Hopf bifurcation and steady-state bifurcation for an autocatalysis reaction–diffusion model

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1. Introduction

In recent years, the autocatalysis models received much attention, which have been used in various forms in the studies of morphogenesis, population dynamics and autocatalysis oxidation reactions. If the initial concentration of reactant is fixed and the reaction rates are the same, then the simplified and non-dimensional form of arbitrary order autocatalysis model can be stated as follows:

$$\begin{align*}
  u_t - d_1 \Delta u &= a - u v^p, & x \in \Omega, \ t > 0, \\
  v_t - d_2 \Delta v &= u v^p - v, & x \in \Omega, \ t > 0, \\
  \partial_\nu u = \partial_\nu v &= 0, & x \in \partial \Omega, \ t > 0, \\
  u(x, 0) &= u_0(x) \geq 0, & \ x \in \Omega, \\
  v(x, 0) &= v_0(x) \geq 0, & \ x \in \Omega,
\end{align*}$$

(1.1)

where \(\Omega\) is a bounded domain in \(\mathbb{R}^N\) (\(N \geq 1\)) with smooth boundary \(\partial \Omega\); the variables \(u, v\) represent the dimensionless densities of reactant and autocatalyst respectively, which are usually considered to be non-negative. The reactor is assumed to be closed, thus no-flux boundary conditions are imposed and \(\partial_\nu = \partial / \partial v\) represents the outer normal derivative. The parameter \(a\) is taken as the initial concentration of reaction precursor, \(p\) denotes the order of the reaction with respect to the autocatalytic species, \(d_1\), \(d_2\) are the diffusion coefficients; \(a\), \(p\), \(d_1\) and \(d_2\) are positive constants. Here, we refer the interested readers to [1,2] and references therein for a more detailed description for the derivation of this model.
It is not difficult to see that (1.1) is very similar to the Sel’kov model

\[
\begin{align*}
\frac{\partial u}{\partial t} - \theta \Delta u &= \lambda (1 - u v^p), & x \in \Omega, & t > 0, \\
\frac{\partial v}{\partial t} - \Delta v &= \lambda (u v^p - v), & x \in \Omega, & t > 0, \\
\frac{\partial u}{\partial \nu} &= \frac{\partial v}{\partial \nu} = 0, & x \in \partial \Omega, & t > 0, \\
u(x, 0) &= u_0(x) \geq 0, & v(x, 0) &= v_0(x) \geq 0, & x \in \Omega.
\end{align*}
\]

(1.2)

In fact, the corresponding elliptic equation of (1.2) is a special case of the corresponding elliptic equation of (1.1) by letting \(a = 1, d_1/d_2 = \theta \) and \(1/d_2 = \lambda \). As we know, the Sel’kov model (1.2) has been extensively studied in the past several years, see [3–5].

In addition, there are many works on other autocatalysis models. For example, see [6–8] for the general Brusselator model, see [9–12] for the Lengyel–Epstein model, see [13–15] for a bimolecular model with saturation law. More specifically, the existence and non-existence of non-constant positive steady-state solutions were studied in [6,7,9,10,14]; Hopf bifurcation analyses were performed in [8,11,12,15]. In particular, both Hopf bifurcation and steady-state bifurcation were considered in [15].

Motivated by the papers [4,5,10–12,15] mentioned above, we will carry out Hopf bifurcation and steady-state bifurcation analyses on the autocatalysis model (1.1). For Hopf bifurcation, the existence and stability of spatially homogeneous and inhomogeneous periodic solutions are obtained; for steady-state bifurcation, we mainly focus on the bifurcation from double eigenvalues. In fact, there are lots of steady-state bifurcation results for many different reaction–diffusion models. But, the assumption of simple eigenvalues was always imposed (see [4,10,15,16]) so that the Crandall–Rabinowitz bifurcation theorem [17] can be directly applied. In the case of double eigenvalues, there are no ready-made theory to our knowledge. Our proof is based on the techniques of space decomposition and implicit function theorem, which have been used in [18] to obtain the existence of positive solutions for predator–prey diffusion equations under Dirichlet boundary conditions. It must be pointed out that the linearized operator in [18] is actually a diagonal matrix, which is much easier to deal with than that in our paper.

The rest of the paper is arranged as follows. In Section 2, the existence and stability of Hopf bifurcating periodic solutions are established; Section 3 is devoted to considering steady-state bifurcations from simple and double eigenvalues; we illustrate our results with numerical simulations in Section 4.

2. Hopf bifurcation analysis

Clearly, \(U^* = (u^*, v^*) = (a^{1-p}, a)\) is the unique positive constant solution of (1.1). In this section, we first derive Turing instability with respect to the equilibrium solution \(U^*\) and then establish the existence and stability of spatially homogeneous and inhomogeneous periodic solutions of (1.1).

We restrict ourselves to the case of one-dimensional spatial domain \(\Omega = (0, \pi)\), for which the structure of the eigenvalues is clear, and consider the diffusive system

\[
\begin{align*}
\frac{\partial u}{\partial t} - d_1 \frac{\partial^2 u}{\partial x^2} &= a - u v^p, & x \in (0, \pi), & t > 0, \\
\frac{\partial v}{\partial t} - d_2 \frac{\partial^2 v}{\partial x^2} &= u v^p - v, & x \in (0, \pi), & t > 0, \\
u(x, 0) &= v(x, 0) = 0, & x = 0, \pi, & t > 0, \\
u(x, 0) &= u_0(x) \geq 0, & v(x, 0) &= v_0(x) \geq 0, & x \in (0, \pi).
\end{align*}
\]

(2.1)

Our results below can be adapted to higher spatial domains.

It is well known that the operator \(u \mapsto -u_{xx}\) with no-flux boundary conditions has eigenvalues and eigenfunctions as follows:

\[
\mu_k = 0, \quad \phi_k(x) = \sqrt{\frac{1}{\pi}}, \quad \mu_k = k^2, \quad \phi_k(x) = \sqrt{\frac{2}{\pi}} \cos kx
\]

for \(k = 1, 2, 3, \ldots\). The linearized system of (2.1) at \(U^*\) has the form

\[
\begin{pmatrix}
u \\
\frac{\partial u}{\partial t} \\
\frac{\partial v}{\partial t}
\end{pmatrix} =
L \begin{pmatrix}
u \\
\frac{\partial u}{\partial x} \\
\frac{\partial v}{\partial x}
\end{pmatrix} := D \begin{pmatrix}u_{xx} \\
v_{xx}
\end{pmatrix} + \begin{pmatrix} -a_p \\
0 \\
-1
\end{pmatrix} \begin{pmatrix}u \\
v
\end{pmatrix},
\]

(2.2)

where \(D = \text{diag}(d_1, d_2)\) is the diffusion matrix.

Consider the characteristic equation \(L(\phi, \psi) = \rho(\phi, \psi)\) and let \((\phi, \psi) = \sum_{k=0}^{\infty} (r_k, s_k) \cos kx\). Then we obtain

\[
\sum_{k=0}^{\infty} (J_k - \rho I)(r_k, s_k)^2 \cos kx = 0,
\]

where \(J_k := J - k^2 D = \begin{pmatrix} -a_p - d_k k^2 \\
a_p \\
-1 - d_k k^2
\end{pmatrix}\).

It is clear to see that all the eigenvalues of \(L\) are given by the eigenvalues of \(J_k\) for \(k = 0, 1, 2, \ldots\). Note that the characteristic equation of \(J_k\) is
\[ \rho^2 - T_k \rho + D_k = 0, \quad k = 0, 1, 2, \ldots, \tag{2.3} \]

\[ T_k = p - 1 - d^2 - (d_1 + d_2)k^2, \quad D_k = d_1 d_2 k^4 + [d_2 a^p - d_1 (p - 1)]k^2 + a^p. \]

By analyzing the distribution of the roots of (2.3), we can obtain the following result.

**Theorem 2.1.**

(i) If \(0 < p \leq 1\), then \(U^*\) is asymptotically stable for (2.1).

(ii) Assume that \(p > 1\) and \(a > a_0\), where \(a_0 = (p - 1)^{1/p}\), so that \(U^*\) is asymptotically stable for the ODE system corresponding to (2.1). Then \(U^*\) is an unstable equilibrium solution of (2.1) if there exists \(k \geq 1\) such that

\[ d_1 d_2 k^4 + [d_2 a^p - d_1 (p - 1)]k^2 + a^p < 0, \tag{2.4} \]

and \(U^*\) is asymptotically stable if

\[ 0 < d_1/d_2 < a^p/(\sqrt{p} - 1)^2 \tag{2.5} \]

or

\[ d_2 a^p - d_1 (p - 1) + d_1 d_2 \geq 0. \tag{2.6} \]

**Proof.** (i) If \(0 < p \leq 1\), then \(T_k < 0\) and \(D_k > 0\) for any \(k = 0, 1, 2, \ldots\). Hence, the roots of (2.3) have negative real parts, which implies that \(U^*\) is asymptotically stable.

(ii) When \(p > 1\) and \(a > a_0\), we have \(T_k < 0\) for any \(k \geq 0\). If there exists \(k \geq 1\) such that (2.4) holds true, then \(D_k < 0\) for this \(k\). Hence, (2.2) has at least a positive root, which implies that \(U^*\) is unstable.

Let \(f(s) = d_1 d_2 s^2 + [d_2 a^p - d_1 (p - 1)]s + a^p\). Denote \(\Delta_f\) by the discriminant of \(f(s)\). If \(\Delta_f = [d_2 a^p - d_1 (p - 1)]^2 - 4d_1 d_2 a^p < 0\), then for any \(s \in \mathbb{R}\), \(f(s) > 0\). At this time, we obtain

\[ |d_2 a^p - d_1 (p - 1)| < 2a^{p/2}\sqrt{d_1 d_2}. \tag{2.7} \]

If \(\Delta_f > 0\), then the equation \(f(s) = 0\) has two real roots, and when these two roots are negative, we have that for any \(s \geq 0\), \(f(s) > 0\). At this time, we obtain

\[ d_2 a^p - d_1 (p - 1) \geq 2a^{p/2}\sqrt{d_1 d_2}. \tag{2.8} \]

It readily follows from (2.7) and (2.8) that if

\[ d_2 a^p - d_1 (p - 1) + 2a^{p/2}\sqrt{d_1 d_2} > 0 \tag{2.9} \]

holds, then \(f(s) > 0\) for any \(s \geq 0\). Since (2.9) is equivalent to

\[ (p - 1)(\sqrt{d_1/d_2})^2 - 2a^{p/2}\sqrt{d_1/d_2} - a^p < 0, \]

noting that \(p > 1\), we have \(f(s) > 0\) for any \(s \geq 0\) if (2.5) holds.

\(f(s)\) can be written as \(f(s) = s[d_1 d_2 s + d_2 a^p - d_1 (p - 1)] + a^p\). Clearly, \(f(\mu_0) = f(0) = a^p > 0\). When (2.6) holds, \(f(\mu_1) = f(1) > a^p > 0\) and for any \(k \geq 2\),

\[ f(\mu_k) \geq \mu_k \left[d_1 d_2 + d_2 a^p - d_1 (p - 1)\right] + a^p \geq a^p > 0. \]

Let \(\zeta_k(\rho) = \rho^2 - T_k \rho + D_k\). From arguments above, if (2.5) or (2.6) holds, then we have \(D_k > 0\) for any \(k \geq 0\). Note that \(T_k < 0\) for any \(k\) since \(p > 1\) and \(a > a_0\). Therefore, for any \(k \geq 0\), \(\zeta_k(\rho) = 0\) has two roots \(\rho_{1,k}\) and \(\rho_{2,k}\) with negative real parts. Next, we shall prove that there exists \(\delta > 0\), which is independent of \(k\), such that \(\text{Re}(\rho_{1,k}), \text{Re}(\rho_{2,k}) \leq -\delta\). Let \(\rho = k^2 \xi\). Then we have \(\zeta_k(\rho) = k^4 \xi^2 - k^2 \xi T_k + D_k = \psi_k(\xi)\). A simple calculation gives

\[ \lim_{k \to \infty} \frac{\psi_k(\xi)}{k^4} = \xi^2 + (d_1 + d_2)\xi + d_1 d_2. \]

It is easy to see that two roots of \(\psi_k(\xi)\) have negative real parts when \(k\) is large enough. Hence, there exists a positive constant \(\delta\) such that \(\text{Re}(\rho_{1,k}), \text{Re}(\rho_{2,k}) \leq -\delta\), which shows that \(U^*\) is asymptotically stable. The proof is completed. \hfill \Box

Now, we look for Hopf bifurcating periodic solutions of (2.1) when \(p > 1\). Let \(\rho = \alpha(a) \pm i\beta(a)\) be the roots of (2.3). Obviously, \(\alpha(a) = T_k(a)/2\) and \(\alpha'(a) = -pa^{p-1}/2 < 0\). If there exists some \(k = 0, 1, 2, \ldots\) such that

\[ (d_1 + d_2)k^2 < p - 1, \tag{2.10} \]

letting \(\alpha_k^H = (p - 1 - (d_1 + d_2)k^2)^{1/p}\) be the roots of \(T_k(a) = 0\), then we have
The transverse condition is always satisfied. We only need to verify whether \( D_i(a_k^H) \neq 0 \) for \( i = 0, 1, 2, \ldots \). Here, we obtain a condition on the parameters so that \( D_i(a_k^H) > 0 \). In fact, if the following inequality
\[
d_1d_2 + (d_2 - d_1)(p - 1) - d_2(d_1 + d_2)k^2 > 0
\] (2.11)
holds, then
\[
D_i(a_k^H) > i^2(d_1d_2 + [d_2(a_k^H)^p - d_1(p - 1)])(a_k^H)^p > 0.
\]
Hence, the condition \((H_1)\) in [16] is satisfied, which implies that (2.1) undergoes a Hopf bifurcation at \( a = a_0^H \). Apparently, \( \alpha = a_0^H = a_0 \) is always the unique value for the Hopf bifurcation of spatially homogeneous periodic solution to (2.1).

**Theorem 2.2.** Assume \( p > 1 \). If there exists some \( k = 0, 1, 2, \ldots \) such that (2.10) and (2.11) hold, then the system (2.1) undergoes a Hopf bifurcation at \( a = a_k^H := (p - 1) - (d_1 + d_2)k^2/p \). Moreover, the bifurcating periodic solutions from \( a = a_0^H \) are spatially homogeneous, which coincides with the periodic solutions of the corresponding ODE system, and the bifurcating periodic solutions from \( a = a_k^H \) (\( k > 1 \)) are spatially inhomogeneous.

Next we consider the direction of Hopf bifurcation and the stability of bifurcating periodic solutions.

**Theorem 2.3.** For the system (2.1), the Hopf bifurcation at \( a = a_0^H \) is subcritical, and the spatially homogeneous bifurcating periodic solutions are asymptotically stable.

**Proof.** From Section 2 in [16], the bifurcation direction and the stability of the bifurcating periodic solutions are determined by \( \rho''(0) \), which is given by
\[
\rho''(0) = -\frac{1}{\alpha'(a_0^H)} \text{Re}(c_1(a_0^H)),
\]
\[
c_1(a_0^H) = \frac{i}{2\beta_0}(q^*_{\alpha}, q_{\alpha}, q_{\alpha}) + |q_{\alpha}|^2 + \sqrt{\frac{1}{2\pi} \int_0^\pi q_{\alpha}^2 q_{\alpha}^2 q_{\alpha} d\alpha}.
\]
And from Theorem 2.1 in [16], the bifurcation is supercritical (resp. subcritical) if
\[
\frac{1}{\alpha'(a_0^H)} \text{Re}(c_1(a_0^H)) < 0 \quad \text{(resp.} \quad > 0) \quad \text{(2.12)}
\]
and in addition, all other eigenvalues of \( L(a_0^H) \) have negative real parts, then the bifurcating periodic solutions are stable (resp. unstable) if \( \text{Re}(c_1(a_0^H)) < 0 \) (resp. \( > 0 \)).

Let \( L^* \) be the conjugate operator of \( L \) defined in (2.2),
\[
L^*(a_0^H) \begin{pmatrix} u \\ v \end{pmatrix} := D \begin{pmatrix} u_{xx} \\ v_{xx} \end{pmatrix} + \begin{pmatrix} 1 - p & p - 1 \\ -p & p - 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} u_{xx} \\ v_{xx} \end{pmatrix}.
\]
Introduce
\[
q := \begin{pmatrix} r_0 \\ s_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 - p - i\sqrt{p^2 - 1} \end{pmatrix}, \quad q^* := \begin{pmatrix} r_0^* \\ s_0^* \end{pmatrix} = \frac{1}{2\pi} \begin{pmatrix} 1 - i\sqrt{p^2 - 1} \\ \frac{-pi}{\sqrt{p^2 - 1}} \end{pmatrix}
\]
satisfying \( \langle q^*, q \rangle = 1, \langle q^*, q \rangle = 0, L(a_0^H)q = i\beta_0 q \) and \( L^*(a_0^H)q^* = -i\beta_0 q^* \), where \( \beta_0 := \beta(a_0^H) = \sqrt{p^2 - 1}, \langle q, q \rangle = \int_0^\pi \overline{q}^2 q d\alpha \) denotes the inner product in \( L^2(0, \pi) \times L^2(0, \pi) \). Recall that in our context, \( f(a, u, v) = -uv^p, g(a, u, v) = uv^p - v \). Then we can get the derivatives at \((a_0^H, U^*)\) as follows:
\[
f_{uv} = f_{vv} = -p(p - 1)^{-1/2} = -g_{uv}, \quad f_{uv} = -p(p - 1)^{-2/2} = -g_{uvv}.
\]
\[
f_{vvv} = -p(p - 2)(p - 1)^{-2/2} = -g_{vvv}, \quad f_{uu} = f_{uv} = g_{uu} = g_{uuv} = g_{uuv} = 0.
\]
Define
\[
Q_{qq} = \begin{pmatrix} c_k \\ d_k \end{pmatrix} \cos^2 kx, \quad Q_{qq} = \begin{pmatrix} e_k \\ f_k \end{pmatrix} \cos^2 kx, \quad C_{qqq} = \begin{pmatrix} g_k \\ h_k \end{pmatrix} \cos^2 kx.
\]
where
Clearly, techniques of space decomposition and implicit function theorem are applied. The bifurcation theorem will be used to derive bifurcations from simple eigenvalues. For the case of double eigenvalues, the calculation of Re \( a \) is subcritical (supercritical) if Re \( a \) is unstable since it is very lengthy.

For \( k = 0 \),
\[
c_0 = \frac{1}{p} (p - 1)^{-\frac{1}{2}} \left( p + 2 + \frac{2i}{\sqrt{p - 1}} \right) = -d_0.
\]
\[
e_0 = (p - 1)^{\frac{1}{2} - \frac{1}{2}} = -f_0.
\]
\[
g_0 = -(p - 1)^{\frac{3}{2} - \frac{1}{2}} (2\sqrt{p - 1} + i) = -h_0.
\]

Further calculations lead to
\[
\langle q^*, Q_{qq} \rangle = \frac{\sqrt{p - 1} - i}{2\sqrt{p - 1}} c_0 = \frac{1}{2} (p - 1)^{-\frac{1}{2}} (p + 1 - i\sqrt{p - 1}),
\]
\[
\langle q^*, Q_{q} q \rangle = \frac{\sqrt{p - 1} - i}{2\sqrt{p - 1}} e_0 = \frac{1}{2} (p - 1)^{\frac{3}{2} - \frac{1}{2}} (\sqrt{p - 1} - i),
\]
\[
\langle \bar{q}^*, Q_{qq} \rangle = \frac{\sqrt{p - 1} + i}{2\sqrt{p - 1}} c_0 = \frac{1}{2p} (p - 1)^{\frac{3}{2} - \frac{1}{2}} \left[ (p + 2)\sqrt{p - 1} - \frac{2}{\sqrt{p - 1}} + (p + 4i) \right].
\]
\[
\langle \bar{q}^*, Q_{q} Q_{q} \rangle = \frac{\sqrt{p - 1} + i}{2\sqrt{p - 1}} e_0 = \frac{1}{2} (p - 1)^{\frac{3}{2} - \frac{1}{2}} (\sqrt{p - 1} + i),
\]
\[
\langle q^*, C_{qq} q \rangle = \frac{\sqrt{p - 1} - i}{2\sqrt{p - 1}} g_0 = -\frac{1}{2} (p - 1)^{\frac{3}{2} - \frac{1}{2}} (2p - 1 - i\sqrt{p - 1}).
\]

Define \( \hat{q} := [2i\beta_0 I - L(a^0_0)^{-1}] w_{20} \), \( \hat{q} := -[L(a^0_0)^{-1}] w_{11} \), where
\[
w_{20} = Q_{qq} - \langle q^*, Q_{qq} \rangle q - \langle \bar{q}^*, Q_{qq} \rangle \bar{q}, \quad w_{11} = Q_{qq} - \langle q^*, Q_{qq} \rangle q - \langle \bar{q}^*, Q_{qq} \rangle \bar{q}.
\]

Direct calculations give \( w_{20} = 0, w_{11} = 0 \). Then we obtain \( \langle q^*, Q_{qq} \rangle = \langle q^*, Q_{qq} \rangle = 0 \). By calculations above and recalling that \( p > 1 \), we get
\[
\text{Re}(c_1(a^0_0)) = -\frac{1}{2\beta_0} \text{Im} \left[ \langle q^*, Q_{qq} \rangle \cdot \langle q^*, Q_{qq} \rangle \right] + \frac{1}{2} \text{Re} \langle q^*, C_{qq} q \rangle
\]
\[
= -\frac{1}{2\sqrt{p - 1}} \text{Im} \left[ \frac{1}{4} (p - 1)^{\frac{3}{2} - \frac{1}{2}} (p + 1 - i\sqrt{p - 1})(\sqrt{p - 1} - i) \right] + \frac{1}{4} (p - 1)^{\frac{3}{2} - \frac{1}{2}} (1 - 2p)
\]
\[
= \frac{1}{4} p(p - 1)^{\frac{3}{2} - \frac{1}{2}} + \frac{1}{4} (1 - 2p)(p - 1)^{\frac{3}{2} - \frac{1}{2}}
\]
\[
= -\frac{1}{4} (p - 1)^{3 - \frac{1}{2}} < 0.
\]

Note that \( \alpha'(a^0_0) < 0 \). Hence, the direction of the Hopf bifurcation is subcritical and the bifurcating periodic solutions are asymptotically stable.

**Remark 2.1.** The spatially inhomogeneous bifurcating periodic solutions obtained in Theorem 2.2 are clearly unstable since the constant steady-state \( U^* \) is unstable. The bifurcation direction is determined by (2.12) with \( a^0_0 \) substituted by \( a^0_0 \). Clearly, \( \alpha'(a^0_0) < 0 \). Then the Hopf bifurcation at \( a = a^0_0 \) is subcritical (supercritical) if Re \( c_1(a^0_0) \) < 0 (> 0). Here, we omit the calculation of Re \( c_1(a^0_0) \) because it is very lengthy.

**3. The steady-state bifurcation**

In this section, we focus on the following steady-state problem
\[
\begin{align*}
-d_1 u'' = a - uv^p, \quad &x \in (0, \pi), \\
-d_2 v'' = u^p - v, \quad &x \in (0, \pi), \\
u' = v' = 0, \quad &x = 0, \pi.
\end{align*}
\]

Taking \( a \) as a parameter, we shall prove the existence of positive solutions bifurcating from \((a, U^*)\). The Crandall–Rabinowitz bifurcation theorem will be used to derive bifurcations from simple eigenvalues. For the case of double eigenvalues, the techniques of space decomposition and implicit function theorem are applied.
Let $X = \{(u, v) \in W^{2, \sigma}(0, \pi) \times W^{2, \sigma}(0, \pi) : u' = v' = 0, \ x = 0, \pi\}$ and $Y = L^\sigma(0, \pi) \times L^\sigma(0, \pi)$. Define the map $F : R^+ \times X \rightarrow Y$ by

$$F(a, U) = \left(\frac{\partial u}{\partial x} + a - u v^p, \ \frac{\partial v}{\partial x} + u v^p - v\right), \ \ U = (u, v).$$

Then the solutions of (3.1) are exactly zeros of this map. Note that $U^* = (u^*, v^*) = (a^{1-p}, a)$ is the unique constant solution of (3.1). We denote $U^*$ by $U^a_0$ and have $F(a, U^a_0) = 0$. The Fréchet derivative of $F$ at $U^a_0$ can be expressed by

$$L(a) = L_{U^a_0}(a, U^a_0) = \left(\frac{d_1 \Delta - a^p}{a p}, \ \frac{d_2 \Delta + p - 1}{1}\right), \ \Delta = \frac{d_2}{dx^2},$$

whose characteristic equation is given by (2.3) in Section 2.

Throughout this section, assume that $p > 1$ and $d_2 < p - 1$. Thus there exists a positive integer $k_0 \geq 1$ such that $p - 1 - d_2 \mu_k > 0$ for $1 \leq k \leq k_0$. Letting $\rho = 0$ in (2.3), we have

$$a = a_k^\rho := \left(\frac{d_1 \mu_k (p - 1 - d_2 \mu_k)}{d_2 \mu_k + 1}\right)^{1/\rho}, \ \mu_k = k^2.$$  \hspace{1cm} (3.2)

We shall prove that there exists a non-constant positive solution of $F(a, U) = 0$ near $(a_j^\rho, U^a_{j_0})$ $(1 \leq j \leq k_0)$. Note that $a_i^\rho$ may be or not be equal to $a_j^\rho$ when $i \neq j$. So our proof deals with two different cases.

**Theorem 3.1.** Assume $p > 1$ and $d_2 < p - 1$. Take $a_j^\rho = \left(\frac{d_1 \mu_j (p - 1 - d_2 \mu_j)}{d_2 \mu_j + 1}\right)^{1/\rho}$ for $1 \leq j \leq k_0$.

(i) If $i \neq j$ implies $a_i^\rho \neq a_j^\rho$ for arbitrary integers $i, j \in [1, k_0]$, then $(a_j^\rho, U^a_{j_0})$ is a bifurcation point of $F = 0$. There is a curve of non-constant solutions $(a(s), (u(s), v(s)))$ of $F = 0$ for $s$ sufficiently small, satisfying $a(0) = a_j^\rho, (u(0), v(0)) = U^a_{j_0}, u(s) = u_{j_0}^s + s \phi_j + a(s^2), v(s) = v_{j_0}^s + b_j \phi_j + o(s^2)$, where $a(s), u(s), v(s)$ are continuous functions and $b_j = -\frac{d_1 \mu_j}{a_j^\rho d_2 \mu_j + 1}$.

(ii) Suppose that there exists a positive integer $i \in [1, k_0]$, $i \neq j$ such that $a_i^\rho = a_j^\rho = \tilde{a}$. Let

$$b_i = -\frac{d_1 \mu_i}{d_2 \mu_i + 1}, \quad b_j^* = \left(\frac{p}{p - 1 - d_2 \mu_i}\right), \quad \Phi_i = \left(\frac{1}{b_i}\right) \phi_i,$$

$$A_1 = -\frac{1}{2} p(p - 1) b_i \tilde{a}^{-1} - b_i \tilde{a}^{-1}, \quad A_3 = -\frac{1}{2} p(p - 1) b_j \tilde{a}^{-1} - b_j \tilde{a}^{-1},$$

$$A_2 = -p(p - 1) b_i b_j \tilde{a}^{-1} - b_i \tilde{a}^{-1} - b_j \tilde{a}^{-1}.$$  \hspace{1cm} (3.3)

$$X_2 := \left\{(y, z) \in X : \int_0^\pi (y + b_i z) \phi_i \ dx = \int_0^\pi (y + b_j z) \phi_j \ dx = 0\right\}. \hspace{1cm} \hspace{1cm} (3.5)$$

If $1 + b_i b_j^* \neq 0, 1 + b_j b_j^* \neq 0$ and $j = 2i$ (resp. $i = 2j$), then $(\tilde{a}, U^a_{j_0})$ is a bifurcation point of $F = 0$. Moreover, there is a curve of non-constant solutions $(a(\omega), U_0^2 + o(\omega)(\cos \omega \Phi_i + \sin \omega \Phi_j + W(\omega)))$ of $F = 0$ for $|\omega - \omega_0|$ sufficiently small, satisfying $a(\omega_0) = \tilde{a}, s(\omega_0) = 0, W(\omega_0) = 0$, where $\omega_0$ is any constant satisfying

$$\cos \omega_0 \neq 0 \quad \text{and} \quad (A_1 + A_2) \cos^2 \omega_0 \neq A_2,$$

$$\left(\text{resp.} \ \sin \omega_0 \neq 0 \quad \text{and} \quad (A_2 + A_3) \sin^2 \omega_0 \neq A_3\right).$$  \hspace{1cm} (3.6)

$$a(\omega), s(\omega), W(\omega) \text{ are continuously differentiable functions with respect to } \omega \text{ and } W(\omega) \in X_2.$$  \hspace{1cm} (3.7)

**Proof.** (i) By the Crandall–Rabinowitz bifurcation theorem about simple eigenvalues in [10,17], $(\tau, U^*)$ is a bifurcation point if the following conditions are satisfied:

(a) $F_0, F_U$ and $F_{duU}$ exist and are continuous;
(b) dim ker $F_U(\tau, U^*) = \text{codim R}(F_U(\tau, U^*)) = 1$;
(c) let ker $F_{duU}(\tau, U^*) = \text{span}\{\Phi\}$, then $F_{duU}(\tau, U^*) \Phi \neq R(F_U(\tau, U^*))$. 


Recall that the operator
\[ L(a_j^5) = F_\mu(a_j^5, U^s_{a_j^5}) = \begin{pmatrix} d_1 \Delta - (a_j^5)^p & -p \\ (a_j^5)^p & d_2 \Delta + p - 1 \end{pmatrix}. \]
It is clear that the linear operators \( F_\mu, F_\sigma \) and \( F_{\alpha U} \) are continuous.

According to the assumption in (i), we have
\[ \ker L(a_j^5) = \text{span}(\Phi_j), \quad \Phi_j = \left( \frac{1}{b_j} \right) \phi_j, \]
where \( b_j = -\frac{d_1 \mu_j + (a_j^5)^p}{p} = -\frac{d_1 \mu_j}{a^2 \mu_j} < 0 \).

The adjoint operator is defined by
\[ L^*(a_j^5) = \begin{pmatrix} d_1 \Delta - (a_j^5)^p & (a_j^5)^p \\ -p & d_2 \Delta + p - 1 \end{pmatrix}. \]
Similarly,
\[ \ker L^*(a_j^5) = \text{span}(\Phi_j^*), \quad \Phi_j^* = \left( \frac{1}{b_j^*} \right) \phi_j, \]
where \( b_j^* = \frac{d_1 \mu_j + (a_j^5)^p}{p} = \frac{p}{\mu_j - a^2 \mu_j} > 1 \).

Because \( R(L) = (\ker L^*)^\perp \), we get
\[ \text{codim } R(L(a_j^5)) = \dim \ker L^*(a_j^5) = 1. \]
Finally, since
\[ F_{\alpha U}(a_j^5, U^s_{a_j^5}) \Phi_j = \begin{pmatrix} -p(a_j^5)^{p-1} & 0 \\ p(a_j^5)^{p-1} & 0 \end{pmatrix} \Phi_j = p(a_j^5)^{p-1} \begin{pmatrix} \phi_j \\ -\phi_j \end{pmatrix} \]
and
\[ \langle F_{\alpha U}(a_j^5, U^s_{a_j^5}) \Phi_j, \Phi_j^* \rangle = p(a_j^5)^{p-1} (1 - b_j^*) \neq 0, \]
we find \( F_{\alpha U}(a_j^5, U^s_{a_j^5}) \Phi_j \notin R(L(a_j^5)). \) The proof of (i) is completed.

(ii) If there exists a positive integer \( i \neq j \) such that \( a_i^5 = a_j^5 = \tilde{a} \), then we have \( \ker L(\tilde{a}) = \text{span}(\Phi_i, \Phi_j) \), \( \ker L^*(\tilde{a}) = \text{span}(\Phi_i^*, \Phi_j^*) \) and
\[ R(L(\tilde{a})) = \left\{ (y, z) \in Y : \int_0^\pi (y + b_j^* z) \phi_i \, dx = \int_0^\pi (y + b_j^* z) \phi_j \, dx = 0 \right\}. \]
which lead to \( \dim \ker L(\tilde{a}) = \text{codim } R(L(\tilde{a})) = 2 \).

Obviously, the condition (b) in (i) is not satisfied and the Crandall–Rabinowitz bifurcation theorem does not work. Now, we resort to the techniques of space decomposition and implicit function theorem to deal with the case of double eigenvalues.

First, we translate \( U_0^s \) to the origin by the translation \( (y, z) = (u, v) - U_0^s \), and thus consider the new map \( \tilde{F} : R^+ \times X \to Y \),
\[ \tilde{F}(a, (y, z)) = \begin{pmatrix} d_1 y'' + a - (y + u_k^0)(z + v_k^0)^p \\ d_2 z'' + (y + u_k^0)(z + v_k^0)^p - (z + v_k^0) \end{pmatrix} = L(a) \begin{pmatrix} y \\ z \end{pmatrix} + \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \]
where \( f_2 = -f_1 \) and
\[ f_1 = -\frac{1}{2} p(p - 1) a^{-1} z^2 - pa^p - 1 y z - \frac{1}{6} p(p - 1)(p - 2) a^{-2} z^3 - \frac{1}{2} p(p - 1) a^{p-2} y z^2 + O(|z|^4, |y|^4, |z|^2). \]
It is clear that \( \tilde{F}(\tilde{a}, (0, 0)) = 0 \) and \( \tilde{F}(y, z)(\tilde{a}, (0, 0)) = L(\tilde{a}) \). And \( (u, v) \) is a solution of (3.1) (or \( F = 0 \)) if and only if \( (y, z) \) satisfies \( F(a, (y, z)) = 0 \). So we only need to find the existence of non-constant pair \( (y, z) \).

Second, we decompose \( X \) as \( X = X_1 \oplus X_2 \) and look for solutions of \( \tilde{F} = 0 \) in the form
\[ (y, z) = s(\cos \omega t_2 + \sin \omega t_1 + W), \quad W = (w_1, w_2)^T \in X_2, \]
where \( X_1 = \text{span}(\Phi_i, \Phi_j), X_2 \) is defined in (3.5), \( s, \omega \in R \) are parameters. For latter use, we define the operator \( P \) on \( Y \) by
\[ P\left( \begin{array}{c} y \\ z \end{array} \right) = \frac{1}{1 + b_1 b_1^*} \int_0^\pi \left( y + b_1^* z \right) \phi_i \, dx \Phi_i + \frac{1}{1 + b_2 b_2^*} \int_0^\pi \left( y + b_2^* z \right) \phi_j \, dx \Phi_j, \]

such that \( P \) is the projection from \( Y \) to \( X_1 \), and then decompose \( Y \) as \( Y = Y_1 \oplus Y_2 \) with \( Y_1 = R(P) = X_1 \) and \( Y_2 = \ker P = R(L(\begin{array}{c} a' \\ b' \end{array})) \).

Next, we apply the implicit function theorem to show the existence of non-constant pair \( (y, z) \). Fix \( \omega_0 \in R \) for the time being and define a nonlinear mapping \( K(a, s, W; \omega) : R \times R \times X_2 \times (\omega_0 - \delta, \omega_0 + \delta) \to Y \) by

\[
K(a, s, W; \omega) = s^{-1} f \left( a, s(\cos \omega \Phi_i + \sin \omega \Phi_j + W) \right)
= L(a)(\cos \omega \Phi_i + \sin \omega \Phi_j + W) + s(\begin{array}{c} f_1 \\ f_2 \end{array}).
\]

Here, \( \begin{array}{c} f_1 \\ f_2 \end{array} = \begin{array}{c} -\frac{1}{2} \left( p - 1 \right) a_1 (b_1 \cos \omega \Phi_i + b_1 \sin \omega \Phi_j + w_2) - \frac{1}{2} \left( p - 1 \right) a_2 (b_1 \cos \omega \Phi_i + b_1 \sin \omega \Phi_j + w_2)^2 - \frac{1}{2} \left( p - 1 \right) a_3 (b_1 \cos \omega \Phi_i + b_1 \sin \omega \Phi_j + w_2)^3 - \frac{1}{2} \left( p - 1 \right) a_4 (b_1 \cos \omega \Phi_i + b_1 \sin \omega \Phi_j + w_2)^4 + O(|s|^2). \end{array} \]

Clearly, \( K(\begin{array}{c} a' \\ b' \end{array}, 0, 0; \omega_0) = 0 \). The Fréchet derivative of \( K(a, s, W; \omega) \) with respect to \( (a, s, W; \omega) \) at \( (a, s, W; \omega) = (\begin{array}{c} a' \\ b' \end{array}, 0, 0; \omega_0) \) is the linear mapping

\[
K_{(a, s, W)}(\begin{array}{c} a' \\ b' \end{array}, 0, 0; \omega_0) = L(\begin{array}{c} a' \\ b' \end{array}) W + \hat{a} \bar{a} \cos \omega \Phi_i - \hat{b} \bar{a} \sin \omega \Phi_j - \hat{a} \bar{a}\cos \omega \Phi_i + \hat{b} \bar{a}\sin \omega \Phi_j \cos \omega \Phi_i + \hat{b} \bar{a}\sin \omega \Phi_j \sin \omega \Phi_j + W_1 \left( b_1 \cos \omega \Phi_i + b_1 \sin \omega \Phi_j + w_2 \right) - \frac{1}{2} \left( p - 1 \right) a_1 (b_1 \cos \omega \Phi_i + b_1 \sin \omega \Phi_j + w_2)^2 - \frac{1}{2} \left( p - 1 \right) a_2 (b_1 \cos \omega \Phi_i + b_1 \sin \omega \Phi_j + w_2)^3 - \frac{1}{2} \left( p - 1 \right) a_3 (b_1 \cos \omega \Phi_i + b_1 \sin \omega \Phi_j + w_2)^4 + O(|s|^2).
\]

where \( A_1, A_2, A_3 \) are given in (3.3) and (3.4).

In order to use the implicit function theorem, it suffices to verify that \( K_{(a, s, W)}(\begin{array}{c} a' \\ b' \end{array}, 0, 0; \omega_0) : R \times R \times X_2 \to Y \) is an isomorphism. For this purpose, try to rewrite

\[
K_{(a, s, W)}(\begin{array}{c} a' \\ b' \end{array}, 0, 0; \omega_0)(a, s, W) = y_1 + y_2 \quad \text{with} \quad y_1 \in Y_1 \quad \text{and} \quad y_2 \in Y_2.
\]

To this end, decompose

\[
\begin{pmatrix} -\phi_i \\ \phi_i \end{pmatrix} = c_1 \phi_i + \begin{pmatrix} y_1 \\ z_1 \end{pmatrix}, \quad \begin{pmatrix} -\phi_j \\ \phi_j \end{pmatrix} = c_2 \phi_j + \begin{pmatrix} y_2 \\ z_2 \end{pmatrix},
\]

where

\[
c_1 = \frac{1 - b_1}{1 + b_1 b_1^*} \neq 0, \quad \begin{pmatrix} y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} (-1 - c_1) \phi_i \\ (1 - c_2) b_1 \phi_i \end{pmatrix} \in Y_2.
\]

\[
c_2 = \frac{1 - b_2}{1 + b_2 b_2^*} \neq 0, \quad \begin{pmatrix} y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} (-1 - c_2) \phi_i \\ (1 - c_2) b_2 \phi_i \end{pmatrix} \in Y_2.
\]

In the following, we need to divide our arguments into two cases \( j = 2i \) and \( i = 2j \).

(A1) \( j = 2i \).

In this case, \( \int_0^\pi \phi_i^2 \phi_1 \, dx = \frac{1}{2\pi} \neq 0 \), \( \int_0^\pi \phi_i^2 \phi_1 \, dx = 0 \). Then \( \begin{pmatrix} \phi_i^2 \\ -\phi_i^2 \end{pmatrix} \in Y_2 \) and we need to decompose

\[
\begin{pmatrix} \phi_i^2 \\ -\phi_i^2 \end{pmatrix} = c_3 \phi_i + \begin{pmatrix} y_3 \\ z_3 \end{pmatrix}, \quad \begin{pmatrix} \phi_i \phi_j \\ -\phi_i \phi_j \end{pmatrix} = c_4 \phi_i + \begin{pmatrix} z_4 \\ y_4 \end{pmatrix},
\]

where

\[
c_3 = \frac{1 - b_1}{1 + b_1 b_1^*} \int_0^\pi \phi_i^2 \phi_1 \, dx = \sqrt{\frac{1}{2\pi} \int_0^\pi \frac{1 - b_1^2}{1 + b_1^2} \neq 0}, \quad \begin{pmatrix} y_3 \\ z_3 \end{pmatrix} = \begin{pmatrix} \phi_i^2 - c_3 \phi_j \\ -\phi_i^2 - c_3 \phi_j \end{pmatrix} \in Y_2.
\]

\[
c_4 = \frac{1 - b_1}{1 + b_1 b_1^*} \int_0^\pi \phi_i^2 \phi_1 \, dx = \sqrt{\frac{1}{2\pi} \int_0^\pi \frac{1 - b_1^2}{1 + b_1^2} \neq 0}, \quad \begin{pmatrix} y_4 \\ z_4 \end{pmatrix} = \begin{pmatrix} \phi_i \phi_j - c_4 \phi_i \\ -\phi_i \phi_j - c_4 \phi_i \end{pmatrix} \in Y_2.
\]

Note that \( L(\begin{array}{c} a' \\ b' \end{array}) \) is an isomorphism from \( X_2 \) to \( Y_2 \). Hence, we can assume
\[ y_1 = (ac_1 \tilde{p}^{-1}_1 \cos \omega_0 + sc_4 A_2 \cos \omega_0 \sin \omega_0) \Phi_i + (ac_2 \tilde{p}^{-1}_2 \sin \omega_0 + sc_3 A_1 \cos^2 \omega_0) \Phi_j, \]
\[ y_2 = L(\tilde{a}) W + ap \tilde{p}^{-1}_0 \cos \omega_0 \left( \frac{y_1}{z_1} \right) + ap \tilde{p}^{-1}_0 \sin \omega_0 \left( \frac{y_2}{z_2} \right) + sA_1 \cos^2 \omega_0 \left( \frac{y_3}{z_3} \right) \]
\[ + sA_2 \cos \omega_0 \sin \omega_0 \left( \frac{y_4}{z_4} \right) + sA_3 \sin^2 \omega_0 \left( \frac{\phi_j^2}{-\phi_j^2} \right). \]

Let
\[ K_{(a,s,W)}(\tilde{a}, 0, 0; \omega_0)(a, s, W) = 0, \]
which is equivalent to \( y_1 = 0 \) and \( y_2 = 0 \). Since \( \cos \omega_0 \neq 0 \) and \((A_1 + A_2) \cos^2 \omega_0 \neq A_2\), we get \( a = 0, s = 0 \) from \( y_1 = 0 \).

For arbitrary \((y, z) \in Y\), we shall find \((a, s, W) \in R \times R \times X_2\) such that
\[ K_{(a,s,W)}(\tilde{a}, 0, 0; \omega_0)(a, s, W) = \begin{pmatrix} y \\ z \end{pmatrix}. \tag{3.8} \]
By the decomposition of \( Y \), there exist \( \alpha, \beta \in R \) and \((y_0, z_0) \in Y_2\) such that
\[ \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} y_0 \\ z_0 \end{pmatrix} + \alpha \Phi_i + \beta \Phi_j. \]

Substituting it into (3.8) yields
\[ \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} y_0 \\ z_0 \end{pmatrix} + \alpha \Phi_i + \beta \Phi_j. \tag{3.9} \]

Since \( \omega_0 \) satisfies (3.6) when \( j = 2i \), we get \( a = \hat{a} \) and \( s = \hat{s} \), where
\[ \hat{a} = \frac{\alpha c_3 A_1 \cos \omega_0 - \beta c_4 A_2 \sin \omega_0}{c_1 c_3 \tilde{p}^{-1}_0(A_1 \cos^2 \omega_0 - A_2 \sin^2 \omega_0)} \quad \hat{s} = \frac{\beta c_1 \cos \omega_0 - \alpha c_2 \sin \omega_0}{c_1 c_3 \cos \omega_0(A_1 \cos^2 \omega_0 - A_2 \sin^2 \omega_0)}. \]

Substituting \( \hat{a}, \hat{s} \) into the third equation of (3.9), we obtain \( W = L^{-1}(\hat{\hat{y}}) \in X_2 \), where
\[ \begin{pmatrix} \hat{\hat{y}} \\ \hat{\hat{z}} \end{pmatrix} = \begin{pmatrix} y_0 \\ z_0 \end{pmatrix} - \hat{a} \tilde{p}^{-1}_0 \cos \omega_0 \left( \frac{y_1}{z_1} \right) - \hat{a} \tilde{p}^{-1}_0 \sin \omega_0 \left( \frac{y_2}{z_2} \right) - \hat{s} A_1 \cos^2 \omega_0 \left( \frac{y_3}{z_3} \right) \]
\[ - \hat{s} A_2 \cos \omega_0 \sin \omega_0 \left( \frac{y_4}{z_4} \right) - \hat{s} A_3 \sin^2 \omega_0 \left( \frac{\phi_j^2}{-\phi_j^2} \right) \in Y_2. \]

Then
\[ (a, s, W) = \left( \hat{a}, \hat{s}, L^{-1}(\hat{\hat{y}}) \right) \]
is the solution of (3.8). This shows that \( K_{(a,s,W)}(\tilde{a}, 0, 0; \omega_0) \) is surjective.

Therefore, \( K_{(a,s,W)}(\tilde{a}, 0, 0; \omega_0) \) is an isomorphism from \( R \times R \times X_2 \) to \( Y \). Using the implicit function theorem for
\[ K(a, s, W; \omega_0) = 0, \tag{3.10} \]
there is a curve of non-constant solutions \((a(\omega), s(\omega), W(\omega))\) of (3.10) (i.e. \( \bar{F} = 0 \)) for \(|\omega - \omega_0|\) sufficiently small, satisfying \( a(\omega_0) = \tilde{a}, s(\omega_0) = 0, W(\omega_0) = 0 \), where \( \omega_0 \) is any constant satisfying (3.6), \( a(\omega), s(\omega), W(\omega) \) are continuously differentiable functions with respect to \( \omega \), and \( W \in X_2 \). Hence, \((a(\omega), U^+_{\omega} + s(\omega)(\cos \omega \Phi_i + \sin \omega \Phi_j + W(\omega))\) are non-constant solutions of \( F = 0 \).

\((A2) \ i = 2j.\)

In this case, \( \int_0^\pi \phi_i^2 \phi_j \ dx = 0, \int_0^\pi \phi_j^2 \phi_i \ dx = 0 \). Then \( (\phi_i^2, -\phi_j^2) \in Y_2 \) and we need to decompose
Numerical simulations of spatially homogeneous and stable periodic solutions to system (2.1) for values of $d_1 = 1$, $d_2 = 3$, $p = 2$ and initial conditions $(u_0, v_0) = (u^* + 0.6068 \cos 10x, v^* + 0.6068 \cos 10x)$. Here, (a) $a = 0.92$; (b) $a = 0.98$. Left: $u$; right: $v$.

\[
\begin{align*}
\left( \phi_i^2 - \phi_j^2 \right) &= c_5 \Phi_i + \left( y_5 \ z_5 \right), \\
\left( \phi_i \phi_j - \phi_i \phi_j \right) &= c_6 \Phi_j + \left( y_6 \ z_6 \right),
\end{align*}
\]

where

\[
\begin{align*}
c_5 &= \frac{1 - b_i^*}{1 + b_j^*} \int_0^\pi \phi_i^2 \phi_i \ dx = \sqrt{\frac{1}{2\pi}} \left( 1 - b_i^* \right) = c_4 \neq 0, \\
c_6 &= \frac{1 - b_j^*}{1 + b_j^*} \int_0^\pi \phi_j^2 \phi_j \ dx = \sqrt{\frac{1}{2\pi}} \left( 1 - b_j^* \right) = c_3 \neq 0,
\end{align*}
\]

Hence, we can assume

\[
\begin{align*}
y_1 &= (a_1 p a_1^{-1} \cos \omega_0 + s c_3 A_3 \sin^2 \omega_0) \Phi_i + (a_2 p a_2^{-1} \sin \omega_0 + s c_3 A_2 \cos \omega_0 \sin \omega_0) \Phi_j, \\
y_2 &= L(\vec{a}) W + a p a^{-1} \cos \omega_0 \left( y_1 \ z_1 \right) + a p a^{-1} \sin \omega_0 \left( y_2 \ z_2 \right) + s A_1 \cos^2 \omega_0 \left( \phi_i^2 - \phi_j^2 \right) + s A_2 \cos \omega_0 \sin \omega_0 \left( y_5 \ z_5 \right) + s A_3 \sin^2 \omega_0 \left( y_6 \ z_6 \right).
\end{align*}
\]

Similar to the arguments in case (A1), $K_{(a, s, W)}(\vec{a}, 0; 0; \omega_0)$ is an isomorphism from $R \times R \times X_2$ to $Y$ if $\omega_0$ satisfies (3.7). By the implicit function theorem, we complete the proof for this case. The whole proof is finished. \(\square\)
4. Numerical simulations

In this section, we present some numerical simulations that verify the analytic results in previous sections by using Matlab programming. Our numerical simulations presented below illustrate the following major outcomes:

(1) For the system (2.1), choose $d_1 = 1, d_2 = 3, p = 2$ and thus $a_0^H = 1$. By Theorem 2.2, a Hopf bifurcation occurs at $a = a_0^H$. And by Theorem 2.3, the bifurcation direction is subcritical and the bifurcating periodic solutions are asymptotically stable. These facts are shown in Fig. 1. Our numerical simulations suggest that the periodicity decreases with respect to the value of $a$, see Fig. 2.
Fig. 5. Numerical simulations of positive steady-state solutions to system (3.1) for values of \( d_1 = 2, d_2 = 0.1, p = 2, a = 1.85 \approx a_S^2 \) and initial condition \((u_0, v_0) = (u^* + 0.4057 \cos 2x, v^* + 0.4057 \cos 2x)\). Left: \( u \); right: \( v \).

Fig. 6. Numerical simulations of positive steady-state solutions to system (3.1) for values of \( d_1 = 4, d_2 = 0.125, p = 27/16, a = 1.50 \approx \tilde{a} \) and initial condition \((u_0, v_0) = (u^* + 0.8132 \cos 5x, v^* + 0.8132 \sin 5x)\). Left: \( u \); right: \( v \).

(2) Choose \( d_1 = 0.01, d_2 = 0.03 \) and \( p = 2 \). It follows from Theorem 2.2 that the system (2.1) undergoes Hopf bifurcations at \( a = a_k^H = \sqrt{1 - 0.04k^2} \), \( k = 1, 2, 3, 4 \), and the bifurcating periodic solutions are spatially inhomogeneous, see Fig. 3 for the case \( k = 2 \). Moreover, our numerical simulations suggest that the spatially inhomogeneous periodic solution in Fig. 3 is unstable, which agrees well with Remark 2.1.

(3) By (3.2), we obtain the curve of steady-state bifurcation

\[
(a_j^S)^p = \frac{d_1 j^2 (p - 1 - d_2 j^2)}{d_2 j^2 + 1} := a^p(j),
\]

and we plot it in Fig. 4, where \( d_1 = 2, d_2 = 0.1, p = 2 \) for (a) and \( d_1 = 4, d_2 = 0.125, p = 27/16 \) for (b). Then we have three simple bifurcation points \( a_1^S = 1.2792, a_2^S = 1.8516, a_3^S = 0.9733 \) in (a) and a double bifurcation point \( a_1^S = a_2^S = \tilde{a} = 1.5080 \) in (b). From Theorem 3.1(i), it follows that the system (3.1) has at least a positive steady-state bifurcation solution in the neighborhood of simple bifurcation point \((a_j^S, U^*_j)\), see Fig. 5 for the case \( j = 2 \). If \((a_j^S, U^*_j)\) is a bifurcation point at the double eigenvalue, then by Theorem 3.1(ii), we know that there still exist positive steady-state solutions of system (3.1), see Fig. 6.

References