The Dual Variational Principle and Elliptic Problems with Discontinuous Nonlinearities*

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INTRODUCTION

The main purpose of this paper is to study elliptic boundary value problems of the type

\[ \begin{cases} -\Delta u = f(u) + p(x), & x \in \Omega \\ u = 0, & x \in \partial \Omega, \end{cases} \tag{\star} \]

where \( \Omega \) is a bounded domain in \( \mathbb{R}^N \) and \( f \) has, possibly, “upward” discontinuities.

The idea is to find solutions of (\star) by using Clarke's Dual Action Principle [5]. This approach has a remarkable smoothing effect, in the sense that it allows one to look for solutions of (\star) as critical points of a functional which, in spite of the discontinuity of \( f \), is \( C^1 \).

The Variational Principle is discussed in Section 1 and is applied in Section 2 to (\star), leading one to proofs of existence and multiplicity results for various kinds of nonlinearities. Among other things, we can find the results of both [4, 5] and [11] in a quite direct way.

Notations. \( \Omega \) is a bounded domain in \( \mathbb{R}^N \), \( N \geq 2 \), with smooth boundary \( \partial \Omega \);

- \( \| \cdot \|_q \) denotes the norm in \( L^q(\Omega) \), \( q \geq 1 \);
- \( (\cdot, \cdot) \) denotes the scalar product in \( L^2(\Omega) \);

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\( \lambda_j \) and \( \phi_j, j \in \mathbb{N}, \) satisfy \( 0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \), \( \phi_j \in W^{1,2}_0(\Omega), \) and

\[ -\Delta \phi_j - \lambda_j \phi_j, \]

where \( \Delta \) stands for the Laplace operator.

We will also take \( \phi_1 > 0 \) and \( |\phi_j|^2 = 1. \)

If \( E \) is a Hilbert space and \( J \in C^1(E; \mathbb{R}), \) \( J'(u) \) will denote the gradient of \( J \) at \( u. \)

\( c_1, c_2, \ldots \) stand for possibly different, positive, constants.

\( \rightharpoonup \) denotes strong and \( \rightarrow \) weak convergence.

1. The Variational Principle

We suppose \( f: \mathbb{R} \to \mathbb{R} \) is a measurable function satisfying

\( (f1) \) there is a set \( A \subset \mathbb{R} \) with no finite accumulation points, such that \( f \in C(\mathbb{R} - A); \)

\( (f2) \) there is an \( m > 0 \) such that \( h(s) := ms + f(s) \) is strictly increasing.

We notice that for a problem such as (*) one can always suppose that \( f \) is bounded from below \([ \text{above} \] as \( s \to +\infty \) \([ - \infty \) \]; otherwise a truncation argument and the maximum principle can be used. Therefore the main restriction imposed in \( (f2) \) is concerned with the discontinuity points. In fact \( (f2) \) implies

\[ f(a-) \leq f(a) \leq f(a+) \quad \forall a \in A, \]

where \( f(a \pm) = \lim_{s \to a \pm} f(s). \)

We set

\[ T_a = [f(a-), f(a+)] \quad (a \in A) \]

and

\[ \hat{f}(s) = \begin{cases} f(s) & \forall s \notin A \\ T_a & \forall a \in A \end{cases} \]

Let \( p \in L^2(\Omega) \) be given and consider the Dirichlet boundary value problem (*)

We say that \( v \) is a solution of (*) if

\[ v \in W^{1,1}_0(\Omega) \cap W^{2,2}(\Omega) \]

and

\[ -\Delta v(x) - p(x) \in \hat{f}(v(x)) \quad \text{a.e. in } \Omega. \quad (1) \]
Without loss of generality, we can suppose that $h$ in (f2) satisfies

$$h(s) \to +\infty \quad [s \to +\infty].$$  \hfill (2)

Moreover, from now on, we will take $A = \{a\}$. This will simplify the notations. The arguments in the general case are quite similar.

By (f2) and (2) it is possible to define a single-valued function $g: \mathbb{R} \to \mathbb{R}$ by setting

$$g(t) = \begin{cases} a, & \text{if } t - ma \in T_a \\ s, & \text{with } h(s) = t \quad \text{if } t - ma \notin T_a. \end{cases}$$

In particular, one has

$$g(t) = \xi \quad \text{iff} \quad t - m \xi \in \tilde{f}(\xi).$$ \hfill (3)

It is easy to verify that $g \in C(\mathbb{R})$. Set $G(t) = \int_0^t g(\tau) d\tau$.

Let $E = L^2(\Omega)$. For all $m \geq 0$ we can define a linear self-adjoint operator $K: E \to E$ by

$$v = K\psi \quad \text{iff} \quad -Av + mv = \psi, \quad v \in W_0^1(\Omega)$$

and a functional $J: E \to \mathbb{R}$ by

$$J(u) = \int_\Omega \left\{ G(u) - \frac{1}{2} uKu - uKp \right\} dx.$$ 

Our main result is:

**Theorem 1.** Let (f1)-(f2) be satisfied. Then:

(i) \hspace{1em} $J \in C^1(E, \mathbb{R})$ and if $J'(u) = 0$ then $v = K(u + p)$ is a solution of $(\ast)$.

(ii) \hspace{1em} If either

\begin{enumerate}
  \item $-p(x) \notin T_a$ for a.e. $x \in \Omega$, or
  \item $u$ is a local minimizer of $J$, then the level set $\Omega_a = \{ x \in \Omega: v(x) = a \}$, $v = K(u + p)$, has Lebesgue measure $|\Omega_a| = 0$ and therefore $v$ satisfies

$$-Av(x) = f(v(x)) + p(x) \quad \text{a.e. in } \Omega.$$  
\end{enumerate}

**Proof:** From (2) it follows that

$$|g(t)| \leq c_1 + c_2 |t|. \quad \hfill (4)$$

Then $|G(t)| \leq c_3 + c_4 |t|^2$ and $G(u) \in L^1(\Omega) \ \forall u \in E$. Moreover, from the regularity theory of elliptic equations one has that $K\psi \in W_0^1(\Omega) \cap W^{2,2}(\Omega)$.
∀ψ ∈ E. Hence J is well defined in E. From (4) it follows [8, Thm. 3.7] that $J ∈ C^1(E, \mathbb{R})$.

Let $u ∈ E$ be such that $J'(u) = 0$ and set $v = K(u + p)$. Then $v ∈ W^{1}_{0}(Ω) \cap W^{2,2}(Ω)$ and

$$g(u) = K(u + p) = v.$$  \hspace{1cm} (5)

The definition of $K$ implies

$$-Δv + mv = u + p.$$  \hspace{1cm} (6)

From (3) and (5) we infer that $u(x) - mv(x) ∈ \hat{f}(v(x))$ a.e. in $Ω$. This and (6) show that $v$ is a solution of $(\ast)$. 

Next, we set

$$Ω_a = \{x ∈ Ω : v(x) = a\}.$$ 

Since $v ∈ W^{2,2}(Ω)$, a theorem of Stampacchia [10] applies and $-Δv(x) = 0$ a.e. in $Ω_a$. From (1) it follows that

$$-p(x) ∈ \hat{f}(v(x)) = T_a \quad \text{a.e. in } Ω_a$$

and this proves (ii) in case (x) holds.

Last, suppose (β) and let $-p(x) ∈ T_a$ a.e. in $Ω_a$.

Set $T_a = [b_1, b_2]$, $T^+ = [b_1, \frac{1}{2}(b_1 + b_2)]$, $T^- = T_a - T^+$, and $Ω^± = \{x ∈ Ω : -p(x) ∈ T^±\}$.

Define $χ ∈ L^2(Ω)$ by

$$χ(x) = \begin{cases} 
1, & x ∈ Ω^+ \\
-1, & x ∈ Ω^- \\
0, & x ∈ Ω - Ω_a.
\end{cases}$$

For $ε > 0$ small enough one has

$$-p(x) + εχ(x) ∈ T_a \quad \text{a.e. in } Ω_a$$  \hspace{1cm} (7)

and

$$\frac{d}{dε} J(u + εχ) = (g(u + εχ), χ) - ε(χ, Kχ) - (χ, K(u + p)).$$

But from $u(x) + p(x) = -Δv(x) + mv(x)$ it follows that $u(x) + p(x) = ma$ a.e. in $Ω_a$ and thus

$$(g(u + εχ), χ) = \int_{Ω_a} g(u + εχ)χ = \int_{Ω_a} g(ma - p + εχ)χ.$$  \hspace{1cm} (8)
From (7), (8) and \( g(t) = a \) if \( t - ma \in T, \) we infer

\[
( g(u + \varepsilon \chi), \chi ) = a \int_{\Omega} \chi.
\]

Moreover

\[
(\chi, K(u + p)) = (\chi, v) = \int_{\Omega_u} \chi v = a \int_{\Omega_u} \chi.
\]

Then we find

\[
\frac{d}{d\varepsilon} J(u + \varepsilon \chi) = -\varepsilon (\chi, K\chi).
\] (9)

Since \( u \) is a minimizer of \( J, \) (9) implies \((\chi, K\chi) = 0.\) Setting \( K\chi = \psi, \) one has \((\chi, K\gamma) = |\text{grad } \varphi|^{2}, \) hence \((\chi, K\gamma) = 0 \) iff \( \chi \equiv 0, \) namely iff \( \text{meas } \Omega_u = 0.\) This completes the proof. \[\]

Remark 2. In all the above arguments the Laplace operator \(-\Delta\) can be substituted by any elliptic variational operator, as well as one can deal with more general nonlinearities like \( f(x, s).\)

Remark 3. In Theorem 1 one can take \( E = L^{2}(\Omega), \) \( z > 1, \) according to the fact that \( G(t) \equiv t^{z} \) as \( |t| \to \infty.\) One would have \( K\psi \in W_{0}^{1}(\Omega) \cap W_{2,2}^{2}(\Omega) \) \( \forall \psi \in L^{2}(\Omega); \) the rest remains unaffected.

Remark 4. Theorem 1 is based on Clarke's Dual Variational Principle \([6].\) Such a principle has been used to overcome the indefiniteness of the Action integral in Hamiltonian systems (see, e.g., \([7]\)); the new feature here is that it allows one to deal with a smooth functional although \( f \) is discontinuous.

A possible interest of our approach is that we can apply to \( J \) the standard critical point theory.

In Section 2 we will indicate how to proceed in the concrete situations. To limit the paper to a reasonable length, we will discuss, rather than all the possible results, some examples only.

2. Examples

For simplicity, in the sequel we will always take \( m = 0.\)

Our first application deals with a case in which \( J \) is coercive and is related to \([11].\)
EXAMPLE 5. Let \( f \) satisfy (f1)-(f2) and let \( k, 0 < k < \lambda_1 \), be such that

\[
(f3) \quad |f(s)| \leq c_1 + k |s|.
\]

We will show that

\[
\forall p \in L^2(\Omega) \text{ (*) has a solution } v \text{ satisfying (3)}.
\]

Indeed, (f3) implies

\[
G(t) \geq \frac{1}{2k} t^2 - c_1 |t|.
\]

Moreover, one has \((u, Ku) \leq (1/\lambda_1) |u|^2 \). Hence

\[
J(u) \geq \frac{1}{2k} |u|^2 - \frac{1}{2\lambda_1} |u|^2 - c_3 |u|_2.
\]

Since \( k < \lambda_1 \), then \( J \) is bounded from below and coercive on \( E \). Since \( K \) is compact in \( E \), then \( \exists u \in E: J(u) = \min_{\mathcal{K}} J \). Applying Theorem 1, the claim follows.

Our next example is a problem at resonance.

EXAMPLE 6. Let \( f \) satisfy (f1)-(f2) and

\[
(f4) \quad f(s) = \lambda_1 s + b(s), \quad \text{with } b_\pm = \lim_{s \to \pm \infty} b(s) \in \mathbb{R}.
\]

Then (\*) has a solution provided

\[
 b_- \int_\Omega \phi_1 - (p, \phi_1) < b_+ \int_\Omega \phi_1. \tag{10}
\]

In this case we shall apply a "linking" theorem. Let \( W = \{w \in E: (w, \phi_1) = 0\} \) and \( E = L^2(\Omega) = \mathbb{R}\phi_1 \oplus W \). Using (f4) and (10) it is easy to check that

\[
\begin{align*}
\{J(t\phi_1) &\to -\infty \quad \text{as } |t| \to \infty \\
\inf_{W} J &> -\infty.
\end{align*}
\]

Moreover, one shows:

**Lemma 7.** \( J \) satisfies

\[(PS)_c \quad \text{if } u_n \in E \text{ is such that } J(u_n) \to c \text{ and } J'(u_n) \to 0 \text{ then } \exists u^* \in E: J(u^*) = c \text{ and } J'(u^*) = 0.
\]
The proof of Lemma 7 requires some technicality and is postponed to the Appendix. Assuming the validity of Lemma 7, we can apply Theorem 1.2 of [9]. Actually in [9] such a theorem is proved under the stronger assumption that $J$ satisfies (PS) (namely: if $J(u_n) \to c$ and $J'(u_n) \to 0$ then $u_n$ has a converging subsequence). However, it is readily verified that (PS), suffices, as already shown in [3] in the case of the Mountain-Pass theorem. Thus $J$ has a critical point which gives rise, through Theorem 1, to a solution of (*). This proves the claim.

**Example 8.** Let $a \neq 0$, $p = 0$, and $f$ satisfy (f1)–(f2) and

$$
(f5) \begin{align*}
\text{at } s = 0, \\
\text{as } |s| \to \infty, 1 < \sigma < (N + 2)/(N - 2).
\end{align*}
$$

Then (*) has a solution $v \neq 0$.

The details of the proof are omitted, because it is based on the arguments of [2] and those of Lemma 7. In fact, letting $E = L^{2}(\Omega)$, $\alpha$ the conjugate exponent of $\sigma + 1$, one shows that: (i) $J$ satisfies (PS); and (ii) the Mountain-Pass theorem [2] applies. As for (i), one first proves as in [2] that $|u_n|_{2} \leq \text{const}$ and then uses the same arguments of Lemma 7.

The following is an example with a continuous but not smooth nonlinearity. The smoothing effect of the Variational Principle will allow one to handle this case in a rather direct way, too.

**Example 9.** Let $k > 1$ and consider the Dirichlet problem

$$
-\Delta u = \text{sign}(u) \cdot |u|^{1/k} \quad \text{in } \Omega, \ u = 0 \text{ on } \partial \Omega.
$$

We will show that (12) has infinitely many solutions.

Here $G(t) = (1/(k + 1)) |t|^{k+1}$. Take $X = L^{k+1}(\Omega)$ and $J(u) = \int_{\Omega} \{ G(u) - \frac{1}{k} u( Ku) \} \ dx$.

Since $\int G(t) - (1/(k + 1)) |t|^{k+1}$, it follows readily that $J$ is coercive and bounded below on $X$. To show that (PS) holds, let $u_n \in X$ be such that $J(u_n) \to c$ and $J'(u_n) \to G'(u_n) - Ku_n \to 0$. By the former we deduce that $|u_n|_{k+1} \leq c$; since $K: X \to W^{2,k+1}(\Omega)$, by the Sobolev embedding theorem it follows that $Ku_n \to z$ in $X^*$, up to a subsequence. Then $G'(u_n) \to z$ and since $G'$ is strictly increasing, we infer that $u_n \to z$, too. This proves (PS).

Next, since $f$ is odd then $J$ is even and the Lusternik–Schnirelman theory applies. We assume the reader is familiar with such a theory and use standard notations (see, for example, [1]).

The critical levels

$$
l_m = \inf_{\gamma(A) \not\subseteq m} \sup_{A} J \quad (m \in \mathbb{N})
$$

($\gamma(A)$ is the "genus" of $A$) carry critical points $u \neq 0$ provided $l_m < 0$. 

Now, let us notice that $G \in C^2$ implies, via a well-known result on Nemitski operators [8, Thms. 3.7 and 3.4], that $J$ is $C^2$ on $X$. Moreover $G''(0) = 0$ yields: $J''(0)[\psi, \psi] = - (\psi, K \psi) \forall \psi \in X$.

For any fixed $m \in \mathbb{N}$, let $X_m = \text{span}\{\phi_1, \ldots, \phi_m\}$. For $\psi \in X_m$, one has

$$(\psi, K \psi) \geq \frac{1}{\lambda_m} |\psi|_2^2,$$

hence ($X_m$ being finite-dimensional)

$$J''(0)[\psi, \psi] \leq -c_1 |\psi|_{k+1}^2 \quad \forall \psi \in X_m.$$

Therefore, $\forall \varepsilon > 0$ small enough, letting $A_{m, \varepsilon} = \{t \in X: |x|_k + 1 = \varepsilon\}$, it follows that

$$\sup_{A_{m, \varepsilon}} J < 0.$$

Since $\gamma(A_{m, \varepsilon}) = m$, then $t_m < 0$. Applying the Lusternik–Schnirelman critical point theory, we conclude that $J$ has infinitely many critical points on $X$, corresponding to solutions of (12) through Theorem 1 and Remark 3.

APPENDIX

Lemma 7 will be proved in several steps. For simplicity, we will take $p = 0$.

First, some remarks are in order. Letting $u_n = t_n \phi_1 + w_n$, $w_n \in W$, and substituting in (11), it follows that $|u_n|_2 \leq c_1$. Hence, up to a subsequence, $u_n \to u^*$ in $L^2(\Omega)$.

Let $v^* = Ku^*$,

$$\Gamma = \{x \in \Omega: v^*(x) = a\}, \quad \Omega^* = \Omega - \Gamma$$

and

$$\psi(x) = \begin{cases} 1 & \text{if } x \in \Gamma \\ 0 & \text{if } x \in \Omega^* \end{cases}.$$

From $J'(u_n) \to 0$ and the compactness of $K$ it follows that

$$g(u_n) \to Ku^* = v^* \quad \text{in } L^2(\Omega) \text{ and a.e. in } \Omega. \quad (13)$$

If we prove that

$$J'(u^*) = g(u^*) - v^* = 0 \quad (A1)$$

$$J(u_n) \to J(u^*), \quad (A2)$$

Lemma 7 will follow.
It is convenient to discuss separately what happens in $\Omega^*$ and in $\Gamma$.
First of all, we claim

$$u_n \to u^* \text{ in } L^2(\Omega^*).$$  \hfill (A3)

In fact $v^*(x) \neq a$ for $x \in \Omega^*$; then $f \in C(\mathbb{R} - \{a\})$ and (13) yield

$$u_n \to f(v^*) \text{ a.e. in } \Omega^*. \hfill (14)$$

Since $f(s) = \lambda_1 s + b(s)$ with $b$ bounded, we deduce that $|u_n| \leq c_1 |g(u_n)| + c_2$. Using also (13), it follows that $|u_n| \leq h$ for some $h \in L^2(\Omega)$. Then (14) yields $u_n \to f(v^*)$ in $L^2(\Omega^*)$. Since $u_n \to u^*$ in $L^2(\Omega)$, (A3) follows.

Since $g$ is asymptotically linear, from (A3) we infer

$$g(u_n) \to g(u^*) \text{ in } L^2(\Omega^*) \hfill (15)$$

and

$$\int_{\Omega^*} G(u_n) \to \int_{\Omega^*} G(u^*). \hfill (16)$$

Next, to study the behaviour on $\Gamma$, we distinguish whether $0 \in T_a$ or not. We first show

$$\text{if } 0 \notin T_a \text{ then } |\Gamma| = 0. \hfill (17)$$

In fact, let $T_a = [b_1, b_2]$ with $b_1 > 0$ (if $b_2 < 0$ the proof is similar). As seen in the proof of Theorem 1(ii), one has

$$u^* = -\Delta v^* = 0 \text{ a.e. in } \Gamma. \hfill (18)$$

Since $u_n \to u^*$ and using (18), we find

$$\int_{\Gamma} u_n = (u_n, \psi) \to (u^*, \psi) = \int_{\Gamma} u^* = 0. \hfill (19)$$

On the other hand, (13) yields, in particular, that $g(u_n) \to a$ a.e. on $\Gamma$. This, the continuity, and the strict monotonicity of $g$ readily imply that $\lim \inf u_n(x) \geq b_1$ for a.e. $x \in \Gamma$. As seen before, $|u_n| \leq h \in L^2(\Omega)$. Then Fatou's lemma yields

$$\lim \inf \int_{\Gamma} u_n \geq b_1 |\Gamma|. \hfill \text{This and (19) prove (17).}$$
Proof of (A1). If $0 \notin T_a$, (15) and (17) imply

$$g(u_n) \to g(u^*) \quad \text{in } L^2(\Omega)$$

and (A1) follows from (13).

If $0 \in T_a$, then (18) implies: $g(u^*(x)) = g(0) = a = v^*(x)$ for a.e. $x \in \Gamma$, and again (A1) holds.

Proof of (A2). If $0 \notin T_a$, (17) holds and (16) becomes

$$\int_\Omega G(u_n) \to \int_\Omega G(u^*).$$

Since $K$ is compact, (A2) follows.

If $0 \in T_a$, then $G(s) = as$ for $s \in T_a$; by arguments similar to those employed before, one shows that

$$|G(u_n) - au_n| \to 0 \quad \text{a.e. in } \Gamma$$

and, as a consequence,

$$\int_\Gamma |G(u_n) - au_n| \to 0. \quad (20)$$

From (16), (19), and (20), we infer

$$\int_\Omega G(u_n) = \int_{\Omega^*} G(u_n) + \int_\Gamma G(u_n) \to \int_{\Omega^*} G(u^*).$$

Since $u^* = 0$ a.e. in $\Gamma$ and $G(0) = 0$, then $\int_{\Omega^*} G(u^*) = \int_\Omega G(u^*)$ and (A2) follows in this case, too.

The proof of Lemma 7 is now complete.

References

