Characterization of $p$-Adic Analytic Groups in Terms of Wreath Products

ANER SHALEV

Mathematical Institute, University of Oxford,
24–29 St. Giles, Oxford OX1 3LB, United Kingdom

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We show that a finitely generated pro-$p$ group is $p$-adic analytic (i.e., can be given the structure of a Lie group over $\mathbb{Q}_p$) if and only if it does not involve arbitrarily large wreath products of the form $C_p \wr C_p$. This result, whose proof applies Zelmanov’s recent solution to the restricted Burnside problem, is in fact equivalent to Zelmanov’s Theorem. © 1992 Academic Press, Inc.

The first criterion for a finitely generated pro-$p$ group $G$ to be $p$-adic analytic (i.e., to have the structure of a Lie group over the $p$-adic field) was given in the mid-sixties by Lazard [L]. It says, essentially, that $G$ is $p$-adic analytic if and only if it possesses an open subgroup $H$ (necessarily of finite index) such that $H' \leq H^p$, where $H'$ is the commutator subgroup of $H$, and $H^p$ denotes the closed subgroup generated by the $p$th powers of $H$ (if $p = 2$, $H^2$ should be replaced by $H^4$). Pro-$p$ groups/finite $p$-groups $H$ with this property are now called powerful.

During the past few years powerful groups and $p$-adic analytic groups have gained much attention, and important work was carried out by A. Lubotzky and A. Mann (see [LM1, LM2, LM4], and also [M3, LM3] for several applications). In particular, they have been able to produce two new characterizations of $p$-adic analytic groups, namely: being of finite rank, and having polynomial subgroup growth. The reader is referred to [LM1, LM4] for the basic definitions. Extensions and some further characterizations are given in [MS]. See also [D, S] and the recent book [DDMS].

The purpose of this note is to give yet another characterization of $p$-adic analytic groups, which has been made possible by the positive solution to the restricted Burnside problem, recently given by Zelmanov [Z]. We say that a group $L$ is involved in a (topological) group $G$, if $L$ is isomorphic to a (closed) section $H/K$ of $G$ (where $K \triangleleft H \triangleleft G$).
THEOREM. A finitely generated pro-p group $G$ is $p$-adic analytic if and only if, for some $n$, the wreath product $C_p \wr C_{p^n}$ is not involved in $G$.

Note that, if $G$ is $p$-adic analytic, then, being of finite rank, it cannot involve arbitrarily large elementary abelian groups $(C_p)^n$, so the "only-if" part follows at once.

The real content of the theorem is that, if $G$ involves $(C_p)^n$ for every $n$, then it must also involve $C_p \wr C_{p^n}$ for every $n$. This may seem reminiscent of Kropholler's theorem [K], stating that a finitely generated soluble group of infinite rank must involve $C_p \wr C_\infty$ for some $p$. It is not clear whether in our (non-soluble) case, infinite wreath products like this must occur.

Before proving the theorem let us mention a consequence which may have some independent interest. Following [M2] (see also [M1]) we say that a $p$-group/pro-$p$ group $G$ is power-closed if, in every (closed) section of $G$, any product of $p^i$th powers is again a $p^i$th power ($i > 0$). Let us extend this definition by saying that $G$ is $n$-power-closed if, in every section of $G$, any (arbitrarily long) product of $p^{i+n}$th powers is a $p^i$th power ($i > 0$). The study of this property was suggested in [M1, Problem 4].

It is well known that, in powerful groups, any product of $p^i$th powers is again a $p^i$th power [LM1, Propositions 1.7 and 4.1.7], but this need not hold in their sections. In fact, there are no inclusion relations between the sets of power-closed $p$-groups and powerful $p$ groups. In spite of this it turns out that, in finitely generated pro-$p$ groups, the properties of being power-closed (or $n$-power-closed) and of being $p$-adic analytic are very much related.

COROLLARY. A finitely generated pro-$p$ group is $p$-adic analytic if and only if it is $n$-power-closed for some $n$.

**Proof.** Suppose $G$ is $p$-adic analytic. Then, by [LM2], $G$ has finite rank, say $r$. Set $n = r \cdot \lceil \log_2 r \rceil$ if $p$ is odd, and $n = r \cdot \lceil \log_2 r \rceil + 1$ otherwise, where $\lceil x \rceil$ denotes the minimal integer greater or equal to $x$. We claim that $G$ is $n$-power-closed.

Indeed, let $P$ be a (closed) section of $G$. Clearly, the rank of $P$ is at most $r$, so, by [LM1, Theorems 1.14 and 4.1.14], $P$ possesses a powerful characteristic subgroup $Q$, such that $(P : Q) \leq p^n$. Obviously, $P^{p^i} \leq Q$, so $P^{p^{i+n}} \leq Q^{p^n}$ for all $i$. Since every element of $Q^{p^n}$ is a $p^n$th power, we conclude that, given any $i > 0$ and $x_1, ..., x_k \in P$, there exists $y \in P$ such that $x_1^{p^{i+n}} \cdot \cdot \cdot x_k^{p^{i+n}} = y^{p^n}$. Hence $G$ is $n$-power-closed.

Conversely, suppose $G$ is $n$-power-closed. It is straightforward to verify that products of $p^{n+i}$th powers in $C_p \wr C_{p^{n+i}}$ need not be $p^i$th powers. Therefore $C_p \wr C_{p^{n+i}}$ is not $n$-power-closed, so it cannot occur as a section of $G$. Applying the theorem, we conclude that $G$ is $p$-adic analytic. \[\square\]
Let us now turn to the proof of the theorem, starting with the part of the argument which does not rely on the positive solution to the restricted Burnside problem.

The following lemma is proved using rather standard arguments.

**Lemma.** Let $G$ be a finite $p$-group, and let $M$ be a normal subgroup which centralizes every normal elementary abelian section of $G$.

(a) Suppose $p$ is odd. Then $M$ is powerful.

(b) Suppose $p = 2$. Then $M^2$, as well as all its subgroups which are normal in $G$, is powerful.

**Proof.** (a) Factoring out $M^p$ we may assume that $M$ has exponent $p$, and we have to show that it is abelian. Let $M_1 \triangleleft G$ be a maximal subgroup of $M$. Arguing by induction, we may assume that $M_1$ is powerful, so it is abelian. By the assumption on $M$ it follows now that $M$ centralizes $M_1$. Therefore $M$ is central-by-cyclic, so it must be abelian.

(b) Let $N \leq M^2$ and suppose $N \triangleleft G$. We will show that $N$ is powerful. So assume $N^4 = 1$, and let us show that $N$ is abelian. Let $N_1 \triangleleft G$ be a maximal subgroup of $N$. Arguing inductively, we may assume that $N_1$ is abelian. The condition on $M$ yields $[M, N_1] \leq N_1^2$ and $[M, [M, N_1]] \leq [M, N_1]^2 \leq (N_1^2)^2 = N_1^4 = 1$. This implies $[M^2, N_1] = [M, N_1]^2 = 1$. Since $N \leq M^2$ it follows that $N_1 \leq Z(N)$. But $N_1$ is maximal in $N$. Therefore $N$ is abelian.

**Proposition.** Let $G$ be a finite $p$-group which does not involve the wreath product $C_p \wr C_p$. Then

(a) $G^{p^n}$ is powerful when $p$ is odd.

(b) $G^{p^{n+1}}$ is powerful when $p = 2$.

**Proof.** In view of the lemma, it is sufficient to show that $G^{p^n}$ centralizes every normal elementary abelian section of $G$. So assume, on the contrary, that for some $x \in G$ and $N \triangleleft G$, $[x^{p^n}, N] \notin N'N^{p^n}$. Factoring out $N'N^{p^n}$ we may assume that $N$ is elementary abelian. Consider $N$ as an $F_p$-module, where $C = \langle x \rangle$. Then $(x^{p^n} - 1)N \neq 0$, so for some indecomposable summand of $N$, $N_1$, $(x^{p^n} - 1)N_1 \neq 0$. Now, $N_1$, as an indecomposable module for a cyclic group $C$, must be uniserial, and it is uniquely determined by its dimension (see, e.g., [A, pp. 24–26]; note that it is sufficient to require that all $|C|$th roots of unit lie in the underlying field). Set $N_2 = N_1/(x^{p^n} - 1)N_1$. Then $N_2$ is a $p^n$-dimensional uniserial module over $F_p[\langle x \rangle/\langle x^{p^n} \rangle] \cong F_p[C_{p^n}]$, so $N_2 \cong F_p[C_{p^n}]$. Returning to $G$, this means that the action of $\langle x \rangle/\langle x^{p^n} \rangle$ on $N_1/[x^{p^n}, N_1]$ is the regular action. Consider the intersection $\langle x \rangle \cap N_1$. It cannot contain $x^{p^n}$, as $x^{p^n}$
does not centralize $N_1$. Therefore $\langle x \rangle \cap N_1 \leq \langle x^{p^{m+1}} \rangle$, and, in particular, it is trivial modulo the normal closure of $\langle x^{p^n} \rangle$ in $N_1 \langle x \rangle$ (which is equal to $[x^{p^n}, N_1] \langle x^{p^n} \rangle$). It follows that $N_1 \langle x \rangle/[x^{p^n}, N_1] \langle x^{p^n} \rangle \cong N_1/[x^{p^n}, N_1] \rtimes \langle x \rangle / \langle x^{p^n} \rangle \cong C_p \wr C_p$, so we are done.

Combining the last proposition with the positive solution to the restricted Burnside problem, our theorem readily follows. Indeed, let $G$ be a finitely generated pro-$p$ group, not involving $C_p \wr C_p$. Set $m = n$ if $p$ is odd, and $m = n + 1$ otherwise. Then, for every finite epimorphic image $P$ of $G$, $P^{p^m}$ is powerful. Hence $G^{p^m}$ is powerful.

Set $d = d(G)$ (the number of generators of $G$ as a pro-$p$ group). Now, given any finite epimorphic image $P$ of $G$, apply Zelmanov's theorem to the $d$-generated group $P/P^{p^m}$ of exponent dividing $p^m$, to conclude that $(P : P^{p^m}) \leq f(d, p^m)$, for some fixed function $f: \mathbb{N} \times \mathbb{N} \to \mathbb{N}$. This clearly yields $(G : G^{p^m}) \leq f(d, p^m)$. Therefore $G$ possesses a powerful subgroup of finite index, so that, by Lazard's criterion, $G$ is $p$-adic analytic.

Remark that our theorem also implies Zelmanov's. Indeed, let $G$ be a finitely generated pro-$p$ group of finite exponent. Then $G$ cannot involve arbitrarily large wreath products of the form $C_p \wr C_p$, so by the theorem, $G$ is $p$-adic analytic. Therefore $G$ has a powerful subgroup $H$ of finite index. Now, $H$, as a finitely generated powerful pro-$p$ group of finite exponent, is easily seen to be finite (see [LM1, Propositions 2.5 and 4.2.5]). This implies that $G$ itself is finite, and Zelmanov's Theorem follows at once (take $G = F_d/F_d^{p^m}$, where $F_d$ is the free pro-$p$ group on $d$-generators).

Thus proving the theorem without relying on Zelmanov's difficult work is paramount to solving the restricted Burnside problem from scratch.

It is natural to ask whether non-$p$-adic analytic groups must involve additional types of finite groups, other than $C_p \wr C_p$, and their sections. It turns out that the answer is negative:

**Example.** Let $G = C_p \wr \mathbb{Z}_p$, where $\mathbb{Z}_p$ are the $p$-adic integers (regarded as an additive group). It is straightforward to verify that $G$ is a 2-generated pro-$p$ group which is not $p$-adic analytic, and any finite section of $G$ is a section of $C_p \wr C_p$ for some $n$. (In particular, $C_p^2 \times C_p^2$ is not involved in $G$).

Therefore the groups $C_p \wr C_p$ may be considered as "the only obstruction" to $G$ being $p$-adic analytic, and our result cannot be strengthened in this direction.

However, it would be interesting to know what other types of residually finite groups of infinite rank necessarily involve arbitrarily large wreath products.
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REFERENCES


