Wave propagation along a non-principal direction in a compressible pre-stressed elastic layer

Priza Kayestha a, Anil C. Wijeyewickrema a,⇑, Kikuo Kishimoto b

a Department of Civil and Environmental Engineering, Tokyo Institute of Technology, 2-12-1, O-okayama, Meguro-ku, Tokyo 152-8552, Japan

b Department of Mechanical Sciences and Engineering, Tokyo Institute of Technology, 2-12-1, Meguro-ku, Tokyo 152-8552, Japan

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ABSTRACT

The dispersive behavior of small amplitude waves propagating along a non-principal direction in a pre-stressed, compressible elastic layer is considered. One of the principal axes of stretch is normal to the elastic layer and the direction of propagation makes an angle with one of the in-plane principal axes. The dispersion relations which relate wave speed and wavenumber are obtained for both symmetric and anti-symmetric motions by formulating the incremental boundary value problem for a general strain energy function. The behavior of the dispersion curves for symmetric waves is for the most part similar to that of the anti-symmetric waves at the low and high wavenumber limits. At the low wavenumber limit, depending on the pre-stress and propagation angle, it may be possible for both the fundamental mode and the next lowest mode to have finite phase speeds, while other higher modes have an infinite phase speed. At the high wavenumber limit, the phase speeds of the fundamental mode and the higher modes tend to the Rayleigh surface wave speed and the limiting wave speeds of the layer, respectively. Numerical results are presented for a Blatz–Ko material and the effect of the propagation angle is clearly illustrated.

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1. Introduction

The problem of time-harmonic wave propagation in an isotropic elastic layer was studied in detail by Mindlin (1960) and the dispersion relations are commonly referred to as the Rayleigh–Lamb equations to recognize the efforts of Rayleigh (1888–1889) and Lamb (1889–1890) who first studied this problem more than a hundred years ago. The efforts of Mindlin (1960) are also discussed in many monographs, see for example Achenbach (1973), Graff (1975) and Miklowitz (1978).

Small amplitude wave propagation along a principal direction in a pre-stressed elastic layer has been studied since the 1960s. Biot (1963, 1965) studied wave propagation in an incompressible pre-stressed elastic layer with an initial stress along the x1-axis. In a series of papers, Willson (1973a,b, 1977a,b) investigated the propagation of waves in a pre-stressed layer for compressible as well as incompressible materials. Later, Ogden and Roxburgh (1993) and Roxburgh and Ogden (1994) investigated the vibration and stability of pre-stressed incompressible and compressible elastic layers, respectively. They derived the dispersion relations for a general form of strain-energy function and studied the effects of pre-stress on the stability of the layer but did not perform a detailed asymptotic analysis of the dispersion relations. Rogerson and Fu (1995) and Rogerson (1997) studied the detailed asymptotic expansions of the dispersion relations for waves propagating in a pre-stressed, incompressible elastic layer. In addition, Sandiford and Rogerson (2000) and Nolde et al. (2004) performed asymptotic analysis of the dispersion relation for waves propagating in a pre-stressed nearly incompressible layer and a pre-stressed compressible elastic layer, respectively. Other related studies of wave propagation in pre-stressed layers are by Prikazchikova et al. (2006), Wijeyewickrema et al. (2008), Rogerson and Prikazchikova (2009) and references therein.

Motivated by the wide usage of layered composites in engineering practice, wave propagation in pre-stressed layered composites has also been studied. Rogerson and Sandiford (1996, 1997, 2000a) considered a pre-stressed, perfectly bonded, incompressible, symmetric four-ply laminate to study the propagation of small amplitude waves along a principal direction. Rogerson and Sandiford (2002) studied wave propagation in a pre-stressed periodically layered incompressible elastic composite. The dispersive behavior of waves propagating in a pre-stressed, imperfectly bonded, incompressible, symmetric laminate was studied by Leungvichcharoen and Wijeyewickrema (2003) and Leungvichcharoen et al. (2004) for symmetric waves and anti-symmetric waves, respectively. The dispersive behavior of small amplitude waves propagating in a pre-stressed, imperfectly bonded, compressible, symmetric

* Corresponding author. Tel.: +81 3 5734 2595; fax: +81 3 5734 3478.
E-mail address: wjeyewickrema.a.aa@m.titech.ac.jp (A.C. Wijeyewickrema).

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lamine was studied by Wijeyewickrema and Leungvichcharoen (2009). Apart from the symmetric laminate, the bi-material laminate has also been studied. The propagation of small amplitude time-harmonic waves along a principal direction in a pre-stressed perfectly bonded bi-material laminate was studied by Rogerson and Sandiford (2000b) for the incompressible case and by Kayestha et al. (2010) for the compressible case. In the case of the bi-material laminate, due to lack of symmetry about the mid-plane, the dispersion relations cannot be decoupled into symmetric and anti-symmetric waves, thus increasing the complexity of the problem compared to the symmetric laminate. All these studies in pre-stressed layers and laminates are limited to wave propagation along a principal direction of pre-stress, with one of the principal axes normal to the surface.

The effects of pre-stress on waves propagating along a non-principal direction in an elastic solid have also been reported. Connor and Ogden (1995) dealt with the influence of homogeneous simple shear and hydrostatic stress on propagation of surface waves in an incompressible elastic half-space. In the case of simple shear the orientation of the principal axes of deformation varies with the pre-stress and the direction of wave propagation is not along the principal direction. Connor and Ogden (1996) studied the effects of simple shear combined with hydrostatic stress in an incompressible elastic layer. Destrade and Ogden (2005) later extended the study to analyze the effects of homogeneous simple shear combined with stretch in an incompressible elastic half-space. Rogerson and Sandiford (1999) investigated small amplitude waves propagating along a non-principal direction in an incompressible elastic layer with one of the principal axes of stretch normal to the surface of the plate. Pichugin and Rogerson (2001, 2002) also considered small amplitude waves propagating along a non-principal direction in an incompressible pre-stressed elastic layer and developed asymptotically consistent theory to describe three-dimensional long-wave high-frequency and long-wave low-frequency motions. Destrade et al. (2005) studied the propagation of surface waves along a non-principal direction in an incompressible, elastic half-space using Taziev’s technique.

In the present paper, the propagation of small amplitude inhomogeneous plane waves along a non-principal direction in a pre-stressed compressible elastic layer with traction-free surfaces is considered. The basic equations of infinitesimal time-harmonic wave propagation in pre-stressed, compressible, elastic media are given in Section 2. In Section 3, the bi-cubic characteristic equation for the attenuation parameter is obtained from the governing equations. The dispersion relations associated with symmetric and anti-symmetric waves are obtained in explicit form, where it can be seen that the dispersion relations differ from each other only through the hyperbolic terms. The special cases \( \theta = 0 \) and \( \pi/2 \), i.e. when wave propagation is along one of the principal directions are also considered. Here, the equations corresponding to the plane strain case are uncoupled from the equation associated with horizontally polarized shear waves. In Section 4, the asymptotic behavior at the low and high wavenumber limits are discussed. Dispersion curves and frequency spectra are presented in Section 5 for Blatz–Ko material. The dispersion curves for propagation angles other than \( \theta = 0 \) and \( \pi/2 \) show that three finite limiting squared phase speeds are possible in the low wavenumber region which is in direct contrast to the two limiting values obtained for wave propagation along a principal direction in a layer as shown in the plot for \( \theta = 0 \) and also in Rogburgh and Ogden (1994) and Wijeyewickrema et al. (2008). The frequency spectra show that the modes which correspond to infinite phase speeds in the dispersion curves tend to cut-off frequencies in the low wavenumber region. In the high wavenumber region, the phase speeds of the fundamental modes tend to the wave speed of the associated Rayleigh surface wave and the phase speeds of higher modes tend to the limiting wave speeds of the layer.

2. Basic equations

The equations for infinitesimal time-harmonic wave propagation in pre-stressed compressible elastic media are given in this section (see Roxburgh and Ogden, 1994 and Ogden and Sotiropoulos, 1998). A homogeneous, compressible isotropic elastic body is considered which when unstressed occupies the configuration \( \mathbf{s}_0 \). This body is subjected to pure homogeneous strains to produce a new pre-stressed equilibrium configuration \( \mathbf{s}_0 \). A Cartesian coordinate system \( \mathbf{Ox}_1x_2x_3 \) is chosen to coincide with the principal axes of the right Cauchy–Green deformation tensor. Let \( \mathbf{u} \) be an infinitesimal time-dependent deformation superimposed on \( \mathbf{s}_0 \). The body thus occupies a new time-dependent deformed configuration \( \mathbf{s} \). The incremental equations of motion in the absence of body forces can be written as

\[
\Delta \mathbf{q}_{ijkl} \mathbf{u}_{ij} = \rho \Delta \mathbf{u}_{ij},
\]

where \( \Delta \mathbf{q}_{ijkl} \) are the components of the fourth-order tensor of first-order instantaneous moduli of compressible isotropic elasticity, which relates the nominal stress increment tensor and the deformation gradient tensor, \( \rho \) is the current material density and the superimposed dot and comma indicate the differentiation with respect to time \( t \) and the spatial coordinate \( x_l \), respectively. The incremental equations of motion given in Eq. (1) can be written as

\[
\Delta \mathbf{x}_{l} + \Delta \mathbf{u}_{2l} + \gamma_{1l} \Delta \mathbf{u}_{1l} + \gamma_{2l} \Delta \mathbf{u}_{22} + \gamma_{3l} \Delta \mathbf{u}_{33} = \rho \Delta \mathbf{u}_{1l},
\]

\[
\Delta \mathbf{x}_{2l} + \Delta \mathbf{u}_{22} + \gamma_{2l} \Delta \mathbf{u}_{1l} + \gamma_{3l} \Delta \mathbf{u}_{22} + \gamma_{2l} \Delta \mathbf{u}_{33} = \rho \Delta \mathbf{u}_{22},
\]

\[
\Delta \mathbf{x}_{3l} + \Delta \mathbf{u}_{22} + \gamma_{2l} \Delta \mathbf{u}_{1l} + \gamma_{3l} \Delta \mathbf{u}_{22} + \gamma_{3l} \Delta \mathbf{u}_{33} = \rho \Delta \mathbf{u}_{33},
\]

where

\[
\Delta \mathbf{x}_l = \Delta \mathbf{q}_{ijkl} \gamma_l \gamma_j = \Delta \mathbf{q}_{ijkl} \delta_{l} \gamma_j \sigma_j = \Delta \mathbf{q}_{ijkl} \sigma_j - \gamma_j \sigma_l, \quad (i, j = 1, 2, 3),
\]

and the properties \( \Delta \mathbf{x}_l = \Delta \mathbf{q}_{ijkl} \sigma_j - \gamma_j \sigma_l = \gamma_l \sigma_j - \sigma_l \sigma_j, \quad (i, \ j = 1, 2, 3) \) have been used in obtaining Eq. (2), where \( \sigma_j \) are the principal Cauchy stresses in \( x_l \)-direction.

The instantaneous elastic moduli \( \Delta \mathbf{q}_{ijkl} \gamma_l \gamma_j \) and principal Cauchy stresses \( \sigma_j \) can be expressed in terms of the strain energy function \( W \) per unit reference volume and principal stretches \( \lambda_i \) as

\[
\mathbf{J} \Delta \mathbf{x}_{ij} \lambda_{ij} \mathbf{W}_{ij} \mathbf{J} \gamma_{ij} = \left( \frac{\omega_{ij}}{\lambda_{ij}} \right)^2 \Lambda_{ij} \gamma_{ij} \mathbf{W}_{ij}, \quad \mathbf{J} \gamma_{ij} \lambda_{ij} \mathbf{W}_{ij}, \quad \mathbf{J} \sigma_{ij} \lambda_{ij} \mathbf{W}_{ij}, \quad (i, j = 1, 2, 3),
\]

where \( \mathbf{W}_{ij} = \partial \mathbf{W} / \partial \mathbf{e}_{ij} \), \( \mathbf{W}_{ij} = \partial ^2 \mathbf{W} / \partial \mathbf{e}_{ij} \partial \mathbf{e}_{jk} \) (Ogden and Sotiropoulos, 1998). In the case of equi-biaxial deformation in the \( x_1x_2 \)-plane when \( \lambda_{1} = \lambda_{2} \), Eq. (4) for \( i, j = 1, 2 \) reduces to

\[
\mathbf{J} \Delta \mathbf{x}_{ij} = \mathbf{J} \gamma_{ij} \lambda_{ij} \mathbf{W}_{ij}, \quad \mathbf{J} \gamma_{ij} = \mathbf{J} \gamma_{21} = \mathbf{J} \gamma_{21} = \mathbf{J} \lambda_{ij} \mathbf{W}_{ij} \mathbf{J} \gamma_{ij},
\]

\[
\mathbf{J} \sigma_{ij} = \mathbf{J} \lambda_{ij} \mathbf{W}_{ij},
\]

and \( \lambda_{1} = \lambda_{2} = \lambda_{12} + 2 \gamma_{21} - \sigma_{2} \). Similar expressions can be obtained for equi-biaxial deformations in the \( x_1x_3 \)-plane and the \( x_2x_3 \)-plane. In addition, in the reference (unstressed) configuration \( \mathbf{s}_0 \), \( \lambda = \lambda = \lambda_0 \), \( \mathbf{J} = \mathbf{J} = \mathbf{J} \), \( (i, j = 1, 2, 3) \) where \( \lambda \) and \( \mu \) are the classical Lamé moduli of the material. Here, the strong ellipticity conditions

\[
\lambda_0 > 0, \quad \gamma_0 > 0, \quad \rho_0/2 + \sqrt{\lambda_0^2 \gamma_0^2} > 0, \quad i, j = 1, 2, 3,
\]
where $\beta_i = \rho_{ij} \rho_j + \gamma_{ij} \gamma_j - \delta_{ij} \delta_j$, $i \neq j$, $(i, j = 1, 2, 3)$ are assumed to hold. These conditions while being necessary conditions are not sufficient conditions, for the three-dimensional case under consideration. Explicit sufficient conditions for the three-dimensional case have not been reported in the literature (see also Section 6.2.7, Ogden, 1984).

The relevant components of the nominal stress increment tensor in configuration $\mathcal{S}$ for traction free boundary conditions can be expressed as

$$
\sigma_{021}(x_1, x_2, x_3, t) = \gamma_{21} u_{12} + (\gamma_{21} - \sigma_2) u_{22,1},
$$

$$
\sigma_{022}(x_1, x_2, x_3, t) = \gamma_{21} u_{11} + \gamma_{22} u_{22} + \gamma_{23} u_{33},
$$

$$
\sigma_{023}(x_1, x_2, x_3, t) = (\gamma_{32} - \sigma_3) u_{22} + \gamma_{23} u_{33}.
$$

(7)

### 3. Formulation of the problem

The compressible pre-stressed elastic infinite layer of thickness $2h$ (see Fig. 1) is assumed to be homogeneous with material parameters $\gamma_{ij}$, $\gamma_{ij}$ $(i, j = 1, 2, 3)$. The Cartesian coordinate system is chosen such that $x_1$-x2- and $x_3$-axes are coincident with the principal axes of the right Cauchy–Green deformation tensor, $x_2$-direction is normal to the free surface of the layer, the direction of time-harmonic wave propagation makes an angle $\theta$ with the $x_1$-axis and the origin $O$ lies at the mid-plane of the layer.

The infinitesimal time-dependent displacements superimposed on the pre-stressed elastic layer may be expressed as

$$
u_j = A_j e^{i(k_1 x_1 + k_3 x_3 + \omega t)}, \quad (j = 1, 2, 3),
$$

where $k$ is the wavenumber, $\nu$ is the phase speed, $A_1, A_2, A_3$ are arbitrary constants and the attenuation parameter $\bar{q}$ is to be determined. Substituting Eq. (8) in Eq. (2) yields a system of three homogeneous equations with three unknowns,

$$(\bar{q}^2 - \gamma_{21}) \cos^2 \theta - \gamma_{31} \sin^2 \theta + \rho \nu^2) A_1
$$

$$+ i \bar{q} \delta_{21} \cos \theta A_2 - \delta_{13} \sin \theta \cos \theta A_3 = 0,
$$

(9)

$$i \bar{q} \delta_{21} \cos \theta A_1 + (\bar{q}^2 \gamma_{22} - \gamma_{12} \cos^2 \theta - \gamma_{12} \sin^2 \theta + \rho \nu^2) A_2
$$

$$+ i \bar{q} \delta_{23} \sin \theta A_3 = 0,
$$

(10)

$$- \delta_{13} \sin \theta \cos \theta A_1 + i \bar{q} \delta_{23} \sin \theta A_2
$$

$$+ (\bar{q}^2 \gamma_{23} - \gamma_{33} \cos^2 \theta - \gamma_{33} \sin^2 \theta + \rho \nu^2) A_3 = 0
$$

(11)

for which a non-trivial solution provided

$$\gamma_{21} \gamma_{23} \bar{q}^2 + \bar{q}^2 \left(\gamma_{22} \gamma_{23} + \gamma_{22} \gamma_{23} + \gamma_{21} \gamma_{23}\right) \rho \nu^2 - \Delta_1
$$

$$+ \bar{q}^2 \left(\gamma_{21} + \gamma_{23} + \gamma_{23}\right) \rho^2 \nu^2 - \Delta_2 \rho \nu^2 + \Delta_3
$$

$$+ (\rho \nu^2 - \Delta_4) (\rho^2 \nu^2 - \Delta_5 \rho \nu^2 + \Delta_6) = 0.
$$

(12)

Next, by making use of Eqs. (9)–(11), the displacements can be given in terms of only the coefficients $A_j^{\nu_0}$ $(n = 1, 2, \ldots, 6)$ as

$$u_1(x_1, x_2, x_3, t) = \sum_{n=1}^{3} k_n \left[ \delta_{12} \delta_{21} \sin^2 \theta + \bar{g}(\bar{q}_n, \rho \nu^2) \delta_{12} \right]
$$

$$ \times \left[ A_2^{(2n-1)} e^{i k_2 x_2} - A_2^{(2n)} e^{-i k_2 x_2} \right] \cos \theta e^{i(k_1 x_1 + k_3 x_3 + \omega t)},
$$

(15a)

$$u_2(x_1, x_2, x_3, t) = \sum_{n=1}^{3} k_n \left[ \delta_{12} \delta_{21} \cos^2 \theta + \bar{f}(\bar{q}_n, \rho \nu^2) \delta_{21} \right]
$$

$$ \times \left[ A_2^{(2n-1)} e^{i k_2 x_2} - A_2^{(2n)} e^{-i k_2 x_2} \right] \sin \theta e^{i(k_1 x_1 + k_3 x_3 + \omega t)},
$$

(15b)

$$u_3(x_1, x_2, x_3, t) = \sum_{n=1}^{3} k_n \left[ \delta_{12} \delta_{21} \cos^2 \theta + \left(\bar{f}(\bar{q}_n, \rho \nu^2) \delta_{21} \right) \sin \theta e^{i(k_1 x_1 + k_3 x_3 + \omega t)} \right]
$$

$$ \times \left[ A_2^{(2n-1)} e^{i k_2 x_2} - A_2^{(2n)} e^{-i k_2 x_2} \right] \sin \theta e^{i(k_1 x_1 + k_3 x_3 + \omega t)},
$$

(15c)

where

$$\bar{f}(\bar{q}_n, \rho \nu^2) = \bar{q}_n^2 \gamma_{21} + \rho \nu^2 - \gamma_{31} \cos^2 \theta - \gamma_{31} \sin^2 \theta,
$$

(16a)

$$\bar{g}(\bar{q}_n, \rho \nu^2) = \bar{q}_n^2 \gamma_{22} + \rho \nu^2 - \gamma_{33} \cos^2 \theta - \gamma_{33} \sin^2 \theta,
$$

(16b)

$$\bar{\theta}(\bar{q}_n, \rho \nu^2) = \bar{q}_n^2 \cos^2 \theta - \bar{f}(\bar{q}_n, \rho \nu^2) \bar{g}(\bar{q}_n, \rho \nu^2). \quad (n = 1, 2, 3).
$$

(16c)

Expressions for the relevant stress components are obtained from Eq. (7) as
\[
\frac{\ddot{u}_2(x_1, t)}{k} = \sum_{n=1}^{3} \frac{\ddot{G}(\bar{q}_n, \rho \nu^2)}{\phi(\bar{q}_n, \rho \nu^2)} (A_{2(2n-1)} e^{\bar{q}_n k x_2} + A_{2(2n)} e^{-\bar{q}_n k x_2}),
\]

(17a)

\[
\frac{\ddot{u}_2(x_1, x_2, t)}{k} = \sum_{n=1}^{3} \frac{\ddot{G}(\bar{q}_n, \rho \nu^2)}{\phi(\bar{q}_n, \rho \nu^2)} (A_{2(2n-1)} e^{\bar{q}_n k x_2} - A_{2(2n)} e^{-\bar{q}_n k x_2}),
\]

(17b)

\[
\frac{\ddot{u}_2(x_1, x_2, t)}{k} = \sum_{n=1}^{3} \frac{\dddot{G}(\bar{q}_n, \rho \nu^2)}{\phi(\bar{q}_n, \rho \nu^2)} (A_{2(2n-1)} e^{\bar{q}_n k x_2} + A_{2(2n)} e^{-\bar{q}_n k x_2}),
\]

(17c)

where

\[
\ddot{G}(\bar{q}_n, \rho \nu^2) = \bar{q}_n^2 \gamma_2 \sin^2 \theta + \delta_{12} \gamma_2 (\bar{q}_n, \rho \nu^2) + (\gamma_{21} - \gamma_2) \bar{q}_n \rho \nu^2,
\]

(18a)

\[
\ddot{H}(\bar{q}_n, \rho \nu^2) = \frac{\pi}{2} \bar{q}_n \cos^2 \theta (\delta_{12} \gamma_2 \sin^2 \theta + \delta_{12} \gamma_2 \sin^2 \theta + \delta_{12} \gamma_2 \cos^2 \theta + \delta_{12} \gamma_2 \cos^2 \theta),
\]

(18b)

\[
\dddot{F}(\bar{q}_n, \rho \nu^2) = \frac{\pi}{2} \bar{q}_n \cos^2 \theta + \frac{\pi}{2} \bar{q}_n \cos^2 \theta + \frac{\pi}{2} \bar{q}_n \cos^2 \theta + \frac{\pi}{2} \bar{q}_n \cos^2 \theta,
\]

(18c)

For the problem being considered i.e. wave propagation along a non-principal direction in a pre-stressed elastic layer, due to the symmetry of the elastic layer about \(x_2 = 0\), the wave motion can be decoupled into symmetric and anti-symmetric modes and consequently it is convenient to analyze the symmetric and anti-symmetric waves separately, and it is sufficient to consider only the upper half of the layer i.e. \(0 \leq x_2 \leq h\). Expressions for the displacements and stresses are given in the Appendix A.

The mid-plane conditions and the incremental traction free upper surface conditions for both symmetric and anti-symmetric waves can be written as,

symmetric waves:

\[
\ddot{u}_2(x_1, 0, x_3, t) = \ddot{u}_2(x_1, 0, x_3, t) = 0,
\]

(19a)

\[
\ddot{u}_2(x_1, h_3, x_3, t) = \ddot{u}_2(x_1, h_3, x_3, t) = 0,
\]

(19b)

anti-symmetric waves:

\[
\ddot{u}_1(x_1, 0, x_3, t) = \ddot{u}_1(x_1, 0, x_3, t) = 0,
\]

(20a)

\[
\ddot{u}_1(x_1, h_3, x_3, t) = \ddot{u}_1(x_1, h_3, x_3, t) = 0.
\]

(20b)

For symmetric waves Eqs. (A1)–(A6) satisfy the mid-plane conditions Eq. (19a) identically. Using Eqs. (A1)–(A6) to satisfy Eq. (19b) results in a system of three homogeneous equations with three unknowns given by

\[
\sum_{n=1}^{3} \frac{\ddot{G}(\bar{q}_n, \rho \nu^2)}{\phi(\bar{q}_n, \rho \nu^2)} B_{1n} C_n = 0,
\]

(21)

\[
\sum_{n=1}^{3} \frac{\ddot{H}(\bar{q}_n, \rho \nu^2)}{\phi(\bar{q}_n, \rho \nu^2)} B_{1n} C_n = 0,
\]

(22)

\[
\sum_{n=1}^{3} \frac{\dddot{F}(\bar{q}_n, \rho \nu^2)}{\phi(\bar{q}_n, \rho \nu^2)} B_{1n} C_n = 0,
\]

(23)

where \(B_{1n}, (n = 1, 2, 3)\) is defined in Eq. (A14). From Eqs. (26)–(28) the dispersion relation is obtained as

\[
\ddot{q}_1 \ddot{H}(\bar{q}_1, \rho \nu^2) \dot{S}_1 \dot{C}_1 \left( \bar{q}_1^2 - \bar{q}_1 \right) \dot{J}(\bar{q}_1, \bar{q}_1, \rho \nu^2) - \ddot{q}_1 \ddot{H}(\bar{q}_1, \rho \nu^2) \dot{C}_1 \dot{S}_1 \left( \bar{q}_1^2 - \bar{q}_1 \right) \dot{J}(\bar{q}_1, \bar{q}_1, \rho \nu^2)
\]

\[
+ \ddot{q}_1 \dddot{H}(\bar{q}_1, \rho \nu^2) \dot{C}_1 \dot{S}_1 \left( \bar{q}_1^2 - \bar{q}_1 \right) \dot{J}(\bar{q}_1, \bar{q}_1, \rho \nu^2) = 0,
\]

(24)

Note that the common factor \(\dot{J}(\bar{q}_1, \bar{q}_1, \rho \nu^2)\) has been removed from the denominator while obtaining the explicit expression given by Eq. (24), which leads to spurious roots in the dispersion relation.

Similarly, for anti-symmetric waves Eqs. (A8)–(A13) satisfy the mid-plane conditions Eq. (20a) identically and using Eqs. (A8)–(A13) to satisfy Eq. (20b) results in a system of three homogeneous equations with three unknowns as

\[
\sum_{n=1}^{3} \frac{\ddot{G}(\bar{q}_n, \rho \nu^2)}{\phi(\bar{q}_n, \rho \nu^2)} B_{2n} C_n = 0,
\]

(26)

\[
\sum_{n=1}^{3} \frac{\ddot{H}(\bar{q}_n, \rho \nu^2)}{\phi(\bar{q}_n, \rho \nu^2)} B_{2n} C_n = 0,
\]

(27)

\[
\sum_{n=1}^{3} \frac{\dddot{F}(\bar{q}_n, \rho \nu^2)}{\phi(\bar{q}_n, \rho \nu^2)} B_{2n} C_n = 0.
\]

(28)

where \(B_{2n}, (n = 1, 2, 3)\) is defined in Eq. (A14). From Eqs. (26)–(28) the dispersion relation is obtained as

\[
\ddot{q}_1 \ddot{H}(\bar{q}_1, \rho \nu^2) \dot{S}_1 \dot{C}_1 \left( \bar{q}_1^2 - \bar{q}_1 \right) \dot{J}(\bar{q}_1, \bar{q}_1, \rho \nu^2) - \ddot{q}_1 \ddot{H}(\bar{q}_1, \rho \nu^2) \dot{C}_1 \dot{S}_1 \left( \bar{q}_1^2 - \bar{q}_1 \right) \dot{J}(\bar{q}_1, \bar{q}_1, \rho \nu^2)
\]

\[
+ \ddot{q}_1 \dddot{H}(\bar{q}_1, \rho \nu^2) \dot{C}_1 \dot{S}_1 \left( \bar{q}_1^2 - \bar{q}_1 \right) \dot{J}(\bar{q}_1, \bar{q}_1, \rho \nu^2) = 0,
\]

(29)

where \(\dot{J}(\bar{q}_1, \bar{q}_1, \rho \nu^2)\) are defined in Eq. (25). Here too, the same common factor \(\dot{J}(\bar{q}_1, \bar{q}_1, \rho \nu^2)\) has been removed from the denominator to obtain the explicit dispersion relation given by Eq. (29), which leads to spurious roots in the dispersion relation.

3.1 Special cases \(\theta = 0\) and \(\theta = \pi/2\)

It is of interest to also consider the special cases of waves propagating along the in-plane principal directions \(\theta = 0\) and \(\theta = \pi/2\). For waves propagating in the direction of the \(x_1\)-axis, i.e. when \(\theta = 0\), Eqs. (9)–(11) yield...
Eq. (12) reads as:

\[ q_i^2 = \sum_{n=1}^{2} \left[ A_{ij} + \rho \varepsilon^2 \right] A_{ij} = 0, \]

\[ i q_i^2 A_1 + (q_i^2 - x_{11} + \rho \varepsilon^2) A_2 = 0, \]

\[ (q_i^2 - x_{11} + \rho \varepsilon^2) A_3 = 0. \]

(Eq. 30) and (31) are coupled and correspond to the plane strain case investigated by Roxburgh and Ogden (1994), while the uncoupled Eq. (32) corresponds to the anti-plane case. The characteristic equation given by Eq. (12) for \( \theta = 0 \) reduces to

\[ x_{11}^2 + 2\rho \varepsilon^2 x_{11} - x_{12}^2 = 0, \]

where \( x_{11} = x_{11} - \rho \varepsilon^2, x_{12} = x_{12} - \rho \varepsilon^2. \) From Eq. (33), it can be seen that either

\[ (q_i^2 - x_{11} + \rho \varepsilon^2) A_2 = 0, \]

or

\[ (q_i^2 - x_{11} + \rho \varepsilon^2) A_2 = 0. \]

(Eq. 34) whose roots are taken as \( q_1^2 \) and \( q_2^2 \) can be shown to be identical to Eq. (13) of Roxburgh and Ogden (1994) and from Eq. (35) \( q_1^2 = (y_1^2 - \rho^2 \varepsilon^2)/y_2^2. \) Making use of Eqs. (8) and (7) the expressions for the relevant displacement and stress components in terms of \( q_i (i = 1, 2, 3) \) are

\[ u_1(x_1, x_2, t) = \sum_{n=1}^{2} \left[ D_1^{(n)} \cosh q_i x_2 + D_2^{(n)} \sinh q_i x_2 \right] e^{ik(x_1 - vt)}, \]

\[ u_2(x_1, x_2, t) = \sum_{n=1}^{2} \left[ -s(q_i) \left( D_1^{(n)} \cosh q_i x_2 + D_2^{(n)} \sinh q_i x_2 \right) \right] e^{ik(x_1 - vt)}, \]

\[ u_3(x_1, x_2, t) = \left( D_2^{(n)} \cosh q_0 x_2 + D_3^{(n)} \sinh q_0 x_2 \right) e^{ik(x_1 - vt)}. \]

For the anti-plane case, equations for the fundamental mode of symmetric waves (i.e. \( n = 0 \)) yields

\[ \rho \varepsilon^2 = \gamma_{11} + \gamma_{23} \left( \frac{n \pi}{k h} \right)^2 \]

\[ n = 0, 1, 2, \ldots \]

respectively, where \( q_2 = i q_1 \). Thus, Eq. (46) results in dispersion relations associated with symmetric and anti-symmetric waves given by

\[ k_h = \frac{\pi}{\sqrt{\rho \varepsilon^2}}, \]

\[ \rho \varepsilon^2 = \gamma_{11} + \gamma_{23} \left( \frac{n \pi}{k h} \right)^2. \]

(Eq. 48) corresponds to horizontally polarized shear waves with displacement in the Ox_2-direction. It can be seen that except for the fundamental mode of symmetric waves (i.e. \( n = 0 \)), the squared phase speed of the other modes (i.e. \( n = 1, 2, \ldots \)) for symmetric waves and \( n = 0, 1, 2, \ldots \) for anti-symmetric waves) depends on the wave-number and are thus dispersive. Similarly when \( \theta = \pi/2 \), for the plane strain case, the dispersion relations analogous to Eqs. (42) and (43) can be obtained as

\[ q_2s(q_1)p(q_2)\tilde{c}_2 s_1 - q_1s(q_2)p(q_1)\tilde{c}_1 s_2 = 0 \]

\[ n = 0, 1, 2, \ldots \]

for symmetric waves and

\[ q_2s(q_1)p(q_2)^2\tilde{c}_2 s_1 - q_1s(q_2)p(q_1)^2\tilde{c}_1 s_2 = 0 \]

\[ n = 0, 1, 2, \ldots \]

for anti-symmetric waves, Eqs. (42) and (43) after some manipulation can be shown to agree with Eqs. (4.24) and (4.22) of Roxburgh and Ogden (1994).

For the plane strain case, when the pre-stress is removed from the elastic layer, it can be shown that Eqs. (42) and (49) reduce to the Rayleigh–Lamb frequency equation for symmetric waves and can be written as
The frequencies equations given by Eqs. (52) and (53) can be shown to be identical to Eq. (4.106) of Roxburgh and Ogden (1994).

4. Analysis of dispersion relation

Prior to considering the low wavenumber limit \( kh \to 0 \) and the high wavenumber limit \( kh \to \infty \), the functions \( \tilde{H}(q, \rho v^2) \), \((m, n, m, n) \) appearing in the dispersion relations given by Eqs. (24) and (29) are further rearranged as quadratic and cubic functions of the non-dimensional squared phase speed \( \zeta = \rho v^2/j_{21} \), as follows

\[
\tilde{H}(q_n, \zeta) = -\tilde{H}_0(q_n) + \tilde{H}_1(q_n) \zeta + \tilde{H}_2(q_n) \zeta^2,
\]

and

\[
\tilde{J}(q_m, \zeta) = \tilde{J}_0(q_m) + \tilde{J}_1(q_m) \zeta + \tilde{J}_2(q_m) \zeta^2 + \tilde{J}_3(q_m) \zeta^3.
\]

in which

\[
U = \gamma_{13} \delta_{12} (\gamma_{12} - \sigma_3) + \gamma_{23} (\delta_{12} - \delta_{31}) (\gamma_{13} + \delta_{23} (1 - \sigma_2),
\]

\[
W = (2 \delta_{12} - \delta_{23})(\gamma_{12} - \sigma_3) - \delta_{23} \gamma_{21} \gamma_{21} (1 - \sigma_2)
\]

and

\[
\delta_0 = \delta_0 / j_{21}, \quad \gamma_0 = \gamma_0 / j_{21}, \quad \sigma_i = \sigma_i / j_{21}, (i,j = 1, 2, 3),
\]

\[
\Lambda_1 = \Lambda_1 / j_{21}^2, \quad \Lambda_2 = \Lambda_2 / j_{21}^2, \quad \Lambda_3 = \Lambda_3 / j_{21}^2,
\]

\[
\Lambda_4 = \Lambda_4 / j_{21}^2, \quad \Lambda_5 = \Lambda_5 / j_{21}^2, \quad \Lambda_6 = \Lambda_6 / j_{21}^2.
\]

4.1. Low wavenumber limit \( kh \to 0 \)

When \( kh \to 0 \), the thickness of the layer is very small compared to the wavelength. Thus by considering small argument expansions of the hyperbolic functions in Eq. (24) and using Eq. (54) yields after some algebraic manipulation for symmetric waves,

\[
p(\zeta) (\tilde{J}_1^{(l+\phi)} + \tilde{H}_1^{(l)} j_0) - j_1 j_0 = 0.
\]

where

\[
p(\zeta) = \zeta^2 - \zeta \Lambda_5 + \Lambda_6, \quad H_0 = \zeta^2 \Lambda_0 - \Lambda_0, \quad H_1 = \Lambda_1 \Lambda_2 \Lambda_3 \Lambda_4 (\tilde{J}_1^{(l+\phi)} + \tilde{H}_1^{(l)} j_0) + j_0 (\Lambda_3 \Lambda_4 \Lambda_5 + \Lambda_4 \Lambda_5 \Lambda_6).
\]

Since Eq. (59) yields a cubic equation in \( \zeta \), it is possible to have three finite limiting squared phase speeds when \( kh \to 0 \). Numerical investigation of Eq. (59) shows that two of the roots correspond to the two finite limiting squared phase speeds \( \zeta^{(0)} \) and \( \zeta^{(2)} \), while the third root corresponds to the non-dispersive spurious root arising from \( \phi(q_1, \rho v^2) \phi(q_2, \rho v^2) \phi(q_3, \rho v^2) = 0 \), the common factor removed from the denominator. Similarly, for anti-symmetric waves considering the small argument expansions of the hyperbolic functions of Eq. (29) and using Eq. (54) yields

\[
p(\zeta) (\tilde{J}_1^{(l+\phi)} + (1 + \gamma_{21}) \zeta^2 - \tilde{H}_0^{(l)} + \tilde{H}_1^{(l)} j_0) + j_0 (\Lambda_3 \Lambda_4 \Lambda_5 + \Lambda_4 \Lambda_5 \Lambda_6) = 0.
\]

Here the factor \( p(\zeta) \) corresponds to two non-dispersive spurious roots and the second factor yields a quadratic equation in \( \zeta \) with two possible limiting squared phase speeds when \( kh \to 0 \). Numerical investigation of the second factor of Eq. (61) shows that one of the roots corresponds to the finite limiting squared phase speed \( \zeta^{(0)} \), while the second root corresponds to the non-dispersive spurious root arising from \( \phi(q_1, \rho v^2) \phi(q_2, \rho v^2) \phi(q_3, \rho v^2) = 0 \), the common factor removed from the denominator.

For \( \theta = 0 \), the limiting squared phase speeds for plane strain case as \( kh \to 0 \) are obtained from Eqs. (42) and (43) as

\[
\zeta^{(0)}_{\theta = 0} = \zeta_{11} - \zeta_{12} / \zeta_{22},
\]

for symmetric waves and

\[
\zeta^{(0)}_{\theta = 0} = \zeta_{12} - (1 - \sigma_2)^2,
\]

for anti-symmetric waves, respectively. Eqs. (62) and (63) agree with (26b) and (27b) of Wijeyewickrema et al. (2008).

For the anti-plane case when \( kh \to 0 \), anti-symmetric waves have no limiting squared phase speed while the limiting squared phase speed for symmetric waves can be written from Eq. (48a) as

\[
\zeta^{(0)}_{\theta = 0} = \gamma_{13}.
\]

Similarly, for \( \theta = \pi/2 \), the limiting squared phase speeds for plane strain case as \( kh \to 0 \) are obtained from Eqs. (49) and (50) as
\[ \zeta^{(2)} = \frac{3}{32} - \left(1 - \sigma_2^2\right)^2. \]  
(65)

for symmetric waves and

\[ \zeta^{(2)} = \frac{3}{32} - (1 - \sigma_2^2)^2. \]  
(66)

for anti-symmetric waves, respectively.

For the anti-plane case when \( kh \to 0 \), anti-symmetric waves have no limiting squared phase speed while the limiting squared phase speed for symmetric waves from Eq. (51a) is

\[ \zeta^{(2)(2)} = \frac{3}{32}. \]  
(67)

When there is no pre-stress, for symmetric waves Eqs. (62) and (65) yield \( \zeta^{(2)} = \frac{3}{32} = \frac{1}{24}\sigma_2 (1 + \sigma_2) \), which agrees with Eq. (15) of Lamb (1917), while for anti-symmetric waves Eqs. (63) and (66) yield \( \zeta^{(2)(2)} = \frac{3}{32} = 0 \), which agrees with Eq. (50) of Lamb (1917).

4.2. High wavenumber limit \( kh \to \infty \)

When \( kh \to \infty \), the thickness of the layer is very large compared to the wavelength and the propagation behavior is similar to waves in a semi-infinite medium. Hence, for both symmetric and anti-symmetric waves the same high wavenumber limit is expected. The behavior of the dispersion relations associated with symmetric and anti-symmetric waves in this region depends on the roots \( \zeta_1, \zeta_2, \zeta_3 \), which may either be real, pure imaginary or complex conjugates which in turn depends on the squared phase speed \( \zeta \) and the parameters \( \eta_0, \zeta_1, \zeta_2, j \).

4.2.1. Roots \( \zeta_1, \zeta_2, \) and \( \zeta_3 \) are real or complex conjugates with non-zero real part

When \( kh \to \infty \), the hyperbolic functions in the dispersion relation \( C_m, S_m \to \infty, (m = 1, 2, 3) \). Dividing the dispersion relations associated with symmetric waves (Eq. (24)) and the anti-symmetric waves (Eq. (29)) by \( C_1C_2C_3 \) and taking the limit \( kh \to \infty \) yields,

\[
\begin{align*}
(\zeta_1 &\zeta_2 + \zeta_3 + \zeta_4) (1 + 2\zeta_1^2 - \zeta_2^2 + b_1^2 + a_1^2 + A) \\
&- \zeta_3^2 \zeta_4^2 (e^{2\zeta_3^2} + e^{2\zeta_4^2})^2 (1 + 2\zeta_3^2 + \zeta_4^2 + \zeta_3^2\zeta_4^2 + \zeta_3^2 + \zeta_4^2 + \zeta_3^2\zeta_4^2 + \zeta_3^2 + \zeta_4^2) \nonumber \\
&+ \zeta_1^2 (1 - \zeta_2^2 + \zeta_3^2 + \zeta_4^2) (\zeta_3^2 + \zeta_4^2 + \zeta_3^2\zeta_4^2 + \zeta_3^2 + \zeta_4^2 + \zeta_3^2\zeta_4^2 + \zeta_3^2 + \zeta_4^2) A \\
&- \zeta_1^2 (1 - \zeta_2^2 + \zeta_3^2 + \zeta_4^2) \nonumber \\
&= 0,
\end{align*}
\]  
(68)

where

\[
A = dH_0, \quad B = dH_1, \quad a = \sqrt{p(\xi)}, \quad b = a_1^2, \quad c = a_2^2, \quad d = \sqrt{q(\xi)}, \quad e = J_1, \quad f = J_2. \]  
(69)

The solution of Eq. (68) results in the wave speed \( \zeta_3 \) which corresponds to the Rayleigh surface wave speed. This agrees with previous studies where the dispersion relation yields the surface wave speed at the high wavenumber limit (see Roxburgh and Ogden, 1994).

For \( \theta = 0 \), dividing the dispersion relations corresponding to plane strain case given in Eqs. (42) and (43) by \( C_1C_2 \) and taking the limit \( kh \to \infty \) yields the secular equation for the squared phase speed of Rayleigh surface wave given by

\[ R_0(\zeta) = \frac{3}{32} - (1 - \sigma_2^2)^2 + (\zeta_2 \zeta_1 - \zeta_1^2) \eta_0. \]  
(70)

where \( \zeta_2 = (\zeta_1 - \zeta), \gamma_1 = (\gamma_2 - \zeta), \eta_0 = \sqrt{\frac{\gamma_{1,2}}{\zeta_2}} \). Eq. (70) agrees with Eq. (5.11) of Dowakih and Ogden (1991) and Eq. (42) of Wijeyewickrema and Leungvichcharoen (2009). The root of Eq. (70) is denoted by \( \zeta_0^{(2)} \).

For the anti-plane case, when \( kh \to \infty \) from Eq. (48) the high wavenumber limit for both symmetric and anti-symmetric waves tend to same value and is given by

\[ \zeta_0^{(2)(3)} = \zeta_{13}. \]  
(71)

Similarly, for \( \theta = \pi/2 \), Eqs. (49) and (50) yields the secular equation for the squared phase speed of Rayleigh surface wave given by

\[ R_{1/2}(\zeta) = \frac{3}{32} - (1 - \sigma_2^2)^2 + (\zeta_2 \zeta_1 - \zeta_1^2) \eta_0, \]  
(72)

where \( \zeta_{13} = (\zeta_2 - \zeta), \gamma_{13} = (\gamma_2 - \zeta), \eta_0 = \sqrt{\frac{\gamma_{1,3}}{\zeta_2}} \). The root of Eq. (72) is denoted by \( \zeta_0^{(2)} \).

For the anti-plane case, when \( kh \to \infty \) from Eq. (51) the high wavenumber limit for both symmetric and anti-symmetric waves tend to same value and is given by

\[ \zeta_0^{(2)(3)} = \zeta_{13}. \]  
(73)

4.2.2. At least one of the roots \( \zeta_1, \zeta_2, \) and \( \zeta_3 \) is pure imaginary

At the high wavenumber limit, in general when one of the roots of Eq. (12) is imaginary, the limiting squared phase speed \( \zeta_{13} \) of the compressible layer is given by

\[ \zeta_{13} = \min\{\zeta_{13}, \zeta_{12}\}, \]  
(74)

where \( \zeta_{13} \) and \( \zeta_{12} \) are the limiting squared body wave speeds obtained from Eq. (12) when \( \bar{q} \to 0 \) and are given by

\[ \zeta_{13} = \frac{1}{2} \left( \lambda_3 - \sqrt{\lambda_3^2 - 4\lambda_6} \right), \quad \zeta_{12} = \lambda_4. \]  
(75)

For some propagation angles depending on the choice of material parameters and pre-stress, the numerical calculations for higher modes show that within a narrow region of squared phase speed, two of the roots \( \zeta_2 \) and \( \zeta_1 \) of Eq. (12) are imaginary and \( \zeta_3 \to \zeta_2 \) when \( kh \to \infty \). Thus, another possible limiting squared phase speed \( \zeta_{12} \) for such a scenario can be written as

\[ \zeta_{12} = \frac{1}{\sqrt{6}}, \]  
(76)

where \( \zeta_2 \) is the limiting squared phase speed obtained from \( \zeta_3 \to \zeta_2 \). Hence, the limiting squared phase speed \( \zeta_3 \) of the compressible elastic layer is

\[ \zeta_3 = \min\{\zeta_{13}, \zeta_{12}\}. \]  
(77)

For \( \theta = 0 \), the limiting squared phase speed \( \zeta_0^{(2)} \) of the layer when \( kh \to \infty \) can be written as

\[ \zeta_0^{(2)(2)} = \min\{\zeta_{01}, \zeta_{02}\}. \]  
(78)

where \( \zeta_{01} \) and \( \zeta_{02} \) are the limiting squared body wave speeds obtained from Eq. (34) when \( \bar{q} \to 0 \) and are given by

\[ \zeta_{01}^{(2)} = \frac{1}{\sqrt{6}}, \quad \zeta_{02}^{(2)} = \frac{1}{\sqrt{6}}. \]  
(79)

In this case too, depending on the choice of material parameters and pre-stress, numerical calculations for higher modes show that for a certain range of squared phase speed both the roots \( \zeta_1 \) and \( \zeta_2 \) of Eq. (34) are imaginary and \( \zeta_3 \to \zeta_2 \) when \( kh \to \infty \), so another possible limiting squared phase speed \( \zeta_0^{(2)} \) can be written as

\[ \zeta_0^{(2)} = \zeta_0^{(2)(2)}. \]  
(80)

where \( \zeta_0^{(2)(2)} \) is the limiting squared phase speed obtained from \( \zeta_1 \to \zeta_2 \). Thus, the limiting squared phase speed \( \zeta_0^{(2)} \) of the compressible layer for \( \theta = 0 \) is

\[ \zeta_0^{(2)} = \min\{\zeta_{01}, \zeta_{02}\}. \]  
(81)
Similarly, for $\theta = \pi/2$, the limiting squared phase speed $c_{l_{12}}^{2}$ of the layer when $kh \to \infty$ can be written as

$$c_{l_{12}}^{2,2} = \min \left\{ c_{l_{1}}^{2}, c_{l_{2}}^{2} \right\},$$

where $c_{l_{1}}^{2}$ and $c_{l_{2}}^{2}$ are the limiting squared body wave speeds obtained from equation analogous to Eq. (34) for $\theta = \pi/2$ when $q = 0$ and are given by

$$c_{l_{1}}^{2} = (\lambda_{3}^{3}, c_{l_{2}}^{2} = \gamma_{3}^{2}.$$  

(83)

Here too, depending on the choice of material parameters and pre-stress, numerical calculations for higher modes show that for a certain range of squared phase speed both the roots $q_{1}$ and $q_{2}$ are imaginary and $|q_{1}| = |q_{2}|$ when $kh \to \infty$, so another possible limiting squared phase speed $c_{l_{12}}^{2}$ can be written as

$$c_{l_{12}}^{2} = \frac{\pi}{4},$$

(84)

where $c_{l_{12}}^{2}$ is the limiting squared phase speed obtained from $q_{1} = q_{2}$. Thus, the limiting squared phase speed $c_{l_{12}}^{2}$ of the compressible layer for $\theta = \pi/2$ is

$$c_{l_{12}}^{2} = \min \left\{ c_{l_{1}}^{2}, c_{l_{2}}^{2} \right\}.$$  

(85)

The squared phase speeds of the fundamental mode and the next higher modes are denoted as $c_{l_{1}}^{(n)}$ and $c_{l_{2}}^{(n)}$ for symmetric waves and $c_{l_{1}}^{(n)}$ and $c_{l_{2}}^{(n)}$ for anti-symmetric waves. Hence, when $\theta \neq 0, \pi/2$, the limiting squared phase speeds when $kh \to \infty$ will be given by

$$c_{l_{1}}^{(n)} \cdot s_{1}^{(n)} \cdot s_{2}^{(n)} \cdot S_{A} \cdot s_{1}^{(n)} \cdot s_{2}^{(n)} \cdot S_{R},$$

(86)

The limiting squared phase speeds for $\theta = 0$ when $kh \to \infty$ are given as

$$c_{l_{1}}^{(n)} \cdot s_{1}^{(n)} \cdot s_{2}^{(n)} \cdot S_{A} \cdot s_{1}^{(n)} \cdot s_{2}^{(n)} \cdot S_{R}.$$  

(87)

Similarly, the limiting squared phase speeds for $\theta = \pi/2$ when $kh \to \infty$ are given as

$$c_{l_{1}}^{(n)} \cdot s_{1}^{(n)} \cdot s_{2}^{(n)} \cdot S_{A} \cdot s_{1}^{(n)} \cdot s_{2}^{(n)} \cdot S_{R}.$$  

(88)

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<td>$\lambda_{1} = 2.0, \sigma_{1} = 0.802\mu_{i}$</td>
<td>$\sigma_{1} = 10.0, \sigma_{2} = 5.0, \gamma_{12} = 4.938, \gamma_{22} = 4.938$</td>
</tr>
<tr>
<td>$\lambda_{2} = 0.9, \sigma_{2} = 0.02\mu_{i}$</td>
<td>$\gamma_{11} = 3.1, \gamma_{12} = 3.1, \gamma_{21} = 3.1, \gamma_{22} = 3.1$</td>
</tr>
<tr>
<td>$\lambda_{3} / 0.7, \sigma_{3} = -0.620\mu$</td>
<td>$\gamma_{11} = 1.0, \gamma_{12} = 0.198\mu, \mu = 1.0$</td>
</tr>
</tbody>
</table>

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<tr>
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<th>Limits of non-dimensional squared phase speed $\zeta$ for Example 1.</th>
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<tr>
<td>$\theta$</td>
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<tr>
<td>$\zeta_{1}$</td>
<td>$\zeta_{2}$</td>
</tr>
<tr>
<td>10$^5$</td>
<td>1.062</td>
</tr>
<tr>
<td>15$^5$</td>
<td>0.848</td>
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<tr>
<td>45$^5$</td>
<td>0.232</td>
</tr>
<tr>
<td>75$^5$</td>
<td>0.854</td>
</tr>
</tbody>
</table>

5. Numerical results

For the numerical examples a Blatz–Ko material which belongs to a class of materials known as restricted Hadamard material is used (Willson, 1973b). The strain energy function of a Blatz–Ko material $W_{c}^{(BK)}$ is expressed as (see Eq. (4.91) of Roebuck and Ogden, 1994).

$$W_{c}^{(BK)} = \frac{\mu}{2} (\lambda_{12}^{2} + \lambda_{22}^{2} + \lambda_{33}^{2} + 2 \lambda_{1} \lambda_{2} \lambda_{3} - 5),$$

(89)

where $\mu$ is the classical Lamé modulus. Using Eqs. (4), (57) and (89), the non-dimensional elastic moduli $33, \gamma_{12} (i, j = 1, 2, 3)$, and the non-dimensional principal Cauchy stress $\sigma_{1}$ are expressed as

$$\bar{\sigma}_{11} = 3, \bar{\sigma}_{12} = \bar{\sigma}_{13} = \bar{\sigma}_{23} = \bar{J} \gamma_{11}, \bar{\sigma}_{22} = 3 \gamma_{12}, \bar{\sigma}_{33} = 3 \gamma_{13},$$

(90)

where $\gamma_{12} = \mu / (\bar{J} \gamma_{11})$.

Two examples have been considered in which Example 1 corresponds to a volume increase while Example 2 corresponds to a volume decrease. The squared phase speed for the first 24 modes (i.e. fundamental mode $c_{1}^{(1)}$ and harmonics $c_{1}^{(n)}, n = 2, 3, \ldots, 24$) are shown in log–log plots to clearly show the behavior of the dispersion curves and the low and the high wavenumber limits.

5.1. Example 1

The primary deformations and elastic moduli that are prescribed ($\lambda_{1}, \lambda_{2}, \lambda_{3}, \mu$) and computed ($\sigma_{1}, \sigma_{2}, \gamma_{12}, \gamma_{13}, \gamma_{23}, \gamma_{11}, \gamma_{22}, \gamma_{33}, \gamma_{13}, \gamma_{23}$) are shown in Table 1 and shows a volume increase of 26%. Dispersion curves are presented for the wave propagation directions $\theta = 10^\circ, 15^\circ, 45^\circ, 75^\circ$ and for in-plane wave propagation $\theta = 0$. The limiting squared phase speeds are listed in Table 2. A typical frequency spectrum is also plotted for $\theta = 15^\circ$.

It was noted by Nolde et al. (2004) for small amplitude wave propagation in a pre-stressed compressible elastic layer that wave fronts can be formed in different ways. In particular, wave fronts that arise from the short wave limit ($kh \to \infty$) of all harmonics are referred to as type 1, while wave fronts that form through the combination of harmonics in a narrow wave speed region are referred to as type 2. Such wave front formation can be seen in the numerical results that follow.

In Figs. 2 and 3 dispersion curves are presented for symmetric and anti-symmetric waves for $\theta = 10^\circ$ and $\theta = 15^\circ$, respectively. It can be seen that when $kh \to 0$ the squared phase speeds of the fundamental mode and the next lowest mode of symmetric waves have distinct finite limits i.e., $c_{l_{1}}^{(1)} = c_{l_{2}}^{(1)}$ (see Table 2), and the squared phase speed of only the fundamental mode has a distinct finite limit i.e., $c_{l_{1}}^{(1)} = c_{l_{2}}^{(1)}$ (see Table 2), and the higher modes have infinite squared phase speeds i.e., $c_{l_{1}}^{(n)} \to \infty, (n = 3, 4, \ldots)$ and $c_{l_{2}}^{(n)} \to \infty, (n = 2, 3, \ldots)$. When $kh \to \infty$, it is noted that the squared phase speeds of the fundamental mode of both symmetric and anti-symmetric waves...
approach the same limit $\zeta_R$, which corresponds to the squared phase speed of the Rayleigh surface wave. All the higher modes of both the symmetric and anti-symmetric waves ($n \geq 2$) show oscillatory behavior in the wave speed region $\zeta_L < \zeta < \zeta_{L_1}$ (i.e. type 2 behavior). For the given pre-stress (see Table 1), this oscillatory region is bounded by $1.740 < \zeta < 2.538$ in Fig. 2 and by $1.637 < \zeta < 2.130$ in Fig. 3. It has been observed that within this region the roots $\tilde{\zeta}_2$ and $\tilde{\zeta}_3$ are imaginary and $|\tilde{\zeta}_2| = |\tilde{\zeta}_3|$ as $kh \to \infty$.

Thus the limiting squared phase speed for the lower bound of this region can be obtained from Eq. (76). This type of oscillatory behavior in the short wave region has been previously observed for both incompressible and compressible pre-stressed elastic layers for wave propagation along principal directions (Rogerson, 1997; Kaplunov et al., 2002 and Nolde et al., 2004).

Figs. 4 and 5 show the dispersion curves for $\theta = 45^\circ$ and $\theta = 75^\circ$, respectively. In the long wave region (i.e. when $kh \to 0$), similar to

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**Fig. 2.** Dispersion curves when $\theta = 10^\circ$ for Example 1. (a) Symmetric waves, (b) anti-symmetric waves.

**Fig. 3.** Dispersion curves when $\theta = 15^\circ$ for Example 1. (a) Symmetric waves, (b) anti-symmetric waves.

**Fig. 4.** Dispersion curves when $\theta = 45^\circ$ for Example 1. (a) Symmetric waves, (b) anti-symmetric waves.
\( \theta = 10^\circ \) and \( 15^\circ \) the squared phase speeds of the fundamental mode and the next lower mode of symmetric waves have distinct finite limits. In the case of anti-symmetric waves, the squared phase speed of the fundamental mode for \( h = 4.5 \) has a distinct finite limit, however for \( h = 7.5 \), the squared phase speed of the fundamental mode \( n_{(1)} \) corresponds to a negative limiting squared phase speed \( n_{(0)} = -2.544 \). When the squared phase speed is negative the elastic layer is unstable as the waves no longer propagate but are standing waves with amplitudes that increase exponentially with time. The other higher modes have infinite squared phase speeds i.e. \( n_{(n)} \rightarrow \infty \) \( (n = 3, 4, \ldots) \) and \( n_{(n)} \rightarrow \infty \) \( (n = 2, 3, \ldots) \). In the short wave region (i.e. when \( kh \rightarrow \infty \)), for both symmetric and anti-symmetric waves the squared phase speed of the fundamental modes correspond to the squared phase speed of Rayleigh surface wave.
and the squared phase speed of the higher modes correspond to the limiting squared phase speed $c_n$ (type 1 behavior).

The three finite squared phase speeds in the high wave region observed for $\theta = 10^\circ$, $15^\circ$ and $45^\circ$ is a unique feature for wave propagation along a non-principal direction and this feature has also been observed for a pre-stressed incompressible layer (Rogerson and Sandiford, 1999).

Fig. 6 shows the dispersion curves for $\theta = 0$, obtained from Eqs. (42), (43) and (48) when wave propagation is along $x_1$-axis. The limiting squared phase speeds are listed in Table 3. When $kh \to 0$, for the plane strain case, the squared phase speeds of the fundamental mode for both symmetric and anti-symmetric waves have distinct finite limits i.e. $c_n^{[2]} \to c_0^{[2]}$, $c_n^{[3]} \to c_0^{[3]}$ (see Table 3), and the higher modes have infinite squared phase speeds. However, for anti-plane case, the squared phase speed of the fundamental mode of only symmetric waves has a distinct finite limit i.e. $c_n^{[1]} \to c_0^{[1]}$ (SH). As noted for $\theta = 10^\circ$ and $15^\circ$ in Figs. 2 and 3, in this case too the higher modes of both symmetric and anti-symmetric waves show oscillatory behavior when $kh \to \infty$ in a narrow wave speed region (type 2 behavior). This wave speed region is bounded by $1.824 < \xi < 3.0$ and within this region the roots $q_1$ and $q_2$ are imaginary with $|q_1| = |q_2|$ as $kh \to \infty$. For the solution arising from the horizontally polarized shear wave, the limiting squared phase speed in short wave region ($kh \to \infty$) corresponds to the squared phase speed $c_n^{[1]}$ of shear wave.

In Fig. 7, frequency spectra for symmetric and anti-symmetric waves for $\theta = 15^\circ$ are presented. It can be seen clearly from Fig. 7(a) that in the long wave region when $kh \to 0$, frequencies of the fundamental mode and the next lowest mode tend to zero i.e., $\Omega_n^{[1]} \to 0$, $\Omega_n^{[2]} \to 0$, corresponding to the finite limiting squared phase speeds $c_{n1} = 0.848$ and $c_{n2} = 5.960$ respectively. However, in Fig. 7(b) frequency of only the fundamental mode tends to zero i.e., $\Omega_n^{[1]} \to 0$, corresponding to the finite limiting squared phase speed $c_{n1}^{[1]} = 3.652$. The frequencies of the other higher modes tend to cut-off frequencies.

### 5.2. Example 2

The primary deformations and elastic moduli that are prescribed ($\sigma_i, \mu$) and computed ($\lambda_1, \lambda_2, \lambda_3, \lambda_l, \gamma_i, l, j \in 1, 2, 3$) are shown in Table 4. In this example, the pre-stressed layer is 

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### Table 3

Limits of non-dimensional squared phase speed $\xi$ for propagation angle $\theta = 0$.  

<table>
<thead>
<tr>
<th>$kh \to 0$</th>
<th>$kh \to \infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_n^{[1]}$</td>
<td>$c_n^{[2]}$</td>
</tr>
<tr>
<td>$c_n^{[1]}$</td>
<td>$c_n^{[2]}$</td>
</tr>
<tr>
<td>$c_n^{[1]}$</td>
<td>$c_n^{[2]}$</td>
</tr>
</tbody>
</table>

### Table 4

Parameters for Example 2.  

<table>
<thead>
<tr>
<th>Primary deformations</th>
<th>Elastic moduli</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_1 = 0.826, \sigma_1 = -0.5\mu$</td>
<td>$\lambda_2 = 3.0, \lambda_{12} = 0.667, \gamma_{12} = 1.0, \gamma_{22} = 1.0$</td>
</tr>
<tr>
<td>$\lambda_2 = 0.826, \sigma_2 = -0.5\mu$</td>
<td>$\lambda_3 = 3.0, \lambda_{13} = 0.667, \gamma_{13} = 1.0, \gamma_{23} = 1.0, \gamma_{33} = 1.0$</td>
</tr>
<tr>
<td>$\lambda_3 = 1.431, \sigma_3 = -0.5\mu$</td>
<td>$\lambda_4 = 3.0, \lambda_{14} = 0.667, \gamma_{14} = 1.0, \gamma_{24} = 1.0, \gamma_{34} = 1.0, \gamma_{44} = 1.0$</td>
</tr>
</tbody>
</table>

---

### Fig. 8

Displacement curves when $\theta = 15^\circ$ for Example 2. (a) Symmetric waves, (b) anti-symmetric waves.

### Fig. 9

Displacement curves when $\theta = 45^\circ$ for Example 2. (a) Symmetric waves, (b) anti-symmetric waves.
equi-biaxially deformed in \((x_1; x_2)\)-plane and the layer undergoes a volume decrease of 2.33%.

**Table 5**

Limits of non-dimensional squared phase speed \(\xi\) for Example 2.

<table>
<thead>
<tr>
<th>(\theta)</th>
<th>(kh \to 0)</th>
<th>(kh \to \infty)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(\xi^{(0)})</td>
<td>(\xi^{(0)})</td>
</tr>
<tr>
<td>15°</td>
<td>0.331</td>
<td>0.3337</td>
</tr>
<tr>
<td>45°</td>
<td>0.367</td>
<td>0.379</td>
</tr>
<tr>
<td>75°</td>
<td>0.637</td>
<td>0.544</td>
</tr>
</tbody>
</table>

**Fig. 10.** Dispersion curves when \(\theta = 75°\) for Example 2. (a) Symmetric waves, (b) anti-symmetric waves.

The dispersive behavior of small amplitude waves propagating along a non-principal direction in a pre-stressed, compressible elastic layer with traction free surfaces is studied. The dispersion relations associated with symmetric and anti-symmetric waves are given in explicit form and differ from each other only through the hyperbolic terms. The limiting squared phase speeds at the low and high wavenumber limits are discussed in detail. In the low wavenumber region, depending on the pre-stress and propagation angle two finite limiting squared phase speeds may exist for symmetric waves, while only one finite limiting squared phase speed may exist for anti-symmetric waves. It is found that the behavior of the dispersion curves for symmetric and anti-symmetric waves is similar in the high wavenumber region, where both the symmetric and anti-symmetric waves tend to same limiting squared phase speeds, viz. the phase speeds of the fundamental mode and the higher modes tend to the phase speeds of Rayleigh surface wave and the limiting phase speeds of the layer, respectively. The special cases \(\theta = 0\) and \(\pi/2\), i.e. when wave propagation is along one of the principal directions are also considered, where the equations corresponding to the plane strain case are uncoupled from the equation associated with horizontally polarized shear waves.

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**Appendix A**

Expressions for the displacements and stresses for the symmetric and anti-symmetric waves can be obtained from Eqs. 15(a–c) and 17(a–c).

Symmetric deformations:

\[
\begin{align*}
\mathbf{u}_1(x_1, x_2, x_3, t) &= \frac{3}{m} \left[ \bar{\mathbf{q}}_b \{ \delta_{13} \} \sin^2 \theta + \bar{g}(Q, \rho \nu^2) \delta_{12} \right] B_0 \cos \theta \sin x_1 \\
&\times \cos \theta \left[ \frac{\delta_{13}}{\cos \theta} \right] \frac{\sin \theta}{\sqrt{\rho_0}} \\
&\times \left[ \sin \theta \sin \theta \right] \\
&\times \left[ \frac{\delta_{13}}{\cos \theta} \right] \frac{\sin \theta}{\sqrt{\rho_0}} \\
&\times \left[ \frac{\delta_{13}}{\cos \theta} \right] \frac{\sin \theta}{\sqrt{\rho_0}} \\
&\times \left[ \frac{\delta_{13}}{\cos \theta} \right] \frac{\sin \theta}{\sqrt{\rho_0}}
\end{align*}
\]

(a1)
\[ u_2(x_1, x_2, x_3, t) = \frac{1}{n-1} \sum \left( \beta_n^2 \sin \phi_n \cos \phi_n \sin \phi_n \right) \left( \frac{\beta_n^2 \sin \phi_n}{\phi_n \beta_n^2} \right) e^{ik_n(x_1 \cos \phi_n + x_3 \sin \phi_n - vt)}, \] (A2)

\[ u_3(x_1, x_2, x_3, t) = \frac{1}{n-1} \sum \left( \frac{\beta_n^2 \sin \phi_n}{\phi_n \beta_n^2} \right) \left( \frac{\beta_n^2 \sin \phi_n}{\phi_n \beta_n^2} \right) e^{ik_n(x_1 \cos \phi_n + x_3 \sin \phi_n - vt)}, \] (A3)

\[ \frac{\sigma_0}{i k_n}(x_{12} x_{22} x_{13}) e^{ik_n(x_1 \cos \phi_n + x_3 \sin \phi_n - vt)}, \] (A4)

\[ \frac{\sigma_0}{i k_n}(x_{12} x_{22} x_{13}) e^{ik_n(x_1 \cos \phi_n + x_3 \sin \phi_n - vt)}, \] (A5)

\[ \frac{\sigma_0}{i k_n}(x_{12} x_{22} x_{13}) e^{ik_n(x_1 \cos \phi_n + x_3 \sin \phi_n - vt)}, \] (A6)

where \( \beta_n^2 = A_2^{(n-1)} A_2^{(2n)}, (n = 1, 2, 3). \)

Anti-symmetric deformations:

\[ u_2(x_1, x_2, x_3, t) = \frac{1}{n-1} \sum \left( \beta_n^2 \sin \phi_n \cos \phi_n \sin \phi_n \right) \left( \frac{\beta_n^2 \sin \phi_n}{\phi_n \beta_n^2} \right) e^{ik_n(x_1 \cos \phi_n + x_3 \sin \phi_n - vt)}, \] (A7)

\[ u_2(x_1, x_2, x_3, t) = \frac{1}{n-1} \sum \left( \beta_n^2 \sin \phi_n \cos \phi_n \sin \phi_n \right) \left( \frac{\beta_n^2 \sin \phi_n}{\phi_n \beta_n^2} \right) e^{ik_n(x_1 \cos \phi_n + x_3 \sin \phi_n - vt)}, \] (A8)

\[ u_2(x_1, x_2, x_3, t) = \frac{1}{n-1} \sum \left( \beta_n^2 \sin \phi_n \cos \phi_n \sin \phi_n \right) \left( \frac{\beta_n^2 \sin \phi_n}{\phi_n \beta_n^2} \right) e^{ik_n(x_1 \cos \phi_n + x_3 \sin \phi_n - vt)}, \] (A9)

\[ u_3(x_1, x_2, x_3, t) = \frac{1}{n-1} \sum \left( \beta_n^2 \sin \phi_n \cos \phi_n \sin \phi_n \right) \left( \frac{\beta_n^2 \sin \phi_n}{\phi_n \beta_n^2} \right) e^{ik_n(x_1 \cos \phi_n + x_3 \sin \phi_n - vt)}, \] (A10)

\[ \frac{\sigma_0}{i k_n}(x_{12} x_{22} x_{13}) e^{ik_n(x_1 \cos \phi_n + x_3 \sin \phi_n - vt)}, \] (A11)

\[ \frac{\sigma_0}{i k_n}(x_{12} x_{22} x_{13}) e^{ik_n(x_1 \cos \phi_n + x_3 \sin \phi_n - vt)}, \] (A12)

\[ \frac{\sigma_0}{i k_n}(x_{12} x_{22} x_{13}) e^{ik_n(x_1 \cos \phi_n + x_3 \sin \phi_n - vt)}, \] (A13)

where \( \beta_n^2 = A_2^{(n-1)} + A_2^{(2n)}, (n = 1, 2, 3). \)

References

