each $n$, the functions $P_{\mathscr{F}}^A(\sigma, n)$, $P_{\mathscr{G}}^A(\sigma, n)$, $k^A(\sigma, n)$, $P_{\mathscr{F}}^B(\sigma, n)$, $P_{\mathscr{G}}^B(\sigma, n)$, and $k^B(\sigma, n)$ change only finitely often.*

*Proof.* Similar to the Friedbergian argument of 3.1.

LEMMA 3.8^A. $(\sigma)(m)[k^A(\sigma, m) < k^A(\sigma, m+1)]$.

*Proof.* By induction on $\sigma$. When $\sigma = \lambda =$ limit ordinal, apply 3.7.

LEMMA 3.9^A. $k^A(\sigma, i) \notin A^\sigma \cup P_{\mathscr{F}}^A(\sigma, m)^- \cup P_{\mathscr{G}}^A(\sigma, m)^-$ *for all $i \geqslant m$ and all $\sigma$.*

*Proof.* By induction on $\sigma$. Suppose $\sigma = \alpha + 1 \equiv 0 (\mathrm{mod}\ 4)$ and $n_\alpha$ is defined. If $i \geqslant m \geqslant n_\alpha$ or $i \geqslant n_\alpha > m$, then the definitions made at stage $\sigma$ insure that all is well; note that

$$\{k^A(\alpha, m)\} \cup P_{\mathscr{F}}{}^A(\alpha, m)^- \cup P_{\mathscr{G}}{}^A(\alpha, m)^- \subseteq P_{\mathscr{F}}{}^A(\sigma, n_\alpha)^-$$

for all $m < n_\alpha$. If $n_\alpha > i \geqslant m$, then the inductive hypothesis is needed.

Suppose $\sigma = \alpha + 1 \equiv 1 (\mathrm{mod}\ 4)$ and $r_\alpha{}^A$ is defined. If $i \geqslant m \geqslant r_\alpha{}^A$ or $i \geqslant r_\alpha{}^A > m$, then all goes smoothly, since

$$P_{\mathscr{G}}{}^A(\alpha, m)^- \cup P_{\mathscr{F}}{}^A(\alpha, m)^- \subseteq P_{\mathscr{G}}{}^A(\sigma, r_\alpha{}^A)^-$$

for all $m \leqslant r_\alpha{}^A$. If $r_\alpha{}^A > i \geqslant m$, then the inductive hypothesis and Lemma 3.8^A are invoked.

If $\sigma = \alpha + 1 \equiv 3 (\mathrm{mod}\ 4)$, then the inductive hypothesis suffices. If $\sigma = \lambda =$ limit ordinal, then an application of Lemma 3.7 is required.

Lemma 3.10^A. *$(A^\sigma, \bigcup \{P_{\mathscr{F}}{}^A(\sigma, m)^- \cup P_{\mathscr{G}}{}^A(\sigma, m)^- \mid m < n\})$ is a coinfinite condition for all m and $\sigma$.*

*Proof.* First observe by induction on $\sigma$ that $A^\sigma \cap (P_{\mathscr{F}}{}^A(\sigma, m)^- \cup P_{\mathscr{G}}{}^A(\sigma, m)^-) = \Phi$ for all $m$ and $\sigma$. Then apply 3.8^A and 3.9^A.

Lemma 3.11^A. *For each m, A satisfies $P_{\mathscr{F}}{}^A(\sigma, m)$ and $P_{\mathscr{G}}{}^A(\sigma, m)$ for all sufficiently large $\sigma$.*

*Proof.* First observe by induction on $\sigma$ that $P_{\mathscr{F}}{}^A(\sigma, m)^+ \cup P_{\mathscr{G}}{}^A(\sigma, m)^+ \subseteq A^\sigma$ for all $m$ and $\sigma$. Then apply 3.7 and 3.10^A.

Lemma 3.12^A. *A is generic with respect to all ranked sentences of $\mathscr{L}_{\omega_1}(\mathfrak{A})$.*

*Proof.* Similar to 3.4. Replace 3.1, 3.2, and 3.3 by 3.7, 3.9^A, and 3.10^A.

Lemma 3.13^A isolates the principal information needed in 3.14 to establish the $\Delta_1^1$-incomparability of $A$ and $B$ over $\omega_1$. There is a curious twist in the proof of 3.13^A. Its very statement makes clear that the genericity of $A$ (supplied by 3.12^A) will be needed for its proof. What is less clear is the reason why in the middle of the proof of 3.13^A it is necessary to note that $A$ is not hyperarithmetic. Thus the proof of the $\Delta_1^1$-incomparability of $A$ and $B$ over recursive $\omega_1$ has to be *preceded* by a proof that neither $A$ nor $B$ is hyperarithmetic. As far as we can tell, this phenomenon has no counterpart in the theory of recursively enumerable subsets of $\omega$. We hope to discuss it further in [13].

By 3.7 it is legitimite to define

$$k^B(n) = \lim\{k^B(\sigma, n) \mid \sigma < \omega_1\},$$
$$P_{\mathscr{G}}{}^A(n) = \lim\{P_{\mathscr{G}}{}^A(\sigma, n)\ \sigma < \omega_1\},$$

for all $n$.

LEMMA 3.13$^{\mathrm{A}}$. *If $\mathscr{G}_n{}^A(x)$ is well-defined, and if $\mathscr{M}(A) \models \mathscr{G}_n{}^A(k^B(n))$, then $P_{\mathscr{G}}{}^A(n) \neq \Phi$.*

*Proof.* Fix $n$. We intend to show that there exist arbitrarily large $\alpha$ such that either $P_{\mathscr{G}}{}^A(\alpha, n) = \Phi$ or $P_{\mathscr{G}}{}^A(\alpha + 1, n) \neq \Phi$. Since $\mathscr{M}(A) \models \mathscr{G}_n{}^A(k^B(n))$, it follows from $3.12^{\mathrm{A}}$ that for some $\underset{\sim}{Q} \neq \Phi$ and satisfied by $A$,

$$\underset{\sim}{Q} \Vdash \mathscr{G}_n{}^A(k^B(n)).$$

Let

$$P^- = \underset{\sim}{Q}^- \bigcup \cup \{P_{\mathscr{G}}{}^A(m)^- \cup P_{\mathscr{F}}{}^A(m)^- \mid m \leqslant n\}.$$

Clearly $P^-$ is hyperarithmetic. $P^- \subseteq \omega - A$ by $3.11^{\mathrm{A}}$. $\omega - A$ is not hyperarithmetic by $3.12^{\mathrm{A}}$ and 2.10. It follows that

$$\omega - (A \cup P^-) \text{ is infinite.}$$

Let $P^+ = \underset{\sim}{Q}^+ \bigcup \cup \{P_{\mathscr{G}}{}^A(m)^+ \cup P_{\mathscr{F}}{}^A(m)^+ \mid m \leqslant n\}$. $P^+ \subseteq A$ by $3.11^{\mathrm{A}}$. Let $P = (P^+, P^-)$. Then $P$ is a coinfinite condition. $P$ meets the following requirements for all sufficiently large $\alpha$:

$$A^\alpha \text{ satisfies } P, \quad \omega - (A^\alpha \cup P^-) \text{ is infinite,}$$
$$P \neq \Phi, \quad P \Vdash \mathscr{G}_n{}^A(k^B(\alpha, n)),$$
$$(m)_{m<n}[P_{\mathscr{G}}{}^A(\alpha, m) \supseteq P],$$
$$(m)_{m\leqslant n}[P_{\mathscr{F}}{}^A(\alpha, m) \supseteq P].$$

By 3.7 $r_\alpha{}^A$ is undefined or $\geqslant n$ for all sufficiently large $\alpha$. But then for all sufficiently large $\alpha$: if $\alpha + 1 \equiv 1 (\text{mod } 4)$ and $P_{\mathscr{G}}{}^A(\alpha, n) = \Phi$, then $r_\alpha{}^A = n$ and $P_{\mathscr{G}}{}^A(\alpha + 1, n) \neq \Phi$.

THEOREM 3.14. *There exist two $\Pi_1{}^1$ sets, each generic with respect to all ranked sentences of $\mathscr{L}_{\omega_1}(\mathfrak{A})$, such that neither is $\Delta_1{}^1$ in the other over recursive $\omega_1$.*

*Proof.* Let $A$ and $B$ be the $\Pi_1^1$ sets constructed above. Let Lemma 3.n$^{B}$ ($9 \leqslant n \leqslant 13$) be the result of replacing $B$ by $A$ in Lemma 3.n$^{A}$. By 3.12$^{A}$ and 3.12$^{B}$, $A$ and $B$ are generic with respect to all ranked sentences of $\mathscr{L}_{\omega_1}(\mathfrak{A})$. Fix $n$ so that $\mathscr{G}_n^A(x)$ is a well-defined $\Sigma_1^1$ formula of $\mathscr{L}_{\omega_1}(\mathfrak{A})$. We intend to show that $\omega - B$ is not $\Sigma_1^1$ in $A$ over recursive $\omega_1$ via $\mathscr{G}_n^A(x)$. We will find a $c$ such that

$$\sim[c \in \omega - B \;\leftrightarrow\; \mathscr{M}(A) \models \mathscr{G}_n^A(\bar{c})].$$

The value of $c$ depends on whether or not $P_{\mathscr{G}}^A(n) = \Phi$.

Suppose $P_{\mathscr{G}}^A(n) \neq \Phi$. Let $\alpha$ be the unique $\delta$ such that $P_{\mathscr{G}}^A(\delta, n) = \Phi$ and

$$(\gamma)_{\gamma > \delta}[P_{\mathscr{G}}^A(\gamma, n) = P_{\mathscr{G}}^A(n)].$$

Clearly, $r_\alpha^A = n$, $k^B(\alpha, n) \in B$, and $P_{\mathscr{G}}^A(n) \Vdash \mathscr{G}_n^A(k^B(\alpha, n))$. By 3.11$^{A}$ $A$ satisfies $P_{\mathscr{G}}^A(n)$, so

$$\mathscr{M}(A) \models \mathscr{G}_n^A(k^B(\alpha, n)).$$

Thus the desired $c$ is $k^B(\alpha, n)$.

Now suppose $P_{\mathscr{G}}^A(n) = \Phi$. By 3.13$^{A}$, we have

$$\mathscr{M}(A) \models \sim\mathscr{G}_n^A(k^B(n)).$$

But $k^B(n) \in \omega - B$ is a consequence of 3.9$^{B}$.

Similarly, $\omega - A$ is not $\Sigma_1^1$ in $B$ over recursive $\omega_1$.

COROLLARY 3.15. *There exist two $\Pi_1^1$ sets such that neither is $\omega_1$-computable from the other.*

*Proof.* 3.14, 2.10, and 2.11(ii).

## 4. EXTENSIONS

Let $\alpha$ be an arbitrary admissible ordinal. Most of the notions of Section 2 generalize routinely from recursive $\omega_1$ to $\alpha$. Let $A$ and $B$ be subsets of $\alpha$. Then $A$ is $\alpha$-computable from $B$ means that $A$ can be computed by means of Kripke's equation calculus for $\alpha$ and computations of height less than $\alpha$. $B$ is $\alpha$-subgeneric means there is an $\alpha$-recursive procedure for replacing computations from $B$ of height $<\alpha$ by $\alpha$-finite computations from $B$. The projectum of $\alpha$ (a concept due to Kripke) is the least $\beta$ such that the range of some total $\alpha$-recursive function is a subset of $\beta$, and is denoted by $\alpha^*$. Thus $\omega_1{}^* = \omega$. The arguments of Section 3 readily generalize to $\alpha$'s such that $\alpha^* = \omega$.

THEOREM 4.1. *If $\alpha^* = \omega$, then there exist two $\alpha$-recursively enumerable subsets of $\omega$ such that neither is $\alpha$-computable from the other.*

*Question* 4.2. Can Theorem 4.1 be strengthened as follows?

If $\alpha$ is admissible then there exist two $\alpha$-recursively enumerable subsets of $\alpha^*$ such that neither is $\alpha$-computable from the other. We conjecture that this last is true if $\alpha$ is countable, but we are completely at sea if $\alpha$ is uncountable. The countability of recursive $\omega_1$ was obviously important in the proof of 2.9, but we are not sure in just what way it was important. A routine generalization of 2.9 from recursive $\omega_1$ to a larger admissible $\alpha$ is successful if $\alpha^* = \omega$. It seems likely to us that a more imaginative generalization, something like MacIntyre's construction [18] of a minimal $\alpha$-degree for each countable admissible $\alpha$, will succeed if $\alpha$ is countable. But we have very little feeling for the case of uncountable $\alpha$, because we don't understand the meaning of the concept of countability within the domain of recursion theory.

The proof of 4.1 includes a proof of 4.2.

THEOREM 4.2. *If $\alpha^* = \omega$, then there exists a non-$\alpha$-recursive, $\alpha$-recursively enumerable, $\alpha$-subgeneric subset of $\omega$.*

In order to prove 4.1 and 4.2, the language $\mathscr{L}_{\omega_1}(\mathscr{T})$ is enlarged to $\mathscr{L}_\alpha(\mathscr{T})$; $\mathscr{L}_\alpha(\mathscr{T})$ is a ramified set-theoretic language which includes numerals for all the ordinals less than $\alpha$ and names for all the sets constructible (in the sense of Gödel [19]) via the ordinals less than $\alpha$. The metafinite, coinfinite forcing conditions of Section 2 become $\alpha$-finite.

COROLLARY 4.3. *There exists a $\Sigma_2{}^1$ subset of $\omega$ which is generic with respect to every sentence of $\Delta_2{}^1$ ordinal rank.*

*Proof.* Let $\delta_2{}^1$ be the least non-$\Delta_2{}^1$ ordinal. Then $\delta_2{}^1$ is admissible and $(\delta_2{}^1)^* = \omega$. A subset of $\omega$ is $\delta_2{}^1$-recursively enumerable if and only if it is $\Sigma_2{}^1$ [9, 20].

The addition of some unappetizing complications results in the following relativizations of 3.6 and 3.15.

THEOREM 4.4. *Let $A$ be a nonhyperarithmetic $\Pi_1^1$ set. There exist two $\Pi_1^1$ sets, each of which is metarecursive in $A$, neither of which is $\omega_1$-computable from the other.*

THEOREM 4.5. *Let $A$ be a nonhyperarithmetic $\Pi_1^1$ set. There exists a nonhyperarithmetic, $\omega_1$-subgeneric $\Pi_1^1$ set $B$ metarecursive in $A$.*

*Question* 4.6. Which metadegrees are metadegrees of $\omega_1$-subgeneric $\Pi_1^1$ sets?

This question is vaguely phrased but nonetheless sensible. By 2.5, if a $\Pi_1^1$ set $B$ is $\omega_1$-subgeneric, then every $\Pi_1^1$ set of the same metadegree is $\omega_1$-subgeneric. There does exist a non-$\omega_1$-subgeneric $\Pi_1^1$ set. Perhaps some Postian [21] condition on the complement of $A$ leads to an improvement of 4.5 in which $A$ is metarecursive in $B$?

The metadegrees of the $\Pi_1^1$ sets are still quite mysterious. It follows from the work of Driscoll [11], Owings [22] and Theorem 5 of [6] that the metadegrees of the $\Pi_1^1$ sets form a dense partial ordering. Not much more than that is known. We know very little about how to lift the following theorem of ordinary recursion theory [23] up into the metadegrees of the $\Pi_1^1$ sets: if $A$ is a nonrecursive, recursively enumerable set, then $A$ can be split into two disjoint recursively enumerable sets such that neither is recursive in the other.

*Question* 4.7. Let $A$ be a non-hyperarithmetic $\Pi_1^1$ set. Can $A$ be split into two disjoint $\Pi_1^1$ sets such that neither is metarecursive in the other?

Owings has shown that if $A$ is simple (touches every infinite $\Pi_1^1$ set), then $A$ can be split as asked. Perhaps every $\omega_1$-generic $A$ can be split.

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