On the Perturbation of Unbounded Linear Operators with Topologically Complemented Ranges

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Let $X$ and $Y$ be normed spaces and let $T: D(T) \subseteq X \to Y$ be a linear transformation having a finite codimensional restriction with a continuous inverse (equivalently, let $T'$ be a $\phi_-$-operator; for example, if $T$ is bounded below). Suppose that $\mathcal{R}(T)$ (or $\mathcal{R}(T)$) is topologically complemented in $Y$. Conditions are obtained under which $\mathcal{R}(T + S)$ (resp. $\mathcal{R}(T + S)$) is topologically complemented whenever $S$ belongs to the class of precompact operators, or to some wider class.

1. INTRODUCTION

Let $T: X \to Y$ be a linear transformation with domain $D(T)$, where $X$ and $Y$ are normed spaces. We denote the class of such operators $T$ by $L(X, Y)$. It is known that if $X$ and $Y$ are Banach spaces and $T$ a $\phi_-$-operator then for any bounded strictly singular operator $S$, the range of $T + S$ is complemented whenever the range of $T$ is complemented ([12]; see also [2, Theorem 4.8(a)]). In the present note we investigate similar stability properties for the wider class of $F_+$-operators: The operator $T$ is called an $F_+$-operator [4, 5]) if it has a finite codimensional restriction having a continuous inverse. The complementation problem for the range $\mathcal{R}(T)$ of $T$ (and more generally for $\overline{\mathcal{R}(T)}$) is related in an obvious way to the existence of algebraic generalised inverses of $T$ satisfying certain continuity requirements. Generalised inverses have applications in approximation theory, optimization theory, systems theory, operator algebras, and other branches of mathematics. The recent paper of Nashed [13] provides a useful list of references. Inner inverses are also treated in the recent book of Harte [9] in the case when $T$ is bounded; then $T$ will have a bounded inner inverse if and only if $T$ is proper, and both $\overline{\mathcal{R}(T)}$ and the null space $\mathcal{N}(T)$ of $T$ are topologically complemented [9, Theorem 3.8.2]. Stability of
complementation under perturbation by finite rank operators (bounded or unbounded) was investigated in [1].

2. PRELIMINARIES

The symbols $X, Y, Z, \ldots$ will denote normed linear spaces and $T$ will always denote an element of $L(X, Y)$. We denote the domain, range and null space of $T$ by $D(T), R(T),$ and $N(T)$, respectively. We call $T$ bounded if $T$ is continuous and $D(T) = X$. If $X$ is a linear subspace of $Y$ then $J_X^Y$ denotes the operator in $L(X, Y)$ that is the natural injection of $X$ into $Y$. The adjoint $T'$, of $T$ is the conjugate of $TJ_{D(T)}^X$ in the sense of [8, II.2.2]. If $X$ and $Y$ are Banach spaces and $T$ is a closed operator then $T$ is called a $\phi_+ (\phi_-)$-operator if $\dim N(T) < \infty$ and $R(T)$ is closed (resp., $\operatorname{codim} R(T) < \infty$). The $F_+$-operators generalise the $\phi_+ $-operators in the following sense: If $X$ and $Y$ are complete and $T$ is closed then $T \in F_+ \iff T' \in \phi_-.$

We have the theorem:

2.1. Theorem [4]. $T \in F_+ \iff T' \in \phi_-.$

The operator $T$ is said to be strictly singular if there is no infinite dimensional subspace $M$ of $D(T)$ for which the restriction $T/M$ has a continuous inverse. This is a generalisation of Kato's definition [11]. A linear subspace $E$ of $X$ is called complemented if there exists a closed subspace $G$ such that $E \cap G = 0$ and $E + G = X$. $E$ is called topologically complemented if there exists a bounded projection on $X$ with range $E$. If $E$ is a subspace of $X$ which is the range of a bounded operator defined on a Banach space and if $X$ is complete then $E$ is complemented if and only if $E$ is topologically complemented, if and only if $E$ is closed and complemented [2, Corollary 2.5].

Let $\Omega$ be a subset of $L(X, Y)$. We denote by $P(\Omega)$ the class of operators in $L(X, Y)$ such that if $T \in \Omega$ and $A \in P(\Omega)$ then $T + A \in \Omega$. $P(\Omega)$ is known as the perturbation class of $\Omega$.

We note the following theorem:

2.2. Theorem [5]. $P(F_+) \text{ coincides with the class of strictly singular operators.}$

3. COMPLEMENTED RANGES

3.1. Lemma. Let $T \in F_+$ and let $R(T)$ be topologically complemented in $Y$. Let $A$ be a continuous operator with $D(A) \supset D(T)$ and $R(A)$ contained in
a subspace topologically complementary to $\overline{R(T)}$. Then $\overline{R(T + A)}$ is topologically complemented in $Y$. Furthermore (i) if $T$ has a continuous inverse then $R(T + A)$ is closed whenever $R(T)$ is closed, and (ii) if $X$ is complete and $T$ is closed then $R(T)$ and $R(T + A)$ are closed.

Proof. Consider first the case when $T$ has a continuous inverse. We assume without loss of generality that $D(T) = X$. Let $P$ be any bounded projection from $Y$ onto $\overline{R(T)}$ with complementary projection $Q$, where $R(Q) \supseteq R(A)$. We shall verify that

$$R(T + A) \subseteq N(Q - AT^{-1}P) \subseteq \overline{R(T + A)}. \quad (*)$$

Let $x \in D(T)$. Then $(Q - AT^{-1}P)(T + A)x = Ax - Ax = 0$, so $R(T + A) \subseteq N(Q - AT^{-1}P)$. Now let $y \in N(Q - AT^{-1}P)$. Then $y = Py + Qy = Py + AT^{-1}Py$. Hence for $y' \in N(T' + A')$ we have $y'y = (Py + AT^{-1}Py)'y'$ (since $y' \in D(A')$ and $T^{-1}P$ is continuous) = $P'y'y - (T^{-1}P)'y'y$. But $(T^{-1}P)'y'y = (T^{-1}P)'y'(Py + Qy) = y'(T^{-1}P)Py + (T'y')(T^{-1}P)Qy = y'Py = P'y'y$. Consequently $y'y = 0$. But $A$ is continuous and hence $(T + A)' = T' + A'$. Therefore $y \in N(T' + A') = \overline{R(T + A)}$ and $(*)$ is established. The bounded projection $P + AT^{-1}P$ has range $N(Q - AT^{-1}P)$. Hence $N(Q - AT^{-1}P) = \overline{R(T + A)}$ is topologically complemented.

We now consider the general case where $T \in F_+$. Again we assume without loss of generality that $D(T) = X$.

There exist closed subspaces (see [5, Theorem 2.2]) $M$ and $W$ with $W$ finite dimensional such that $M \oplus W \oplus N(T) = X$ for which $T/M$ has a continuous inverse. Since $\overline{R(T)} = \overline{R(T/M)} + R(T/W)$, where $R(T/W)$ is finite dimensional, it is clear that $\overline{R(T/M)}$ is topologically complemented and, furthermore, that $R(A)$ is contained in a closed subspace complementary to $\overline{R(T/M)}$. Hence by the first part, $\overline{R(T/M + A)} = \overline{R(T/M + A/M)}$ is topologically complemented. Since $W$ and $N(T)$ are finite dimensional, it follows that $\overline{R(T + A)} = \overline{R(T/M + A/M) + R(T/W + N(T) + A/W + N(T))} = \overline{R(T/M + A/M) + R(T/W) + R(A/W + N(T))}$ is topologically complemented in $Y$.

(i) Suppose $T$ has a continuous inverse and let $R(T)$ be closed. Let $P, Q$ be as above, let $y \in \overline{R(T + A)}$ and let $(x_n)$ be a sequence in $D(T)$ such that $(T + A)x_n \to y$. Then $PTx_n + PAx_n \to Py$, i.e., $Tx_n \to Py$ and $QTx_n + QAx_n \to Qy$, i.e., $Ax_n \to Qy$. Since $R(T)$ is closed, the continuity of $T^{-1}$ now gives $x_n \to T^{-1}Py$ whence $Ax_n \to AT^{-1}Py = Qy$. Hence $y = Py + Qy = Py + AT^{-1}Py = (T + A)(T^{-1}Py)$. Therefore $R(T + A)$ is closed.

(ii) Suppose $T$ is closed and $X$ is complete. Then $T \in \phi_+$, [4], so $R(T)$ is closed. Let $M$ and $W$ be defined as before and let $P$ and $Q$ be com-
complementary continuous projections of $M \oplus W$ onto $M$ and $W$, respectively (see, e.g., [6]). Then $T \overline{P}$ is a closed $F_+$-operator in $L(X, Y)$ and since $X$ is complete and $T \overline{P} = T - TQ$ we have $R(T)$ closed $\Rightarrow R(T \overline{P})$ closed [4, Theorem 16]. Since $(T/M)^{-1}$ is continuous, $R(T/M + A/M)$ is closed by part (i). Since $R(T + A) = R(T/M + A/M) + R((T + A)/W + N(T))$, we have $R(T/M + A/M) \subset R(T + A) \subset R(T/M + A/M) + F$, where $\dim F < \infty$ and where the two subspaces on the left and right are closed. Therefore $R(T + A)$ is closed. 

Lemma 3.1 and Corollary 3.2 below are generalisations of Holub [10, Proposition 2].

3.2. Corollary. Let $X$ and $Y$ be Banach spaces and let $T$ be a (closed) $\phi_+$-operator with complemented range. If $A \in L(X, Y)$ is a bounded operator with range contained in a subspace complementary to $R(T)$ then $R(T + A)$ is closed and complemented.

3.3. Proposition. $T'$ is Fredholm if and only if $T \in F_+$ and $\text{codim } R(T) < \infty$.

Proof. Let $T \in F_+$ and $\text{codim } \overline{R(T)} < \infty$. By Theorem 2.1, $T' \in \phi_-$. Also $\dim N(T') = \dim \overline{R(T)} = \dim \{Y/\overline{R(T)}\} < \infty$. Hence $T'$ is Fredholm. The converse follows from earlier remarks (see Section 1).

3.4. Theorem. Let $T \in F_+$ with $\text{codim } \overline{R(T)} < \infty$ and let $S$ be a continuous operator with $D(S) \supset D(T)$ having a strictly singular adjoint. Then $\text{codim } \overline{R(T + S)} < \infty$. If in addition $T + S$ is closed and $X$ is complete, then $R(T + S)$ is closed.

Proof. Without loss of generality assume that $D(T) = X$. We have $T' \in F_+$ by Proposition 3.3. Hence $T' + S' \in F_+$ by Theorem 2.2 and since $S$ is continuous we have $(T + S)' = T' + S'$. Then $\text{codim } \overline{R(T + S)} = \dim Y/\overline{R(T + S)} = \dim \overline{R(T + S)} = \dim \overline{R(T + S)} = \dim N((T + S)') = \dim N(T' + S') < \infty$.

Now suppose that $T + S$ is closed and $X$ is complete. Let $A$ be the operator $T + S$ regarded as an element of $L(X, \overline{R(T + S)})$. Then $A$ is closed and $\overline{R(A')} = \overline{R((T + S)')}$.

Moreover, $A'$ is injective. Since $T' + S' \in \phi_+$ (see Section 2), $R(A')$ is closed. Consequently $A'$ has a continuous inverse by the closed graph theorem. Hence $A$ is surjective [8, II.4.11], or equivalently, $R(T + S)$ is closed.

Various special cases of Theorem 3.4 are known. Examples can be found in [11; 8, Chap. V].

3.5. Theorem. Let $T \in F_+$ with $\overline{R(T)}$ topologically complemented in $Y$. 

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Let $S$ be a continuous strictly singular operator having a strictly singular adjoint, and let $D(S) \supset D(T)$. Then $R(T + S)$ is topologically complemented.

**Proof.** We shall assume without loss of generality that $D(S) = D(T)$. Let $P$ be a bounded projection with domain $Y$ and range $R(T)$. Since $(PS)' = S'P'$ is strictly singular, $R(T + PS)$ has finite codimension in $R(T)$ by Theorem 3.4. Therefore, $R(T + PS)$ is topologically complemented in $Y$. But $T + S = (T + PS) + (I - P)S$, where $(I - P)S$ is a continuous operator with range contained in $R(I - P)$ which is complementary to $R(T)$. Hence by Lemma 3.1, $R(T + S)$ is topologically complemented. 

If $Y$ is complete the continuity of $S$ can be dropped. We have

3.6. **COROLLARY.** Let $Y$ be complete, let $T \in F_+$ and let $R(T)$ be complemented. Let $S$ be a strictly singular operator having a strictly singular adjoint and let $D(S) \supset D(T)$. Then $R(T + S)$ is complemented.

**Proof:** Again we assume $D(S) = D(T)$. Write $S = A + F$, where $A$ is continuous and $F$ has finite rank [3]. Then $T + F \in F_+$ by Theorem 2.2 and $R(T + F)$ is complemented (see [1, Proposition 20]). Since $A$ and $A'$ are both strictly singular the corollary follows from Theorem 3.5. 

Suppose $D(S) = X$. Examples where both $S$ and $S'$ are strictly singular are:

(i) $S$ closed and $S'$ precompact.

(ii) $S$ partially continuous [5] and $S'$ both strictly singular and strictly cosingular (see [3]). For example, if $S$ is a (bounded or unbounded) finite rank operator.

(iii) $X$ and $Y$ Banach spaces with $Y$ subprojective, $S$ bounded and $S'$ strictly singular. Further examples of this type are found in [14].

(iv) $S$ weakly compact and $X$ and $Y'$ both Dunford Pettis spaces.

(v) $X$ a Dunford Pettis space, $Y$ a reflexive space, and $S$ any bounded operator (see, e.g., [8, III.3]).

(vi) $S$ bounded and the pairs $X$, $Y$ and $X'$, $Y'$ both totally incomparable. For example, if $X$ is reflexive and $Y$ has no infinite dimensional reflexive subspace or quotient.

3.7. **THEOREM.** Let $T$ be a closed $F_+$-operator with topologically complemented range and let $X$ be complete. Let $S$ be a bounded strictly singular operator having a strictly singular adjoint. Then $R(T + S)$ is topologically complemented.

**Proof.** Since $R(T + S)$ is topologically complemented by Theorem 3.5,
it is sufficient to show that \( R(T + S) \) is closed. But \( T + S \in F_+ \) (Theorem 2.2) and \( T + S \) is closed since \( S \) is bounded. Therefore, \( R(T + S) \) is closed by Theorem 3.4.

REFERENCES