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Modified star-products beyond the large-B limit

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Abstract

Derivative corrections to the Wess–Zumino couplings of open-string effective actions are computed at all orders in derivatives, taking the open-string metric into account. This leads to a set of deformed star-products beyond the Seiberg–Witten limit, and allows to reinterpret the couplings in terms of a deformed integration prescription along a Wilson line in the non-commutative set-up. Moreover, the recursive definition of the star-products induces deformations of $U(1)$ non-commutative Yang–Mills theory.

1. Introduction

Non-commutative field theory on the world-volume of a D-brane has been developed in a peculiar limit where a large constant background B-field is turned on [1,2]. This is called the Seiberg–Witten limit and amounts to $\alpha' \to 0$ together with a scaling of the metric, $g_{ij} \sim \alpha'^2$, while open-string parameters are kept fixed. Duality properties have been studied, and an explicit mapping between ordinary and non-commutative gauge fields exhibited [3–5]. This inspiring correspondence has been successfully extended to the Ramond–Ramond couplings in the Seiberg–Witten limit, leading to an infinite set of derivative corrections [6–8], since the corrections are suppressed by powers of α' in the non-commutative set-up. The series of corrections are expressed in terms of modified star-products named $*_p$ (for integer p), that arise naturally from an integration prescription along an open Wilson line. This prescription originates from the requirement of gauge invariance of observables in non-commutative field theory [9–15]. Computations at tree level in presence of a single Euclidean D9-brane in commutative string theory provided successful checks [16,17], confirming that these corrections are leading in the Seiberg–Witten limit. This suggested that the correspondence could be extended by string computations beyond this limit. Mukhi and Suryanarayana [18] derived the first correction in terms of the open-string metric to the coupling of quadratic order in the field strength, and generalized it to all orders (in the metric) by using a disk amplitude computed by Liu and Michelson [20]. This led to a deformation of the $*_2$-product by a differential operator t constructed out of

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the open-string metric G:

$$C^{(6)}(-k) \wedge \int dx\, \langle F \wedge F \rangle_{*_2} e^{\mathrm{i}kx} \mapsto C^{(6)}(-k) \wedge \int dx\, \langle F \wedge F \rangle_{*_2(t)} e^{\mathrm{i}kx},$$

where the $*_2(t)$ has the expected Seiberg–Witten limit

$$t := \alpha' \partial G \partial', \qquad a := \frac{\partial \theta \partial'}{2\pi}, \qquad *_2(t) = \frac{\Gamma(1+2t)}{\Gamma(1-a+t)\Gamma(1+a+t)}, \qquad *_2(0) = \frac{\sin \pi a}{\pi a} = *_2.$$

The deformed star-product $*_2(t)$ received an interpretation in terms of a deformed smearing prescription along an open Wilson line, that parallels the one that had given rise to the $*_2$-product, since

$$\int dx\, \langle F \wedge F \rangle_{*_2(t)} e^{\mathrm{i}kx} = \int dx \int_0^1 d\tau\, F(x) * (t) \wedge F(x + \theta k \tau) * e^{\mathrm{i}kx},$$

$$\text{where} \quad *(t) = * \times \frac{*_2(t)}{*_2} = * \times \frac{\Gamma(1-a)\Gamma(1+a)\Gamma(1+2t)}{\Gamma(1-a+t)\Gamma(1+a+t)}.$$

In this Letter we shall derive the contribution of the open-string metric to the amplitude

$$S_{\mathrm{CS}} + \Delta S_{\mathrm{CS}} = \langle C | \exp\left(-\frac{\mathrm{i}}{2\pi\alpha'} \int d\sigma\, d\theta\, D\phi^\mu A_\mu(\phi) \right) |B\rangle_R,$$

where $\phi^\mu = X^\mu + \theta \psi^\mu$ denotes a superfield, and D a derivative in superspace. This will enable us to derive the prescription of [18] and to extend it to larger orders in the field strength.

The recursive definition of the modified star-products allows to address the question of the correct definition of gauge transformation laws beyond the large-B limit. We shall work with a single Euclidean D9-brane in the description where the open-string metric G is defined by

$$\left(\frac{1}{g + 2\pi\alpha' B} \right)^{ij} = \frac{\theta^{ij}}{2\pi\alpha'} + G^{ij}.$$

We shall first work out the deformation of the star-products in presence of the symmetric part of the two-point functions of world-sheet scalars, and then interpret the results in terms of a deformed gauge-invariant smearing prescription in the non-commutative set-up.

2. Taking the open-string metric into account

2.1. Quadratic order in the field strength

As noticed in previous investigations of Ramond–Ramond couplings for small field strength, we are instructed to compute couplings whose order in the field strength is half of the degree as a differential form. Writing the result in terms of the differential form F, one has to make the substitution

$$\frac{1}{2} \psi_0^\mu \psi_0^\nu F_{\mu\nu} \mapsto -\mathrm{i}\alpha' F.$$

The only role played by the fermions in our computation will therefore be to provide us with the suitable number of zero modes, in order to build the grading of the coupling. As we are dealing with the Ramond–Ramond sector, we may forget the first part of the following expression, because it does not contribute to the grading:

$$\int d\sigma\, d\theta\, D\phi^{\mu} A_{\mu}(\phi) = -\int d\sigma\, d\theta \sum_{k\geqslant 0} \frac{1}{(k+1)!}\frac{k+1}{k+2} D\tilde{\phi}^{\nu}\tilde{\phi}^{\mu}\tilde{\phi}^{\mu_1}\cdots\tilde{\phi}^{\mu_k}\partial_{\mu_1}\cdots\partial_{\mu_k}F_{\mu\nu}(x)$$
$$-\int d\sigma \left(\tilde{\psi}^{\mu}\psi_0^{\nu} + \psi_0^{\mu}\psi_0^{\nu}\right)\sum_{k\geqslant 0}\frac{1}{k!}\tilde{X}^{\mu_1}\cdots\tilde{X}^{\mu_k}\partial_{\mu_1}\cdots\partial_{\mu_k}F_{\mu\nu}(x).$$

The computation is along the lines of the work by Wyllard [19] and amounts to contracting pairs of scalars using the open-string propagator

$$D^{a,b}(\sigma) = \alpha'\left[\frac{\theta^{a_i b_i}}{2\pi\alpha'}\log\left(\frac{1-e^{-\epsilon+\mathrm{i}\sigma}}{1-e^{-\epsilon-\mathrm{i}\sigma}}\right) + G^{a_i b_i}\log\left|1-e^{-\epsilon+\mathrm{i}\sigma}\right|^2\right].$$

Each of the propagators contributing to the regular part of the coupling comes with two derivatives acting on two different field strengths: $D^{a,b}\partial_a\partial'_b|_{x'=x}$. At order $2n$ in derivatives, the regular part of the coupling to $C^{(6)}$ reads

$$\frac{\alpha'^n}{n!}\int_0^{2\pi}\frac{d\sigma}{2\pi}\prod_{i=1}^{n}\left\{\frac{\theta^{a_i b_i}\partial_{a_i}\partial'_{b_i}}{2\pi\alpha'}\log\left(\frac{1-e^{-\epsilon+\mathrm{i}\sigma}}{1-e^{-\epsilon-\mathrm{i}\sigma}}\right) + \left(G^{a_i b_i}\partial_{a_i}\partial'_{b_i}\right)\log\left|1-e^{-\epsilon+\mathrm{i}\sigma}\right|^2\right\}F(x)\wedge F(x').$$

Let us expand the product in the above expression in terms of the symmetric and antisymmetric parts of the propagator, and remove the regulator:

$$\sum_{p=0}^{n} C_n^p \frac{1}{(2\pi\alpha')^p}\theta^{a_1 b_1}\cdots\theta^{a_p b_p} i^p(\sigma-\pi)^p \times G^{a_{p+1}b_{p+1}}\cdots G^{a_n b_n}\left(\log\left|1-e^{\mathrm{i}\sigma}\right|^2\right)^{n-p}.$$

As we expand the gauge coupling to all orders in derivatives, we have to sum these contributions over n:

$$\sum_{n\geqslant 0}\frac{\alpha'^n}{n!}\int_0^{2\pi}\left[\frac{d\sigma}{2\pi}\left(\frac{\partial\theta\,\partial'}{2\pi\alpha'}\mathrm{i}(\sigma-\pi) + \partial G\partial'\log\left|1-e^{-\mathrm{i}\sigma}\right|^2\right)^n\right]$$
$$= \int_0^{2\pi}\frac{d\sigma}{2\pi}\exp\left[\alpha'\left(\frac{\partial\theta\,\partial'}{2\pi\alpha'}\mathrm{i}(\sigma-\pi) + \partial G\partial'\log\left|1-e^{-\mathrm{i}\sigma}\right|^2\right)\right].$$

The last integral expression therefore equals

$$\int_0^1 d\tau\,\left|2\sin(\pi\tau)\right|^{2t}\exp\left((\mathrm{i}a\pi)(2\tau-1)\right) = \frac{\Gamma(1+2t)}{\Gamma(1-a+t)\Gamma(1+a+t)}.$$

This is a recipe for going beyond the Seiberg–Witten limit at quadratic order in the field strength. It confirms the prescription of [18], where the first order in G thereof was computed, and where larger orders were included by requiring consistency with [20].

2.2. Higher orders in the field strength

The derivation [17] of the regular part of the couplings of larger orders in the field strength relies on symmetry factors and not on the precise form of the propagator. It may, therefore, be applied here using the full open-string propagator with derivatives included, together with the following notations:

$$Q_{ij} := \mathrm{i}a_{ij}\big(\sigma_{ij} - \epsilon(\sigma_{ij})\big) + t_{ij}\log\left|2\sin\left(\frac{\sigma_{ij}}{2}\right)\right|^2,$$

$$\sigma_{ij} := \sigma_i - \sigma_j, \qquad a_{ij} := \frac{\partial_{i,\mu}\theta^{\mu\nu}\partial_{j,\nu}}{2\pi}, \qquad t_{ij} := \alpha'\partial_{i,\mu}G^{\mu\nu}\partial_{j,\nu}.$$

The coupling to a mode of $C^{(10-2p)}$ is going to be expressed as the image of F^p by some differential operator $\tilde{*}_p$, so that $\tilde{*}_2$ is the $*_2(t)$ of [18]. Furthermore, if all the metric-dependent coefficients t_{ij} are set to 0, the kernel $\tilde{*}_p$ will reduce to the modified star-product $*_p$. The only change with respect to the derivation in the Seiberg–Witten limit comes from the symmetric part of the propagator, which is going to insert a factor of $|2\sin(\sigma_{ij}/2)|^{2t_{ij}}$ in the integral for each pair of labels $\{i, j\}$ (with $i \neq j$, since extracting the regular part of the coupling prohibits self-contractions).

At cubic order in the field strength one can write explicitly:

$$\begin{aligned}\tilde{*}_3 &= \sum_{A,B,C\geqslant 0}\frac{1}{A!}\frac{1}{B!}\frac{1}{C!}\int_0^{2\pi}\frac{d\sigma_1}{2\pi}\int_0^{2\pi}\frac{d\sigma_2}{2\pi}\int_0^{2\pi}\frac{d\sigma_3}{2\pi}Q_{12}^A Q_{23}^B Q_{31}^C\\ &= \int_0^1 d\tau_1\int_0^1 d\tau_2\int_0^1 d\tau_3\exp\big\{\mathrm{i}a_{12}\pi\big(2\tau_{12}-\epsilon(\tau_{12})\big) + 2t_{12}\log\big|2\sin(\pi\tau_{12})\big|\\ &\qquad + \mathrm{i}a_{23}\pi\big(2\tau_{23}-\epsilon(\tau_{23})\big) + 2t_{23}\log\big|2\sin(\pi\tau_{23})\big|\\ &\qquad + \mathrm{i}a_{31}\pi\big(2\tau_{31}-\epsilon(\tau_{31})\big) + 2t_{31}\log\big|2\sin(\pi\tau_{31})\big|\big\}.\end{aligned}$$

As the symmetry factors have been shown to keep the same structure for an arbitrary number of operators, the desired operator is seen to be the following for any integer p:

$$\tilde{*}_p = \int_0^1 d\tau_1\cdots\int_0^1 d\tau_p\exp\left\{\sum_{i<j}\big(\mathrm{i}\pi a_{ij}\big(2\tau_{ij}-\epsilon(\tau_{ij})\big) + 2t_{ij}\log\big|2\sin(\pi\tau_{ij})\big|\big)\right\}.$$

Since we have disregarded from the very beginning the contact terms that can arise from insertion of operators at the same point, we missed the explicit counterpart of commutators that show up in the corresponding computations in the non-commutative description [20] (what we derived is just the deformation of the differential operator $*_p$). These contact terms are naturally related to point-splitting regularization and therefore to non-commutative gauge theory. However, the existence of well-established Ramond–Ramond couplings in the Seiberg–Witten limit will allow us to complete the field strength into the one of non-commutative Yang–Mills by an educated guess, and to investigate compatibility with the kernels computed above. On the other hand, the lack of explicit commutative treatment of these terms will restrict the range of our discussion of scalar couplings to deformations of the Seiberg–Witten map for transverse scalars.

3. Effect on the non-commutative action

3.1. How to modify the smearing prescription

It is possible to adapt the above derivation to the non-commutative set-up, by inserting a factor of $2\sin(\pi\tau_{ab})^{2t_{ab}}$ at each of the points at which the operators are inserted along the Wilson line, since the full operator entering the coupling of degree $2p$ reads

$$\tilde{*}_p = \int_0^1 d\tau_1 \cdots \int_0^1 d\tau_p \prod_{1\leqslant i<j\leqslant p} \exp\bigl\{i\pi a_{ij}\bigl(2\tau_{ij} - \epsilon(\tau_{ij})\bigr)\bigr\}\bigl(2\sin(\pi\tau_{ij})\bigr)^{2t_{ij}}.$$

In order to see whether the couplings can be rewritten in terms of a smearing prescription, ordered with respect to a deformed star-product along a Wilson line, we are urged to find a recursive definition of the deformed star-products. It should be compatible the one derived by Liu [4] between the modified star-products in the Seiberg–Witten limit:

$$i\theta^{ij}\partial_i\langle f_1,\dots,f_p,\partial_j f_{p+1}\rangle_{*_{p+1}} = \sum_{i=1}^{p}\langle f_1,\dots,[f_i,f_{p+1}],\dots,f_p\rangle_{*_p},$$

where the commutator is understood with respect to the star-product. As noted in [18], the commutator can still be expressed in terms of $*_2$ after deformation:

$$i\theta^{ij}\langle\partial_i f,\partial_j g\rangle_{*_2(t)} = [f,g]_{*(t)},$$

where $*(t)$ is the deformed version of the star-product defined by the prescription of integration along an open Wilson line for two observables. Therefore, we are inclined to look for a deformed version of the recursive formula using derivatives for some of the arguments. Let us consider the following expression:

$$i\theta^{ij}\partial_i\langle f_1,\dots,f_p,\partial_j f_{p+1}\rangle_{\tilde{*}_{p+1}},$$

and show how the recursion is organized for one of the terms in the above derivative. It is of the general form of multiple convolutions (denoted by $\circ$) between operators O_i smeared along a line, where the kernel K is translation-invariant.

$$\begin{aligned}
&i\theta^{ij}\partial_i O_1\circ\partial_j O_2\circ\cdots\circ O_{p+1}\\
&\quad= i\theta^{ij}\int_0^1 d\tau_1\cdots\int_0^1 d\tau_p\,\partial_i O_1(0)K(\tau_1)\partial_j O_2(\tau_1)K(\tau_2)O_3(\tau_1+\tau_2)K(\tau_3)O_4(\tau_1+\tau_2+\tau_3)\cdots\\
&\qquad\times K(\tau_{p-1})O_p(\tau_1+\cdots+\tau_{p-1})K(\tau_p)O_{p+1}(\tau_1+\cdots+\tau_p)\delta(\tau_1+\cdots+\tau_p-1)\\
&\quad=\left(i\theta^{ij}\int d\tau_1\,\partial_i O_1(0)K(\tau_1)\partial_j O_2(\tau_1)\right)\\
&\qquad\times\int_0^1 d\tau_2\cdots\int_0^1 d\tau_p\,K(\tau_2)O_3(1-\tau_3-\cdots-\tau_p)K(\tau_3)\cdots O_p(1-\tau_p)K(\tau_p)O_{p+1}(1)\\
&\quad=\int_0^1 d\tau_2\cdots\int_0^1 d\tau_p\,[O_1,O_2]_{*(t_{12})}(0)K(\tau_2)O_3(\tau_2)K(\tau_3)O_4(\tau_2+\tau_3)\cdots\\
&\qquad\times O_p(\tau_2+\cdots+\tau_{p-1})K(\tau_p)O_{p+1}(\tau_2+\cdots+\tau_p)\delta(\tau_2+\cdots+\tau_p-1)\\
&\quad=[O_1,O_2]_{*(t_{12})}\circ O_3\circ\cdots\circ O_p\circ O_{p+1}.
\end{aligned}$$

This allows us to open up one more integration interval over an intermediate time and to write the recursive definition of the deformed smearing prescription

$$i\theta^{ij}\partial_i\langle f_1,\dots,f_p,\partial_j f_{p+1}\rangle_{\tilde{*}_{p+1}} = \sum_{i=1}^{p}\langle f_1,\dots,[f_i,f_{p+1}]_{*(t_{i,p+1})},\dots,f_p\rangle_{\tilde{*}_p}.$$

Let us write the third rank differential operator explicitly:

$$\tilde{*}_3 = \frac{a_{32}}{a_{31}+a_{32}} \frac{\Gamma(1+2t_{32})}{\Gamma(1-a_{32}+t_{32})\Gamma(1+a_{32}+t_{32})} \frac{\Gamma(1+2t_{12}+2t_{13})}{\Gamma(1-a_{12}-a_{13}+t_{12}+t_{13})\Gamma(1+a_{12}+a_{13}+t_{12}+t_{13})}$$
$$+ \frac{a_{31}}{a_{31}+a_{32}} \frac{\Gamma(1+2t_{31})}{\Gamma(1-a_{31}+t_{31})\Gamma(1+a_{31}+t_{31})} \frac{\Gamma(1+2t_{32}+2t_{12})}{\Gamma(1-a_{32}-a_{12}+t_{32}+t_{12})\Gamma(1+a_{32}+a_{12}+t_{32}+t_{12})},$$

whose Seiberg–Witten limit is recognized as $*_3$:

$$*_3 = \frac{\sin(\pi a_{32})\sin(\pi(a_{12}+a_{13}))}{\pi(a_{31}+a_{32})\pi(a_{12}+a_{13})} + \frac{\sin(\pi a_{31})\sin(\pi(a_{32}+a_{12}))}{\pi(a_{31}+a_{32})\pi(a_{32}+a_{12})} = \lim_{\alpha'\to 0} \tilde{*}_3.$$

3.2. Deformed non-commutative gauge transformations

The previous investigation of deformed star-products at larger degree allows to derive the deformation of non-commutative field strength and gauge transformation required to ensure gauge invariance of the deformed smeared expression. These are as announced in [18]:

$$\hat{F}_{ij} = \partial_i \hat{A}_j - \partial_j \hat{A}_i - \mathrm{i}[\hat{A}_i, \hat{A}_j]_{*(t)}, \qquad \delta\hat{A}_i = \partial_i\hat{\lambda} - \mathrm{i}[\hat{A}_i, \hat{\lambda}]_{*(t)}.$$

The recursive formula was the custodian of gauge invariance in the Seiberg–Witten limit. Extending our prescriptions to larger orders in the gauge potentials by expanding the deformed Wilson line, we see that the deformed smeared prescription plays exactly the same role. The Ramond–Ramond couplings $Q(k)$ are given, for some mode k, by a smeared integral along a straight open Wilson line W_k (of extension $\theta^{\mu\nu}k_\nu$), of operators O_i of the form $(\theta - \theta\hat{F}\theta)^{\mu\nu}$, transforming as

$$O_i \mapsto -\mathrm{i}[O_i, \hat{\lambda}]_{\tilde{*}},$$

so that the gauge invariance of the couplings can be checked on an expansion in terms of the gauge potential, using the recursive definition of the deformed star-products:

$$Q(k) = \sum_{m\geqslant 0} Q_m(k),$$
$$Q_m(k) = \frac{1}{m!}(\theta\partial)^{\mu_1}\cdots(\theta\partial)^{\mu_m}\langle O_1,\ldots,O_p,\hat{A}_{\mu_1},\ldots,\hat{A}_{\mu_m}\rangle_{\tilde{*}_{p+m}}.$$

The gauge variation of one of the terms in the above expansion reads

$$\delta Q_m = -\frac{\mathrm{i}}{m!}(\theta\partial)^{\mu_1}\cdots(\theta\partial)^{\mu_m}\sum_{i=1}^{p}\langle O_1,\ldots,[\hat{\lambda},O_i]_{\tilde{*}},\ldots,O_p,\hat{A}_{\mu_1},\ldots,\hat{A}_{\mu_m}\rangle_{\tilde{*}_{p+m}}$$
$$-\frac{\mathrm{i}}{m!}(\theta\partial)^{\mu_1}\cdots(\theta\partial)^{\mu_m}\sum_{i=1}^{m}\langle O_1,\ldots,O_p,\hat{A}_{\mu_1},\ldots,[\hat{\lambda},\hat{A}_{\mu_i}]_{\tilde{*}},\ldots,\hat{A}_{\mu_m}\rangle_{\tilde{*}_{p+m}}$$
$$+\frac{1}{(m-1)!}(\theta\partial)^{\mu_1}\cdots(\theta\partial)^{\mu_m}\langle O_1,\ldots,O_p,\hat{A}_{\mu_1},\ldots,\hat{A}_{\mu_{m-1}},\partial_{\mu_m}\hat{\lambda}\rangle_{\tilde{*}_{p+m}},$$

so that the gauge variation of the field strengths in Q_m is compensated by the gauge variation of the gauge potentials in Q_{m+1}. The quantity $Q(k)$ is therefore gauge-invariant, and the deformed smearing prescription is consistent in the non-commutative set-up, provided the commutators of non-commutative Yang–Mills theory are also deformed. Furthermore, we may infer deformations of the Seiberg–Witten map for the transverse scalars, by considering a lower-dimensional brane and identifying the coefficients of the transverse momentum of the Wilson line in both descriptions. This results in the substitution of deformed star-products to the usual ones in the corresponding result of Mukhi and Suryanarayana [6].

The family of differential operators (and the deformed gauge theory) we have just worked out induce deformations of the expressions of the form

$$\int dx\, L_*\left(\sqrt{\det(1-\theta\hat{F})}\left(\hat{F}\frac{1}{1-\theta\hat{F}}\right)^p W_k(x)\right) * e^{\mathrm{i}kx},$$

by replacing star-products of various ranks (and field strengths) with their deformations. A few more terms in the commutative set-up can back this proposal. During the computation on a commutative space, we ignored terms that involved other tensor structures than derivatives of $F \wedge F$. But an important class of such terms are predicted by the modified smearing prescription, since the Wilson line not only gives rise to an ordering of the observables, but can be expanded, generating forms of degree four and of cubic order in the gauge field, even once expressed back in the commutative language. The Seiberg–Witten limit of these forms has been worked out in [8]. Terms from the expansion of the open Wilson lines that are cubic in the field strength arise through the four-form

$$\frac{1}{2}\theta^{\mu\nu}\partial_\nu\langle\hat{A}_\mu, \hat{F}_{\alpha\beta}, \hat{F}_{\gamma\delta}\rangle_{\tilde{*}_3}.$$

The commutative counterpart [19] (at low order in derivatives) of these terms with all the form indices carried by two field strengths is in the four-derivative four-form term

$$\theta^{\mu\nu}\theta^{\rho\kappa}\theta^{\sigma\tau}F_{\rho\mu}\partial_\sigma\partial_\nu F_{\alpha\beta}\partial_\kappa\partial_\tau F_{\gamma\delta},$$

and modifications thereof beyond the Seiberg–Witten limit. The relevant modifications are quadratic in the metric, since the two-forms commute with each other, forcing the two differential operators acting on them to have the same symmetry. The relevant tensor structure is therefore as follows:

$$\theta^{\mu\nu}F_{\mu\rho}G^{\rho\kappa}G^{\sigma\tau}\partial_\nu\partial_\sigma F_{\alpha\beta}\partial_\kappa\partial_\tau F_{\gamma\delta}.$$

Now, to be consistent on the non-commutative side, we must take into account the contribution from the factor $\sqrt{\det(1-\theta\hat{F})}$ to the four-form coupling with the relevant index structure

$$-\frac{1}{4}\theta^{\mu\nu}\langle\hat{F}_{\nu\mu}, \hat{F}_{\alpha\beta}, \hat{F}_{\gamma\delta}\rangle_{\tilde{*}_3},$$

and the cubic part of the Seiberg–Witten image of $\langle\hat{F}_{\alpha\beta}, \hat{F}_{\gamma\delta}\rangle_{\tilde{*}_2}$, which read

$$-\theta^{\mu\nu}\big\langle\big(\langle\hat{A}_\mu, \partial_\nu\hat{F}_{\alpha\beta}\rangle_{\tilde{*}_2} - \langle\hat{F}_{\alpha\mu}, \hat{F}_{\beta\nu}\rangle_{\tilde{*}_2}\big), \hat{F}_{\gamma\delta}\big\rangle_{\tilde{*}_2}.$$

We may note that this expression includes deformations due the ones of *gauge transformations*, while the new terms from the open Wilson line are direct consequences of the deformation of the *star-products*. The two levels of our previous discussion are therefore tied together. Let us trace the modifications of the commutative terms to the deformations of the open Wilson line. They can only come from the terms in the open Wilson line where the overall derivative acts on one of the field strengths. The first contribution of the differential operator t is quadratic, as could be awaited:

$$\tilde{*}_2 - *_2 = \frac{\pi^2}{3}t^2 + o(t^2),$$

$$\tilde{*}_3 - *_3 = \frac{\pi^2}{6}\big(t_{13}^2 + (t_{12}+t_{23})^2\big) + o(t^2).$$

In order to make a field strength out of the gauge field $\hat{A}_i$, one has to use derivatives under the disguise of t^2. This produces terms where each of the two-forms bears a pair of derivatives, and where one of these pairs contains two derivatives contracted with inverse open-string metrics:

$$\theta^{\mu\nu}\partial_\kappa \hat{A}_\mu G^{\tau\sigma}\partial_\tau\partial_\nu \hat{F}_{\alpha\beta} G^{\kappa\rho}\partial_\rho\partial_\sigma \hat{F}_{\gamma\delta}.$$

The gauge-invariant completion comes from the expansion of the external $\tilde{*}_2$ in the Seiberg–Witten map. The index structure of the commutative candidate is recognized after rearranging, and the removal of hats is consistent with our cubic prescription.

4. Conclusion

In order to obtain results beyond the Seiberg–Witten limit, the full open-string propagator has been taken into account in the computation of the Ramond–Ramond couplings for small $U(1)$ field strength at all orders in derivatives. The resulting differential operators acting on powers of the field strength are deformations of the modified star-products previously derived in the Seiberg–Witten limit. The expression of $\tilde{*}_2$ is consistent with known string amplitudes. Moreover, the recursive definition of the modified star-products enables us to express any of them as rational functions of differential operators containing open-string parameters, and Gamma functions thereof. The results can be reformulated in the non-commutative set-up in terms of a deformed smearing prescription along an open Wilson line. This confirms earlier proposals at all orders in the open-string metric at disk level, and extends them to couplings of higher degrees. Furthermore, the deformation of the commutators induced in non-commutative Yang–Mills theory by $\tilde{*}_2$ has been shown to lead to gauge-invariant couplings.

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