Maximal, Minimal, and Primary Invariant Subspaces

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Let $X$ be a complex infinite dimensional Banach space. An operator $L$ on $X$ is called of subcritical class, if \[ \sum_{n=1}^{\infty} n^{-3/2} \log^+ |L^n| < \infty. \] Assume that $T$ is an operator on $X$ whose iterates have norms of polynomial growth. We prove that if $T$ has a range of finite codimension and a left inverse of subcritical class, then every maximal invariant subspace of $T$ has codimension one, and if $T$ has a finite dimensional kernel and a right inverse of subcritical class, then every minimal invariant subspace of $T$ is one dimensional. Using these results we obtain new information on the invariant subspace lattices of shifts and backward shifts on a wide class of Banach spaces of analytic functions on the unit disc. We also introduce the notion of primary invariant subspaces, and determine their structure for a large class of shifts.

1. INTRODUCTION AND MAIN RESULTS

Throughout this paper, $X$ will denote a complex Banach space of dimension greater than one, and $\mathcal{L}(X)$ the algebra of bounded linear operators on $X$. The term operator on $X$ will mean an element of $\mathcal{L}(X)$. An invariant subspace of an operator $T$ on $X$ is a closed subspace $M$ of $X$ such that $TM \subset M$; if $M \neq \{0\}$ and $M \neq X$, then $M$ is called a nontrivial invariant subspace of $T$. The collection of all invariant subspaces of $T$ is denoted by \( \text{Lat} T \). A maximal invariant subspace of $T$ is an element of $\text{Lat} T$ that is different from $X$, and is not properly included in any element of $\text{Lat} T$ other than $X$. A minimal invariant subspace of $T$ is an element of $\text{Lat} T$ that is different from the zero space, and the only element of $\text{Lat} T$ which is properly included in it, is the zero space.

It is clear that every element of $\text{Lat} T$ which has codimension one is a maximal invariant subspace of $T$, and every element of $\text{Lat} T$ which has dimension one is a minimal invariant subspace of $T$.

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A natural step in an attempt to understand the action of an operator, is to determine the structure of its invariant subspaces. However, there is only a handful of operators on an infinite dimensional Banach space, for which a complete and useful description of their invariant subspace lattice is known. Even for a concrete operator such as the shift on the classical Bergman space, such a description is far out of scope (see [6, 11]). For most operators $T$, it is a more realistic task, to try to get some partial information on $\text{Lat}T$. A natural question is whether every maximal (resp. minimal) invariant subspace of $T$ has codimension (resp. dimension) one.

Even these seemingly simple questions, often encounter serious obstacles. To illustrate this point, assume that $X$ is an infinite dimensional Banach space, such that each of its infinite dimensional closed subspaces, includes another closed subspace which is (topologically) isomorphic to $X$. For example, the spaces $\ell^p$, $1 \leq p < \infty$ and $c_0$ have this property (see [31, Chap. 2]). Let $T$ be an operator on $X$ which has a right inverse (equivalently, is surjective and its kernel is complemented) and infinite dimensional kernel. Then by a result of Caradus [18], every operator on $X$ is similar to a constant multiple of the restriction of $T$ to an element of $\text{Lat}T$. Hence, if every minimal invariant subspace of $T$ is one dimensional, then every operator on $X$ has a nontrivial invariant subspace. Thus, if $X$ is a Hilbert space, the question whether every minimal invariant subspace of $T$ is one dimensional, is equivalent to the general invariant subspace problem on Hilbert space.

By a duality argument, we obtain from these observations, that if $X$ is one of the spaces $\ell^p$, $1 < p < \infty$ or $c_0$, and $T$ is an operator on $X$ which has a left inverse and range of infinite codimension, and every maximal invariant subspace of $T$ has codimension one, then every operator on $X$ has a nontrivial invariant subspace.

Another natural question concerning the structure of the invariant subspace lattice of an operator $T$ (which is considerably harder than the questions above), is whether for every two elements $M_1$ and $M_2$ of $\text{Lat}T$ such that such $M_1 \subset M_2$ and $\dim M_2/M_1 > 1$, there exists another element $M$ of $\text{Lat}T$ such that $M_1 \subsetneq M \subsetneq M_2$. An equivalent formulation is, whether for every $M \neq \{0\}$ in $\text{Lat}T$, every maximal invariant subspace of the operator $T|_M$ has codimension one in $M$. If this is the case, we shall say that $\text{Lat}T$ has no proper gaps.

It follows from the preceding discussion, that if $T$ is an operator on a separable Hilbert space which has a left inverse (or equivalently is injective and has closed range) and there exists an element $M$ of $\text{Lat}T$ such that $\dim(M \oplus TM) = \infty$, then $\text{Lat}T$ has no proper gaps, if and only if, the invariant subspace problem on Hilbert space has a positive answer.

We now introduce a class of operators that will play an important role in the sequel.
Definition. An operator $L$ on $X$ will be called of subcritical class if

$$\sum_{n=1}^{\infty} n^{-3/2} \log^2 + \|L^n\| < \infty.$$ 

This terminology was inspired by a paper of Beurling [13] that will be discussed in Section 3.

We shall say that the iterates of an operator $T$ on $X$ have polynomial growth if

$$\|T^n\| = o(n^\alpha), \quad n \to \infty,$$

for some $\alpha > 0$.

Our first main result is

**Theorem 1.1.** Assume that $T$ is an operator on $X$ whose iterates have norms of polynomial growth.

(a) If $T$ has range of finite codimension and a left inverse of subcritical class, then every maximal invariant subspace of $T$ has codimension one.

(b) If $T$ has a finite dimensional kernel and a right inverse of subcritical class, then every minimal invariant subspace of $T$ is one dimensional.

**Remarks.** (1) It is clear that the class of operators which satisfy the conditions of the theorem is preserved under similarity and finite direct sums.

(2) A duality argument shows that if $X$ is reflexive, both parts of the theorem are equivalent.

We describe next, in a general setting, applications of the theorem to shifts and backward shifts on Banach spaces of complex and vector valued holomorphic functions on the unit disc. Concrete applications will be given in Sections 2 and 4.

We first need some notations and definitions.

The open unit disc in $\mathbb{C}$ will be denoted by $D$. In what follows, $K$ denotes a nonzero complex Banach space, and $\text{Hol}(D, K)$ the vector space of $K$-valued holomorphic functions on $D$. The identity function on $D$ is denoted by $z$.

**Definition.** A Banach space $E$ of $K$-valued functions on $D$ is called admissible if the following conditions hold.

(a) $E$ is a nonzero vector subspace of $\text{Hol}(D, K)$.

(b) For every $\lambda$ in $D$, the linear transformation from $E$ to $K$, defined by $f \mapsto f(\lambda)$, $f \in E$, is continuous.
(c) If $f$ is in $E$, then the function $zf$ is also in $E$.

(d) If $f$ is in $E$, then the holomorphic function $f_0$ on $D$ defined by

$$f_0(\zeta) = \frac{f(\zeta) - f(0)}{\zeta}, \quad \zeta \in D \setminus \{0\},$$

is also in $E$.

**Remark.** If $E$ is an admissible Banach space of complex functions, then the definition implies that it contains the polynomials.

It follows from condition (b) and the closed graph theorem, that the transformations $f \to zf$ and $f \to f_0$, $f \in E$, are in $\mathcal{L}(E)$. The first is called the **shift** on $E$ and will be denoted by $S_E$, and the second is called the **backward shift** on $E$ and will be denoted by $B_E$.

It is clear that $B_E$ is a left inverse of $S_E$ and that $S_E B_E$ is a projection on the range of $S_E$ along the kernel of $B_E$, which consists of the constant functions in $E$. Thus, if $K$ is finite dimensional, then the kernel of $B_E$ has finite dimension and the range of $S_E$ has finite codimension. Hence, an immediate consequence of Theorem 1.1 is

**Theorem 1.2.** Assume that $K$ is finite dimensional and that $E$ is an admissible Banach space of $K$-valued functions on $D$.

(a) If the iterates of $S_E$ have norms of polynomial growth and $B_E$ is of subcritical class, then every maximal invariant subspace of $S_E$ has codimension one.

(b) If the iterates of $B_E$ have norms of polynomial growth and $S_E$ is of subcritical class, then every minimal invariant subspace of $B_E$ is one dimensional.

Many concrete applications of the theorem will be given in Section 4. Using a result of Aleman et al. in [5], we shall show in Section 2, that if $S$ is the shift on the classical Bergman space, and $M$ is a nonzero element of $\text{Lat } S$, then the operator $S |_{M}$ has a left inverse of subcritical class. Thus, by Theorem 1.1(a), if $M \oplus SM$ is finite dimensional, then every maximal invariant subspace of the operator $S |_{M}$ has codimension one in $M$. When $M$ is the whole space, this follows directly from Theorem 1.1(a), and was also proved by Hedenmalm [23].

Interesting applications of Theorem 1.1 can be obtained from the following simple observation. If $T$ is an operator on $X$ which is bounded below by one (that is, $\|Tx\| \geq \|x\|$, $\forall x \in X$) and there exists a projection of norm one on $TX$, then $T$ has a left inverse $L$, which is a contraction (i.e., $\|L\| \leq 1$), hence of subcritical class. In fact, assume that there exists such a
projection, and denote its kernel by $Y$. Then it is easily verified that the operator $L$ on $X$, defined by $Ly = 0$ for $y$ in $Y$ and $LTx = x$ for $x$ in $X$, and by linearity for an arbitrary element in $X$, has these properties.

It follows from this, that every operator on a Hilbert space, which is bounded below by one has a left inverse of subcritical class, and the same is true for the restrictions of that operator to its nonzero invariant subspaces. Hence Theorem 1.1 implies

**Theorem 1.3.** Assume that $X$ is a Hilbert space, and that $T$ is an operator on $X$ with range of finite codimension, which is bounded below by one, and its iterates have norms of polynomial growth. Then every maximal invariant subspace of $T$ has codimension one. If in addition, for every element $M$ of Lat$T$, the space $M \ominus TM$ has finite dimension, then Lat$T$ has no proper gaps.

In Section 4 we shall show that the theorem implies that the invariant subspace lattices of shifts on a wide class of Dirichlet type spaces, have no proper gaps.

It follows in particular from Theorem 1.3, that if $T$ is an isometry on a Hilbert space, with range of finite codimension, then every maximal invariant subspace of $T$ has codimension one. When $T$ is a unilateral shift of finite multiplicity, this result is well known. Two different proofs can be found in [25, Theorem 16; 38, Theorem 3.32 and Corollary 6.18]. It is possible that the arguments in these proofs can be extended to general isometries with range of finite codimension, but it is clear that they cannot be extended to isometries on $\ell^p$, $1 \leq p < \infty$, and $c_0$. However, from Theorem 1.1(a) we can get the result also for isometries on these spaces. If $X$ is $\ell^p$, $1 \leq p < \infty$, or $c_0$, and $T$ is an isometry on $X$, then by a result of Pelczynski [37, Th. 2] there exists a projection of norm one on $TX$. Hence by the preceding observations, $T$ has a left inverse which is a contraction. Thus from Theorem 1.1(a) we obtain

**Theorem 1.4.** If $X$ is one of the spaces $\ell^p$, $1 \leq p < \infty$ or $c_0$, and $T$ is an isometry on $X$ with range of finite codimension, then every maximal invariant subspace of $T$ has codimension one.

If $E$ is an admissible Hilbert space of complex holomorphic functions on $\mathbb{D}$ such that $S_E$ is bounded below by one, and $\dim M/S_E M < \infty$ for every $M$ in Lat$S_E$, we can deduce from Theorem 1.3 that Lat$S_E$ has no proper gaps. But when $E$ is not a Hilbert space, we cannot in general apply Theorem 1.1(a) to reach this conclusion. However, if $E$ is a Banach algebra in which the polynomials are dense, we can prove the assertion by a different approach. More precisely, we have the following
Theorem 1.5. Assume that $E$ is an admissible Banach space of complex holomorphic functions on $\mathbb{D}$ that satisfies the hypotheses of Theorem 1.2(a), which is also a Banach algebra with respect to pointwise multiplication. If the polynomials are dense in $E$, then $\text{Lat}S_E$ has no proper gaps.

The hypothesis that the polynomials are dense in $E$, implies that $\text{Lat}S_E$ is the collection of closed ideals in $E$. The theorem applies to a wide class of Banach algebras of holomorphic functions on $\mathbb{D}$, in particular the Banach algebra of absolutely convergent Taylor series, in which the structure of closed ideals is widely open and very complicated; as shown in [9], there exist closed ideals in this algebra that are not finitely generated (topologically). We shall deduce the theorem in Section 2 from a more general result (Theorem 2.18) which also provides some new information on the structure of closed ideals in the Banach algebra $H^\infty$. In Section 4 we shall deduce from the theorem that if $B$ is the backward shift on the Bergman space

$$A^p = \left\{ f \in \text{Hol}(\mathbb{D}) : \|f\|_p^p = \frac{1}{\pi} \int_{\mathbb{D}} |f|^p \, dx \, dy < \infty \right\}$$

for some $1 < p < 2$, then $\text{Lat}B$ has no proper gaps. This will also be proved for $p = 2$ by a different argument.

We now turn to the second class of invariant subspaces considered in this paper, namely, the primary invariant subspaces. To define this concept we need some notations.

For a closed subspace $M$ of $X$, we shall denote by $\pi_M$ the canonical map of $X$ onto the quotient space $X/M$. If $T$ is an operator on $X$ and $M$ is in $\text{Lat}T$, we shall denote by $T^M$ the quotient operator induced by $T$ on $X/M$, that is, the operator defined by the identity $T^M \pi_M = \pi_M T$. When $X$ is a Hilbert space, $T^M$ can be identified with the compression of $T$ to $X \ominus M$. The spectrum of an operator $T$ will be denoted as usual by $\sigma(T)$.

Definition. A primary invariant subspace of an operator $T$ on $X$, is an element $M$ of $\text{Lat}T\setminus\{X\}$ such that $\sigma(T^M)$ is a singleton. If the element of this singleton is $\lambda$, we shall say that $M$ is a primary invariant subspace of $T$ at $\lambda$.

Remark. We shall see in Section 3 that if $M$ is a primary invariant subspace of $T$, then $\sigma(T^M) \subset \sigma(T)$.

This definition was motivated by the concept of primary ideal in a Banach algebra. We recall that if $X$ is a commutative Banach algebra with unit, then a closed ideal in $X$ is called primary, if it is included in a single
maximal ideal. If $X$ is generated by a single element $x$ (that is, the polynomials in $x$ and the unit element are dense in $X$) and $T$ is the operator on $X$ defined by $Ty = xy$, $y \in X$, then the elements of $\text{Lat} T \setminus \{X\}$ are the closed ideals in $X$, and it follows from the Gelfand theory (see [21]) that the primary invariant subspaces of $T$ are the primary ideals in $X$.

We shall determine the primary invariant subspaces of $S_E$ for a large class of admissible Banach spaces $E$ of complex holomorphic functions on $\mathbb{D}$. It is easy to show (see Section 3) that if $\sigma(B_E) \subset \mathbb{D}$, then the primary invariant subspaces of $S_E$ at a point $\lambda$ in $\mathbb{D}$ are given by

$$M^{(n)}(\lambda) = \{ f \in E : f^{(j)}(\lambda) = 0, \ j = 1, \ldots, n-1 \}, \quad n = 1, 2, \ldots.$$ 

The main problem is to determine the primary invariant subspaces of $S_E$ at a point in $\partial \mathbb{D}$. Before stating our result, we introduce some notations and make a few preliminary observations.

We denote by $H^\infty$ the Banach algebra of all bounded holomorphic functions on $\mathbb{D}$ with the norm

$$\| f \|_\infty = \sup_{\zeta \in \mathbb{D}} |f(\zeta)|, \quad f \in H^\infty,$$

and by $H^\infty_1$ the Banach algebra of all holomorphic functions $f$ on $\mathbb{D}$, such that $f$ and $f'$ are in $H^\infty$, with the norm

$$\| f \|_{\infty, 1} = \| f \|_\infty + \| f' \|_\infty, \quad f \in H^\infty_1.$$

The Banach algebra of absolutely convergent Taylor series will be denoted by $W^+$. That is, $W^+$ consists of all holomorphic functions $f$ on $\mathbb{D}$ such that

$$\| f \|_{W^+} = \sum_{n=0}^{\infty} \frac{|f^{(n)}(0)|}{n!} < \infty.$$

It is easy to see that $H^\infty_1 \subset W^+$, and that the imbedding is continuous.

For every $a > 0$, we shall denote by $f_a$ the holomorphic function on $\mathbb{D}$ defined by

$$f_a(\zeta) = (\zeta - 1)^2 \exp \frac{a}{\zeta - 1}, \quad \zeta \in \mathbb{D}.$$

This function is in $H^\infty_1$, hence also in $W^+$.

We assume next that $E$ is an admissible Banach space of complex holomorphic functions on $\mathbb{D}$ and that $S_E$ is a contraction. This implies that
the sequence \( \{ |z^n| \}_{n \in \mathbb{Z}} \) is bounded, and therefore \( W^+ \subset E \), and the
imbedding is continuous.

For a function \( f \) in \( E \), we shall denote by \([ f ]\) the closed linear span in \( E \) of the functions \( z^nf, \ n = 0,1, \ldots \).

For every \( a > 0 \), we shall denote by \( M_a \) the space of all functions \( f \) in \( E \) such that

\[
\limsup_{r \to 1^-} (1 - r) \log |f(r)| \leq -a,
\]

and by \( M_0 \), the space \([ z - 1 ]\). It is clear that if \( M_0 \) is different from \( E \), then it is a primary invariant subspace of \( S_E \) at \( 1 \).

We now state our result on the primary invariant subspaces of \( S_E \) at \( 1 \). A similar result holds for every \( \lambda \) in \( \partial D \).

**Theorem 1.6.** Assume that the polynomials are dense in \( E \), that \( B_E \) is of subcritical class and that one of the following conditions holds.

1. \( 1 \) is not an eigenvalue of \( S_E^* \).
2. \( 1 \) is an eigenvalue of \( S_E^* \), and \( \sup_{0 < x < 1} |f(x)| < \infty \) for every \( f \) in \( E \).

Then the primary invariant subspaces of \( S_E \) at \( 1 \), are the spaces \( \{ M_a, a > 0 \} \) in the first case, and \( \{ M_a, a \geq 0 \} \) in the second case. Furthermore, \( M_a = [ f_a ] \) for all \( a > 0 \), and if \( 0 < a < b \), then \( M_b \subset M_a \).

Many examples of Banach spaces that satisfy the conditions of the theorem will be given in Section 4. They include the disc algebra, the algebra \( W^+ \), the Hardy spaces \( H^p, 1 \leq p < \infty \), and a large class of weighted Bergman spaces. For the disc algebra the result was proved by Silov [46], for the algebra \( W^+ \) by Feldman [20] and independently by Kahane [27], and for the Bergman space \( A^2 \) by Hedenmalm [23]. The proof of the general result requires different methods than those employed by these authors.

Another class of invariant subspaces that is of interest, and arises naturally in the study of primary invariant subspaces, is the class of generalized root spaces.

**Definition.** A nonzero invariant subspace \( V \) of an operator \( T \) on \( X \) is called a *generalized root space* of \( T \), if \( \sigma(T)_V \) is a singleton. If the element of this singleton is \( \lambda \), we shall say that \( V \) is a generalized root space of \( T \) at \( \lambda \).
It follows by a duality argument that if $T$ is an operator on $X$, then an element $M$ of $\text{Lat} T$ is a primary invariant subspace of $T$ at $\lambda$, if and only if, $M^\perp$ (the annihilator of $M$ in the dual space $X^*$) is a generalized root space of the adjoint operator $T^*$ at $\lambda$. Thus, the mapping $M \to M^\perp$ defines a bijection between the set of primary invariant subspaces of $T$ at $\lambda$ and the set of $w^*$-closed generalized root spaces of $T^*$ at $\lambda$.

It is easily verified (see Section 3) that if $V$ is a generalized root space of $T$, then $\sigma(T^*) \subset \sigma(T)$.

We shall determine all the $w^*$-closed generalized root spaces of the operator $S^*_E$ when $E$ satisfies the conditions of Theorem 1.6. In the sequel we assume that $E$ is such a space.

It is easy to determine the generalized root spaces of $S^*_E$ at a point in $\mathbb{D}$ (see Theorem 3.3), and the main problem is to determine these spaces at a point in $\partial \mathbb{D}$. For simplicity, we state the result for $\lambda = 1$. First we need some notations.

For every $\varphi$ in $E^*$ we shall denote by $\hat{\varphi}$ the sequence $\hat{\varphi}(n) = \langle z^n, \varphi \rangle$, $n \in \mathbb{Z}_+$.

Since the polynomials are dense in $E$, the mapping $\varphi \to \hat{\varphi}$ is injective. If $\mathcal{F}$ is a set of functions on $\mathbb{C}$, we shall denote the set of their restrictions to $\mathbb{Z}_+$ by $\mathcal{F}|_{\mathbb{Z}_+}$.

For every $0 \leq \tau < \infty$ and $0 \leq \rho < \infty$, we shall denote by $\mathcal{E}_{\rho, \tau}$ the vector space of all entire functions $f$ such that

$$\sup_{\zeta \in \mathbb{C}} |f(\zeta)| \exp[-(\tau + \varepsilon) |\zeta|^\rho] < \infty, \quad \forall \varepsilon > 0.$$  

That is, $\mathcal{E}_{\rho, \tau}$ consists of all entire functions of order at most $\rho$ and type at most $\tau$.

Finally, for every $0 \leq \tau < \infty$ we denote

$$V_{\tau} = \{ \varphi \in E^* : \hat{\varphi} \in \mathcal{E}_{1/2, \tau}|_{\mathbb{Z}_+} \}.$$  

**Theorem 1.7.** The $w^*$-closed generalized root spaces of $S^*_E$ at 1, are the spaces $\{ V_{\tau}, \tau > 0 \}$ if (C.1) holds, and the spaces $\{ V_{\tau}, \tau \geq 0 \}$ if (C.2) holds. Furthermore, $M^a = V_{\tau} \cap \mathbb{D}$ for all $a \geq 0$.

**Remark.** For $E = W^+$ the conclusion of the theorem is implicitly contained in [20].
The rest of the paper is organized as follows. In Section 2 we prove Theorem 1.1 and Theorem 1.5 along with some related results. We also establish there the result on the Bergman shift that was mentioned before. In Section 3 we prove Theorem 1.6 and Theorem 1.7, and in Section 4 we give applications to shifts and backward shifts on concrete spaces. In Section 5 we present some additional results.

We have described the results on primary invariant subspaces in a conference on operator theory at CIRM, Luminy (France) in April 1993, and in a workshop on Bergman spaces and related topics at NTNU, Trondheim (Norway) in October 1996.

2. MAXIMAL AND MINIMAL INVARIANT SUBSPACES

The proof of Theorem 1.1 requires some intermediate results. We begin with some simple observations.

Assume that $T$ is an operator on $X$, and that $M$ is an element of $\text{Lat}(T \setminus \{X\})$. It is easily verified that the mapping

$$V \mapsto \pi_{M}^{-1}(V), \quad V \in \text{Lat}T$$

is a bijection between $\text{Lat}T$ and the set of elements in $\text{Lat}T$ which include $M$, and that $V$ is a nontrivial invariant subspace of $T$, if and only if, $M \subseteq \pi_{M}^{-1}(V) \subseteq X$. Therefore, the assertion that every maximal invariant subspace of $T$ has codimension one, is equivalent to the assertion that for every element $M$ of $\text{Lat}T$ with codimension greater than one, the operator $T$ has a nontrivial invariant subspace. The next result shows that under some conditions on $T$, it suffices to prove the assertion when $\sigma(T) \cap \mathbb{D}$ is empty.

**Proposition 2.1.** Assume that $T$ is an operator on $X$ with range of finite codimension, which has a left inverse $L$ such that $\sigma(L) \subseteq \overline{\mathbb{D}}$. If $M$ is an element in $\text{Lat}T$ of codimension greater than one, and $\sigma(T) \cap \mathbb{D}$ is not empty, then $T$ has a nontrivial invariant subspace.

**Proof.** Since $\sigma(L) \subseteq \overline{\mathbb{D}}$, the operator $I - iL$ is invertible for every $\lambda$ in $\mathbb{D}$. Let $K$ denote the kernel of $L$, and $\tilde{L}$ the operator function on $\mathbb{D}$ defined by

$$\tilde{L}(\lambda) = (I - \lambda L)^{-1} L, \quad \lambda \in \mathbb{D}. \quad (2.1)$$
Since $LT = I$, we obtain that for every $\lambda$ in $\mathbb{D}$, $\tilde{L}(\lambda)(T - \lambda) = I$, and therefore the operator $(T - \lambda) \tilde{L}(\lambda)$ is a projection on $(T - \lambda)X$ along $K$. This shows that

$$\text{codim}(T - \lambda)X = \dim K, \quad \forall \lambda \in \mathbb{D}, \tag{2.2}$$

and by the assumption that $TX$ has finite codimension, we get that $(T - \lambda)X$ has finite codimension for every $\lambda$ in $\mathbb{D}$. Remembering that $(T^M - \lambda)\pi_M = \pi_M(T - \lambda)$, we conclude that for every $\lambda$ in $\mathbb{D}$, the range of the operator $T^M - \lambda$ has finite codimension, and therefore is closed. Thus, if $\lambda \in \mathbb{D} \cap \sigma(T^M)$ and $T^M - \lambda \neq 0$, we get from the assumption that $\text{codim } M > 1$, that either the kernel or the range of $T^M - \lambda$ is a nontrivial invariant subspace of $T^M$. If $T^M - \lambda = 0$, the assertion is clear. $\square$

An analogous result holds for operators with right inverse.

**Proposition 2.2.** Assume that $T$ is an operator with finite dimensional kernel, which has a right inverse $R$ such that $\sigma(R) \subset \mathbb{D}$. If $M$ is an element in $\text{Lat } T$ of dimension greater than one, and $\sigma(T|_M) \cap \mathbb{D}$ is not empty, then $T|_M$ has a nontrivial invariant subspace.

**Proof.** A similar argument as in the previous proof, shows that the assumptions imply that for every $\lambda$ in $\mathbb{D}$, the operator $T - \lambda$ is surjective and has finite dimensional kernel. Therefore, (see [28, Chap. 4, Lemma 5.29]) for every $\lambda$ in $\mathbb{D}$ the operator $T|_M - \lambda$ has closed range. From this the assertion follows as before. $\square$

If $T$ is an operator on $X$ such that $\sigma(T) \subset \mathbb{D}$, it follows from the spectral radius formula, that for every $M$ in $\text{Lat } T$, $\sigma(T|_M) \subset \mathbb{D}$ and $\sigma(T^M) \subset \mathbb{D}$. It also follows from that formula, that if the iterates of $T$ have norms of polynomial growth, or more generally, if $T$ is of subcritical class, then $\sigma(T) \subset \mathbb{D}$. Thus from Propositions 2.1 and 2.2 we see that Theorem 1.1 is equivalent to

**Proposition 2.3.** Assume that $T$ is an operator on $X$ whose iterates have norms of polynomial growth.

(a) If $T$ has range of finite codimension and a left inverse of subcritical class, and $M$ is an element in $\text{Lat } T$ of codimension greater than one such that $\sigma(T^M) \subset \mathbb{D}$, then $T^M$ has a nontrivial invariant subspace.

(b) If $T$ has finite dimensional kernel and a right inverse of subcritical class, and $M$ is an element in $\text{Lat } T$ of dimension greater than one such that $\sigma(T|_M) \subset \mathbb{D}$, then $T|_M$ has a nontrivial invariant subspace.
The proof of the proposition will be based on the following result, which is an immediate consequence of [7, Theorem 1 and Lemma 2].

**Theorem 2.4.** Assume that $T$ is an operator on $X$ whose iterates have norms of polynomial growth, and $o(T) \subset \partial \Omega$. If for some positive constants $a$ and $C$

$$\| (\lambda - T)^{-1} \| \leq C \exp \frac{a}{1 - |\lambda|}, \quad \forall \lambda \in \Omega,$$

(2.3)

then $T$ has a nontrivial invariant subspace.

Proposition 2.3 will be established by showing that the resolvents of the operators $T^M$ and $T|_M$ satisfy the growth condition (2.3). For this we need estimates of the growth of vector valued holomorphic functions on $\Omega$, which are the quotient of a vector function and a scalar function. Before stating the result that provides this estimate, we introduce some notations and definitions.

If $G$ is a function on $\Omega$ with values in $\mathbb{C}$ or in a Banach space, we shall use the notation

$$M_G(r) = \sup_{|\zeta| < r} \| G(\zeta) \|, \quad 0 \leq r < 1.$$ 

If

$$\limsup_{r \to 1-} (1 - r) \log^+ M_G(r) < \infty,$$

we shall say that $G$ has **exponential growth**, and if the left hand side of this inequality is zero, we shall say that $G$ has **minimal exponential growth**. If

$$\int_0^1 \left( \frac{\log^+ M_G(t)}{1 - t} \right)^{1/2} dt < \infty,$$

we shall say that $G$ has **moderate growth**. The inequality

$$\int_r^1 \left( \frac{\log^+ M_G(t)}{1 - t} \right)^{1/2} dt \geq \left[ (1 - r) \log^+ M_G(r) \right]^{1/2}, \quad 0 \leq r < 1,$$

shows that if $G$ has moderate growth, then it has minimal exponential growth.
If \( g \) is a complex holomorphic function on \( D \), we shall denote
\[
A_g(r) = \left[ \int_0^r \left( \frac{\log^+ M_g(t)}{1-t} \right)^{1/2} \, dt \right]^2, \quad 0 \leq r < 1.
\]

**Proposition 2.5.** If \( g \) is a complex holomorphic function on \( \mathbb{D} \) such that \( g(0) = 1 \), and \( F \) is a holomorphic function on \( \mathbb{D} \) with values in a Banach space, then
\[
\log M_F(r) \leq \log M_{gF}(r) + c A_g(r) \log^+ M_g(r), \quad 0 \leq r < 1,
\]
where \( c \) is an absolute positive constant.

**Proof.** Fix \( 0 < r < 1 \). A result of Momm (the Corollary in [34] with \( \epsilon = \frac{1}{4} \)) asserts that if \( \psi \) is a positive increasing function on \( [1, \infty) \), and \( f \) is a (complex) holomorphic function on \( D \) such that \( f(0) = 1 \), and
\[
\log |f(\zeta)| \leq \psi \left( \frac{1}{1-|\zeta|} \right), \quad \zeta \in D,
\]
then there exists a Jordan curve \( \Gamma \) in the annulus \( r < |\zeta| < \frac{3r+1}{4} \), such that
\[
\log |f(\zeta)| \geq -c_0 \left[ \int_1^{5/4} \left( \psi(t) \right)^{1/2} \frac{dt}{t^r} \right]^2, \quad \zeta \in \Gamma,
\]
where \( c_0 \) is an absolute positive constant. Applying this with \( f = g \) and \( \psi(t) = \log M_g(1 - \frac{1}{r}) \), we get by a change of variables and a trivial estimate that
\[
\log |g(\zeta)| \geq -c \left[ \frac{r+1}{2} A_g \left( \frac{r+1}{2} \right) \right], \quad \zeta \in \Gamma,
\]
with \( c = 4c_0/3 \). Therefore, by the maximum principle
\[
\log M_F(r) \leq \log M_{gF}(r) + c A_g(r) \left( \frac{r+1}{2} \right),
\]
and the proposition is proved. \( \blacksquare \)

An immediate consequence of the proposition is

**Corollary 2.6.** Assume that \( g \neq 0 \) is a complex holomorphic function on \( \mathbb{D} \) of moderate growth, and that \( F \) is a holomorphic function on \( \mathbb{D} \) with
values in a complex Banach space. If the function $gF$ has exponential growth, then the same is true for $F$.

**Remark.** For the proof of Proposition 2.3, we need only the fact that the conclusion of the corollary holds when the function $gF$ has moderate growth. For this particular case, the assertion can be deduced also from a theorem of Matsaev and Mogulskii [33, Theorem 1].

We shall need the following characterization of operators of subcritical class.

**Lemma 2.7.** An operator $L$ on $X$ is of subcritical class, if and only if, $\sigma(L) \subset \mathbb{D}$ and the operator function $G$ on $\mathbb{D}$ defined by

$$G(\lambda) = (I - \lambda L)^{-1}, \quad \lambda \in \mathbb{D},$$

has moderate growth.

**Proof.** The lemma can be proved by the arguments of Beurling in his proof of Lemma 2 in [14]. (See also [22, Lemma 3.5], where a similar result is proved.) We indicate a direct proof.

Assume first that $L$ is of subcritical class. As already mentioned before, the spectral radius formula implies that $\sigma(L) \subset \mathbb{D}$. Consider the function $u$ on the interval $(0, 1)$ defined by

$$u(r) = \sup_{n \in \mathbb{Z}_+} r^n \|L^n\|, \quad 0 < r < 1.$$

It follows from the proof of Theorem 4 in [47, p. 359], that the assumption on $L$ implies that

$$\int_{1/2}^{1} \left( \frac{\log u(t)}{1-t} \right)^{1/2} dt < \infty.$$ 

This shows that $G$ is of moderate growth, since from its Taylor expansion and the Schwartz inequality, it follows that

$$\|G(\lambda)\| \leq (1 - |\lambda|)^{-1} u(|\lambda|^{1/2}), \quad \lambda \in \mathbb{D}.$$ 

Conversely, assume that $\sigma(L) \subset \mathbb{D}$ and that $G$ is of moderate growth. By the standard estimates of Taylor coefficients

$$\|L^n\| \leq \inf_{0 < r < 1} r^{-n} M_d(r), \quad n \in \mathbb{Z}_+,$$

and this implies by [35, Sect. 2.6, Lemma 1], that $L$ is of subcritical class. \[\square\]
Remark. It follows from [47, Lemma 4.3], that if $L$ is of subcritical class, then
\[
\log^+ \|L^n\| = o(n^{1/2}), \quad n \to \infty.
\] (2.5)

We shall also need

**Lemma 2.8.** An operator $L$ on $X$ satisfies the condition
\[
\log^+ \|L^n\| = 0(n^{1/2}), \quad n \to \infty,
\] (2.6)
if and only if, $\sigma(L) \subset \mathbb{D}$ and the operator function $G$ on $\mathbb{D}$ defined by (2.4) has exponential growth.

The proof is identical to the proof of Lemma 2 in [7]. That proof also shows that (2.5) holds, if and only if, $\sigma(L) \subset \mathbb{D}$ and the operator function $G$ has minimal exponential growth.

If $X_1$ and $X_2$ are Banach spaces, we shall denote as usual by $\mathcal{L}(X_1, X_2)$, the Banach space of all bounded linear operators from $X_1$ to $X_2$. The last preliminary result needed for the proof of Proposition 2.3 is

**Lemma 2.9.** Assume that $X_1$ and $X_2$ are nonzero complex Banach spaces such that $X_2$ is finite dimensional and that $\Phi: \mathbb{D} \to \mathcal{L}(X_1, X_2)$ is a holomorphic operator function such that for some $\lambda_0$ in $\mathbb{D}$, the operator $\Phi(\lambda_0)$ is surjective. Then there exists a holomorphic operator function $\Psi: \mathbb{D} \to \mathcal{L}(X_1, X_2)$, and a complex holomorphic function $g$ on $\mathbb{D}$, such that $g(\lambda_0) = 1$ and
\[
\Phi(\lambda) \Psi(\lambda) = g(\lambda) I_2, \quad \forall \lambda \in \mathbb{D},
\] (2.7)
where $I_2$ is the identity operator on $X_2$. If in addition, the function $\Phi$ has moderate growth, the same holds for the functions $\Psi$ and $g$.

**Proof.** Assume that the dimension of $X_2$ is $d$, and let $U = \{u_1, u_2, ..., u_d\}$ be a basis for that space. Since $\Phi(\lambda_0)$ is surjective, there exist vectors $v_1, v_2, ..., v_d$ in $X_1$, such that $\Phi(\lambda_0) v_i = u_i$, $i = 1, 2, ..., d$. Let $Y$ be the linear span of these vectors in $X_1$. Then $V = \{v_1, v_2, ..., v_d\}$ is a basis for $Y$. For every $\lambda$ in $\mathbb{D}$, let $A(\lambda) = (a_{ij}(\lambda))_{i,j=1}^{d}$ denote the matrix of the transformation $\Phi(\lambda)|_Y: Y \to X_2$ with respect to the pair of bases $V$ and $U$, and denote by $B(\lambda) = (b_{ij}(\lambda))_{i,j=1}^{d}$ the algebraic adjoint of $A(\lambda)$. That is, for every $i, j = 1, 2, ..., d$, $b_{ij}(\lambda)$ is the cofactor of the entry $a_{ij}(\lambda)$ in the matrix $A(\lambda)$.

Let $\Psi: \mathbb{D} \to \mathcal{L}(X_1, X_1)$ be the operator function defined by
\[
\Psi(\lambda) u_j = \sum_{i=1}^{d} b_{ij}(\lambda) v_i, \quad \lambda \in \mathbb{D}, \quad j = 1, 2, ..., d.
\]
That is, for every $\lambda$ in $\mathbb{D}$, $\Psi(\lambda)$ is the linear transformation from $X_2$ to $Y$, defined by the matrix $B(\lambda)$ and the pair of bases $U$ and $V$. Consider the function $g$ on $\mathbb{D}$ defined by

$$g(\lambda) = \det A(\lambda), \quad \lambda \in \mathbb{D}.$$ 

It follows from these definitions that the functions $\Psi$ and $g$ satisfy (2.7) and $g(\lambda_0) = 1$. If $U^* = \{u_1^*, u_2^*, \ldots, u_d^*\}$ is the dual basis of $U$ in $X_2^*$ (the dual space of $X_2$), then

$$a_{i,j}(\lambda) = \langle \Phi(\lambda) v_j, u_i^* \rangle, \quad \lambda \in \mathbb{D}, \quad i, j = 1, 2, \ldots, d.$$ 

This shows that the functions $\Psi$ and $g$ are holomorphic, and that there exists a positive constant $c$ such that

$$|g(\lambda)| + \|\Psi(\lambda)\| \leq c \|\Phi(\lambda)\|^d, \quad \lambda \in \mathbb{D},$$ 

and therefore, if $\Phi$ has moderate growth, the same is true for $\Psi$ and $g$. □

In the sequel, we shall denote for a subspace $Y$ of $X$, by $i_Y$ the injection map of $Y$ into $X$.

Proof of Proposition 2.3(a). Since the iterates of $T$ have norms of polynomial growth, the same holds for the iterates of $T^M$, and therefore in view of Theorem 2.4, it suffices to show that the resolvent of $T^M$ satisfies the growth condition (2.3) in $\mathbb{D}$. We keep all the notations in the proof of Proposition 2.1. Thus $K$ is the kernel of $L$, and $L$ is the operator function defined in (2.1). The assumption that $TX$ has finite codimension, implies by (2.2) that $K$ has finite dimension. Let $Q$ denote the operator function on $\mathbb{D}$ defined by

$$Q(\lambda) = I - (T - \lambda) \tilde{L}(\lambda), \quad \lambda \in \mathbb{D}. \quad (2.8)$$

As observed in the proof of Proposition 2.1, for every $\lambda$ in $\mathbb{D}$, the operator $(T - \lambda) \tilde{L}(\lambda)$ is a projection on $(T - \lambda) X$ along $K$, and therefore, $Q(\lambda)$ is a projection on $K$ along $(T - \lambda) X$. It follows from (2.8) that

$$\pi_M = \pi_M Q(\lambda) - (\lambda - T^M) \pi_M \tilde{L}(\lambda), \quad \lambda \in \mathbb{D},$$

and therefore by the assumption that $\sigma(T^M) \subset \partial \mathbb{D}$,

$$(\lambda - T^M)^{-1} \pi_M = (\lambda - T^M)^{-1} \pi_M Q(\lambda) - \pi_M \tilde{L}(\lambda), \quad \lambda \in \mathbb{D}. \quad (2.9)$$

Assume first that $K$ is the zero space. This means that $T$ is invertible and $L = T^{-1}$. In this case the conclusion of the proposition (hence of
Theorem 1.1(a)) holds under the weaker assumption that $L$ satisfies condition (2.6). In fact, the assumption on $K$ implies that $Q = 0$, and therefore by (2.9)

$$(\lambda - T^M)^{-1} \pi_M = -\pi_M \tilde{L}(\lambda), \quad \lambda \in \mathbb{D}.$$  

This shows by Lemma 2.8, that if $L$ satisfies (2.6), then the operator function $\lambda \mapsto (\lambda - T^M)^{-1} \pi_M$, $\lambda \in \mathbb{D}$, has exponential growth, hence noting that 

$$\| (\lambda - T^M)^{-1} \pi_M \| = \| (\lambda - T^M)^{-1} \|$$  

(since $\pi_M$ maps the unit ball of $X$ onto the unit ball of $X/M$), we conclude that the resolvent of $T^M$ satisfies condition (2.3) in $\mathbb{D}$.

Assume now that $K$ is not the zero space, and consider the holomorphic operator function $F: \mathbb{D} \to \mathcal{L}(K, X/M)$ defined by

$$F(\lambda) = (\lambda - T^M)^{-1} \pi_M i_K, \quad \lambda \in \mathbb{D}.$$  

Since $Q(\lambda) X = K$, $\forall \lambda \in \mathbb{D}$, it follows from (2.9) that

$$(\lambda - T^M)^{-1} \pi_M = F(\lambda) Q(\lambda) - \pi_M \tilde{L}(\lambda), \quad \lambda \in \mathbb{D}. \quad (2.10)$$  

The assumption that $L$ is of subcritical class implies by Lemma 2.7 that the operator functions $\tilde{L}$ and $Q$ have moderate growth, hence (minimal) exponential growth. Therefore, it follows from (2.10), that in order to prove that the resolvent of $T^M$ satisfies condition (2.3) in $\mathbb{D}$, it suffices to show that the operator function $F$ has exponential growth. To this end, multiply both sides of (2.10) from the right by $i_M$. Using the fact that $\pi_M i_M = 0$, we get that

$$F(\lambda) Q(\lambda) i_M = \pi_M \tilde{L}(\lambda) i_M, \quad \lambda \in \mathbb{D}. \quad (2.11)$$  

Since for every $\lambda$ in $\mathbb{D}$ the operator $T^M - \lambda$ is surjective,

$$M + (T - \lambda) X = X, \quad \forall \lambda \in \mathbb{D},$$  

and therefore, remembering that $Q(\lambda)$ is a projection on $K$ along $(T - \lambda) X$, we get that

$$Q(\lambda) M = K, \quad \forall \lambda \in \mathbb{D}.$$  

That is, the operator $Q(\lambda) i_M; M \to K$, is surjective for all $\lambda$ in $\mathbb{D}$. Therefore, since $K$ is finite dimensional and is not the zero space, it follows from Lemma 2.9, that there there exists a holomorphic operator function $\Psi: \mathbb{D} \to \mathcal{L}(K, M)$, and a holomorphic complex function $g$ on $\mathbb{D}$, which is not identically zero, both of moderate growth, such that

$$Q(\lambda) i_M \Psi(\lambda) = g(\lambda) i_K, \quad \lambda \in \mathbb{D}.$$
Therefore by (2.11),
\[ g(\lambda) F(\lambda) = H(\lambda), \quad \lambda \in \mathbb{D}, \]  
(2.12)
where \( H \) denotes the operator function on \( \mathbb{D} \) defined by
\[ H(\lambda) = \pi_M \bar{L}(\lambda) i_M \Psi(\lambda), \quad X \in \mathbb{D}. \]

Since \( H \) is a product of operator functions of moderate growth, it has also moderate growth, and therefore by (2.12) and Corollary 2.6, \( F \) has exponential growth. This concludes the proof.

Proof of Proposition 2.3(b). Again, it suffices to show that \( T|_M \) satisfies the conditions of Theorem 2.4. For this, consider the operator \( T^* \). It is easy to see that the assumptions on \( T \) imply that \( T^* \) satisfies the conditions of part (a). Let \( N \) denote the annihilator of \( M \) in \( X^* \). The adjoint of \( T|_M \) can be identified with the operator \( T^{*N} \). Since \( \sigma(T|_M) \subset \partial \mathbb{D} \), also \( \sigma(T^{*N}) \subset \partial \mathbb{D} \), and therefore by the proof of part (a), the operator \( T^{*N} \) satisfies the conditions of Theorem 2.4. Since an operator satisfies these conditions if and only if its adjoint does, the assertion is proved.

We show next that if \( T \) is an operator on \( X \) which has a left inverse of subcritical class, and \( M \) is a nonzero invariant subspace of \( T \), then under some conditions on \( M \), the operator \( T|_M \) also has a left inverse of subcritical class. When \( T \) is the Bergman shift, this will imply by a result in [5], that the restriction of \( T \) to any nonzero invariant subspace, has a left inverse of subcritical class.

Note that if \( T \) is an operator on \( X \) which is bounded below (equivalently, is injective and has closed range) and \( TX \) is complemented by a closed subspace \( K \), then the linear operator \( L \) on \( X \) defined by \( Ly = 0 \) for \( y \) in \( K \) and \( LTx = x \) for \( x \) in \( X \), is a (bounded) left inverse of \( T \) with kernel \( K \).

We shall need the following simple fact.

**Lemma 2.10.** Assume that \( T \) is an operator on \( X \) such that the operator \( T - \lambda \) is bounded below for every \( \lambda \) in \( \mathbb{D} \), and that there exists a closed subspace \( K \) of \( X \) which is complemented to all the spaces \( (T - \lambda) X, \lambda \in \mathbb{D} \). If \( L \) is the left inverse of \( T \) with kernel \( K \), then \( \sigma(L) \subset \bar{\mathbb{D}} \).

**Proof.** For every \( \lambda \) in \( \mathbb{D} \), let \( J(\lambda) \) denote the left inverse of \( T - \lambda \) with kernel \( K \). Using the assumption that \( K + (T - \lambda) X = X \) for every \( \lambda \) in \( \mathbb{D} \), we get by a simple computation that
\[ (I - \lambda L)(I + \lambda J(\lambda)) = (I + \lambda J(\lambda))(I - \lambda L) = I, \quad \forall \lambda \in \mathbb{D}, \]
and this shows that \( \sigma(L) \subset \bar{\mathbb{D}} \).
Remark. An alternative proof of the lemma appears (implicitly) in [5, p. 289].

Assume now that $T$ is an operator on $X$ which has a left inverse $L$ such that $\sigma(L) \subset \mathbb{D}$, and that $M$ is a nonzero invariant subspace of $T$. Set $T_0 = T|_M$. The proof of Proposition 2.1 shows that the assumption on $L$ implies, that for every $\lambda$ in $\mathbb{D}$, the operator $T - \lambda$ has a left inverse; therefore it is bounded below, hence the same is true for the operator $T_0 - \lambda$. Assume further, that there exists a closed subspace $K$ of $M$, that is complemented in $M$ to all the spaces $(T_0 - \lambda)M$, $\lambda \in \mathbb{D}$, and denote by $L_0$ the left inverse of $T_0$ with kernel $K$. By Lemma 2.10, $\sigma(L_0) \subset \mathbb{D}$. Denote by $\tilde{L}_0$ the operator function on $\mathbb{D}$ defined by $\lambda \to (I_0 - \lambda L_0)^{-1} L_0$ (where $I_0$ is the identity operator on $M$) and by $Q_0$ the operator function from $\mathbb{D}$ to $\mathcal{L}(M)$ whose value at a point $\lambda$ in $\mathbb{D}$ is the projection on $K$ along $(T_0 - \lambda)M$.

Finally, let $\tilde{L}$ denote the operator function on $\mathbb{D}$ defined by (2.1).

**Lemma 2.11.** With the notations above, we have that

$$
|\tilde{L}_0(\lambda)| \leq 2 \|\tilde{L}(\lambda)\| \|Q_0(\lambda)\|, \quad \forall \lambda \in \mathbb{D}. \quad (2.13)
$$

**Proof.** Let $\lambda \in \mathbb{D}$. Since the operators $\tilde{L}(\lambda)$ and $\tilde{L}_0(\lambda)$ are left inverses of the operators $T - \lambda$ and $T_0 - \lambda$ respectively, their restrictions to the subspace $(T_0 - \lambda)M$ coincide. Thus, remembering that the range of $Q_0(\lambda)$ is $K$ (which is the kernel of $\tilde{L}_0(\lambda)$) and that the range of $I_0 - Q_0(\lambda)$ is $(T_0 - \lambda)M$, we get that

$$
\tilde{L}_0(\lambda) = \tilde{L}_0(\lambda)(I_0 - Q_0(\lambda)) = \tilde{L}(\lambda)(I_0 - Q_0(\lambda)),
$$

and this implies (2.13). $\Box$

From Lemma 2.7 and Lemma 2.11 we obtain

**Corollary 2.12.** Assume that $L$, $L_0$ and $Q_0$ are as above. If $L$ is of subcritical class and $Q_0$ has moderate growth, then $L_0$ is of subcritical class.

Combining this with Theorem 1.1(a) we get

**Proposition 2.13.** Let $T$ be an operator on $X$ whose iterates have norms of polynomial growth, which has a left inverse of subcritical class. Assume that $M$ is a nonzero invariant subspace of $T$, which has a finite dimensional subspace $K$ that is complemented in $M$ to all the spaces $(T - \lambda)M$, $\lambda \in \mathbb{D}$, and that the operator function from $\mathbb{D}$ to $\mathcal{L}(M)$ whose value at a point $\lambda$ in $\mathbb{D}$ is the projection on $K$ along $(T - \lambda)M$ has moderate growth. Then every maximal invariant subspace of $T|_M$ has codimension one in $M$.  

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We now apply this result to the Bergman shift. Recall that the Bergman space $A^2$ consists of all holomorphic functions $f$ on $D$ for which the norm \( \left( \frac{1}{\pi} \iint_D |f|^2 \, dx \, dy \right)^{1/2} \) is finite. It is well known that with respect to this norm, $A^2$ is a Hilbert space, which is admissible (in the sense of the definition in Section 1). Let $S$ and $B$ denote the shift and backward shift on $A^2$ respectively. It is clear that $S$ is a contraction and that $B$ is a left inverse of $S$. It is easy to verify (see Section 4) that 
\[
\| B^n \| = (n+1)^{1/2}, \quad n \in \mathbb{Z}_+.
\]

Let $M$ be a nonzero invariant subspace of $S$, and set $K = M \ominus SM$. It is shown in [5, Lemma 3.1] that for every $\lambda$ in $D$, $K$ is complemented in $M$ to $(S-\lambda)M$, and that the norm of the corresponding projection from $M$ onto $K$ is at most $2(1-|\lambda|^2)^{-1}$. Thus from Proposition 2.13 we obtain

**Proposition 2.14.** If $M$ is a nonzero invariant subspace of $S$ such that $M \ominus SM$ has finite dimension, then every maximal invariant subspace of $S|_M$ has codimension one in $M$.

**Remarks.** (1) As already mentioned in Section 1, for $M = A^2$ this result was proved by Hedenmalm [23].

(2) As shown in [6, 11], if $n$ is a positive integer or $n = \infty$, then there exists an invariant subspace $M$ of $S$ such that $\dim(M \ominus SM) = n$. If $M$ is an invariant subspace such that $\dim(M \ominus SM) = \infty$, then by the observations in Section 1, the question whether every maximal invariant subspace of $S|_M$ has codimension one in $M$, is equivalent to the general invariant subspace problem on Hilbert space.

(3) In Section 5 we shall show that if $T$ is an operator that satisfies the conditions of Proposition 2.13 and \( \bigcap_{n=0}^\infty T^n X = \{0\} \), and $M$ is an invariant subspace of $T$ which contains a one-dimensional subspace that is complemented in $M$ to all the spaces $(T-\lambda)M$, $\lambda \in \mathbb{D}$, then every maximal invariant subspace of $T|_M$ has codimension one in $M$. This implies in particular that if $E$ is an admissible Banach space which satisfies the conditions of Theorem 1.2(a), and every invariant subspace of $S_E$ is singly generated, then $\text{Lat} S_E$ has no proper gaps. (See Corollary 5.4.)

We prove next a general result about representations of Banach algebras of holomorphic functions on $D$, that has several applications to invariant subspaces of shifts on admissible Banach spaces, and also implies Theorem 1.5. Before stating it, we introduce some definitions and make a few observations.

If $E$ is an admissible Banach space of complex holomorphic functions on $D$, which is also a Banach algebra with respect to pointwise multiplication, we shall say that it is an admissible Banach algebra.
If $E$ is an admissible Banach algebra, then $E \subset H^\infty$ and the imbedding is continuous. This follows from the fact that the evaluations at the points of $\mathbb{D}$ are multiplicative linear functionals on $E$, and therefore have norm at most 1. Every admissible Banach algebra has a unit, since as is easily verified, an admissible Banach space of complex holomorphic functions contains the constant functions (hence also the polynomials).

The term **bounded unital representation** of an admissible Banach algebra $E$ on the Banach space $X$, will mean as usual, a bounded homomorphism of $E$ into the Banach algebra $\mathcal{L}(X)$, that takes the unit of $E$ into the identity operator on $X$.

If $T$ is an operator on $X$, we shall denote by $T'$ its commutant (that is, the algebra of all operators on $X$ that commute with $T$). Recall that a **hyperinvariant subspace** of $T$, is a closed subspace of $X$ which is invariant for all the operators in $T'$. If this subspace is different from the zero space and from $X$, it is called a **nontrivial hyperinvariant subspace** of $T$. The collection of all hyperinvariant subspaces of $T$ will be denoted by $H\text{Lat} T$. The identity function on $\mathbb{D}$ will be denoted as before by $z$.

**Theorem 2.15.** Let $E$ be an admissible Banach algebra such that

$$\log^+ \|B_n\| = O(n^{1/2}), \quad n \to \infty. \quad (2.14)$$

Assume that $U$ is a bounded unital representation of $E$ on $X$ with nontrivial kernel, and that the iterates of the operator $T = U(z)$ have norms of polynomial growth. If $T$ is not a scalar multiple of the identity operator, then it has a nontrivial hyperinvariant subspace.

For the proof of the theorem and its applications we need a lemma. In the sequel we shall denote for every holomorphic function $f$ on $\mathbb{D}$ (with complex or vector values) and every $* \in \mathbb{D}$, by $f*$ the holomorphic function on $\mathbb{D}$ defined by the equation

$$f*f(z) = (z - *)f. \quad (2.15)$$

**Lemma 2.16.** Assume that $E$ is an admissible Banach space of holomorphic functions on $\mathbb{D}$, with complex values or values in a complex Banach space. Let $r$ denote the spectral radius of $B_E$, and $D_0$ the open disc in $\mathbb{C}$ with center at the origin and radius $\min\{r^{-1}, 1\}$, and denote by $\tilde{B}_E$ the operator function $\lambda \mapsto (I - \lambda B_E)^{-1}B_E$ on $D_0$. Then:

(a) For every $f \in E$ and $\lambda \in D_0$,

$$\tilde{B}_E(\lambda)f = f_\lambda \quad (2.16)$$

hence in particular, $f_\lambda$ is in $E$. 

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If for every \( f \) in \( E \) and \( \lambda \) in \( \mathbb{D} \) the function \( f_\lambda \) is in \( E \), then \( \sigma(B_E) \subset \mathbb{D} \).

**Proof.** (a) A simple computation shows that for every \( f \) in \( E \) and \( \lambda \) in \( \mathbb{D}_0 \), the unique solution in \( E \) of the equation \((I - \lambda B_E)g = B_E f\), is \( g = f_\lambda \).

(b) Let \( f \) be in \( E \) and \( \lambda \) in \( \mathbb{D} \). By the hypotheses the function \((S_\lambda f)_\lambda\) is in \( E \) and a computation gives

\[
(I - \lambda B_E)(S_\lambda f)_\lambda = f.
\]

This shows that the operator \( I - \lambda B_E \) is surjective; it is easily verified that it is also injective, and therefore \( \sigma(B_E) \subset \mathbb{D} \).

**Proof of Theorem 2.15.** Condition (2.14) implies by the spectral radius formula that \( \sigma(B_E) \subset \mathbb{D} \), and therefore by Lemma 2.16, for every \( f \) in \( E \) and \( \lambda \) in \( \mathbb{D} \), the function \( f_\lambda \) is in \( E \) and (2.16) holds. Hence, if \( f \) is in \( E \) we obtain from (2.15) that

\[
U(f) - f(\lambda) I = U(f_\lambda)(T - \lambda) = (T - \lambda) U(f_\lambda), \quad \lambda \in \mathbb{D}, \quad (2.17)
\]

and this implies that

\[
f(\sigma(T) \cap \mathbb{D}) \subset \sigma(U(f)).
\]

Thus in particular, if \( f \in \ker U \), then

\[
\sigma(T) \cap \mathbb{D} \subset f^{-1}(0). \quad (2.18)
\]

We show first that the assertion of the theorem holds when the set \( \sigma(T) \cap \mathbb{D} \) is not empty. Assume that \( \lambda \) is in this set, and let \( f \) be a nonzero function in the kernel of \( U \). Denote the order of the zero of \( f \) at \( \lambda \) by \( j \). It follows from (2.18) that \( j \geq 1 \). Let \( g \) by the holomorphic function on \( \mathbb{D} \) such that

\[
f = (z - \lambda)^j g. \quad (2.19)
\]

Then \( g(\lambda) \neq 0 \), and from (2.16) we obtain that \( B_E f(\lambda) = g \), and therefore \( g \) is in \( E \). Since \( f \) is in the kernel of \( U \), we get from (2.19) that

\[
(T - \lambda)^j U(g) = 0. \quad (2.20)
\]

The fact that \( g(\lambda) \neq 0 \) implies by (2.18) that \( U(g) \neq 0 \), and therefore it follows from (2.20) that \( \lambda \) is an eigenvalue of \( T \). Hence if \( T \neq \lambda I \), then the kernel of \( T - \lambda \) is a nontrivial hyperinvariant subspace of \( T \). Assume now
that the set $\sigma(T) \cap D$ is empty, and consider again a nonzero function $f$ in the kernel of $U$. It follows from (2.17) that
\[ f(\lambda)(\lambda - T)^{-1} = U(f_{\lambda}), \quad \forall \lambda \in D. \tag{2.21} \]

By Lemma 2.8, condition (2.14) implies that the operator function $\tilde{B}_E$ defined in Lemma 2.16 has exponential growth, and therefore since the representation $U$ is bounded, we obtain from (2.16) that the operator function $\lambda \mapsto U(f_{\lambda})$ on $D$ also has exponential growth. Thus, remembering that $E \subset H^\infty$, we obtain from (2.21) and [7, Lemma 5(a)] that the operator function $\lambda \mapsto (\lambda - T)^{-1}$ on $D$, has exponential growth, and therefore by [7, Lemma 2(b)],
\[ \log^+ \|T^{-n}\| = O(n^{1/2}), \quad n \to \infty. \]

Combining this with the assumption that the iterates of $T$ have norms of polynomial growth, we obtain the desired conclusion from [7, Theorem 1].

In the rest of this section, $E$ will denote an admissible Banach space of complex holomorphic functions on $D$. In order to apply Theorem 2.15 to invariant and hyperinvariant subspaces of $S_E$, we have to consider the multiplier algebra of $E$.

**Definition.** A complex function $\varphi$ on $D$ is called a multiplier of $E$, if $\varphi E \subset E$.

The set of multipliers on $E$ will be denoted by $M(E)$. It is clear that $E$ contains the polynomials, and since $E$ contains the constants, $M(E)$ is included in $E$. It follows from the closed graph theorem, that the multiplication operator defined by a function $\varphi$ in $M(E)$ is continuous; we shall denote this operator also by $\varphi$, and its norm in $L(E)$ by $\|\varphi\|_E$. It is known and easy to show (see [17]) that for every $\varphi$ in $M(E)$,
\[ |\varphi(\lambda)| \leq \|\varphi\|_E, \quad \forall \lambda \in D. \tag{2.22} \]

This implies that with respect to the operator norm and pointwise addition and multiplication, $M(E)$ is a Banach algebra that is continuously embedded in $H^\infty$, and is isometrically isomorphic to a subalgebra of $S_E$. We claim that this subalgebra coincides with $S_E$. To see this, assume that $T$ is in $S_E$, and consider the function $\varphi = T1$. Let $D_0$ be the disc defined in Lemma 2.16, and let $f$ be in $E$. By Lemma 2.16, $f_{\lambda}$ is in $E$ for every $\lambda$ in $D_0$, and therefore by (2.15),
\[ Tf = f(\lambda) \varphi + (S_E - \lambda) T(f_{\lambda}), \quad \forall \lambda \in D_0. \]
This shows that the functions \( T f \) and \( \varphi f \) that are holomorphic on \( \mathbb{D} \), coincide on \( \mathbb{D}_0 \), so they coincide also on \( \mathbb{D} \). Therefore \( T \) is the multiplication operator defined by \( \varphi \), and the claim is proved.

It follows from these observations that a closed subspace \( V \) of \( E \) is a hyperinvariant subspace of \( S_E \), if and only if, \( \varphi V \subseteq V \) for every \( \varphi \) in \( M(E) \). Thus in particular, if \( E \) is an admissible Banach algebra, then \( \text{HLat}_S E \) consists of all closed ideals in \( E \) (since in this case \( M(E) = E \)). It is also clear that if the polynomials are dense in the Banach algebra \( E \), then \( \text{Lat}_S E = \text{HLat}_S E \).

The Banach algebra \( M(E) \) is admissible. This follows from the fact that for any two holomorphic functions \( \varphi \) and \( f \) on \( \mathbb{D} \)

\[
\varphi, f = (\varphi f) - \varphi(\lambda) f, \quad \forall \lambda \in \mathbb{D},
\]

and therefore if \( \varphi \) is in \( M(E) \), then \( \varphi_0 \) is also in \( E \).

Let \( B \) denote the backward shift on \( M(E) \), and assume that \( \sigma(B_E) \subseteq \mathbb{D} \). This implies by Lemma 2.16 and (2.23) that \( \sigma(B) \subseteq \mathbb{D} \). Denote by \( \hat{B} \) the operator function \( \lambda \rightarrow (I - \lambda B)^{-1} B \) on \( \mathbb{D} \). It follows from (2.16), (2.22), and (2.23) that

\[
\|\hat{B}(\lambda)\| \leq 2 \|\hat{B}_0(\lambda)\|, \quad \forall \lambda \in \mathbb{D},
\]

and therefore if \( \hat{B}_E \) has exponential growth, the same is true for \( \hat{B} \). Thus from Lemma 2.8 we obtain that if \( B_E \) satisfies (2.14), then \( B \) also satisfies this condition.

Using these observations and Theorem 2.15 we can now prove

**Theorem 2.17.** Assume that the iterates of \( S_E \) have norms of polynomial growth, that \( B_E \) satisfies (2.14), that \( M(E) \) is dense in \( E \), and that every non-zero hyperinvariant subspace of \( S_E \) contains a nonzero element of \( M(E) \). Then for any two elements \( V \) and \( W \) of \( \text{HLat}_S E \) such that \( V \subseteq W \) and \( \dim W/V > 1 \), either there exists an element \( K \) of \( \text{HLat}_S E \) such that \( V \backsimeq K \backsimeq W \), or \( (z - \lambda) W \subseteq V \) for some \( \lambda \) in \( \partial \mathbb{D} \). In the second case, there exists an element \( K \) of \( \text{Lat}_S E \) such that \( V \backsimeq K \backsimeq W \). Thus if \( \text{Lat}_S E = \text{HLat}_S E \), then \( \text{Lat}_S E \) has no proper gaps.

**Proof.** By the preceding observations, the hypothesis that \( B_E \) satisfies (2.14) implies that \( B_{M(E)} \) also satisfies this condition, and therefore \( M(E) \) is an admissible Banach algebra that satisfies the conditions of Theorem 2.15. Assume now that \( V \) and \( W \) are hyperinvariant subspaces of \( S_E \) such that \( \{0\} \neq V \subseteq W \) and \( \dim W/V > 1 \), and set \( Y = W/V \). Let \( U \) be the mapping from \( M(E) \) to \( \mathcal{L}(Y) \) defined by

\[
U(\varphi) \pi_Y f = \pi_Y (\varphi f), \quad \varphi \in M(E), \quad f \in W.
\]
It is easily verified that \( U \) is a bounded unital representation of \( M(E) \) on \( Y \), and the assumption that the iterates of \( S_E \) have norms of polynomial growth, implies that the same is true for the operator \( U(z) = (S|_W)^V \). We claim that

\[
V \cap M(E) \subset \ker U.
\]

To show this assume that \( \psi \) is in \( V \cap M(E) \). Since \( V \) is a hyperinvariant subspace of \( S_E \), \( M(E) \psi \) is included in \( V \), and this implies by the assumption that \( M(E) \) is dense in \( E \), that \( \psi E \) is also included in \( V \). Therefore, \( \psi \in \ker U \), and the claim is proved. Hence by the assumption on \( V \), we get that \( \ker U \neq \{0\} \). Thus, by Theorem 2.15, if \( (S|_W)^V \) is not a scalar multiple of the identity operator, it has a nontrivial hyperinvariant subspace, say \( G \).

It is readily verified that the subspace \( K = \pi_V^{-1}(G) \) is in \( \text{Lat} S_E \), and \( V \varsubsetneq K \varsubsetneq W \).

Assume now that \( (S|_W)^V \) is a multiple of the identity operator by the complex number \( \lambda \). This means that

\[
(z - \lambda) W \subset V. \tag{2.24}
\]

Consequently, every closed subspace that includes \( V \) and is included in \( W \), is an invariant subspace of \( S_E \). Therefore, since \( \dim W/V > 1 \), there exists an invariant subspace \( K \) of \( S_E \) such that \( V \varsubsetneq K \varsubsetneq W \).

We claim that \( \lambda \) is in \( \partial D \). First, since the iterates of \( (S|_W)^V \) have norms of polynomial growth, \( \lambda \) is in \( \overline{D} \). Assume that \( \lambda \) is in \( D \). Since the operator \( B_E \) satisfies condition (2.14), its spectrum is included in \( \overline{D} \), and this implies by Lemma 2.16(a) that \( E \) satisfies conditions (1.1)–(1.4) in [39]. Hence by [39, Proposition 3.6] and the assumption that \( W \cap M(E) \neq \{0\} \), we get that \( \dim W/zW = 1 \). But as shown in [39] the dimension of the space \( W/(z - \zeta) \) \( W \) is the same for all \( \zeta \) in \( D \), so we obtain that \( \dim W/(z - \lambda) W = 1 \). By (2.24), this contradicts the assumptions that \( V \subset W \) and \( \dim W/V > 1 \), and the claim is proved.

This completes the proof of the theorem when \( V \neq \{0\} \). If \( V = \{0\} \), the assertion is trivial, since if \( \zeta \) is a point in \( D \) that is not a common zero of all the functions in \( W \), then the space of all functions in \( W \) that vanish at \( \zeta \), is a nonzero hyperinvariant subspace of \( S_E \) which is strictly included in \( W \).

An immediate consequence of Theorem 2.17 is the following result, whose last part is an equivalent statement of Theorem 1.5.

**Theorem 2.18.** Assume that \( E \) is an admissible Banach algebra such that \( B_E \) satisfies (2.14) and for some \( \alpha \geq 0 \),

\[
\|z^n\|_E = O(n^\alpha), \quad n \to \infty.
\]
Then for any two closed ideals \( J_1 \) and \( J_2 \) in \( E \) such that \( J_1 \subseteq J_2 \) and \( \dim J_2/J_1 > 1 \), either there exists a closed ideal \( J \) in \( E \) such that \( J_1 \subseteq J \subseteq J_2 \), or \((z - \lambda)J_2 \subset J_1 \) for some \( \lambda \) in \( \mathbb{C} \). If the polynomials are dense in \( E \), then the first alternative always holds.

The theorem applies in particular to \( H^\infty \) and \( W^+ \) (see Section 4) and provides new information on the collection of closed ideals in these algebras. For the algebra \( W^+ \) the first alternative always holds, since the polynomials are dense in it. It would be interesting to know whether the first alternative always holds also for the algebra \( H^\infty \).

We close this section with some comments on maximal invariant subspaces of codimension one. If \( T \) is an operator on \( X \), then it is clear that the maximal invariant subspaces of \( T \) which have codimension one, are the kernels of the eigenvectors of \( T^* \). Thus, if \( \sigma_p(T^*) \) (the point spectrum of \( T^* \)) is empty, and \( T \) satisfies the conditions of Theorem 1.1(a), then the conclusion of the theorem is that \( T \) has no maximal invariant subspaces. This is possible only when \( T \) is invertible. In fact, if \( T \) is not invertible and has a left inverse \( L \), then the open set

\[
G_L = \{ \lambda \in \mathbb{C} : I - \lambda L \text{ is invertible} \}
\]

is included in \( \sigma_p(T^*) \), as follows from the identity

\[
(T^* - \lambda)(I - \lambda L)^{-1}(I - L^*T^*) = 0, \quad \forall \lambda \in G_L.
\]

that can be verified by a simple calculation.

Assume now that \( E \) is an admissible Banach space of holomorphic functions on \( \mathbb{D} \) with values in a complex Banach space \( K \), and that \( E \) contains the vector space of all constant \( K \)-valued functions on \( \mathbb{D} \) (that we identify with \( K \)).

For every \( \lambda \) in \( \mathbb{D} \) and \( \varphi \neq 0 \) in \( K^* \), let \( \delta_{\lambda, \varphi} \) denote the linear functional on \( E \) defined by

\[
\langle f, \delta_{\lambda, \varphi} \rangle = \langle f(\lambda), \varphi \rangle, \quad f \in E.
\]

It follows from the properties of \( E \), that \( \delta_{\lambda, \varphi} \) is in \( E^* \) and is an eigenvector of \( S_K^* \) with eigenvalue \( \lambda \). If \( \sigma(B_K) \subset \mathbb{D} \), then all the eigenvectors of \( S_K^* \) with eigenvalue in \( \mathbb{D} \) are of this form. To see this, assume that \( v \) is an eigenvalue of \( S_K^* \) with eigenvalue \( \lambda \) in \( \mathbb{D} \). Then \( v \) annihilates the range of \( S_K - \lambda \). Let \( \varphi \) denote the restriction of \( v \) to \( K \). The assumption that \( \sigma(B_K) \subset \mathbb{D} \) implies by Lemma 2.16(a), that for every \( f \) in \( E \) the function \( f - f(\lambda) \) is in the range of \( S_K - \lambda \), and therefore

\[
\langle f, v \rangle = \langle f(\lambda), v \rangle = \langle f(\lambda), \varphi \rangle,
\]
and this shows that $v = \delta_{\lambda, \psi}$. From these observations we get in particular

**Proposition 2.19.** Assume that $E$ satisfies the conditions above, that $\sigma(B_2) \subseteq \overline{D}$ and $\sigma(S_0^E) \subseteq \overline{D}$. Then the invariant subspaces of $S_E$ of codimension one, are the spaces

$$M_{\lambda, \psi} = \{ f \in E : \langle f(\lambda), \psi \rangle = 0 \},$$

where $0 \neq \psi \in K^*$ and $\lambda \in D$.

3. PRIMARY INVARIANT SUBSPACES AND GENERALIZED ROOT SPACES

The main part of this section is devoted to the proofs of Theorems 1.6 and 1.7, but first we prove some simple facts on primary invariant subspaces and generalized root spaces of general operators.

It is well known that if $T$ is an operator on $X$ and $V$ is an invariant subspace of $T$, then the boundary of $\sigma(T|_V)$ is included in $\sigma(T)$ [38, Theorem 0.8]. This implies that if $V$ is a generalized root space of $T$ at $\lambda$, then $\lambda$ is in $\sigma(T)$. It also follows from this, that if $M$ is a primary invariant subspace of $T$ at $\lambda$, then $\lambda$ is in $\sigma(T)$, since the adjoint of $T^M$ is the operator $T^*|_{M^*}$.

The following two propositions will enable us to determine the primary invariant subspaces of $S_E$ and the generalized root spaces of $S_0^E$ at the points in $D$, when $E$ is an admissible Banach space of complex holomorphic functions on $D$, such that $\sigma(B_2) \subseteq \overline{D}$.

**Proposition 3.1.** Assume that $T$ is an operator on $X$ and that $\lambda$ is in $\sigma(T)$. If the range of $T - \lambda$ has codimension one, then the primary invariant subspaces of $T$ at $\lambda$ are the spaces $(T - \lambda)^n X$, $n = 1, 2, \ldots$.

**Proof.** The assumption implies that each of these spaces is different from $X$ and has codimension at most $n$. Therefore if $M = (T - \lambda)^n X$ for some $n$, then $M$ is an invariant subspace of $T$ and $(T^M - \lambda)^n = 0$. Thus $\sigma(T^M) = \{ \lambda \}$, so $M$ is a primary invariant subspace of $T$ at $\lambda$.

Conversely, assume that $M$ is a primary invariant subspace of $T$ at $\lambda$. Since $(T - \lambda) X$ has codimension one, the range of the operator $T^M - \lambda$ has codimension at most one, and therefore is closed. Hence the assumption that $\sigma(T^M) = \{ \lambda \}$, implies by [28, Chap. IV, Theorem 5.30], that the space $X/M$ has finite dimension, say $n$. Therefore $(T^M - \lambda)^n = 0$ (by the Caley–Hamilton theorem) and this means that $(T - \lambda)^n X \subseteq M$. Since $(T - \lambda)^n X$ has codimension at most $n$, we conclude that $M = (T - \lambda)^n X$, and the proposition is proved.
Proposition 3.2. Assume that $T$ is an operator on $X$ and that $\lambda$ is in $\sigma(T)$. If the operator $T - \lambda$ has closed range and one-dimensional kernel, then the generalized root spaces of $T$ at $\lambda$, are the spaces $\ker(T - \lambda)^n$, $n = 1, 2, \ldots$.

Proof. It is clear that each of these spaces is a generalized (actually an ordinary) root space of $T$ at $\lambda$. Conversely, assume that $V$ is a generalized root space of $T$ at $\lambda$. The assumptions imply by [28, Chap. IV, Theorem 5.29] that the operator $T|_V - \lambda|_V$ has closed range, and therefore, since $\sigma(T|_V) = \{\lambda\}$, we get from [28, Chap. IV, Theorem 5.30], that $V$ has finite dimension, say $n$. This implies as before that $(T|_V - \lambda)^n = 0$, hence $V \subset \ker(T - \lambda)^n$, and therefore by a comparison of dimensions we get that $V = \ker(T - \lambda)^n$, which completes the proof.

In what follows $E$ denotes an admissible Banach space of complex holomorphic functions on $D$. To simplify notations, we shall denote the operators $S_E$ and $B_E$ by $S$ and $B$, respectively.

For every $j$ in $\mathbb{Z}_+$ and $\lambda \in D$, we denote by $\delta^{(j)}(\lambda)$ the linear functional on $E$ defined by

$$\langle f, \delta^{(j)}(\lambda) \rangle = f^{(j)}(\lambda), \quad f \in E.$$ 

One can show by standard arguments that these functionals are in $E^*$.

Let $n$ be a positive integer and $\lambda \in D$. We shall denote by $V^{(n)}(\lambda)$ the linear span in $E^*$ of the functionals $\delta^{(j)}(\lambda)$, $j = 0, 1, \ldots, n - 1$, and by $M^{(n)}(\lambda)$ its pre-annihilator in $E$, that is,

$$M^{(n)}(\lambda) = \{ f \in E : f^{(j)}(\lambda) = 0, j = 0, 1, \ldots, n - 1 \}.$$ 

Since $V^{(n)}(\lambda)$ is finite dimensional

$$V^{(n)}(\lambda) = M^{(n)}(\lambda)^\perp. \quad (3.1)$$

As observed in Section 1, $M^{(n)}(\lambda)$ is a primary invariant of $S$ at $\lambda$, and therefore by (3.1), $V^{(n)}(\lambda)$ is a generalized root space of $S^*$ at $\lambda$. This can also be seen directly by observing that this space is annihilated by the operator $(S^* - \lambda)^n$.

Theorem 3.3. Assume that $\sigma(B) \subset \overline{D}$ and that $\lambda$ is in $D$. Then:

(a) The primary invariant subspaces of $S$ at $\lambda$, are the spaces $M^{(n)}(\lambda)$, $n = 1, 2, \ldots$.

(b) The generalized root spaces of $S^*$ at $\lambda$, are the spaces $V^{(n)}(\lambda)$, $n = 1, 2, \ldots$. 


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Proof of (a). The assumption on \( B \) implies by Lemma 2.16(a) that
\[
M^{(n)}(\lambda) = (S - \lambda)^n E, \quad n = 1, 2, \ldots
\] (3.2)
In particular, this implies that \((S - \lambda)E\) has codimension one, hence the assertion follows from Proposition 3.1.

Proof of (b). Since the operator \( S - \lambda \) is injective and by (3.2) its range has codimension one, the operator \( S^* - \lambda \) is surjective and has one-dimensional kernel. Thus, since for every positive integer \( n \), the kernel of \((S^* - \lambda)^n\) is the annihilator of \((S - \hat{\lambda})^n E\), the desired conclusion follows from (3.1), (3.2), and Proposition 3.2.

We assume next that the polynomials are dense in \( E \) and that \( S \) is a contraction. The following result gives a sufficient condition on an element \( M \) of \( \text{Lat} S \{ E \} \) to be a primary invariant subspace of \( S \) at a point \( \hat{\lambda} \) in \( \hat{D} \).

To state it, we need two notations.

If \( f \) is a function in \( W^+ \), then it admits a continuous extension to \( \hat{D} \), that we also denote by \( f \), and by \( f^{-1}(0) \), we mean the set \( \{ \hat{\lambda} \in \hat{D} : f(\hat{\lambda}) = 0 \} \).

If \( T \) is a contraction in \( \mathcal{L}(X) \) and \( f \) is a function in \( W^+ \), then the series \( \sum_{n=0}^\infty (f^{(n)}(0)/n!) T^n \) converges in \( \mathcal{L}(X) \) to an operator on \( X \), that we denote by \( f(T) \).

**Proposition 3.4.** If \( M \) is an element in \( \text{Lat} S \{ E \} \) that contains a function \( f \neq 0 \) in \( W^+ \), then \( \sigma(SM) \subset f^{-1}(0) \). Thus in particular, if \( f^{-1}(0) = \{ \hat{\lambda} \} \), then \( M \) is a primary invariant subspace of \( S \) at \( \hat{\lambda} \).

**Proof.** Since \( f(S) 1 = f \) and \( M \) is in \( \text{Lat} S \), the assumption that \( f \) is in \( M \) implies that \( f(S) S^n \) is in \( M \) for every integer \( n \geq 0 \), and since the polynomials are dense in \( E \), it follows that \( f(S) E \subset M \). Thus noting that
\[
\pi_M f(S) = f(SM) \pi_M
\]
we get that \( f(SM) = 0 \), and as shown in the proof of [7, Theorem 4], this implies that \( \sigma(SM) \subset f^{-1}(0) \).

Applying the proposition to the functions introduced in Section 1 we get

**Corollary 3.5.** If \( a > 0 \) and the subspace \( [f_a] \) is different from \( E \), then it is a primary invariant subspace of \( S \) at 1.

We turn now to the proofs of Theorems 1.6 and 1.7. In the sequel, we assume that \( E \) satisfies the conditions of Theorem 1.6, that is, the polynomials are dense in \( E \), \( S \) is a contraction, and \( B \) is of subcritical class. (It is easy to see that the last two assumptions imply that \( \sigma(S) = \hat{D} \).)
One can deduce from the arguments of Beurling in his proof of Theorem 1 in [13], that the functions \( f_a, a > 0 \), satisfy the assumption of Corollary 3.5. However, for the proof of Theorem 1.6, we need the stronger assertion that \( [f_a] \subseteq [f_b] \) for \( 0 < a < b \). This will be established by a refinement of Beurling’s proof, that leads to the following results. (We use the notation introduced in Section 1.)

**Theorem 3.6.** If \( 0 < a < b \), then:

1. \( V_a \leq V_b \).
2. \( [f_a] = V_{a; b} \). 

**Proof of (1).** Let \( v \) denote the function on \([0, \infty)\) that is linear on the intervals \([n, n+1], n \in \mathbb{Z}_+\), and

\[
v(n) = (n^2 + 1) \|B^n\|, \quad n \in \mathbb{Z}_+.
\]

Since \( B^n = B^{n+1} S, \forall n \in \mathbb{Z}_+ \), and \( S \) is a contraction, the sequence \( \|B^n\|, n = 0, 1, \ldots \), is nondecreasing, and therefore \( v \) is also nondecreasing. By the assumption that \( B \) is of subcritical class, this implies that

\[
\int_0^\infty (\log v(t)/t^{3/2}) \, dt < \infty,
\]

and by a change of variables we get that

\[
\int_0^\infty (\log v(x^2)/x^3) \, dx < \infty.
\]

Thus by a classical result of Paley-Wiener [36, Chap. 1] and Levinson [30, Theorem 26], there exists an even function \( F \neq 0 \) in \( \mathcal{E}_{1,a} \) such that

\[
\sup_{-\infty < x < \infty} |v(x^2) F(x)| < \infty.
\]

Set \( c = \lim \sup_{y \to \infty} y^{-1} \log |F(iy)|. \) Since \( F \) is in \( \mathcal{E}_{1,a} \) and is not identically zero it follows [15, Theorem 5.4.2], that \( -\infty < c \leq a \). Consider the even entire function

\[
F_1(z) = F(z) \sin \frac{b-c}{2} z, \quad z \in \mathbb{C}.
\]

It is clear that this function is in \( \mathcal{E}_{1,a+b-c} \), and since \( v(x) \geq v(0) \geq 1 \) for every \( x \geq 0 \), it is bounded on the real line. Since \( F_1 \) is even and

\[
\lim_{y \to \infty} y^{-1} \log |F_1(iy)| = b
\]


it follows from \[15, \text{Theorem } 6.2.4\] that \( F_1 \) is in \( E \). Therefore, the entire function \( g \) defined by

\[ g(z^2) = F_1(z), \quad z \in \mathbb{C}, \]

is in \( E \), and by (3.5) it is not in \( \mathfrak{E} \).

Let \( \delta(0) \) denote the functional on \( E \) of evaluation at 0. It follows from (3.3) and (3.4) that \( \sum_{n=0}^{\infty} |g(n)| \|B^n\| < \infty \), and therefore, the linear functional \( \varphi \) defined on \( E \) by

\[ \langle f, \varphi \rangle = \sum_{n=0}^{\infty} g(n)\langle B^nf, \delta(0) \rangle, \quad f \in E, \]

is in \( E^* \). It is easy to verify that

\[ \varphi(n) = g(n), \quad \forall n \in \mathbb{Z}_+, \]

(recall that \( \varphi(n) = \langle z^n, \varphi \rangle \), \( n \in \mathbb{Z}_+ \)), and we conclude that \( \varphi \) is in \( V \) and not in \( V_a \). Since the inclusion \( V_a \subset V_b \) is clear, the assertion is proved.

**Proof of (2).** For the proof we need an equivalent definition of the spaces \( V_a, d > 0. \) It follows from \[32\] that a sequence of complex numbers \( \{a_n\}_{n \in \mathbb{Z}_+} \) is in \( E \), if and only if there exists an entire function \( G \) in \( E \) such that

\[ G \left( \frac{1}{1-z} \right) = \sum_{n=0}^{\infty} \frac{a_n}{z^{n+1}}, \quad z \in \mathbb{C} \setminus \mathbb{D}. \]

In that reference, the result is stated in terms of the Taylor coefficients of the function \( G \left( \frac{1}{1-z} \right) \) in \( \mathbb{D} \), but an obvious change of variables shows that both formulations are equivalent.

Assume now that \( \varphi \) is in \( E^* \). Since \( S \) is a contraction, the sequences \( \varphi \) is bounded, and therefore the series \( \sum_{n=0}^{\infty} (\varphi(n)/z^{n+1}) \) converges in \( \mathbb{C} \setminus \mathbb{D} \), and defines a holomorphic function on this domain, that we denote in the sequel by \( \varphi \). From the preceding observation, it follows that \( \varphi \) is in \( V_a \), if and only if, \( \varphi \) has a holomorphic extension \( \varphi_a \) to \( \mathbb{C} \setminus \{1\} \) such that

\[ \sup_{z \neq 1} |\varphi_a(z)| \exp \left[ -\frac{a + \varepsilon}{|z-1|} \right] < \infty, \quad \forall \varepsilon > 0. \quad (3.6) \]

We now prove that \( V_a \subset [f_a]^\perp \). Assume that \( \varphi \) is in \( V_a \). We have to show that

\[ \langle S^nf, \varphi \rangle = 0, \quad \forall n \in \mathbb{Z}_+. \]
It is easy to see that $\lim_{d \to a} f_d = f_a$ in $H^p$, and therefore, since this space is continuously imbedded in $W^+$, which in turn is continuously imbedded in $E$, it suffices to show that

$$\langle S^n f_a, \varphi \rangle = 0, \quad \forall n \in \mathbb{Z}_+,$$

(3.7)

for every $d > a$. To prove this, fix $d > a$, and denote the sequence of Taylor coefficients of $f_d$ by $\{b_j\}_{j \in \mathbb{Z}_+}$, and the sequence $\hat{\varphi}$ by $\{e_j\}_{j \in \mathbb{Z}_+}$. Since $f_d$ is in $W^+$ and the sequence $\{||z||^k\}_{j \in \mathbb{Z}_+}$ is bounded, the series $\sum_{j=0}^{\infty} b_j z^j$ converges to $f_d$ in $E$, and therefore (3.7) is equivalent to the claim that

$$\sum_{j=0}^{\infty} b_j e_{n+j} = 0, \quad \forall n \in \mathbb{Z}_+,$$

which by the Parseval formula, is equivalent to the assertion that

$$\lim_{r \to 1^+} \int_0^{2\pi} f_d(e^{i\theta}) \hat{\varphi}(re^{i\theta}) e^{i(n+1)\theta} d\theta = 0, \quad \forall n \in \mathbb{Z}_+.$$ 

(3.8)

To prove the assertion, consider the holomorphic extension $\hat{\varphi}_*$ of $\hat{\varphi}$ to $C \setminus \{1\}$. Since the sequence $\hat{\varphi}$ is bounded,

$$\sup_{z \in C \setminus \{1\}} |z - 1| |\hat{\varphi}_* (z)| < \infty,$$

and this together with (3.6) implies by [49, Lemma 5.8 and Lemma 5.9], that

$$\sup_{1 < |z| < 2} |1 - z|^2 |\hat{\varphi}_* (z)| < \infty.$$ 

(3.9)

This implies that there exists a constant $C > 0$, such that

$$|f_d(e^{i\theta}) \hat{\varphi}_* (re^{i\theta})| \leq C, \quad 0 \leq \theta < 2\pi, \quad 1 < r < 2,$$

and therefore by the dominated convergence theorem, (3.8) is equivalent to the assertion that

$$\int_0^{2\pi} f_d(e^{i\theta}) \hat{\varphi}_*(e^{i\theta}) e^{i(n+1)\theta} d\theta = 0, \quad \forall n \in \mathbb{Z}_+.$$ 

(3.10)

To prove it, consider the function $h$ on $\mathbb{D} \setminus \{1\}$ defined by

$$h(z) = f_d(z) \hat{\varphi}_*(z), \quad z \in \mathbb{D} \setminus \{1\}.$$
By (3.9), \( h \) is bounded on \( \partial \mathbb{D} \backslash \{1\} \), and since \( d > a \), it follows from (3.6) that it is also bounded on \([0, 1)\). Thus, an application of the Phragmén–Lindelöf principle (as in the proof of Lemma 4 in [7]) shows that \( h \) is bounded on \( \mathbb{D} \). Hence the Fourier coefficients with negative indices, of the restriction of \( h \) to \( \partial \mathbb{D} \backslash \{1\} \) are zero, and (3.10) is proved.

We prove next that \( [f_a]^{-1} \subset V_{\alpha \beta} \). Assume that \( \varphi \) is in \([f_a]^{-1}\). We have to show that \( \hat{\varphi} \) has a holomorphic extension \( \hat{\varphi}_* \) to \( \mathbb{C} \backslash \{1\} \) that satisfies (3.6). Set \( [f_a] = M \). Since \( V_{\alpha \beta} \neq \{0\} \) by part (1), and \( V_{\alpha \beta} \subset M^+ \) by the preceding proof, it follows that \( M \neq E \), and therefore by Corollary 3.5, \( M \) is a primary invariant subspace of \( S \) at 1.

We claim that

\[
\sup_{z \neq 1} \| (z - S^M)^{-1} \| \exp \left[ -\frac{a + \beta}{|1 - z|} \right] < \infty, \quad \forall \epsilon > 0. \tag{3.11}
\]

To see this, consider the function \( g \) on \( \mathbb{D} \) defined by

\[ g(z) = (z - 1)^2 f_a(z), \quad z \in \mathbb{D}, \]

and let \( \{d_n\}_{n \in \mathbb{Z}^+} \) be the sequence of its Taylor coefficients. Since the functions \( g, g' \) and \( g'' \) are in \( H^\omega \), it follows that \( \sum_{n=0}^{\infty} |d(n) (n+1) < \infty \), and this implies that for every \( \lambda \) in \( \mathbb{D} \), the function \( g_\lambda \) (defined by Eq. (2.15)) is in \( W^\omega \) and that \( \sup_{z \in \mathbb{D}} \| g_\lambda \|_{W^\omega} < \infty \) (see [7, p. 36]). Therefore, since \( S^M \) is a contraction, we get that

\[
\sup_{\lambda \in \mathbb{D}} \| g_\lambda(S^M) \| < \infty. \tag{3.12}
\]

Since \( g \) is in \( M \), we obtain from the proof of Proposition 3.4, that \( g(S^M) = 0 \). This implies by (3.12) and [7, Lemma 1] that

\[
\sup_{z \in \mathbb{D}} \| g(z)(z - S^M)^{-1} \| < \infty
\]

and consequently

\[
\sup_{z \in \mathbb{D}} |(z - 1)^2 \exp \left[ -\frac{a}{|z - 1|} \right] \| (z - S^M)^{-1} \| < \infty. \tag{3.13}
\]

Combining this with the trivial estimate

\[
\sup_{|z| > 1} \| (z - 1) \| (z - S^M)^{-1} \| < \infty
\]
(which follows from the fact that $S^M$ is a contraction) we obtain from [7, Lemma 3] that

$$\sup_{1<|z|<2} |1-z|^2 \|(z-S^M)^{-1}\| < \infty.$$  

This implies (3.11), by (3.13) and the fact that $\sigma(S^M) = \{1\}$.

Now consider the functional $\psi$ on $X/M$ defined by

$$\langle \pi_M f, \psi \rangle = \langle f, \varphi \rangle, \quad f \in E.$$  

Since $\varphi$ is in $M^\perp$, $\psi$ is properly defined and bounded. Observing that

$$\hat{\psi}(z) = \langle (z-S)^{-1} 1, \varphi \rangle, \quad |z| > 1,$$

we see that the function

$$\hat{\psi}_a(z) = \langle (z-S^M)^{-1} \pi_M 1, \psi \rangle, \quad z \neq 1$$

is a holomorphic extension of $\hat{\psi}$ to $C \setminus \{1\}$, and by (3.11) it satisfies (3.6). This completes the proof of the theorem.  

**Remarks.** (1) The device employed in the last step of the proof is an adaptation of the so-called Carleman transform method that is usually used in the study of closed ideals in a Banach algebra, and goes back to Carleman and Beurling. A systematic development of the method in the Banach algebra setting was given by Domar [19].

(2) We use this occasion to indicate a minor correction needed in the statement of Lemma 3 in [7]. The domain $|z| > 1$ in inequality (11) there, has to be replaced by the domain $1 < |z| < 2$. This has no effect on the results of the paper, since the inequality is used only in this smaller domain.

From Corollary 3.5, Theorem 3.6, and the Hahn–Banach theorem we obtain

**Corollary 3.7.** The spaces $[f_c]$, $0 < c < \infty$, are primary invariant subspaces of $S$ at 1, and if $0 < a < b$, then $[f_a] \subset [f_b]$.  

The proof of Theorem 1.6 requires several additional preliminary results.

**Lemma 3.8.** If $f$ is in $M_\psi$, then $f$ is in $[(z-1)f]$.  

**Proof.** For every positive integer $n$, let $v_n$ denote the polynomial $(n+1)^n$. Elementary estimates show that

$$\|(z-1)v_n\|_{W^*} = O(n^{-1/2}), \quad n \to \infty,$$  

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$$\|(z-1)v_n\|_{W^*} = O(n^{-1/2}), \quad n \to \infty,$$
and therefore, since $S$ is a contraction, we get that
\[
\lim_{n \to \infty} \|v_n(S)(S - 1)\| = 0.
\]
Thus, noting that \(\|v_n(S)\| \leq 1\), \(n = 1, 2, \ldots\), we obtain from the assumption on \(f\), that \(\lim_{n \to \infty} v_n(S)f = 0\). This implies the assertion, since for every integer \(n \geq 1\), \(v_n(S)f\) is in \([(z - 1)f]\).

**Lemma 3.9.** If \(f\) is in \(E\) and \(\lim_{x \to 1^-} f(x) = 0\), then \(f\) is in \(M_o\).

**Proof.** Since the polynomials are dense in \(E\), \(M_o\) is the closure of the range of \(S - I\), so if condition (C.1) holds then \(M_o = E\) and the assertion is clear.

Assume now that condition (C.2) holds. Then \(M_o\) is the kernel of an element \(\varphi\) in \(E^*\) such that \(\langle 1, \varphi \rangle = 1\). We have to show that \(\langle f, \varphi \rangle = 0\).

The assumption on \(\varphi\) implies that for every polynomial \(p\)
\[
\langle p, \varphi \rangle = p(1)
\]
and the second part of (C.2) implies by the Banach–Steinhaus theorem, that there exists a constant \(c > 0\), such that
\[
\sup_{0 < x < 1} |g(x)| \leq c \| g \|, \quad \forall g \in E. \quad (3.15)
\]
Let \(p_n, n = 1, 2, \ldots\), be a sequence of polynomials that converges to \(f\) in \(E\). It follows from (3.15) that \(p_n(x) \to f(x)\), uniformly on \((0, 1)\), and this implies by (3.14) and the assumption on \(f\), that \(\langle f, \varphi \rangle = 0\).

**Lemma 3.10.** If \(M\) is a primary invariant subspace of \(S\) at 1, then
\[
\sup_{\lambda \in \mathcal{D}} |1 - \lambda| \log \| (\lambda - S^M)^{-1} \| < \infty, \quad (3.16)
\]
and
\[
\sup_{1 < |\lambda| < 2} |1 - \lambda|^2 \| (\lambda - S^M)^{-1} \| < \infty. \quad (3.17)
\]

**Proof.** Since \(B\) is of subcritical class and is a left inverse of \(S\), it follows from the proof of Proposition 2.3(a), that
\[
\sup_{\lambda \in \mathcal{D}} (1 - |\lambda|) \log \| (\lambda - S^M)^{-1} \| < \infty, \quad (3.18)
\]
and since $S$ is a contraction we have the trivial estimate

$$\sup_{|\lambda| > 1} (|\lambda| - 1) \| (\lambda - S^M)^{-1} \| < \infty. \quad (3.19)$$

Since $\sigma(S^M) = \{1\}$, the estimates (3.18) and (3.19) imply (3.16) and (3.17) by [7, Lemma 3].

Before stating the last preliminary result, we introduce some notations. We denote by $\tilde{B}$ and $Q$ the operator functions on $D$ defined by

$$\tilde{B}(\lambda) = (I - \lambda B)^{-1} B, \quad \lambda \in \mathbb{D},$$

and

$$Q(\lambda) = I - (S - \lambda) \tilde{B}(\lambda), \quad \lambda \in \mathbb{D}. \quad (3.20)$$

It follows from Lemma 2.16(a), that $Q$ can also be represented in the form

$$Q(\lambda) = 1 \otimes \delta(\lambda), \quad \lambda \in \mathbb{D}, \quad (3.21)$$

where $\delta(\lambda)$ is the functional on $E$ of evaluation at $\lambda$.

If $M$ is a primary invariant subspace of $S$ at 1, we denote

$$\alpha(M) = \limsup_{x \to 1^{-}} (1 - x) \log \|(x - S^M)^{-1}\|,$$

and

$$\gamma(M) = \limsup_{x \to 1^{-}} (1 - x) \log \|(x - S^M)^{-1} \pi_M 1\|.$$ 

It follows from (3.16) that $\alpha(M) < \infty$.

**Lemma 3.11.** If $M$ is a primary invariant subspace of $S$ at 1 and $\alpha(M) > 0$, then

$$\alpha(M) = \gamma(M). \quad (3.22)$$

**Proof.** Using (3.20) and (3.21), and the identity (2.9) with $T$ and $L$ replaced by $S$ and $B$, respectively, we get that

$$(\lambda - S^M)^{-1} \pi_M = (\lambda - S^M)^{-1} \pi_M 1 \otimes \delta(\lambda) - \pi_M \tilde{B}(\lambda), \quad \lambda \in \mathbb{D}. \quad (3.23)$$

Since $B$ is of subcritical class, it follows from Lemma 2.7, that the operator function $\tilde{B}$ has moderate growth, and therefore has minimal exponential
growth. Hence by (3.20) and (3.21), the vector function \( \lambda \to \delta(\lambda) \), from \( \mathbb{D} \) to \( E^* \), also has minimal exponential growth. Therefore, since by (3.23)

\[
\| (\lambda - S_M)^{-1} \| \leq \| \delta(\lambda) \| \| (\lambda - S_M)^{-1} \pi_M 1 \| + \| \delta(\lambda) \|, \quad \lambda \in \mathbb{D},
\]

we obtain that, if \( \gamma(M) \leq 0 \) then \( \alpha(M) \leq 0 \), and if \( \alpha(M) > 0 \) then \( \alpha(M) \leq \gamma(M) \). Since the opposite inequality is clear, the assertion is proved.

**Remark.** Since the vector function \( \lambda \to \delta(\lambda) \) has minimal exponential growth, the same is true for the functions in \( E \).

**Proof of Theorem 1.6.** Let \( M \) be a primary invariant subspace of \( S \) at 1, and set \( \alpha(M) = a \). We assume first that \( a > 0 \), and show that \( M = M_a \).

We begin with the inclusion \( M_a \subset M \). Assume that \( f \) is in \( M_a \), and consider the holomorphic vector function \( g \) on \( \mathbb{C} \setminus \{1\} \) defined by

\[
g(\lambda) = (\lambda - 1)^2 (\lambda - S_M)^{-1} \pi_M f, \quad \lambda \in \mathbb{C} \setminus \{1\}.
\]

It follows from (3.17) that \( g \) is bounded on \( \partial \mathbb{D} \setminus \{1\} \), and from (3.16), that

\[
\sup_{\lambda \in \mathbb{D}} |1 - \lambda| \log \| g(\lambda) \| < \infty. \quad (3.24)
\]

Using the fact that \( \bar{B} \) has minimal exponential growth (see the proof of Lemma 3.11) and the assumption that \( f \) is in \( M_a \), we obtain from (3.23) that

\[
\limsup_{x \to -1} |x - 1| \log \| g(x) \| \leq 0. \quad (3.25)
\]

Let \( U \) be the upper half plane \( \text{Im} \, z > 0 \), and denote by \( G \) the vector function in \( U \) defined by

\[
G(z) = g \left( \frac{z-i}{z+i} \right), \quad z \in \bar{U}.
\]

It is continuous on \( \bar{U} \), holomorphic on \( U \), and bounded on the real line. From (3.24) and (3.25) we get that

\[
\sup_{z \in U} (|z| + 1)^{-1} \log \| G(z) \| < \infty,
\]

and

\[
\limsup_{y \to \infty} y^{-1} \log \| G(iy) \| \leq 0.
\]
Hence by the version of [15, Theorem 6.24] for vector functions (that follows from the scalar case by a standard application of the Hahn–Banach theorem) we obtain that $G$ is bounded, and therefore $g$ is bounded on $\mathbb{D}\setminus\{1\}$. Since by (3.17), $g$ is also bounded on the annulus $1 < |\lambda| < 2$, we infer that it has a removable singularity at $\lambda = 1$. Therefore the vector function

$$\lambda \mapsto (\lambda - S^M)^{-1} \pi_M f, \quad \lambda \in \mathbb{C}\setminus\{1\},$$

has a pole at $\lambda = 1$, which has order at most one by (3.19). Thus, from the expansion

$$(\lambda - S^M)^{-1} = - \sum_{n=0}^{\infty} (1 - S^M)^n (1 - \lambda)^{-n-1}, \quad \lambda \in \mathbb{C}\setminus\{1\},$$

we obtain that $(1 - S^M) \pi_M f = 0$. This means that $(z - 1) f$ is in $M$, and therefore $[(z - 1) f] \in M$. On the other hand, since $f$ is in $M_a$ and $a > 0$, we have that $\lim_{x \to -1} f(x) = 0$, so the inclusion above implies by Lemma 3.8 and Lemma 3.9 that $f$ is in $M$, and we conclude that $M_a \subset M$.

We next show that $M \subset M_a$. The proof is based on the Ahlfors–Heins theorem [2] and requires some preparations. Let $\Omega$ denote the half plane $\text{Re } z > -1/2$, and consider the vector functions $q$ on $\mathbb{D}\setminus\{1\}$ and $H$ on $\bar{\Omega}$ defined by

$$q(\lambda) = (\lambda - 1)^2 (\lambda - S^M)^{-1} \pi_M 1, \quad \lambda \in \mathbb{D}\setminus\{1\},$$

and

$$H(z) = q \left( \frac{z}{z+1} \right), \quad z \in \bar{\Omega}.$$ 

The function $H$ is continuous on $\bar{\Omega}$, holomorphic on $\Omega$, and by (3.16) and (3.17) it is bounded on the line $\text{Re } z = -1/2$, and satisfies the condition

$$\sup_{z \in \mathbb{D}} (|z| + 1)^{-1} \log \|H(z)\| < \infty.$$ 

It follows from (3.22) that

$$\limsup_{x \to -\infty} x^{-1} \log \|H(x)\| = a.$$  

(3.26)
Thus, again using the vector version [15, Theorem 6.24], we obtain that
\[ |H(z)| \leq m \exp(a \Re z), \quad z \in \Omega, \quad (3.27) \]
where \( m = e^{a^2} \sup_{-\infty < y < \infty} |H(-1/2 + iy)|. \)

Let \( P \) denote the half plane \( \Re z > 0 \), and consider the function \( u \) on \( P \) defined by
\[ u(z) = \log |H(z)| - \log m, \quad z \in P. \]

Since \( H \) is holomorphic on \( P \), the function \( u \) is subharmonic (cf. [45, p. 74, Theorem A]) and by (3.26) and (3.27),
\[ \sup_{z \in P} \frac{u(z)}{\Re z} = a, \quad \text{and} \quad \limsup_{z \to \zeta} u(z) \leq 0, \quad \forall \zeta \in \partial P. \]

Therefore, by the Ahlfors–Heins theorem [2],
\[ \lim_{r \to \infty} \frac{u(re^{i\theta})}{r} = a \cos \theta, \quad \text{a.e. on} \quad (-\pi/2, \pi/2). \quad (3.28) \]

Now assume that \( f \) is in \( M \), and denote by \( F \) the function on \( \bar{P} \) defined by
\[ F(z) = f\left(\frac{z}{z+1}\right), \quad z \in \bar{P}. \]

This function is continuous on \( P \) and holomorphic on \( P \). Since the functions in \( E \) have exponential growth (see the remark following the proof of Lemma 3.11), we have that
\[ \sup_{|z| \leq \pi/4} (|z|+1)^{-1} \log |F(z)| < \infty. \quad (3.29) \]

Consider the function \( h \) on \( [-\pi/4, \pi/4] \) defined by
\[ h(\theta) = \limsup_{r \to \infty} r^{-1} \log |F(re^{i\theta})|, \quad -\pi/4 \leq \theta \leq \pi/4. \]

In order to prove that \( f \) is in \( M_a \), we have to show that \( h(0) \leq -a \).

If \( h(0) = -\infty \), the inequality is clear, and if \( h(0) > -\infty \), it follows from (3.28) by [15, Theorem 5.14], that \( h \) is continuous. Since \( f \) is in \( M \), we get from (3.23) that
\[ f(\lambda) q(\lambda) = (\lambda - 1)^2 \pi_M \tilde{B}(\lambda) f, \quad \lambda \in \mathbb{D}. \]
Therefore, since $B$ has minimal exponential growth, the same is true for the vector function $f \cdot q$ on $D$, and this implies that

$$
\limsup_{r \to \infty} r^{-1} \left[ \log |F(re^{i\theta}) + o(re^{i\theta})| \right] \leq 0, \quad -\pi/2 < \theta < \pi/2,
$$
hence by (3.28),

$$
h(\theta) + a \cos \theta \leq 0, \quad \text{a.e. on } [-\pi/4, \pi/4].
$$

Since $h$ is continuous the inequality holds everywhere on $[-\pi/4, \pi/4]$, in particular for $\theta = 0$. Thus (3.30) holds, and $f$ is in $M_a$. This concludes the proof of the equality $M = M_a$ when $a > 0$.

Assume now that $\sigma(M) \leq 0$. We claim that in this case $M = M_0$. This can be seen as follows. Replacing in the proof of the first part, the vector function $g$ by the operator function $\lambda \to (\lambda - 1)^2 (\lambda - S^M)^{-1}$, $\lambda \in \mathbb{C} \setminus \{1\}$, we obtain by the same arguments, that the assumption that $\sigma(M) \leq 0$, implies that $S^M$ is the identity operator, which means that $(S - I)E \subset M$, and therefore, $M_0 \subset M$. The opposite inclusion follows from the fact that $M_0$ has codimension at most 1, since the polynomials are dense in $E$.

The proof of the claim also shows that if condition (C.1) holds, and $M$ is a primary invariant subspace of $S$ at 1, then $\sigma(M) > 0$ (since if 1 is not an eigenvalue of $S^M$, then $M_0 = E$). Combining all the facts proved so far, we obtain the first two assertions of the theorem.

We show next that $M_a = [f_a]$ for every $a > 0$. To see this, fix $a > 0$, and set $\sigma([f_a]) = c$. Let $0 < d < a$. By Lemma 3.9 the function $f_d$ is in $M_0$, and by Corollary 3.7 it is not in $[f_a]$. Therefore by the preceding assertions, $c > 0$ and $[f_a] = M_c$. The fact that $M_c$ contains $f_d$ and does not contain $f_a$, implies that $d < c < a$. Since this holds for every $d$ in $(0, a)$, we conclude that $a = c$, and the desired equality is proved.

To prove the last claim of the theorem, assume that $0 < a < b$, and let $a < d < b$. Then by Corollary 3.7, Lemma 3.9, and the previous assertion, $M_b \supseteq M_a \subset M_a$. This completes the proof of the theorem.

A simple consequence of Theorem 1.6 is

**Corollary 3.12.** If for some $a > 0$, the function

$$
u_a(z) = \exp \frac{a}{z - 1} \quad \text{is in } E, \text{ then } M_a = [\nu_a].
$$

**Proof.** The assertion follows from Theorem 1.6 and the obvious inclusions $[f_a] \subset [\nu_a] \subset M_a$. 

In a similar way we obtain that if for some \( a > 0 \), the function \( v_a = (z - 1) u_a \) is in \( E \), then \( M_a = [v_a] \).

**Proof of Theorem 1.7.** In view of Theorems 1.6 and 3.6, it suffices to show that if condition (C.2) holds then \( M^*_0 = V_0 \). (In fact the equality holds also if (C.1) holds, and in this case both spaces coincide with the zero space.) The proof is simple. It is well known that a function in \( E^* \) that is bounded on \( \mathbb{Z}_+ \) is constant (this can be deduced from [15, Theorem 10.2.11] by an obvious change of variables). Thus, remembering that if \( \varphi \) is in \( E^* \) then the sequence \( \varphi \) is bounded, we see that \( V_0 \) is the set of all elements \( \varphi \) in \( E^* \) such that the sequence \( \varphi \) is constant. Since the polynomials are dense in \( E \), \( M^*_0 \) also coincides with this set, and the proof is complete.

4. APPLICATIONS TO SHIFTS AND BACKWARD SHIFTS

In this section we apply the preceding results to shifts and backward shifts on concrete Banach spaces of holomorphic functions on \( D \). In what follows, \( K \) will denote a nonzero complex Hilbert space.

We consider first spaces determined by growth conditions on the Taylor coefficients. For every \( 1 \leq p \leq \infty \), we denote by \( \ell^p_K \) the \( K \)-valued \( \ell^p \) space on \( \mathbb{Z}_+ \), that is, the Banach space of all sequences \( a: \mathbb{Z}_+ \to K \) such that the sequence \( \{ |a(n)| \}_{n \in \mathbb{Z}_+} \) is in \( \ell^p \), with the norm of \( a \) defined by \( (\sum_{n=0}^{\infty} |a(n)|^p)^{1/p} \), if \( p < \infty \), and \( \sup_{n \in \mathbb{Z}_+} |a(n)| \), if \( p = \infty \).

In the sequel \( \gamma \) will denote a sequence of positive numbers on \( \mathbb{Z}_+ \) that satisfies the following conditions:

\[
\gamma(0) = 1; \quad 0 < \inf_{n \in \mathbb{Z}_+} \frac{\gamma(n+1)}{\gamma(n)} \leq \sup_{n \in \mathbb{Z}_+} \frac{\gamma(n+1)}{\gamma(n)} < \infty,
\]

and \( \lim_{n \to \infty} \gamma(n)^{1/n} = 1 \).

If \( f \) is a \( K \)-valued holomorphic function on \( D \) we shall denote by \( \hat{f} \) the sequence of its Taylor coefficients.

For every \( 1 \leq p \leq \infty \), we denote by \( W^p_K(\gamma) \) the vector space of all holomorphic \( K \)-valued functions \( f \) on \( D \) such that the norm

\[
\|f\|_{W^p_K(\gamma)} = \|\hat{f}\|_{\ell^p_K}
\]

is finite. If \( K = \mathbb{C} \) we denote this space by \( W^p(\gamma) \). It is easily verified that with this norm, \( W^p_K(\gamma) \) is an admissible Banach space, and for \( 1 \leq p < \infty \),
its dual can be identified with the space $W^q_K(\frac{1}{2})$, where $\frac{1}{p} + \frac{1}{q} = 1$, via the pairing

$$\langle f, g \rangle = \sum_{n=0}^{\infty} (\hat{f}(n), \hat{g}(n)), \quad f \in W^p_K(\gamma), \quad g \in W^q_K(\frac{1}{\gamma}).$$

With respect to this representation of the dual space, the adjoint of the shift (resp. backward shift) on $W^p_K(\gamma)$, is the backward shift (resp. shift) on $W^q_K(\frac{1}{2})$, where $\frac{1}{p} + \frac{1}{q} = 1$.

We fix now $1 \leq p \leq \infty$, and set $E = W^p_K(\gamma)$. It is easy to show that

$$\|S^*_E\| = \sup_{j \in \mathbb{Z}_+} \frac{\gamma(n + j)}{\gamma(j)}, \quad n \in \mathbb{Z}_+,$$

and

$$\|B^*_E\| = \sup_{j \in \mathbb{Z}_+} \frac{\gamma(j)}{\gamma(n + j)}, \quad n \in \mathbb{Z}_+.$$  

This implies in particular that if $\gamma$ is submultiplicative (that is, $\gamma(n + j) \leq \gamma(n) \gamma(j), \forall n, j \in \mathbb{Z}_+$) then

$$\|S^*_E\| = \gamma(n), \quad n \in \mathbb{Z}_+,$$

and if $\frac{1}{\gamma}$ is submultiplicative, then

$$\|B^*_E\| = \frac{1}{\gamma(n)}, \quad n \in \mathbb{Z}_+.$$  

It is also clear that if $\gamma$ is nonincreasing then $S_E$ is a contraction, and if $\gamma$ is nondecreasing, then $S_E$ is bounded below by 1 and $B_E$ is a contraction.

From these observations and Theorem 1.2(a) we obtain

**Theorem 4.1.** Assume that $K$ is finite dimensional and that one of the following conditions holds.

(a) $\gamma$ is nondecreasing, submultiplicative, and has polynomial growth.

(b) $\gamma$ is nonincreasing, $\frac{1}{\gamma}$ is submultiplicative, and

$$\sum_{n=1}^{\infty} n^{-3/2} \log \frac{1}{\gamma(n)} < \infty.$$

Then every maximal invariant subspace of $S_E$ has codimension one.
Similarly, from Theorem 1.2(b) we get

**Theorem 4.2.** Assume that $K$ is finite dimensional and that one of the following conditions holds.

(a) $\gamma$ is nonincreasing, $\frac{1}{2}$ is submultiplicative and has polynomial growth.

(b) $\gamma$ is nonincreasing, submultiplicative, and $\sum_{n=1}^{\infty} n^{-3/2} \log \gamma(n) < \infty$.

Then every minimal invariant subspace of $B_E$ is one dimensional.

**Examples.** (1) If $x$ is a real number, then the sequence

$$\gamma(n) = (n+1)^x, \quad n \in \mathbb{Z}_+$$

satisfies condition (a) of Theorem 4.1 when $x \geq 0$, and condition (a) of Theorem 4.2 when $x \leq 0$.

(2) If $0 \leq \beta < 1$, then the sequence

$$\gamma(n) = \exp(-n^\beta), \quad n \in \mathbb{Z}_+,$$

satisfies condition (b) of Theorem 4.1, if and only if $\beta < 1/2$, and in this case the sequence

$$\gamma(n) = \exp(n^\beta), \quad n \in \mathbb{Z}_+,$$

satisfies condition (b) of Theorem 4.2.

From Theorem 4.1 and Theorem 4.2 we obtain

**Corollary 4.3.** If $K$ is finite dimensional and $\gamma$ is as in Example 1, for some real number $x$, then every maximal invariant subspace of $S_K$ has codimension one, and every minimal invariant subspace of $B_K$ is one dimensional.

We examine now more closely the spaces described in the corollary. For $1 \leq p < \infty$ and $-\infty < x < \infty$, we shall denote by $D_{x,K}^p$ the space $W_{x,K}^p(\gamma)$ with $\gamma(n) = (n+1)^x$, $n \in \mathbb{Z}_+$. That is,

$$D_{x,K}^p = \left\{ f \in \text{Hol}(\mathbb{D}, K) : \sum_{n=0}^{\infty} \|f(n)\|^p (n+1)^x < \infty \right\}.$$ 

The Hilbert spaces $D_{x,K}^2$ will be denoted by $D_{x,K}$. If $K = \mathbb{C}$, we denote the spaces $D_{x,K}^p$ and $D_{x,K}$ by $D_k^p$ and $D_k$, respectively.

It is known (cf. [48]) that for $x < 0$,

$$D_{x,K} = \left\{ f \in \text{Hol}(\mathbb{D}, K) : \left( \int_\mathbb{D} |f(z)|^2 (1-|z|^2)^{-x-1} \, dA(z) < \infty \right) \right\}. \quad (4.1)$$
and it is clear that for every real \( \alpha \),
\[
D_{\alpha, K} = \{ f \in \text{Hol}(\mathbb{D}, K) : f' \in D_{\alpha-2, K} \}.
\] (4.2)

If \( K \) is finite dimensional and \( E = D_{\alpha, K}^p \) for some real \( \alpha \) and \( 1 \leq p < \infty \),
then by Corollary 4.3, every maximal invariant subspace of \( S_E \) has codimension one. For \( p = 2 \) and \( \alpha \geq 0 \), we have a considerably stronger result.

**Theorem 4.4.** If \( E = D_{\alpha, K} \) with \( \alpha \geq 0 \) and \( K \) finite dimensional, then \( \text{Lat}S_E \) has no proper gaps.

The theorem is an immediate consequence of Theorem 1.3 and the following

**Proposition 4.5.** If \( E \) is as in Theorem 4.4 and \( M \) is an invariant subspace of \( S_E \), then \( \dim(M \cap S_E M) \leq \dim K \).

If \( \alpha = 0 \), then \( E \) is the Hardy space \( H^2_K \), and in this case the proposition is well known (see [38, Lemma 3.24]). For \( \alpha = 1 \), the proposition is proved in [40, Theorem 2], by using the following properties of the Dirichlet space \( D_1 \).

1. Every function in \( D_1 \) is the quotient of two functions in \( D_1 \cap H^\infty \).
2. If \( f \) is in \( D_1 \) and \( \psi \) is in \( D_1 \cap H^\infty \), then if the function \( \psi f \) is in \( D_1 \),
it is also in the closed linear span in \( D_1 \) of the functions \( z^n \psi f, n \in \mathbb{Z}_+ \).
3. \( D_1 \cap H^\infty \) is an algebra with respect to pointwise multiplication.

The first two properties were proved [42] and the third follows from (4.1) and (4.2). Now all these properties remain true if \( D_1 \) is replaced by \( D_\alpha \) for some \( \alpha > 0 \). For \( \alpha > 1 \) this follows trivially from the fact that in this case \( D_\alpha \) is a Banach algebra with respect to pointwise multiplication, that is included in the disc algebra, and the polynomials are dense in it. (See [45, p. 99].) For \( 0 < \alpha < 1 \) the first two properties are proved in [3], and the third one follows again from (4.1) and (4.2). From this, it can easily be seen that the arguments in the proof of Theorem 2 in [40] apply to the spaces \( D_{\alpha, K}, \alpha > 0 \).

From Theorem 4.4 we obtain by duality

**Theorem 4.6.** If \( E = D_{\alpha, K} \) with \( \alpha \leq 0 \) and \( K \) finite dimensional, then \( \text{Lat}B_E \) has no proper gaps.

**Remark.** If \( E = D_\alpha \) for some \( \alpha < 0 \), then it follows from the results in [11, Sect. 10] and [6, Sect. 3] that there exists an invariant subspace \( M \) of \( S_E \) such that the space \( M \cap S_E M \) is infinite dimensional. Concrete
constructions of such invariant subspaces are given in [24] and in a more
general setting in [1, 16]. Thus by the comments in Section 1 (see also the
discussion [24]) the question whether \( \text{Lat} \mathcal{S}_E \) has no proper gaps is equiv-
alent to the general invariant subspace problem in Hilbert space. By
duality we obtain that if \( E = D_\alpha \) for some \( \alpha > 0 \), then the question whether
\( \text{Lat} \mathcal{B}_E \) has no proper gaps is also equivalent to that problem.

Some of the spaces \( D^p_\alpha \) are admissible Banach algebras. This is clearly the
case for the spaces \( D^1_\alpha, \alpha \geq 0 \) (note that \( D^1_0 = W^+ \)), and as observed before
also for the spaces \( D_\alpha, \alpha > 1 \). The proof of the second fact given in [45]
shows that spaces \( D^p_\alpha, \alpha > p - 1 \) are also admissible Banach algebras. (See
also [1, Sect. 10].) Since the polynomials are dense in all of these spaces,
we obtain from Theorem 1.5

Theorem 4.7. If \( E \) is one of the spaces \( D^1_\alpha, \alpha \geq 0 \), or \( D^p_\alpha, \alpha > p - 1 \), then
\( \text{Lat} \mathcal{S}_E \) has no proper gaps.

By duality we get

Theorem 4.8. If \( E = D^p_\alpha \), with \( 1 < p < \infty \) and \( \alpha > -1 \), then \( \text{Lat} \mathcal{B}_E \) has
no proper gaps.

Remark. If \( E = D^p_\alpha \) for some \( 1 \leq p < \infty \), the question whether \( \text{Lat} \mathcal{S}_E \) has
no proper gaps is of particular interest. By Theorem 4.4, or directly by
Beurling’s theory [12], the answer is positive for \( p = 2 \). By Theorem 4.7,
the answer is also positive for \( p = 1 \). For other values of \( p \), the problem is
open and seems very hard. In this connection it is worth mentioning, that
it was proved in [1], that for \( 2 < p < \infty \), there exist invariant subspaces \( M \)
of \( \mathcal{S}_E \) such that \( \dim M/\mathcal{S}_EM = \infty \).

If \( E = W^p_\alpha(\gamma) \) for some \( 1 \leq p < \infty \), it is easy to describe the invariant
subspaces of \( \mathcal{S}_E \) that have codimension one. Let \( q = \frac{p}{p-1} \). If \( \frac{1}{q} \) is not in \( \ell^q \), then
\( \mathcal{S}^*_E \) has no eigenvalues in \( \partial \mathbb{D} \), and the invariant subspaces of codimension one are given by (2.25). If \( \frac{1}{q} \) is in \( \ell^q \), then every function \( f \) in \( E \) has a con-
tinuous extension \( f^* \) to \( \mathbb{D} \), and in this case we have to add also the spaces

\[ \left\{ f \in E : \langle f^*(\lambda), \varphi \rangle = 0 \right\}, \quad 0 \neq \varphi \in K, \quad \lambda \in \partial \mathbb{D}. \]

If \( E = W^p_\alpha(\gamma) \) for some \( 1 \leq p < \infty \), then the polynomials are dense in \( E \), and therefore, if \( \gamma \) satisfies the conditions of Theorem 4.1(b), then the
primary invariant subspaces of \( \mathcal{S}_E \) at \( 1 \) are given by Theorem 1.6, and the
generalized root spaces of \( \mathcal{S}^*_E \) at \( 1 \) are given by Theorem 1.7. If \( q = \frac{p}{p-1} \),
then condition (C.1) holds when \( \frac{1}{q} \notin \ell^q \) and condition (C.2) holds when
\( \frac{1}{q} \in \ell^q \).
We turn now to the weighted Bergman spaces. In what follows, \( w \) denotes a positive continuous function on \( D \) such that \( \int_D w \, dA < \infty \) (where \( dA \) is as before the normalized area measure \( \frac{1}{2} \, dx \, dy \) on \( D \)). For every \( 1 \leq p < \infty \), we denote by \( A^p_K(w) \) the vector space of all holomorphic \( K \)-valued functions \( f \) on \( D \) such that
\[
\| f \|_{p, w} = \int_D \| f \|^p w \, dA < \infty.
\]
If \( w \) is bounded, we denote by \( A^K_\infty(w) \) the vector space of all \( K \)-valued holomorphic functions \( f \) on \( D \) such that
\[
\| f \|_{\infty, w} = \sup_{\zeta \in D} \| f(\zeta) \| w(\zeta) < \infty.
\]
The spaces \( A^K_\infty(w) \) will be denoted by \( A^K_p(w) \).

It is well known that with respect to these norms, the spaces \( A^K_p(w) \) are admissible Banach spaces (this also follows from the proof of Lemma 4.10 below). It is clear that if \( E \) is one of these spaces then \( S_E \) is a contraction, and if \( f \) is in \( E \), then for every \( \lambda \) in \( D \), the function \( f_\lambda \) is also in \( E \), and therefore by Lemma 2.16(b), the spectrum of \( B_E \) is included in \( D \).

For the applications of the results of Sections 2 and 3 to these space we need

**Proposition 4.9.** If the function \( \frac{1}{w} \) has moderate growth and \( E = A^K_p(w) \) for some \( 1 \leq p \leq \infty \), then the operator \( B_E \) is of subcritical class.

In view of Lemma 2.7 and Lemma 2.16(a), the proposition follows from

**Lemma 4.10.** If \( 1 \leq p \leq \infty \), there exists a positive constant \( C \), such that for every \( f \) in the unit ball of \( A^K_p(w) \) and for every \( \lambda \) in \( D \),
\[
\| f_\lambda \|_{p, w} \leq C (1 - |\lambda|)^{-p-2} M_{1/w} \left( \frac{|\lambda| + 1}{2} \right)
\]
(4.3)
if \( p < \infty \), and
\[
\| f_\lambda \|_{\infty, w} \leq C (1 - |\lambda|)^{-1} M_{1/w} \left( \frac{|\lambda| + 1}{2} \right).
\]
(4.4)

**Proof.** We give the proof for \( K = \mathbb{C} \). The general case is proved in the same way, and can also be deduced from the scalar case by a standard application of the Hahn–Banach Theorem. To simplify notations we set \( v = \frac{1}{2} \).

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Assume first that $1 \leq p < \infty$, and fix a function $f$ in the unit ball of $A^p(w)$, and a point $\lambda$ in $D$. Let $0 < \delta < \frac{1}{|\lambda|}$, and denote by $U(\delta)$ the disc $|z-\lambda| < \delta$. Using the subharmonicity of the function $|f|^p$, we get
\[ |f(\lambda)|^p \leq \delta^{-2} \int_{U(\delta)} |f|^p dA \leq \delta^{-2} M_p(|\lambda| + \delta). \]
A simple estimate (that uses the Cauchy formula for derivatives, and is valid for every holomorphic function on $D$) shows that
\[ |f_\lambda(z)| \leq 2\delta^{-1} M_p(|\lambda| + 2\delta), \quad \forall z \in U(\delta). \] (4.5)
On the other hand it is clear that
\[ |f_\lambda(z)| \leq \delta^{-1}(|f(z)| + |f(\lambda)|), \quad \forall z \in D \setminus U(\delta). \] (4.6)
Combining these estimates, we get
\[ \|f_\lambda\|_{p,w}^p \leq 2\delta^{-p} - 2 M_p(|\lambda| + 3\delta)(1 + w(0) + a), \]
where $a = \int_D w dA$. Setting $\delta = \frac{1}{|\lambda|}$, we obtain (4.3) with $C = 36 \cdot 12^p(1 + w(0) + a)$.

Using (4.5) and (4.6) with $\delta = \frac{1}{|\lambda|}$, we get (4.4) with $C = 8 \sup_{z \in D} w(z)$.

One can show that if $\rho$ is a positive nonincreasing function on $[0, 1)$ such that $\log \rho$ is concave, and $w$ is the function on $D$ defined by $w(z) = \rho(|z|)$, $z \in D$, then the condition of Proposition 4.9 is also necessary for the backward shift on $A^p_K(w)$, $1 \leq p < \infty$, to be of subcritical class. So if for some $\beta \geq 0$
\[ \omega(z) = \exp \left[ -\frac{1}{(1 - |z|)^\beta} \right], \quad z \in D, \]
then the backward shift on these spaces is of subcritical class, if and only if $\beta < 1$.

From Theorem 1.2(a) and Proposition 4.9 we obtain

**Theorem 4.11.** If $1 \leq p < \infty$ and $E = A^p_K(w)$, with $K$ finite dimensional and $\frac{1}{w}$ of moderate growth, then any maximal invariant subspace of $S_K$ has codimension one.

The space $A^p_K(w)$ with $w(z) = (1 - |z|^2)^\alpha$ for some $\alpha > -1$ (and if $p = \infty$ for $\alpha \geq 0$) will be denoted in the sequel by $A^p_K$, and if $K = \mathbb{C}$, by $A^p$. The space $A^{p,0}$ will also be denoted by $A^p$.

The spaces $A^{p,*}$ are often called **standard weighted Bergman spaces**.
It follows from Proposition 4.9 that the backward shift on the spaces $A^p_{\mathbb{K}}$ is of subcritical class. In fact, using Lemma 4.10 and standard estimates of Taylor coefficients, it is easy to see that the iterates of the backward shift on any of these spaces have polynomial growth. Thus from Theorem 1.2 we obtain

**Theorem 4.12.** If $E = A^p_{\mathbb{K}}$, with $1 \leq p < \infty$ and $p > -1$, or $p = \infty$ and $p \geq 0$, and $\mathbb{K}$ is finite dimensional, then every maximal invariant subspace of $S_E$ has codimension one, and every minimal invariant subspace of $B_E$ is one dimensional.

For the backward shifts on the spaces $A^2_{\mathbb{K}}$ we have a stronger result.

**Theorem 4.13.** If $E = A^2_{\mathbb{K}}$, with $p > -1$ and $\mathbb{K}$ finite dimensional, then $\text{Lat}_B$ has no proper gaps.

This is just a reformulation of Theorem 4.6, since by (4.1), $A^2_{\mathbb{K}} = D_{-\alpha-1}$, if $p > -1$.

If $\mathbb{K} = \mathbb{C}$, then the result can be extended also to some other standard weighted Bergman spaces. To see this, we have to describe the dual spaces of the spaces $A^p_{\mathbb{K}}$ for $1 < p < \infty$, and some of their properties. These facts can be found in [29] and the references listed there. (See also [4, Sect. 5].)

Assume that $p > -1$ and that $1 < q < \infty$. Let $n$ denote the smallest integer that is not less than $p$, and set $\beta = (n - p) + q$. The vector space $X_{\alpha, q} = \{ f \in \text{Hol}(\mathbb{D}) : f^{(n-1)} \in A^{\alpha, \beta} \}$ can be identified with the dual of $A^{p, q}$ where $p = \frac{1}{q-1}$, via the pairing

$$
\langle f, g \rangle = \lim_{\tau \to 1} \frac{1}{\tau} \int_{|z|=1} f(\tau z) \overline{g(\tau z)} \, |dz|,
$$

$f \in A^{p, q}, \quad g \in X_{\alpha, q}$.

With respect to dual norm, $X_{\alpha, q}$ is an admissible Banach space in which the polynomials are dense, and the adjoint of the shift (resp. backward shift) on $A^{p, q}$ is the backward shift (resp. shift) on $X_{\alpha, q}$. So if $E = X_{\alpha, q}$, then $B_E$ is a contraction and the iterates of $S_E$ have norms of polynomial growth.

It is also known (see [29]) that for $p > \frac{2}{q-2}, X_{\alpha, q}$ is a Banach algebra with respect to pointwise multiplication. Thus from Theorem 1.5 we obtain

**Theorem 4.14.** If $E = X_{\alpha, q}$, with $1 < q < \infty$ and $p > \frac{2}{q-2}$ then $\text{Lat}_S$ has no proper gaps.
By duality we get

**Theorem 4.15.** If \( E = A^{p', \infty} \) with \( p > p - 2 \), then \( \text{Lat}B_E \) has no proper gaps.

Combining this result with Theorem 4.13 we have

**Corollary 4.16.** If \( 1 < p \leq 2 \) and \( E = A^p \), then \( \text{Lat}B_E \) has no proper gaps.

If \( E = A^p_K(w) \) for some \( 1 \leq p < \infty \), then \( S^*_E \to 0 \) in the strong operator topology, and therefore \( S^*_E \) has no eigenvalues in \( \partial D \). Hence the invariant subspaces of \( S_E \) of codimension one are given by (2.25).

It follows from Proposition 4.9, that if the polynomials are dense in \( A^p(w) \) (a necessary condition for this is that \( p < \infty \)) and \( \frac{1}{n} \) has moderate growth, then the primary invariant subspaces of \( S_E \) at \( 1 \) are given by Theorem 1.6. By the previous observation, these spaces satisfy condition (C.1) of the theorem. It is known that if \( w \) is a radial function (that is of the form \( w(z) = h(|z|) \), \( z \in D \), where \( h \) is a function on \( [0, 1) \)), then the polynomials are dense in \( A^p(w) \), for \( 1 \leq p < \infty \). In fact, if \( f \) is a function in that space, then \( \sum_{j=0}^{n} (1 - \frac{j}{n}) \hat{f}(j) z^j \to f \) in the norm topology.

We next consider the Hardy spaces

\[
\mathcal{H}^p_K = \left\{ f \in \text{Hol}(D, K) : \|f\|_p = \sup_{0 < r < 1} \frac{1}{2\pi} \int_{|\zeta|=1} |f(r\zeta)|^p |d\zeta| < \infty \right\}
\]

for \( 1 \leq p < \infty \), and

\[
\mathcal{H}^\infty_K = \left\{ f \in \text{Hol}(D, K) : \|f\|_\infty = \sup_{\zeta \in D} |f(\zeta)| < \infty \right\}.
\]

We denote by \( A_K \) the Banach space of all \( K \)-valued continuous functions on \( D \) that are holomorphic in \( D \) equipped with the maximum norm. If \( K = \mathbb{C} \), we denote these spaces as usual by \( \mathcal{H}^p \) and \( A \).

It is well known that all these spaces are admissible Banach spaces, and the shift is an isometry on all of them. If \( E \) is one of these spaces then

\[
\|B^n_E\| = 0(\log n), \quad n \to \infty,
\]

since \( S_E \) is an isometry, and for every \( n \in \mathbb{Z}_+ \), \( I - S^{n+1}_E B^{n+1}_E \) is the \( n \)th partial sum operator, which can be represented as convolution with the Dirichlet kernel. Thus from Theorem 1.2 we obtain

**Theorem 4.17.** If \( K \) has finite dimension and \( E = \mathcal{H}^p_K \) for some \( 1 \leq p \leq \infty \), or \( E = A_K \), then every maximal invariant subspace of \( S_E \) has codimension one, and every minimal invariant subspace of \( B_E \) is one dimensional.
As already mentioned before, the result is well known for the space $H^2_{K}$. The proofs (see [25, Theorem 16; 38, Theorem 3.32 and Corollary 6.18]) rely strongly on the Hilbert space structure of that space, and do not work for $p \neq 2$. If $E$ is the disc algebra $A$ the first assertion follows also from the fact that $\text{Lat} S_E$ consists of the closed ideals in $A$, and in a commutative Banach algebra with unit, every maximal ideal has codimension one. This argument does not work for $H^p$, since the invariant subspaces of the shift on that space are not the closed ideals in that algebra. If $E = H^p$ for $1 \leq p < \infty$, the assertion follows from the known structure of $\text{Lat} S_E$ (see [12; 25, Lecture IV]) which actually shows that it has no proper gaps. This fact is also true for the algebra $A$ by Theorem 1.5, and also follows from the known structure of its closed ideals (see [25, p. 28]).

If $E = H^p_K$ for some $1 \leq p < \infty$, then the invariant subspaces of $S_E$ that have codimension one are given by (2.25), and if $E = A_K$ one needs to add also the spaces $\{ f \in A_K : \langle f(\lambda), \varphi \rangle = 0 \}, 0 \neq \varphi \in K, \lambda \in \partial D$.

If $E = H^p$ for $p < \infty$ or $E = A$, then the primary invariant subspaces of $S_E$ at 1 are given by Theorem 1.6, and the generalized root spaces of $S_E^*$ at 1, by Theorem 1.7.

We conclude this section with an application of Theorem 1.3 to the generalized Dirichlet spaces introduced by Richter in [41]. These spaces are defined as follows. Assume that $\mu$ is a positive Borel measure on the unit circle $\partial D$, and denote by $P\mu$ its Poisson integral. This is a positive harmonic function on $D$. If $f$ is a holomorphic function on $D$, then its Dirichlet integral with respect to $\mu$ is defined by

$$D_\mu(f) = \left| \int_D |f'|^2 P\mu \, dA \right|,$$

where $dA$ denotes again normalized area measure on $D$.

Let $D(\mu)$ denote the vector space of all holomorphic functions $f$ on $D$ such that $D_\mu(f)$ is finite. When $\mu$ is normalized Lebesgue measure on $\partial D$, this space coincides with the classical Dirichlet space $D_1$.

It is shown in [41] that $H^2 \subset D(\mu)$, and that $D(\mu)$ is a Hilbert space with respect to the norm

$$\|f\|_{D(\mu)}^2 = \|f\|_2^2 + D_\mu(f), \quad f \in D(\mu).$$

It is also proved in [41] that the operator $S_\mu$ of multiplication by $z$ is a bounded linear operator on $D(\mu)$, that is bounded below by 1. Furthermore, it is shown in [40] that

$$\|S_\mu^n\| = O(n^{1/2}), \quad n \to \infty,$$
and in [43], that if $M$ is a nonzero invariant subspace of $S_+$, then the space $M \ominus S_+M$ is one dimensional. Thus from Theorem 1.3 we get

**Theorem 4.18.** If $\mu$ is a positive Borel measure on $\partial \mathbb{D}$, then $\operatorname{Lat} S_+ \mu$ has no proper gaps.

## 5. FURTHER RESULTS

In this section we prove some additional results. We begin with an extension of Theorem 1.1(a) that has some interesting applications to shifts on admissible Banach spaces.

**Theorem 5.1.** The conclusion of Theorem 1.1(a) remains true, if the condition that $T$ has a left inverse of subcritical class, is replaced by the assumption that $T$ has a left inverse $L$ with spectrum included in $\mathbb{D}$, such that for some complex holomorphic function $g \neq 0$ on $\mathbb{D}$ of moderate growth, the operator function on $\mathbb{D}$ defined by $\lambda \mapsto g(\lambda)(I - \lambda L)^{-1}$ has moderate growth.

**Proof.** First note that by Lemma 2.7, the assumptions of the theorem are satisfied if $L$ is of subcritical class.

It suffices to show that the conclusion of Proposition 2.3(a) remains true under these weaker assumptions. We keep all the notations in the proof of that proposition. The assumptions imply by Corollary 2.6 that $\bar{L}$ has exponential growth. Hence if $T$ is invertible, the conclusion follows from the proof of the first part of the proposition. Assume now that $T$ is not invertible. Since $\bar{L}$ has exponential growth, the same is true for $Q$ and therefore by (2.9) it suffices to show that $F$ has exponential growth. To see this, consider the operator functions $\bar{L}_1 = g \bar{L}$ and $Q_1 = gQ$. It follows from the assumptions, that both have moderate growth. Multiplying both sides of (2.11) by $g$, we get that

$$F(\lambda) Q_1(\lambda) i_M = \pi_M \bar{L}_1(\lambda) i_M, \quad \lambda \in \mathbb{D}.$$ 

Hence, noting that for every $\lambda$ in $\mathbb{D}$ such that $g(\lambda) \neq 0$, the operator $Q_1(\lambda) i_M$ from $M$ to $K$ is surjective, we obtain by the same arguments as in the proof of Proposition 2.3(a), that $F$ has exponential growth. $\blacksquare$

Following [40] we shall say that an operator $T$ on $X$ is **analytic** if

$$\bigcap_{n=0}^{\infty} T^n X = \{0\}.$$
By using Theorem 5.1 we can prove

**Theorem 5.2.** Assume that $T$ is an analytic operator on $X$ that satisfies the conditions of Theorem 1.1(a). If $M$ is an invariant subspace of $T$ which has a one dimensional subspace that is complemented in $M$ to all the spaces $(T-\lambda)M$, $\lambda \in \mathbb{D}$, then every maximal invariant subspace of $T|_M$ has codimension one in $M$.

*Proof.* Let $u$ be a unit vector in $M$ such that the one dimensional subspace $K = Cu$ is complemented in $M$ to all the spaces $(T-\lambda)M$, $\lambda \in \mathbb{D}$. Let $L_0$ denote the left inverse of $T|_M$ that annihilates $K$. It follows from the assumptions, that $T-\lambda$ is bounded below for every $\lambda$ in $\mathbb{D}$ (see the comments following the proof of Lemma 2.10), and therefore by Lemma 2.10, $\sigma(L_0) \subset \mathbb{D}$. Thus by Theorem 5.1, the assertion will be proved, if we show that there exists a holomorphic function $g \neq 0$ on $\mathbb{D}$ of moderate growth, such that the operator function on $\mathbb{D}$ defined by $\lambda \mapsto g(\lambda)(I_0-\lambda L_0)^{-1}$ has moderate growth. ($I_0$ denotes the identity operator on $M$). To show this, consider the operator functions $\tilde{L}$ and $Q$ associated with $L$ and $T$ as in the proof of Proposition 2.3(a), and the operator functions $\tilde{L}_0$ and $Q_0$ associated with $T|_M$ and $K$, as in Lemma 2.11. Since by Corollary 2.6 the operator function $\tilde{L}$ has moderate growth, it follows from (2.13) that it suffices to show that there exists a holomorphic function $g \neq 0$ on $\mathbb{D}$ of moderate growth, such that the operator function $gQ_0$ has moderate growth.

From the definitions of $K$ and $Q_0$, it follows that there exists a vector function $\pi: \mathbb{D} \to M^*$ such that

$$Q_0(\lambda) = u \otimes \pi(\lambda), \quad \lambda \in \mathbb{D},$$

(5.1)

Since the range of $I_0 - Q_0(\lambda)$ is $(T-\lambda)M$, and this subspace is included in the kernel of $Q(\lambda)$, we get from (5.1) that

$$Q(\lambda)|_M = Q(\lambda) u \otimes \pi(\lambda), \quad \lambda \in \mathbb{D},$$

(5.2)

We claim that the vector function $\lambda \mapsto Q(\lambda) u$, $\lambda \in \mathbb{D}$, is not identically zero. Assume on the contrary that it is identically zero. Using the fact that

$$Q(\lambda) = \sum_{n=0}^{\infty} (I - TL)^n \lambda^n, \quad \lambda \in \mathbb{D}$$

(that follows easily from the definition of $Q$), the assumption implies that

$$L^* u = TL^{n+1} u, \quad \forall n \in \mathbb{Z}_+,$$

and from this we get by an induction argument that $u \in \bigcap_{n=0}^{\infty} T^n X$, which is a contradiction, since $T$ is analytic and $\|u\| = 1$. So that the above vector
function is not identically zero. Therefore, by the Hahn–Banach theorem, there exists an element \( v \) in \( X^* \), such that the holomorphic function \( g \) on \( \mathbb{D} \) defined by

\[
g(\lambda) = \langle Q(\lambda) u, v \rangle, \quad \lambda \in \mathbb{D},
\]

is not identically zero. From (5.2) and (5.3) we get that

\[
g(\lambda) x(\lambda) = (Q(\lambda)|_M)^* v, \quad \lambda \in \mathbb{D},
\]

and combining this with (5.1), we see that

\[
g(\lambda) Q_0(\lambda) = u \otimes (Q(\lambda)|_M)^* v, \quad \lambda \in \mathbb{D}.
\]

Since by Corollary 2.6, \( Q \) has moderate growth, we conclude from (5.3) and (5.4), that the functions \( g \) and \( gQ_0 \) have moderate growth, and the proof is complete.

Next we give several applications of the theorem. First we introduce some terminology and make a few comments.

Recall that an operator \( T \) on \( X \) is called cyclic if there exists a vector \( v \) in \( X \) such that \( \bigvee_{n \geq 0} T^n v = X \). Such a vector is called a cyclic vector of \( T \).

It is clear that if \( T \) has closed range and \( v \) is a cyclic vector of \( T \), then \( C_v + TX = X \), so if \( T \) is not surjective, then \( C_v \) is complemented to \( TX \).

If \( T \) is not invertible and \( T - \lambda \) is bounded below for every \( \lambda \) in \( \mathbb{D} \), then by Fredholm theory, for every \( \lambda \) in \( \mathbb{D} \), \( T - \lambda \) is not invertible, hence if \( v \) is a cyclic vector for \( T \) (so also for \( T - \lambda \), \( \forall \lambda \in \mathbb{C} \)) then \( C_v \) is complemented to all the spaces \( (T - \lambda) X, \lambda \in \mathbb{D} \).

An invariant subspace \( M \) of an operator \( T \) is called singly generated if the operator \( T|_M \) is cyclic; a cyclic vector of this operator is called a generator of \( M \). It follows from the comments above, that if \( T \) is an analytic operator on \( X \) such that \( T - \lambda \) is bounded below for every \( \lambda \) in \( \mathbb{D} \), and \( M \) is a singly generated invariant subspace of \( T \) with generator \( x \), then \( C_x \) is complemented in \( M \) to all the spaces \( (T - \lambda) M, \lambda \in \mathbb{D} \).

As observed in Section 2, if \( T \) has a left inverse whose spectrum is included in \( \mathbb{D} \), then \( T - \lambda \) is bounded below for every \( \lambda \) in \( \mathbb{D} \).

Combining these facts with Theorem 5.2 we get

**Theorem 5.3.** If \( T \) is an analytic operator on \( X \) that satisfies the conditions of Theorem 1.1(a) and \( M \) is a nonzero singly generated invariant subspace of \( T \), then every maximal invariant subspace of \( T|_M \) has codimension one in \( M \).
A direct consequence of the theorem is

**Corollary 5.4.** Assume that $E$ is an admissible Banach space that satisfies the conditions of Theorem 1.2(a). If $M$ is a nonzero singly generated subspace of $S_E$, then every maximal invariant subspace of $S_E|_M$ has codimension one in $M$. Thus, if every invariant subspace of $S_E$ is singly generated, then $\operatorname{Lat}S_E$ has no proper gaps.

If $E$ consists of complex functions then the conclusion of the corollary holds under weaker conditions. To describe them, we need some additional terminology.

If $f$ is a holomorphic function on $\mathbb{D}$ that is not identically zero, we shall denote by $\partial f$ the function on $\mathbb{D}$ whose value at a point $\lambda$ is the order of the zero of $f$ at $\lambda$. Let $E$ be an admissible Banach space of complex holomorphic functions on $\mathbb{D}$, and set $S_E = S$. Assume that $M$ is a nonzero invariant subspace of $S$, and denote by $v_M$ the function on $\mathbb{D}$ defined by

$$v_M(\lambda) = \inf \{ \partial f(\lambda) : f \in M \}, \quad \lambda \in \mathbb{D}.$$ 

If $g$ is a function in $M$ such that $\partial g = v_M$, we shall say that $g$ has the same zero set as $M$. It is clear that if $M$ is generated by $g$, then $g$ has the same zero set as $M$, but the converse is not true in general.

It is easily verified that $M$ has a one dimensional subspace that is complemented to all the spaces $(S - \lambda) M$, $\lambda \in \mathbb{D}$, if and only if, there exists a function $g$ in $M$ that has the same zero set as $M$, and $\dim M/(S - \lambda) M = 1$ for every $\lambda$ in $\mathbb{D}$. If $\sigma(B_E) \subset \mathbb{D}$, then it follows from Lemma 2.16(a) and Fredholm theory (see [39]) that the second condition above is equivalent to the condition $\dim M/SM = 1$. Thus from Theorem 5.2 we obtain

**Proposition 5.5.** Assume that $E$ is an admissible Banach space of complex holomorphic functions on $\mathbb{D}$ that satisfies the conditions of Theorem 1.2(a), and set $S_E = S$. If $M$ is a nonzero invariant subspace of $S$ such that $\dim M/SM = 1$, and there exists a function $g$ in $M$ that has the same zero set as $M$, then every maximal invariant subspace of $S|_M$ has codimension one in $M$.

A concrete application of the proposition is

**Proposition 5.6.** If $E = A^{p, \infty}$, with $1 \leq p < \infty$ and $\alpha \geq 0$, and $M$ is a nonzero invariant subspace of $S_E$ such that $\dim M/S_E M = 1$, then every maximal invariant subspace of $S_E|_M$ has codimension one in $M$.

This follows from Proposition 4.9 and Proposition 5.5, since (as observed in [39, Sect. 5.3]) the proof of Theorem 7.9 in [26] implies that every
nonzero invariant subspace $M$ of $S_E$, contains a function that has the same zero set as $M$.

If $T$ is an operator on $X$, then it is clear that every maximal invariant subspace of $T$ of codimension one, is a primary invariant subspace of $T$. It is natural to ask whether the same is true for every maximal invariant subspace of $T$. We do not know the answer in the general case. However, using the methods of Section 2 and a theorem of Wermer [50], one can prove the following.

**Theorem 5.7.** Assume that $T$ is a contraction in $L(X)$ which has range of finite codimension and a left inverse $L$ such that

$$\sum_{n=1}^{\infty} n^{-2} \log \|L^n\| < \infty.$$ 

Then every maximal invariant subspace of $T$ is a primary invariant subspace of $T$.

We will not give the proof here.

If $E$ is an admissible Banach space of complex holomorphic functions on $\mathbb{D}$ such that $\sigma(S_E) = \mathbb{D}$, it is natural to ask whether $S_E$ has any primary invariant subspaces at 1, or equivalently, whether $S_E^*$ has any $w^*$-closed generalized root spaces at 1. In the case that the polynomials are dense in $E$, and $S_E$ is a contraction, and $B_E$ is of subcritical class, Theorems 1.6 and 1.7 provide a positive answer, and give even a complete description of these spaces when (C.1) or (C.2) holds. If we omit the condition that $B_E$ is of subcritical class, the problem becomes much harder, and we do not know the answer in the general case. However, for the spaces $W^p(\gamma)$, $1 < p < \infty$, we have a positive answer if $\gamma$ satisfies the following regularity conditions.

1. The sequence $\log \gamma$ is concave, that is, $\gamma(n-1) \gamma(n+1) \leq (\gamma(n))^2$, $n \in \mathbb{Z}_+$.

2. $\sup_{n \in \mathbb{Z}_+} (\gamma(n))^2 \gamma(n-1) \gamma(n+1) < \infty$.

This result will appear in [10]. An outline of related results can be found in [8].

We now describe a concrete example of this type. Assume that $1/2 < \alpha < 1$, and consider the sequence $\gamma(n) = \exp(-n^\alpha)$, $n \in \mathbb{Z}$. Let $1 < p < \infty$, and set $E = W^p(\gamma)$. As observed in Section 4 (Example 2), in this case $B_E$ is not of subcritical class. The dual space is $W^q(\frac{1}{\gamma})$, where $q = \frac{p}{p-1}$. Let $v$ be the function $x \mapsto \exp(x^\alpha)$ on $[0, \infty)$, and for every real
number \(c\), denote by \(u\), the function \(x \mapsto \exp(-|x|^c/\cos \pi c) - e^{-|x|^{1/2}}\) on \((-\infty, 0]\). Consider for every \(-\infty < c < \infty\), the vector space

\[ Y_c = \{ f \in \delta_{1,0} : f|_{(0, \infty)} \in L^p(0, \infty), f|_{(-\infty, 0]} \in L^p_{(-\infty, 0]} \}. \]

It is shown in [10] that with respect to the norm

\[ \|f\|_{Y_c} = \|f|_{L^q(0, \infty)} \] \( f \in Y_c \),

\(Y_c\) is a Banach space which is invariant under differentiation, the differentiation operator is quasinilpotent, convergence in \(Y_c\) implies uniform convergence on compact subsets of \(\mathbb{C}\), and the space

\[ V_c = \{ \phi \in E^* : \phi \in Y_c|_{Z_z} \} \]

is a generalized root space of \(S^k_E\) at \(1\). (Note that since \(E\) is reflexive, every closed subspace of \(E^*\) is \(\sigma\)-closed). We conjecture that every generalized root space of \(S^k_E\) at \(1\) is of this form. This would imply by Theorem 5.7, that every maximal invariant subspace of \(S_E\) has codimension one.

The appearance of spaces of entire functions included in \(E^{1,0}\), in the description of generalized root spaces at \(1\) of the operators \(S^k_E\) (when \(E\) is an admissible Banach space in which the polynomials are dense), is not accidental. In fact, it is in the nature of things. To see this, assume that \(E\) is an admissible Banach space of complex holomorphic functions on \(D\), in which the polynomials are dense, and denote by \(E^*\) the sequence space \(\{ \phi : \phi \in E^* \}\) equipped with the norm that makes the mapping \(\phi \mapsto \phi\) an isometry. Then \(E^*\) is a Banach space which is isometrically isomorphic to \(E^*\). Assume that \(V\) is a generalized root space of \(S^k_E\) at \(1\). Then by the standard functional calculus, there exists an operator \(A\) in \(L^p(V)\) such that \(\sigma(A) = \{0\}\) and \(S^k_E|_V = e^A\). Consider the mapping \(J : V \to \delta_{1,0}\) defined by

\[ J\phi(z) = \langle 1, e^{iz}\phi \rangle, \quad \phi \in V, \quad z \in \mathbb{C}. \]

The function \(J\phi\) is in \(\delta_{1,0}\) since \(\sigma(A) = \{0\}\). (See [7]). Since \(J\phi|_{Z_z} = \phi\), \(\forall \phi \in V\), the mapping \(J\) is injective. Let \(Y_V\) denote the vector space \(JV\), equipped with the norm that makes the mapping \(J\) an isometry. Then \(Y_V\) is a Banach space that is isometrically isomorphic to \(V\), and it is clear that

\[ V = \{ \phi \in E^* : \phi \in Y_V|_{Z_z} \}. \]

It is easily verified that the operator on \(Y_V\) that corresponds to the operator \(A\) by the mapping \(J\) is the differentiation operator \(\frac{d}{dz}\). So \(Y_V\) is invariant under differentiation and this operator is quasinilpotent. It is also easy to see that convergence in \(Y_V\) implies uniform convergence on compact subsets of \(\mathbb{C}\).
This process can also be reversed. If $Y$ is a Banach space of functions in $\ell^1_0$ such that convergence in $Y$ implies uniform convergence on compact subsets of $C$, the differentiation operator leaves $Y$ invariant, is quasinilpotent, and $Y|_{Z_+}$ is a closed subspace of $E^*$, then it is easy to show that the space

$$V_Y = \{ \varphi \in E^* : \tilde{\varphi} \in Y|_{Z_+} \}$$

is a generalized root space of $S^*_E$ at 1, that is isometrically isomorphic to $Y$.

If $T$ is an operator on $X$, one can define in an obvious way the concepts of maximal hyperinvariant subspace and minimal hyperinvariant subspace of $T$. By using the same arguments as in the proof of Theorem 1.1 one can prove the following:

**Theorem 5.8.**  
(a) If $T$ satisfies the conditions of Theorem 1.1(a), then every maximal hyperinvariant subspace of $T$ has codimension one.

(b) If $T$ satisfies the conditions of Theorem 1.1(b), then every minimal hyperinvariant subspace of $T$ is one dimensional.

**REFERENCES**