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Oscillation Criteria for Second Order Nonlinear Differential Equations Involving General Means

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Consider the second order nonlinear differential equation (E) $y'' + a(t)f(y) = 0$ where $a(t) \in C[0,\infty)$, $f(y) \in C^1(-\infty,\infty)$, $f'(y) \geq 0$, and $yf(y) > 0$ for $y \neq 0$. Furthermore, $f(y)$ also satisfies either a superlinear or a sublinear condition, which covers the nonlinear function $f(y) = y|y|^{\gamma-1}$ with $\gamma > 1$ and $0 < \gamma < 1$, respectively, commonly known as the Emden–Fowler case. Here the coefficient function $a(t)$ is allowed to be negative for arbitrarily large values of t. Kamenev type oscillation criteria involving integral averages for the linear equations (L) $y'' + a(t)y = 0$ are extended to the nonlinear equation (E) by using more general means. The results extend similar results on general means by Philos for the linear equation (L) and also results based upon Kamenev's integral averaging method concerning the nonlinear equation (E).

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Consider the second order nonlinear differential equation

$$y'' + a(t)f(y) = 0, \qquad t \in [0,\infty), \tag{1}$$

where $a(t) \in C[0,\infty)$ and $f(y) \in C^1(-\infty,\infty)$, $f'(y) \geq 0$ for all y, and $yf(y) > 0$ if $y \neq 0$. The prototype of Eq. (1) is the so-called Emden–Fowler equation

$$y'' + a(t)|y|^{\gamma}\text{sgn}\, y = 0, \qquad \gamma > 0. \tag{2}$$

Here we are interested in the oscillation of solutions of (1) when $f(y)$ satisfies, in addition, the sublinear condition

$$0 < \int_0^{\varepsilon} \frac{dy}{f(y)}, \qquad \int_{-\varepsilon}^{0} \frac{dy}{f(y)} < \infty \qquad \text{for all } \varepsilon > 0, \tag{F$_1$}$$

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which corresponds to the special case $f(y) = |y|^{\gamma}\operatorname{sgn} y$ when $0 < \gamma < 1$, and also the superlinear condition

$$0 < \int_{\varepsilon}^{\infty} \frac{dy}{f(y)}, \qquad \int_{-\infty}^{-\varepsilon} \frac{dy}{f(y)} < \infty \qquad \text{for all } \varepsilon > 0, \tag{F$_2$}$$

which corresponds to the special case $f(y) = |y|^{\gamma}\operatorname{sgn} y$ when $\gamma > 1$. The coefficient $a(t)$ is allowed to be negative for arbitrarily large values of t. Under these circumstances, in general not every solution to the second order nonlinear differential equation (1) is continuable throughout the entire half real axis. For this reason, we confine ourselves to those solutions of (1) that exist and can be continued on some interval of the form $[t_0, \infty)$, where $t_0 \geq 0$ may depend on the particular solution. A solution $y(t)$ is said to be oscillatory if it has arbitrarily large zeros; i.e., for each $t \in [t_0, \infty)$, there exists $t_1 \geq t$ such that $y(t_1) = 0$. Equation (1) is called oscillatory if all continuable solutions are oscillatory. Here we are concerned with sufficient conditions on $a(t)$ so that all solutions of (1) are oscillatory.

In the linear case, i.e., Eq. (2) when $\gamma = 1$, the most important simple oscillation criterion is the well known Fite–Wintner Theorem which states that if $a(t)$ satisfies

$$\lim_{T\to\infty} A(T) = \lim_{T\to\infty} \int_0^T a(t)\, dt = +\infty, \tag{A$_0$}$$

then Eq. (2) is oscillatory when $\gamma = 1$. Fite [5] assumed in addition that $a(t)$ is non-negative, while Wintner [17] in fact proved a stronger result which required a weaker condition on $a(t)$ and involved the integral average of $A(t)$, namely,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T A(t)\, dt = +\infty. \tag{A$_1$}$$

Clearly, (A_0) implies (A_1). Wintner's result was later improved by Hartman [6] who proved that (A_1) can be replaced by the two weaker conditions

$$\liminf_{T\to\infty} \frac{1}{T} \int_0^T A(t)\, dt = -L > -\infty, \qquad L > 0, \tag{A$_2$}$$

and

$$\limsup_{T} \frac{1}{T} \int_0^T A(t)\, dt = +\infty. \tag{A$_3$}$$

Obviously, (A_1) implies both (A_2) and (A_3).

Another extension of Wintner's theorem is due to Kamenev [9] who proved that the linear equation, Eq. (2) with $\gamma = 1$, is oscillatory if for some integer $n > 1$,

$$\limsup_{T\to\infty} \frac{1}{T^n} \int_0^T (T-t)^n a(t)\, dt = +\infty. \qquad (A_4)$$

In an earlier paper [21], we showed that (A_4) together with another condition similar to (A_2), namely,

$$\liminf_{T\to\infty} \int_0^t a(s)\, ds = -L > -\infty, \qquad L > 0, \qquad (A_5)$$

is sufficient for the oscillation of the Emden–Fowler Eq. (2) for all $\gamma > 0$. Note that condition (A_1) is equivalent to

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T (T-t) a(t)\, dt = +\infty,$$

which in turn implies that for any real $\alpha > 1$,

$$\lim_{T\to\infty} \frac{1}{T^\alpha} \int_0^T (T-t)^\alpha a(t)\, dt = +\infty. \qquad (A_6)$$

Condition (A_6) implies (A_4) trivially, so Kamenev's condition (A_4) is weaker than the original Wintner criteria (A_1).

More recently, Philos [14] introduced the concept of general means and obtained further extensions of the Kamenev type oscillation criterion (A_4) for the linear equation, i.e., (2) when $\gamma = 1$. The subject of extending oscillation criteria for the linear equation to that of the Emden–Fowler Eq. (2) and more generally Eq. (1) has been of considerable interest in the past 30 years. It was at first Waltman [15] who showed that the Fite–Wintner condition (A_0) remained valid for the oscillation of Eq. (2) and his result was extended to the more general equation by Bhatia [1] and independently using different methods by Wong [18]. The extension of the Wintner condition (A_1) to Eq. (2) in the sublinear case $0 < \gamma < 1$ was given by Kamenev [7], while for the superlinear case $\gamma > 1$ it was finally proved in a substantive paper by Butler [2] (see also Kamenev [8]). The extension of Butler's result to the more general Eq. (1) was only resolved satisfactorily in recent papers by this author [22, 23]. On the other hand, Kamenev's condition (A_4) is known to be valid also for the oscillation of (2) in the sublinear case, i.e., $0 < \gamma < 1$; see [20]. In the superlinear case, it was recently shown (see [23]) that (A_4) together with the following condition that for some integer $p > 1$ such that

$$\liminf_{T\to\infty} \frac{1}{T^p} \int_0^T (T-t)^p a(t)\, dt = -L > -\infty, \qquad (A_7)$$

is sufficient for oscillation of (2) when $\gamma > 1$. Clearly, (A_5) implies (A_7). Also, (A_6) implies both (A_4) and (A_7) and is in itself sufficient for the oscillation of Eq. (2) for all $\gamma > 0$. This extends the Wintner's original oscillation criterion (A_1) to the nonlinear Eq. (2) in a way rather different from that given by Butler [2]. On the other hand, conditions (A_2) and (A_3) imply (A_7) and (A_4), respectively, hence Hartman's oscillation criterion was also shown to be valid for the oscillation of the nonlinear Eq. (2).

Our earlier result [23] was actually proved for the more general Eq. (1) subject to (F_1) and for some positive constant c

$$f'(y)F(y) \geq c^{-1} > 0, \qquad \text{for all } y, \tag{F_3}$$

where $F(y) = \int_0^y dv/f(v)$ in the sublinear case and also to (F_2) and for some positive constant d

$$f'(y)G(y) \geq d > 1, \qquad \text{for all } y, \tag{F_4}$$

where $G(y) = \int_y^\infty dv/f(v)$ in the superlinear case. The purpose of this paper is to generalize conditions (A_4) and (A_7) by allowing more general means along the lines given in [14]. Oscillation criteria in terms of more general means have also been obtained for second order linear matrix equations involving symmetric matrix coefficients by Erbe *et al.* [4] and by Coles and Kinyon [3]. For other related results see [24], and for general discussion on oscillation criteria for Eq. (1), we refer to [20].

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Following Philos [14], we consider a non-negative kernel function $h(t, s)$ defined on $D = \{(t, s) : t \geq s \geq t_0\}$. Denote $D_1 = \{(t, s) : t > s \geq t_0\} \subseteq D$. We shall assume that $h(t, s)$ is sufficiently smooth in both variables t and s so that the following conditions are satisfied

(H_1) $h(t, t) \equiv 0 \qquad \text{for } t \geq t_0,$

(H_2) $\dfrac{\partial h}{\partial s}(t, s) \leq 0 \qquad \text{for } t \geq s \geq t_0,$

(H_3) $\left.\dfrac{\partial h}{\partial s}(t, s)\right|_{s=t} \equiv 0 \qquad \text{for } t \geq t_0,$

(H_4) $\dfrac{\partial^2 h}{\partial s^2}(t, s) \geq 0 \qquad \text{for } t \geq s \geq t_0,$

(H_5) $-h^{-1}(t, t_0)\left.\dfrac{\partial h}{\partial s}(t, s)\right|_{s=t_0} \leq M_0 \qquad \text{for } t \geq t_0,$

$$(\mathrm{H}_6)\quad 0 < b_0 \le \lim_{t\to\infty} \frac{h(t,s)}{h(t,t_0)} \le B_0 < \infty \qquad \text{for } s \ge t_0,$$

where M_0, b_0, B_0 are constants depending only on t_0,

$$(\mathrm{H}_7)\quad -h^{-1}(t,\tau)\frac{\partial h}{\partial s}(t,s)\Big|_{s=\tau} = o(1) \qquad \text{for all } \tau \ge t_0, \text{ as } t\to\infty,$$

$$(\mathrm{H}_8)\quad \frac{\partial^2}{\partial s\,\partial t} h(t,s) = \frac{\partial^2 h}{\partial t\,\partial s}(t,s), \qquad \text{for all } (t,s)\in D,$$

$$(\mathrm{H}_9)\quad \lambda(t,s) = h^{-1}(t,s)\frac{\partial h}{\partial s}(t,s) \text{ is nondecreasing in } t$$
for all $(t,s)\in D_1$.

Our first result is a simple generalization of Kamenev's oscillation criterion (A_4) for Eq. (1) in the sublinear case.

THEOREM 1. *Let $f(y)$ satisfy* (F_1) *and* (F_3). *Suppose that there exists a non-negative kernel function $h(t,s)$ on D satisfying* (H_1)–(H_5) *such that*

$$\limsup_{t\to\infty} \frac{1}{h(t,t_0)} \int_{t_0}^{t} h(t,s)a(s)\,ds = +\infty, \tag{3}$$

and Eq. (1) *is oscillatory.*

Proof. We shall prove this by contradiction. Let $y(t)$ be a nonoscillatory solution of (1) which we can assume, in view of (F_1) and the sign condition that $yf(y) > 0$ whenever $y \ne 0$, to be positive on $[t_0,\infty)$. Define $z(t) = F(y(t))$ and so $z'(t) = y'(t)/F(y(t))$. By (1), we find that $v(t) = z'(t)$ satisfies the Riccati differential equation

$$v'(t) + f'(y)v^2(t) + a(t) = 0, \qquad t \ge t_0. \tag{4}$$

Since $f'(y) \ge 0$, Eq. (4) implies that $v(t)$ satisfies the simple first order differential inequality $v'(t) + a(t) \le 0$. Multiplying $v' + a \le 0$ through by $h(t,s)$ and integrating from t_0 to t, we obtain

$$\int_{t_0}^{t} h(t,s)v'(s)\,ds + \int_{t_0}^{t} h(t,s)a(s)\,ds \le 0. \tag{5}$$

Now integrate the first integral by parts twice and we find by (H_1), (H_2), (H_3)

$$\int_{t_0}^{t} h(t,s)v'(s)\,ds = -h(t,t_0)v(t_0) + \frac{\partial h}{\partial s}(t,s)\Big|_{s=t_0} F(x(t_0))$$
$$+ \int_{t_0}^{t} \frac{\partial^2 h}{\partial s^2}(t,s)z(s)\,ds. \tag{6}$$

Substituting (6) into (5) and dividing through by $h(t, t_0)$, we arrive at

$$-v(t_0) + h^{-1}(t, t_0) \frac{\partial h}{\partial s}(t, s)\bigg|_{s=t_0} F(x(t_0))$$

$$+ h^{-1}(t, t_0) \int_{t_0}^{t} h(t, s) a(s)\, ds \leq 0. \tag{7}$$

Here we have dropped the integral involving $\partial^2 h(t, s)/\partial s^2$ in (6) because of (H_4). We can now use (H_5) and (3) to deduce from (7) a desired contradiction upon taking lim sup as $t \to \infty$. Thus Eq. (1) is oscillatory in the sublinear case. ▮

Remark 1. Let $h(t, s) = (t - s)^{\alpha}$, $\alpha > 1$. It is easy to see that $h(t, s)$ satisfies all of (H_1)–(H_9). In this case, condition (3) becomes the familiar Kamenev's criterion (A_4). Theorem 1 also shows that assumption (A_7) is not required in our earlier result [21] for the sublinear case, namely.

COROLLARY 1. *Let $f(y)$ satisfy* (F_1) *and* (F_3). *Then condition* (A_4) *is sufficient for the oscillation of Eq.* (1).

In fact, n in (A_4) also does not have to be an integer but any real number greater than 1.

Remark 2. Let $h(t, s) = (\ln(t/s))^{\alpha}$, $\alpha > 1$. It is easily verified that $h(t, s)$ satisfies (H_1)–(H_8) in this case. To see that $h(t, s)$ also satisfies (H_9), note that $\lambda(t, s) = -\alpha[(\ln t/s)s]^{-1}$ which is non-decreasing in t for $t > s \geq t_0$. We therefore have the following

COROLLARY 2. *Let $f(y)$ satisfy* (F_1) *and* (F_3). *If $a(t)$ satisfies for some* $\alpha > 1$

$$\limsup_{t \to \infty} \left(\ln \frac{t}{s}\right)^{-\alpha} \int_{t_0}^{t} \left(\ln \frac{t}{s}\right)^{\alpha} a(s)\, ds = \infty, \tag{8}$$

then Eq. (1) *is oscillatory.*

We now consider the situation when condition (3) fails. Here, we require that the kernel function satisfy the additional hypothesis (H_6), (H_7), (H_8), and (H_9).

THEOREM 2. *Let $f(y)$ satisfy* (F_1) *and* (F_3). *Suppose that there exists a non-negative kernel function $h(t, s)$ satisfying* (H_1)–(H_9), *and also there exists a function $\varphi \in C[t_0, \infty)$ such that*

$$\limsup_{t \to \infty} \frac{1}{h(t, \tau)} \int_{\tau}^{t} h(t, s) a(s)\, ds \geq \varphi(\tau) \tag{9}$$

for all $\tau \geq t_0$ *and that* $\tau^{-1}\varphi_+^2(\tau) \notin L^1[t_0, \infty)$, *where* $\varphi_+(\tau) = \max(\varphi(\tau), 0)$, *i.e.*,

$$\int_{t_0}^{\infty} s^{-1}\varphi_+^2(s)\, ds = \infty. \tag{10}$$

Then Eq. (1) *is oscillatory.*

Proof. We proceed as in the proof of Theorem 1 and return to estimate (7). Writing τ for t_0 in (7) and taking lim sup as $t \to \infty$ in (7), we find by (H_7) and (9) that

$$\varphi(\tau) \leq \limsup_{t \to \infty} h^{-1}(t, \tau) \int_{\tau}^{t} h(t, s) a(s)\, ds \leq v(\tau). \tag{11}$$

Multiply (4) through by $h(t, s)$ and integrate from t_0 to t to obtain by (6)

$$\begin{aligned}
&\int_{t_0}^{t} h(t, s) f'(y(s)) v^2(s)\, ds + \int_{t_0}^{t} h(t, s) a(s)\, ds \\
&\quad \leq h(t, t_0) v(t_0) - \left(\frac{\partial h}{\partial s}(t, s)\bigg|_{s = t_0} F(x(t_0)) \right) \\
&\quad\quad - \int_{t_0}^{t} \frac{\partial^2 h}{\partial s^2}(t, s) F(x(s))\, ds.
\end{aligned} \tag{12}$$

The last integral involving $\partial^2 h(t, s)/\partial s^2$ is non-negative by (H_4). We now claim that the limit of the ratio

$$h^{-1}(t, t_0) \int_{t_0}^{t} h(t, s) f'(y) v^2(s)\, ds \tag{13}$$

exists, possibly infinite, by showing that the ratio in (13) is non-decreasing in t. It suffices to show that

$$\frac{\partial}{\partial t} \int_{t_0}^{t} \frac{h(t, s)}{h(t, t_0)} f'(y(s)) v^2(s)\, ds \geq 0.$$

Since $f'(y) \geq 0$, the above is true provided that

$$\frac{\partial}{\partial t} \frac{h(t, s)}{h(t, t_0)} = \frac{h(t, s)}{h(t, t_0)} \left\{ \frac{\partial}{\partial t} \ln h(t, s) - \frac{\partial}{\partial t} \ln h(t, t_0) \right\} \geq 0.$$

This then reduces to proving that $\partial \ln h(t, s)/\partial t$ is non-decreasing in s. By (H_8) we interchange partial differential with respect to s and t and find by

(H_9) that

$$\frac{\partial}{\partial s}\frac{\partial}{\partial t}\ln h(t,s) = \frac{\partial}{\partial t}\left[\frac{\partial}{\partial s}\ln h(t,s)\right] = \frac{\partial}{\partial t}\left[h^{-1}(t,s)\frac{\partial h}{\partial s}(t,s)\right] = \frac{\partial \lambda}{\partial t}(t,s) \geq 0.$$

Thus the limit of the expression in (13) exists. Now, dividing (12) through by $h(t,t_0)$ and taking lim sup as $t \to \infty$, we obtain by (H_5) and (11) that

$$\varphi(t_0) + \lim_{t\to\infty} h^{-1}(t,t_0)\int_{t_0}^{t} h(t,s)f'(y)v^2(s)\,ds \leq v(t_0) + M_0 F(x(t_0)),$$

from which it follows that the limit in (13) must be finite and bounded by the constant $M_1 = M_0 F(x(t_0)) + v(t_0) - \varphi(t_0)$. We now suppose that $f'(y)v^2 \notin L^1[t_0,\infty)$. For any $B > 0$, there exists $t_1 > t_0$ such that $\int_{t_0}^{t} f'(y)v^2 \geq B$ for all $t \geq t_1$. Observe that by (H_2)

$$\begin{aligned} h^{-1}(t,t_0)&\int_{t_0}^{t} h(t,s)f'(y)v^2(s)\,ds \\ &= h^{-1}(t,t_0)\int_{t_0}^{t}\left[-\frac{\partial h}{\partial s}(t,s)\right]\left(\int_{t_0}^{s} f'(y)v^2\right)ds \\ &\geq h^{-1}(t,t_0)\int_{t_1}^{t} B\left[-\frac{\partial h}{\partial s}(t,s)\right]ds = Bh^{-1}(t,t_0)h(t,t_1). \end{aligned} \tag{14}$$

Letting $t \to \infty$ in (14) and by (H_6), we find

$$\lim_{t\to\infty} h^{-1}(t,t_0)\int_{t_0}^{t} h(t,s)f'(y)v^2(s)\,ds \geq Bb_0 > 0. \tag{15}$$

Since B is arbitrary, (15) contradicts that the limit in (13) is finite, so it must be the case that $f'(y)v^2 \in L^1[t_0,\infty)$. Let $K_0 = \int_{t_0}^{\infty} f'(y)v^2$. Using the Schwarz inequality, we find by (F_3)

$$\begin{aligned} F(y(t)) - F(y(t_0)) &= \int_{t_0}^{t}\frac{y'(s)}{f(y(s))}\,ds \leq \left(\int_{t_0}^{t} f'(y)v^2\right)^{\frac{1}{2}}\left(\int_{t_0}^{t}\frac{1}{f'(y)}\right)^{\frac{1}{2}} \\ &\leq K_0^{\frac{1}{2}}\sqrt{c}\left(\int_{t_0}^{t} F(y(s))\,ds\right)^{\frac{1}{2}}. \end{aligned} \tag{16}$$

If $\int_{t_0}^{\infty} F(y(t))\,dt < \infty$, then (16) yields $|F(y(t))| \leq M_2$ for some constant M_2 independent of t. Otherwise, there exists $t_2 > t_0$ such that (16) gives

$$F(y(t)) \leq 2K_0^{\frac{1}{2}}\sqrt{c}\left(\int_{t_0}^{t} F(y(s))\,ds\right)^{\frac{1}{2}}, \qquad t \geq t_2. \tag{17}$$

Upon a quadrature of (17), we obtain a constant M_3 such that for $t \geq t_2$

$$\left(\int_{t_0}^{t} F(y(s))\,ds\right)^{\frac{1}{2}} \leq \left(\int_{t_0}^{t_1} F(y(s))\,ds\right)^{\frac{1}{2}} + 2K_0^{\frac{1}{2}}\sqrt{c}\,(t - t_1) \leq M_3 t. \tag{18}$$

Using (18) in (17), we find that $|F(y(t))| \leq M_4 t$ holds for some constant M_4 for all $t \geq t_2$, and this also holds when $F(y(t)) \in L^1[t_0, \infty)$. From (17), (11), and ($F_3$), we obtain

$$\begin{aligned}\int_{t_2}^{\infty} \tau^{-1}\varphi_+^2(\tau)\,d\tau &\leq \int_{t_2}^{\infty} \tau^{-1}v^2(\tau)\,d\tau \leq M_4\int_{t_2}^{\infty}[F(y(s)]^{-1}v^2(s)\,ds\\ &\leq cM_4\int_{t_2}^{t} f'(y)v^2 \leq cM_4\int_{t_0}^{\infty} f'(y)v^2 = cM_4K_0,\end{aligned}$$

which contradicts (10). Thus the existence of non-oscillatory solution $y(t)$ is ruled out, so Eq. (1) is oscillatory and the proof is complete. ∎

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We now turn to the superlinear Eq. (1) where $f(y)$ satisfies (F_2) and (F_4) and prove an extension of our earlier result using Kamenev type conditions (A_4) and (A_7). Our main result is

THEOREM 3. *Let $f(y)$ satisfy* (F_2) *and* (F_4). *Suppose that there exist non-negative kernel functions $h_1(t, s)$ and $h_2(t, s)$ on D satisfying* (H_1)–(H_7) *such that*

$$\limsup_{t\to\infty} \frac{1}{h_1(t, t_0)}\int_{t_0}^{t} h_1(t, s)a(s)\,ds = \infty, \tag{19}$$

and also

$$\liminf_{t\to\infty} \frac{1}{h_2(t, t_0)}\int_{t_0}^{t} h_2(t, s)a(s)\,ds = -L > -\infty, \qquad L > 0. \tag{20}$$

Furthermore, if $h_2(t,s)$ *satisfies in addition*

$$\left|\frac{\partial h_2}{\partial s}(t,s)\right|^2 \le d_1 \frac{\partial^2 h_2}{\partial s^2}(t,s)h_2(t,s), \tag{H_{10}}$$

with $d_1 < d$, *then Eq.* (1) *is oscillatory.*

Theorem 3 extends our result [23] as follows:

COROLLARY 3. *Let* $f(y)$ *satisfy* (F_2) *and* (F_4). *Suppose that there exists* $\alpha > 1$ *and* $\beta \ge 0$ *such that*

$$\limsup_{T\to\infty} \frac{1}{T^\alpha}\int_0^T (-t)^\alpha a(t)\,dt = \infty, \tag{A_4}'$$

and

$$\liminf_{T\to\infty} \frac{1}{T^\beta}\int_0^T (T-t)^\beta a(t)\,dt = -L > -\infty, \qquad L > 0. \tag{A_7}'$$

Then Eq. (1) *is oscillatory.*

Corollary 3 improves our earlier result where α, β are assumed to be integers both of which must be greater than 1. The proof given in [23] is based upon iterated integration so α, β must be integers and the question was raised as to the validity of the theorem when α, β are simply real numbers; see [23, p. 90, Remark 1]. Our proof of Theorem 3 to be given below not only allows α, β to be any real numbers but β is only required to be ≥ 0. It was mentioned in [23] that if there is any positive integer then (A_7) remains valid when p is replaced by any integer $q > p$ by an inductive argument which requires p to be an integer. However, it is easily verified that (A_7)′ also remains valid when β is replaced by any real $\sigma > \beta$ and β is any real number ≥ 0. To see this, we observe from an integration by parts

$$\int_0^T (T-t)^{\beta+\sigma-\beta} a(t)\,dt$$

$$= (\sigma-\beta)\int_0^T (T-s)^{\sigma-\beta-1}\int_0^s (T-t)^\beta a(t)\,dt\,ds. \tag{21}$$

By (A_7)′ there exists $T_0 > 0$ such that $\int_0^s (T-t)^\beta a(t)\,dt \ge -Ls^\beta$ for $s \ge T_0$.

Using this and the fact that $\sigma > \beta$, we can estimate as

$$\int_0^T (T-t)^{\sigma-\beta-1} \int_0^s (T-t)^{\beta} a(t)\, dt\, ds$$

$$\geq \int_{T_0}^T (T-s)^{\sigma-\beta-1} [-\lambda s^{\beta}]\, ds$$

$$\geq -LT^{\beta} \int_0^T (T-s)^{\sigma-\beta-1}\, ds = -\frac{L}{\sigma-\beta} T^{\beta}(T-T_0)^{\sigma-\beta}. \quad (22)$$

Substituting (22) into (21) and dividing (21) through by T^{σ}, we find

$$T^{-\sigma} \int_0^T (T-t)^{\sigma} a(t)\, dt \geq -LT^{\beta-\sigma}(T-T_0)^{\sigma-\beta},$$

which gives

$$\liminf_{T\to\infty} T^{-\sigma} \int_0^T (T-t)^{\sigma} a(t)\, dt \geq -L \lim_{T\to\infty} T^{\beta-\sigma}(T-T_0)^{\sigma-\beta}$$
$$= -L > -\infty.$$

This observation allows us to formulate Corollary 3 thereby answering the question raised earlier. In particular, when $\beta = 0$ Corollary 3 reduces to our original result in [23].

To see Corollary 3 follows from Theorem 3, we let $h_1(t,s) = (t-s)^{\alpha}$, $\alpha > 1$, and $h_2(t,s) = (t-s)^{\sigma}$, $\sigma \geq \beta \geq 0$ where σ is chosen to be large so that $\sigma(\sigma-1)^{-1} < d$. It is easy to see that for such a choice of $h_1(t,s)$ and $h_2(t,s)$, (H_1)–(H_7) are again satisfied. For $\sigma \geq \beta > 0, h_2(t,s) = (t-s)^{\sigma}$ also satisfies (H_{10}) with $d_1 = \sigma(\sigma-1)^{-1}$ since

$$\left|\frac{\partial h_l}{\partial s}(t,s)\right|^2 = \sigma^2 (t-s)^{2\sigma-2} \leq \left(\frac{\sigma}{\sigma-1}\right) h_2(t,s) \frac{\partial^2 h_2}{\partial s^2}(t,s).$$

Thus conditions $(A_4)'$, $(A_7)'$ imply conditions (19) and (20), respectively, so Corollary 3 follows as a special case of Theorem 3.

Proof of Theorem 3. Since $f(y)$ satisfies (F_2), we can define $w(t) = G(y(t))$ for a nonoscillatory solution of (1) $y(t)$ which can without loss of generality be assumed to be positive on $[t_0, \infty)$. In view of (1), $w(t)$ satisfies

$$w''(t) = a(t) + f'(y) w'^2(t), \qquad t \geq t_0. \quad (23)$$

Denote $u(t) = w'(t)$, then (23) becomes a first order Riccati differential equation in $u(t)$,

$$u'(t) = a(t) + f'(y) u^2(t). \quad (24)$$

We consider two separate cases, that of $f'(y)u^2 \in L^1[t_0, \infty)$ and that of $f'(y)u^2 \notin L^1[t_0, \infty)$. Multiplying (24) through by $h_1(t, s)$ and integrating by parts, we find

$$-h_1(t, t_0)u(t_0) - \int_{t_0}^{t} \frac{\partial h_1}{\partial s}(t, s)u(s)\, ds$$

$$= \int_{t_0}^{t} h_1(t, s)a(s)\, ds + \int_{t_0}^{t} h_1(t, s)f'(y)u^2(s)\, ds. \tag{25}$$

Let $K_0 = \int_{t_0}^{\infty} f'(y)u^2$ and observe by the Schwarz inequality and by (F_4) that

$$\begin{aligned} G(y(t)) &= G(y(t_0)) - \int_{t_0}^{t} \frac{y'(s)}{f(y(s))}\, ds \\ &\le G(y(t_0)) + \left\{ \int_{t_0}^{t} f'(y)u^2 \right\}^{\frac{1}{2}} \left\{ \int_{t_0}^{t} \frac{1}{f'(y)} \right\}^{\frac{1}{2}} \\ &\le G(y(t_0)) + \sqrt{K_0 d^{-1}} \left\{ \int_{t_0}^{t} G(y(s))\, ds \right\}^{\frac{1}{2}}. \end{aligned}$$

Suppose that $\int_{t_0}^{\infty} G(y(s))\, ds < \infty$; then it follows that $|G(y(t))| \le B_1$ for some constant B_1 depending only on t_0. Otherwise, there exists $t_1 \ge t_0$ such that

$$G(y(t)) \le 2\sqrt{K_0 d^{-1}} \left\{ \int_{t_0}^{t} G(y(s))\, ds \right\}^{\frac{1}{2}}, \qquad t \ge t_1$$

which upon a quadrature yields $\int_{t_0}^{t} G(y(s))\, ds \le K_0 d^{-1}(t - t_0)^2 \le K_0 d^{-1} t^2$. Thus $G(y(t)) \le 2K_0 d^{-1} t$ for $t \ge t_1$, and in any case there exists a constant B_2 such that $G(y(t)) \le B_2 t$ for all $t \ge t_0$ where B_2 depending only on t_0.

We now return to (25) and integrate by parts to obtain

$$\begin{aligned} -\int_{t_0}^{t} \frac{\partial h_1}{\partial s}(t, s)u(s)\, ds &= -\int_{t_0}^{t} \frac{\partial^2 h_1}{\partial s^2}(t, s)\{G(y(t_0)) - G(y(s))\}\, ds \\ &\le \int_{t_0}^{t} \frac{\partial^2 h_1}{\partial s^2}(t, s)G(y(s))\, ds \\ &\le B_2 \int_{t_0}^{t} \frac{\partial^2 h_1}{\partial s^2}(t, s)\, ds \\ &= -B_2 \left\{ \left. \frac{\partial h_1(t, s)}{\partial s} \right|_{s = t_0} \right\}. \end{aligned} \tag{26}$$

Using (26) in (25) and dividing through by $h_1^{-1}(t, t_0)$, we find by (H_5)

$$\begin{aligned} h_1^{-1}(t,t_0)\int_{t_0}^{t} h_1(t,s)a(s)\,ds &\le -u(t_0) - h_1^{-1}(t,t_0)\int_{t_0}^{t}\frac{\partial h_1}{\partial s}(t,s)u(s)\,ds \\ &\le -u(t_0) - B_2 h_1^{-1}(t,t_0)\frac{\partial h_1}{\partial s}(t,s)\bigg|_{s=t_0} \\ &\le -u(t_0) + B_2 M_0 < \infty. \end{aligned} \tag{27}$$

Taking lim sup as $t \to \infty$ in (27), we obtain the desired contradiction to (19). This completes the proof in the case when $f'(y)u^2 \in L^1[t_0, \infty)$.

Now we return to the case when $f'(y)u^2 \notin L^1[t_0, \infty)$. On this occasion, we multiply (24) by $h_2(t, s)$ and proceed to obtain a similar statement to (25) with h_2 substituting for h_1

$$\begin{aligned} -h_2(t,t_0)u(t_0) &= \int_{t_0}^{t}\frac{\partial h_2}{\partial s}(t,s)u(s)\,ds + \int_{t_0}^{t} h_2(t,s)f'(y)u^2\,ds \\ &\quad + \int_{t_0}^{t} h_2(t,s)a(s)\,ds. \end{aligned} \tag{28}$$

Claim that for any chosen $0 < \lambda < 1$ such that $d_1\lambda^{-1} < d$ (this is possible since $d_1 < d$) there exists a sequence $\{t_n\}$, $t_n \to \infty$, satisfying

$$\int_{t_0}^{t_n} h_2(t_n,s)f'(y)u^2(s)\,ds \ge -\frac{1}{\lambda}\int_{t_0}^{t_n}\frac{\partial h_2}{\partial s}(t_n,s)u(s)\,ds. \tag{29}$$

Now suppose that (29) holds. Using this in (28), we find

$$\begin{aligned} -\int_{t_0}^{t_n} h_2(t_n,s)a(s)\,ds - h_2(t_n,t_0)u(t_0) \\ \ge (1-\lambda)\int_{t_0}^{t_n} h_2(t_n,s)f'(y)u^2(s)\,ds. \end{aligned} \tag{30}$$

Since $f'(y)u^2 \notin L^1[t_0, \infty)$, we note from

$$\int_{t_0}^{t_n} h_2(t_n,s)f'(y)u^2(s)\,ds = -\int_{t_0}^{t_n}\frac{\partial h_2}{\partial s}(t_n,s)\left[\int_{t_0}^{s} f'(y)u^2\right]ds$$

and (H_6) that

$$\lim_{t_n\to\infty} h_2^{-1}(t_n,t_0)\int_{t_0}^{t_n} h_2(t_n,s)f'(y)u^2(s)\,ds = \infty. \tag{31}$$

Applying (31) to (30) and since $0 < \lambda < 1$, we obtain

$$\lim_{n\to\infty} \int_{t_0}^{t_n} h_2(t_n s) a(s)\, ds / h_2(t_n, t_0) = -\infty,$$

which contradicts (20). It remains to prove assertion (29). Suppose that no such sequence $\{t_n\}$ exists; then there must exist $\bar{t} \geq t_0$ so that for all $t \geq \bar{t}$ we have

$$\mathscr{I}(t) = \int_{t_0}^{t} h_2(t,s) f'(y) u^2(s)\, ds \leq -\frac{1}{\lambda} \int_{t_0}^{t} \frac{\partial h_2}{\partial s}(t,s) u(s)\, ds. \quad (32)$$

Note that

$$\int_{t_0}^{t} \frac{\partial^2 h_2}{\partial s^2}(t,s) G(y(s))\, ds = -\frac{\partial h_2}{\partial s}(t,s)\Big|_{s=t_0} G(y(t_0)) - \int_{t_0}^{t} \frac{\partial h_2}{\partial s}(t,s) u(s)\, ds. \quad (33)$$

Denote $N_0(t) = G(y(t_0))[-(\partial h_2/\partial s)(t,s)|_{s=t_0}]$. We can now use (33), (H_{10}), and Schwarz's inequality to obtain

$$\begin{aligned}
&\left| \int_{t_0}^{t} \frac{\partial h_2}{\partial s}(t,s) u(s)\, ds \right|^2 \\
&\leq d_1 \left(\int_{t_0}^{t} h_2(t,s) f'(y) u^2(s)\, ds \right) \left(\int_{t_0}^{t} \frac{\partial^2 h_2}{\partial s^2}(t,s) \frac{1}{f'(y)} \right) \\
&\leq d^{-1} d_1 \mathscr{I}(t) \left(\int_{t_0}^{t} \frac{\partial^2 h_2}{\partial s^2}(t,s) G(y(s))\, ds \right) \\
&\leq d^{-1} d_1 \mathscr{I}(t) \left\{ N_0(t) - \int_{t_0}^{t} \frac{\partial h_2}{\partial s}(t,s) u(s)\, ds \right\}. \quad (34)
\end{aligned}$$

Using (32) in (34) above, we have

$$\left| \int_{t_0}^{t} \frac{\partial h_2}{\partial s}(t,s) u(s)\, ds \right| \leq \frac{d^{-1} d_1}{\lambda} \left\{ N_0(t) - \int_{t_0}^{t} \frac{\partial h_2}{\partial s}(t,s) u(s)\, ds \right\},$$

or

$$\left(1 - \frac{d^{-1}d_1}{\lambda}\right)\left|\int_{t_0}^{t} \frac{\partial h_2}{\partial s}(t,s)u(s)\,ds\right| \le \frac{d^{-1}d_1}{\lambda}N_0(t). \tag{35}$$

Combining (35) and (32) we find

$$\mathscr{I}(t) \le \frac{1}{\lambda}\left(1 - \frac{d^{-1}d_1}{\lambda}\right)^{-1}\frac{d^{-1}d_1}{\lambda}N_0(t). \tag{36}$$

Since (H_5) implies that $\limsup h^{-1}(t,t_0)N_0(t) = B_2 < \infty$ for some constant B_2, dividing (36) by $h(t,t_0)$ and taking lim sup produces a desired contradiction to (31). This completes the proof in the case when $f'(y)u^2 \notin L^1[t_0,\infty)$ and concludes the proof of Theorem 3. ∎

An immediate consequence of Corollary 3 is that condition (A_6) alone implies oscillation of Eq. (1) which improves upon our earlier result when α is assumed to be an integer. For $\beta = 0$, condition $(A_7)'$ becomes (A_5), so Corollary 3 also includes our first result in [21].

Remark 3. Our results differ from that of Philos [14] in that we impose conditions on $(\partial^2 h/\partial s^2)(t,s)$ whereas Philos assumed a growth condition on $(\partial^2 h/\partial s^2)(t,s)$. In the special case of $h(t,s) = (t-s)^\alpha$, $\alpha > 1$, both our conditions and that of Philos are satisfied. Philos' assumptions on the kernel function are more effective for the linear equation and do not seem to carry over readily to the more general nonlinear Eq. (1).

Remark 4. We now apply our results to the following Emden–Fowler Equation

$$y'' + t^\sigma \sin t|y|^{\gamma-1}y = 0, \qquad \gamma > 0. \tag{37}$$

When $\gamma = 1$, i.e., the linear equation, a complete classification with respect to oscillation can be found in Willett [16] and Wong [19]. For the superlinear equation, i.e., $\gamma > 1$, Eq. (37) is oscillatory if and only if $\sigma \ge -1$. For $\sigma > 0$, it was proved by Butler [2] by a completely different method and the proof is rather complicated. For the sublinear equation, i.e., $0 < \gamma < 1$, Butler [2] also proved that Eq. (37) is oscillatory provided $\sigma \ge 1$. In this case, it is known that Eq. (37) is oscillatory if and only if $\sigma \ge -\gamma$. Kura [10] and independently Kwong and Wong [11] showed that Eq. (37) is oscillatory if $\sigma > -\gamma$. Oscillation of (37) when $\sigma = -\gamma$ was settled by Kwong and Wong [12] and also independently using different methods by Onose [13]. When $\sigma > 1$, $a(t) = t^\sigma \sin t$ fails to satisfy (A_0), (A_1), (A_2), and (A_3), and also (A_6). However, we can apply Corollary 1 to show that Eq. (37) is oscillatory if $\sigma > 1$ in the sublinear case, i.e.,

$0 < \gamma < 1$. In the superlinear case, i.e., $\gamma > 1$, we can apply Corollary 3 and note that condition $(A_4)'$ is satisfied for any $\alpha > 1$ (in fact for any $\alpha > 0$). On the other hand, choosing $\beta > \sigma$, it is easily shown that $t^\sigma \sin t$ also satisfies $(A_7)'$ so that Eq. (37) is oscillatory for all $\sigma > 1$ and $\gamma > 0$. This offers an alternative approach considerably simpler than that of Butler [2], Kura [10], and Kwong and Wong [11].

Remark 5. It is easy to give examples of $f(y)$ satisfying (F_1) and (F_3) which are not of the form of Emden–Fowler type. Consider

$$y'' + \sigma \sin t |y|^{\frac{1}{2}}(1 + |y|)\operatorname{sign} y = 0. \tag{38}$$

Here $f(y) = \sqrt{y}(1 + y)$ for $y > 0$ and $f(y) = -f(-y)$. Note that $F(y) = 2\tan^{-1}\sqrt{y}$ and for all y

$$f'(y)F(y) = \frac{1}{\sqrt{y}}(1 + 3y)\tan^{-1} y \geq 1.$$

(In fact, $f'(y)F(y) \to 1$ as $y \to 0$.) Applying Corollary 1 to Eq. (38), we deduce that it is oscillatory for all $\sigma > 1$. Examples in the superlinear case, i.e., $f(y)$ satisfying (F_2) and (F_4), can be formulated in a similar way.

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