DISCRETE MATHEMATICS

Note

Some formulae for partitions into squares

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Received 17 February 1999; accepted 1 March 1999

Abstract

We consider the new problem of determining the number of partitions of a number into a fixed number k of squares, and find explicit formulae in the cases $k = 2,\ 3,\ 4$. © 2000 Elsevier Science B.V. All rights reserved.

AMS classification: 11P81; 05A17

Keywords: Partitions; Squares

1. Introduction and statement of results

We consider the problem of determining the number of partitions of a number into a fixed number of squares. This problem is distinct from, but not unrelated to, the corresponding classical problem of determining the number of representations of a number as the sum of a fixed number of squares. Whereas the classical problem has received an enormous amount of attention over the years, see Dickson [3, Vol. II, Chapters VI–IX], it appears that this (restricted) partition problem has not previously been investigated. Hardy and Ramanujan [4, p. 305] and Baxter [1] have looked at the problem of determining the number of partitions of a number into (an unrestricted number of) squares, Hardy and Ramanujan in the context of number theory, where they give a transformation for the generating function and the dominant term in the asymptotic expansion, and Baxter in the context of statistical mechanics, where he gives the same transformation formula and uses it to prove three identities found earlier by D. Kim.

To see why one might want to examine the partition problem, consider the following.

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PII: S0012-365X(99)00159-4

Jacobi's four square theorem states that the number of representations of $n > 1$ as the sum of four squares is given by

$$r_4(n) = 8 \sum_{d \mid n, 4 \nmid d} d.$$

This is in fact the number of representations of $n > 1$ as the sum of four squares of integers positive, negative or zero, with order relevant. Thus, for example, $r_4(30) = 8(1 + 2 + 3 + 5 + 6 + 10 + 15 + 30) = 576$. The expression

$$30 = 4^2 + 3^2 + 2^2 + 1^2$$

accounts for $4! \times 2^4 = 384$ of the representations (permute the four numbers in 4! ways, place a $\pm$ sign in front of each of the four numbers independently in 2^4 ways), while the expression

$$30 = 5^2 + 2^2 + 1^2 + 0^2$$

accounts for $4! \times 2^3 = 192$ of the representations. But 30 has only 2 partitions into four squares of non-negative integers, the two representations given above.

Thus, we see that counting the number of partitions avoids an enormous amount of 'undesirable' duplication.

In this note we shall find expressions for $p_{2\square}(n)$, $p_{3\square}(n)$ and $p_{4\square}(n)$, the numbers of partitions of n into two, three and four squares of non-negative integers, in terms of divisors of n and of numbers less than n.

In order to state our results, we need to define a few terms.

Let $\delta(n)$ be given for $n \geqslant 1$ by

$$\delta(n) = \begin{cases} 1 & \text{if } n \text{ is a square,} \\ 0 & \text{otherwise,} \end{cases}$$

and let

$$\phi(q) = \sum_{-\infty}^{\infty} q^{n^2} = \frac{(-q;q^2)_\infty (q^2;q^2)_\infty}{(q;q^2)_\infty (-q^2;q^2)_\infty} = 1 + 2 \sum_{n \geqslant 1} \delta(n) q^n.$$

Let $d_{r,m}(n)$ denote the number of divisors d of n with $d \equiv r \pmod{m}$, and let

$$e_{r,s,m}(n) = d_{r,m}(n) - d_{s,m}(n),$$

be the excess of divisors d of n with $d \equiv r \pmod{m}$ over those with $d \equiv s \pmod{m}$.

We shall show that

$$\sum_{n \geqslant 0} p_{2\square}(n) q^n = \sum_{i \geqslant j \geqslant 0} q^{i^2+j^2} = \frac{1}{8}(\phi(q)^2 + 2\phi(q) + 2\phi(q^2) + 3), \tag{1}$$

$$\begin{aligned} \sum_{n \geqslant 0} p_{3\square}(n) q^n &= \sum_{i \geqslant j \geqslant k \geqslant 0} q^{i^2+j^2+k^2} \\ &= \frac{1}{48}(\phi(q)^3 + 3\phi(q)^2 + 6\phi(q)\phi(q^2) + 9\phi(q) \\ &\quad + 6\phi(q^2) + 8\phi(q^3) + 15) \end{aligned} \tag{2}$$

and

$$\sum_{n\geqslant 0} p_{4\square}(n)q^n = \sum_{i\geqslant j\geqslant k\geqslant l\geqslant 0} q^{i^2+j^2+k^2+l^2}$$
$$= \frac{1}{384}(\phi(q)^4 + 4\phi(q)^3 + 12\phi(q)^2\phi(q^2) + 18\phi(q)^2 + 24\phi(q)\phi(q^2) + 12\phi(q^2)^2 + 32\phi(q)\phi(q^3) + 60\phi(q) + 36\phi(q^2) + 32\phi(q^3) + 48\phi(q^4) + 105). \quad (3)$$

If we now use the facts [2, (3.2.23), (9.1.9)]; [5, Theorems 312, 385]; [6] that

$$\phi(q) = 1 + 2\sum_{n\geqslant 1}\delta(n)q^n,$$

$$\phi(q)^2 = 1 + 4\sum_{n\geqslant 1} e_{1,3,4}(n)q^n \quad \text{(Jacobi, 1828)},$$

$$\phi(q)\phi(q^2) = 1 + 2\sum_{n\geqslant 1}(e_{1,7,8}(n) + e_{3,5,8}(n))q^n \quad \text{(Dirichlet, 1840)},$$

$$\phi(q)\phi(q^3) = 1 + \sum_{n\geqslant 1}(2e_{1,2,3}(n) + 4e_{1,2,3}(n/4))q^n \quad \text{(Lorenz, 1871)},$$

$$\phi(q)^3 = \left(1 + 4\sum_{n\geqslant 1} e_{1,3,4}(n)q^n\right)\left(1 + 2\sum_{n\geqslant 1}\delta(n)q^n\right)$$
$$= 1 + \sum_{n\geqslant 1}(4e_{1,3,4}(n) + 8\sum_{1\leqslant k^2<n} e_{1,3,4}(n-k^2) + 2\delta(n))q^n,$$

$$\phi(q)^2\phi(q^2) = \left(1 + 4\sum_{n\geqslant 1} e_{1,3,4}(n)q^n\right)\left(1 + 2\sum_{n\geqslant 1}\delta(n)q^{2n}\right)$$
$$= 1 + \sum_{n\geqslant 1}(4e_{1,3,4}(n) + 8\sum_{1\leqslant 2k^2<n} e_{1,3,4}(n-2k^2) + 2\delta(n/2))q^n$$

and

$$\phi(q)^4 = 1 + 8\sum_{n\geqslant 1}(\sigma(n) - 4\sigma(n/4))q^n \quad \text{(Jacobi, 1829)}$$

we find that for $n\geqslant 1$

$$p_{2\square}(n) = \tfrac{1}{2}(e_{1,3,4}(n) + \delta(n) + \delta(n/2)), \quad (4)$$

$$p_{3\square}(n) = \frac{1}{12}(4e_{1,3,4}(n) + 2\sum_{1\leqslant k^2<n} e_{1,3,4}(n-k^2) + 3e_{1,7,8}(n) + 3e_{3,5,8}(n) + 5\delta(n) + 3\delta(n/2) + 4\delta(n/3)) \quad (5)$$

and

$$p_{4\Box}(n) = \frac{1}{48}(\sigma(n) - 4\sigma(n/4) + 17e_{1,3,4}(n) + 4\sum_{1\leqslant k^2<n} e_{1,3,4}(n-k^2)$$
$$+12\sum_{1\leqslant 2k^2<n} e_{1,3,4}(n-2k^2) + 6e_{1,7,8}(n) + 6e_{3,5,8}(n) + 6e_{1,3,4}(n/2)$$
$$+8e_{1,2,3}(n) + 16e_{1,2,3}(n/4) + 16\delta(n) + 12\delta(n/2) + 8\delta(n/3) + 12\delta(n/4)). \quad (6)$$

2. Proofs

Let x_i, $i = 0, \ldots, N$ be arbitrary.

It is easy to verify that

$$2\sum_{i\geqslant j} x_i x_j = \left(\sum_{i=0}^{N} x_i\right)^2 + \sum_{i=0}^{N} x_i^2, \quad (7)$$

$$6\sum_{i\geqslant j\geqslant k} x_i x_j x_k = \left(\sum_{i=0}^{N} x_i\right)^3 + 3\sum_{i=0}^{N} x_i \sum_{i=0}^{N} x_i^2 + 2\sum_{i=0}^{N} x_i^3 \quad (8)$$

and

$$24\sum_{i\geqslant j\geqslant k\geqslant l} x_i x_j x_k x_l$$
$$= \left(\sum_{i=0}^{N} x_i\right)^4 + 6\left(\sum_{i=0}^{N} x_i\right)^2 \sum_{i=0}^{N} x_i^2 + 3\left(\sum_{i=0}^{N} x_i^2\right)^2$$
$$+8\sum_{i=0}^{N} x_i \sum_{i=0}^{N} x_i^3 + 6\sum_{i=0}^{N} x_i^4. \quad (9)$$

If in (7)–(9) we set $x_i = q^{i^2}$, let $N \to \infty$ and use the fact that

$$\sum_{i\geqslant 0} q^{i^2} = \frac{1}{2}(\phi(q) + 1),$$

we obtain (1)–(3).

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