The Hyers–Ulam stability constants of first order linear differential operators

Sin-Ei Takahasi\textsuperscript{a}, Hiroyuki Takagi\textsuperscript{b}, Takeshi Miura\textsuperscript{a,∗}, Shizuo Miyajima\textsuperscript{c}

\textsuperscript{a} Department of Basic Technology, Applied Mathematics and Physics, Yamagata University, Yonezawa 992-8510, Japan
\textsuperscript{b} Department of Mathematical Sciences, Faculty of Science, Shinshu University, Matsumoto 390-8621, Japan
\textsuperscript{c} Department of Mathematics, Faculty of Science, Science University of Tokyo, Shinjuku-ku Wakamiya 26, Tokyo 162-8601, Japan

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Abstract

Let $X$ be a complex Banach space, $h$ a complex-valued continuous function on the real line $\mathbb{R}$ and $T_h : C^1(\mathbb{R}, X) \to C(\mathbb{R}, X)$ the linear differential operator defined by $T_h u = u' + hu$. We completely determine the Hyers–Ulam stability constant of $T_h$.

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1. Introduction and result

In 1940, S.M. Ulam posed the well-known stability problem, and in the next year, D.H. Hyers gave an answer to this problem for linear mappings between two Banach spaces (cf. [2,8,9]). The stability problems of this type have been studied by many mathematicians. We are concerned with the Hyers–Ulam stability constants of linear differential operators.
Let $X$ be a complex Banach space. We denote by $C(\mathbb{R}, X)$ the linear space of all $X$-valued continuous functions on the real line $\mathbb{R}$. Also, we denote by $C^1(\mathbb{R}, X)$ the linear space of all $u \in C(\mathbb{R}, X)$ which are strongly differentiable and whose derivatives $u'$ are continuous on $\mathbb{R}$. For any $u \in C(\mathbb{R}, X)$, we define

$$
\|u\|_\infty = \sup_{t \in \mathbb{R}} \|u(t)\|,
$$

admitting the value $\infty$, where $\|\cdot\|$ denotes the norm of $X$.

Let $h$ be a complex-valued continuous function on $\mathbb{R}$. We define the linear differential operator $T_h : C^1(\mathbb{R}, X) \to C(\mathbb{R}, X)$ by

$$(Tu)(t) = u'(t) + h(t)u(t)$$

for all $t \in \mathbb{R}$ and $u \in C^1(\mathbb{R}, X)$.

**Definition.** The operator $T_h$ is said to have the Hyers–Ulam stability, if there exists a constant $K \geq 0$ with the following property:

For any $\varepsilon \geq 0$, $u \in C^1(\mathbb{R}, X)$ and $v \in C(\mathbb{R}, X)$ satisfying $\|Tu - v\|_\infty \leq \varepsilon$, there exists $u_0 \in C^1(\mathbb{R}, X)$ such that $Tu_0 = v$ and $\|u_0 - u\|_\infty \leq K\varepsilon$.

We call such constant $K$ a HUS constant for $T_h$. We write $K_{T_h}$ for the infimum of all HUS constants for $T_h$ (if $T_h$ does not have the Hyers–Ulam stability, we understand $K_{T_h} = \infty$). If $K_{T_h}$ is finite and is a HUS constant, then we call $K_{T_h}$ the HUS constant for $T_h$ (cf. [5,6]).

The case that $X = \mathbb{R}$ and $h(t) = -1$ ($t \in \mathbb{R}$) was considered by C. Alsina and R. Ger [1]. More results of the Hyers–Ulam stability problem for linear differential operators can be found in [3–7].

The purpose of this paper is to determine the HUS constants for $T_h$. To do this, we introduce three constants: For any complex-valued continuous function $h$ on $\mathbb{R}$, we define $\tilde{h}(t) = \exp\left\{\int_0^t h(s) \, ds\right\}$ for all $t \in \mathbb{R}$, and set

$$
Ch = \sup_{t \in \mathbb{R}} \frac{1}{|\tilde{h}(t)|} \int_t^\infty |\tilde{h}(s)| \, ds,
$$

$$
Dh = \sup_{t \in \mathbb{R}} \frac{1}{|\tilde{h}(t)|} \int_{-\infty}^t |\tilde{h}(s)| \, ds,
$$

and

$$
Eh = \sup_{t \in \mathbb{R}} \frac{1}{|\tilde{h}(t)|} \left| \int_0^t |\tilde{h}(s)| \, ds \right|.
$$

In [6, Remark 2.1], it is shown that only one of $Ch$, $Dh$ and $Eh$ can be finite. In other words, the possible cases are precisely the following four:

(a) $Ch < \infty$ and $Dh = Eh = \infty$;
(b) $Dh < \infty$ and $Ch = Eh = \infty$;
(c) $Eh < \infty$ and $Ch = Dh = \infty$;
(d) $Ch = Dh = Eh = \infty$. 

For example, if \( h \) is a polynomial \( a_0 t^n + a_1 t^{n-1} + \cdots + a_{n-1} t + a_n \) with real coefficient, then each case occurs as follows: If \( n \) is even and \( a_0 < 0 \), then (a) holds; if \( n \) is even and \( a_0 > 0 \), then (b) holds; if \( n \) is odd and \( a_0 > 0 \), then (c) holds; if \( n \) is odd and \( a_0 < 0 \), then (d) holds (see [6, Corollary 2.4 and Example 2.1]).

Now, we state the main theorem of this paper.

**Theorem.** Let \( h \) be a complex-valued continuous function on \( \mathbb{R} \). Then \( Th \) has the Hyers–Ulam stability if and only if one of \( C_h \), \( D_h \) and \( E_h \) is finite. Moreover, the HUS constant for \( Th \) is determined as follows:

(i) If \( C_h \) is finite, then \( C_h \) is the HUS constant for \( Th \).
(ii) If \( D_h \) is finite, then \( D_h \) is the HUS constant for \( Th \).
(iii) If \( E_h \) is finite, then \( E_h \) is the HUS constant for \( Th \).

This theorem says that \( Th \) does not have the Hyers–Ulam stability if and only if \( C_h = D_h = E_h = \infty \). Also, it answers the question in [6, Remark 2.5].

2. **Proof of Theorem**

All but (iii) have been already proved in [6, Theorem 2.2, Corollary 2.3 and Remark 2.4]. We here show (iii). For its proof, we need the following two lemmas.

**Lemma 1.** Let \( C \) be a symmetric set, that is \( C = -C \), in a Banach space \( B \). For each \( y \in B \), we have

\[
\sup_{x \in C} \| y + x \| \geq \sup_{x \in C} \| x \|.
\]

**Lemma 2.** For a complex-valued continuous function \( h \) on \( \mathbb{R} \), we have

\[
K_{Th} = \inf_{x \in X} \sup_{w \in C(\mathbb{R}, X), \| w \|_{\infty} \leq 1} \left\| \frac{1}{h(t)} \left( x + \int_0^t h(s) w(s) \, ds \right) \right\|.
\]

**Proof of Lemma 1.** Put \( R = \sup_{x \in C} \| x \| \leq \infty \), and pick \( y \in B \) arbitrarily. If \( R = \infty \), then \( \sup_{x \in C} \| y + x \| = \infty \) is clearly true. Thus we consider the case that \( R < \infty \). Pick \( \varepsilon > 0 \) arbitrarily. There is \( x_0 \in C \) such that \( \| x_0 \| > R - \varepsilon \). Then we get

\[
2 \max \left\{ \| y + x_0 \|, \| y - x_0 \| \right\} \geq \| y + x_0 \| + \| y - x_0 \| \geq 2\| x_0 \| > 2(R - \varepsilon).
\]

Since \( C \) is symmetric, \( -x_0 \) is in \( C \). We thus obtain

\[
\sup_{x \in C} \| y + x \| \geq \max \left\{ \| y + x_0 \|, \| y - x_0 \| \right\} \geq R - \varepsilon.
\]

Since \( \varepsilon > 0 \) was arbitrary, it follows that \( \sup_{x \in C} \| y + x \| \geq R \).
The key of the proof of Lemma 2 is the following fact: For any \( v \in C(\mathbb{R}, X) \), the general solution of the equation \( T_h u = v \) is of the form

\[
    u(t) = \frac{1}{h(t)} \left( x_0 + \int_0^t \hat{h}(s)v(s)\,ds \right) \quad (t \in \mathbb{R}),
\]

where \( x_0 \) is an arbitrary element of \( X \) (cf. [6, p. 137]). This fact implies that \( T_h : C^1(\mathbb{R}, X) \rightarrow C(\mathbb{R}, X) \) is surjective.

**Proof of Lemma 2.** For each \( x \in X \), we define

\[
    K_0(x) = \sup_{w \in C(\mathbb{R}, X)} \sup_{\|w\|_{\infty} \leq 1} \left\| \frac{1}{h(t)} \left( x + \int_0^t \hat{h}(s)w(s)\,ds \right) \right\|,
\]

admitting the value \( \infty \). We must show that \( K = \inf_{x \in X} K_0(x) \).

We first show that \( K \geq \inf_{x \in X} K_0(x) \). If \( K = \infty \), then there is nothing to prove, and so we assume that \( K < \infty \). Let \( K \) be an arbitrary HUS constant for \( T_h \). Then, for any \( w \in C(\mathbb{R}, X) \) with \( \|w\|_{\infty} \leq 1 \), there exists \( u_0 \in C^1(\mathbb{R}, X) \) such that \( T_h u_0 = w \) and \( \|u_0\|_{\infty} \leq K \). By the above key fact, \( u_0 \) has the form

\[
    u_0(t) = \frac{1}{h(t)} \left( x_0 + \int_0^t \hat{h}(s)w(s)\,ds \right) \quad (t \in \mathbb{R})
\]

for some \( x_0 \in X \), and hence

\[
    K \geq \|u_0\|_{\infty} = \sup_{t \in \mathbb{R}} \left\| \frac{1}{h(t)} \left( x + \int_0^t \hat{h}(s)w(s)\,ds \right) \right\|.
\]

This holds for any \( w \in C(\mathbb{R}, X) \) with \( \|w\|_{\infty} \leq 1 \) and we get

\[
    K \geq K_0(x_0) \geq \inf_{x \in X} K_0(x).
\]

Since \( K \) was an arbitrary HUS constant for \( T_h \), it follows that \( K = \inf_{x \in X} K_0(x) \).

We next show the inequality \( K \leq \inf_{x \in X} K_0(x) \). We may assume that \( \inf_{x \in X} K_0(x) < \infty \). Take an arbitrary element \( x \) of \( X \) so that \( K_0(x) < \infty \). Let us show that \( K_0(x) \) is a HUS constant for \( T_h \). For this end, it suffices to show that for any \( \varepsilon > 0 \), \( u \in C^1(\mathbb{R}, X) \) and \( v \in C(\mathbb{R}, X) \) with \( \|T_h u - v\|_{\infty} \leq \varepsilon \), there exists \( u_0 \in C^1(\mathbb{R}, X) \) such that \( T_h u_0 = v \) and \( \|u_0 - u\|_{\infty} \leq K_0(x) \varepsilon \). Let \( u \) and \( v \) be such functions and put \( \varepsilon w = T_h u - v \). Then \( \|w\|_{\infty} \leq 1 \) and \( T_h u = v + \varepsilon w \). Hence our key fact gives \( x_1 \in X \) such that

\[
    u(t) = \frac{1}{h(t)} \left( x_1 + \int_0^t \hat{h}(s)v(s)\,ds + \varepsilon \int_0^t \hat{h}(s)w(s)\,ds \right) \quad (t \in \mathbb{R}).
\]
Now define a function $u_0$ on $\mathbb{R}$ by

$$u_0(t) = \frac{1}{h(t)} \left( x_1 - \varepsilon x + \int_0^t \tilde{h}(s) v(s) \, ds \right) \quad (t \in \mathbb{R}).$$

Then $u_0 \in C^1(\mathbb{R}, X)$ and $T_h u_0 = v$. Moreover, we have

$$\|u_0 - u\|_\infty = \sup_{t \in \mathbb{R}} \left| \frac{1}{h(t)} \left( x_1 - \varepsilon x + \int_0^t \tilde{h}(s) w(s) \, ds \right) \right| \leq K_0(x) \varepsilon,$$

where the last inequality deduces from $\|w\|_\infty \leq 1$ and the definition of $K_0(x)$. Thus $K_0(x)$ is a HUS constant for $T_h$. Hence we have $K_{T_h} \leq K_0(x)$. Since $x$ was arbitrary, we conclude that $K_{T_h} \leq \inf_{x \in X} K_0(x)$. Thus the lemma is proved.

We are now in a position to prove the theorem. In the proof, we deal with the space

$$C^b(\mathbb{R}, X) = \{ f \in C(\mathbb{R}, X) : \|f\|_\infty < \infty \},$$

which is a Banach space with norm $\| \cdot \|_\infty$.

**Proof of Theorem (iii).** Suppose that $E_h$ is finite. Then we see from [6, Theorem 2.1] that $T_h$ has the Hyers–Ulam stability and that $E_h$ is a HUS constant for it. Hence $K_{T_h} \leq E_h$.

Once we show $K_{T_h} \geq E_h$, we get $K_{T_h} = E_h$ and $E_h$ becomes the HUS constant. Thus it suffices to show that $K_{T_h} \geq E_h$.

Define a linear operator $S : C^b(\mathbb{R}, X) \to C(\mathbb{R}, X)$ by

$$(Su)(t) = \frac{1}{h(t)} \int_0^t \tilde{h}(s) u(s) \, ds$$

for all $t \in \mathbb{R}$ and $u \in C^b(\mathbb{R}, X)$. Then we have

$$\|Su\|_\infty = \sup_{t \in \mathbb{R}} \left| \frac{1}{h(t)} \int_0^t \tilde{h}(s) u(s) \, ds \right| \leq \sup_{t \in \mathbb{R}} \frac{1}{|h(t)|} \|u\|_\infty \left| \int_0^t |\tilde{h}(s)| \, ds \right| = E_h \|u\|_\infty < \infty$$

for all $u \in C^b(\mathbb{R}, X)$. Hence $S$ is a bounded linear operator of $C^b(\mathbb{R}, X)$ into itself and $\|S\| \leq E_h$. Moreover, if $x_0$ is a unit element of $X$ and $u_0(t) = (|h(t)|/\tilde{h}(t))x_0$ for $t \in \mathbb{R}$, then $u_0 \in C^b(\mathbb{R}, X)$, $\|u_0\|_\infty = 1$ and $\|Su_0\|_\infty = E_h$. Thus we obtain

$$\|S\| = E_h.$$  \(1\)
We next observe that \(1/\tilde{h}\) is a bounded function on \(\mathbb{R}\). Since \(\tilde{h}\) is continuous on \(\mathbb{R}\) and \(\tilde{h}(0) = 1\), there is \(\delta > 0\) such that \(|\tilde{h}(t)| \geq 1/2\) for \(|t| \leq \delta\). Hence if \(|t| \leq \delta\), then \(1/|\tilde{h}(t)| \leq 2\). While, if \(|t| > \delta\), then

\[
E_h \geq \frac{1}{|\tilde{h}(t)|} \left| \int_0^t |\tilde{h}(s)| \, ds \right| \geq \frac{1}{|\tilde{h}(t)|} \left| \int_0^\delta |\tilde{h}(s)| \, ds \right| \geq \frac{1}{|\tilde{h}(t)|} \frac{\delta}{2},
\]

or \(1/|\tilde{h}(t)| \leq 2E_h/\delta\). Thus \(1/\tilde{h}\) is bounded.

Now, pick \(x \in X\) arbitrarily. Then the observation above implies that \((1/\tilde{h})x \in \text{Cb}(\mathbb{R}, X)\). Noting that the range \(S(\{w \in \text{Cb}(\mathbb{R}, X): \|w\|_\infty \leq 1\})\) is a symmetric set of \(\text{Cb}(\mathbb{R}, X)\), we apply Lemma 1 to obtain

\[
\sup_{w \in \text{Cb}(\mathbb{R}, X)} \sup_{\|w\|_\infty \leq 1} \left\| \frac{1}{\tilde{h}(t)} \left( x + \int_0^t \tilde{h}(s)w(s) \, ds \right) \right\| = \sup_{w \in \text{Cb}(\mathbb{R}, X)} \sup_{\|w\|_\infty \leq 1} \left\| \frac{1}{\tilde{h}(t)} x + (Sw)(t) \right\|
\]

\[
= \sup_{\|w\|_\infty \leq 1} \left\| \frac{1}{\tilde{h}(t)} x + Sw \right\| \geq \sup_{\|w\|_\infty \leq 1} \|Sw\|_\infty = \|S\|.
\]

Since this holds for all \(x \in X\), we have

\[
\inf_{x \in X} \sup_{w \in \text{Cb}(\mathbb{R}, X)} \sup_{\|w\|_\infty \leq 1} \left\| \frac{1}{\tilde{h}(t)} \left( x + \int_0^t \tilde{h}(s)w(s) \, ds \right) \right\| \geq \|S\|.
\]

By Lemma 2 and (1), we obtain \(K_{Th} \geq E_h\), which is to be proved.

Finally we pose some questions. Let \(C(\mathbb{R}, \mathbb{C})\) be the linear space of all complex-valued functions on \(\mathbb{R}\), and write \(C_{\text{HUS}}(\mathbb{R}, \mathbb{C})\) for those \(h \in C(\mathbb{R}, \mathbb{C})\) such that \(T_h\) has the Hyers–Ulam stability. Then it is natural to ask what properties the set \(C_{\text{HUS}}(\mathbb{R}, \mathbb{C})\) possesses as a subset of \(C(\mathbb{R}, \mathbb{C})\). For example, is it closed under some algebraic operations or in some topology? It would be also interesting to investigate the properties of the mapping \(h \mapsto K_{Th}\) on \(C_{\text{HUS}}(\mathbb{R}, \mathbb{C})\).

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