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Note

Distinguishing labeling of group actions

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Abstract

Suppose Γ is a group acting on a set X. An r-labeling $f : X \to \{1, 2, \ldots, r\}$ of X is distinguishing (with respect to Γ) if the only label preserving permutation of X in Γ is the identity. The distinguishing number, $D_\Gamma(X)$, of the action of Γ on X is the minimum r for which there is an r-labeling which is distinguishing. This paper investigates the relation between the cardinality of a set X and the distinguishing numbers of group actions on X. For a positive integer n, let $D(n)$ be the set of distinguishing numbers of transitive group actions on a set X of cardinality n, i.e., $D(n) = \{D_\Gamma(X) : |X| = n \text{ and } \Gamma \text{ acts transitively on } X\}$. We prove that $|D(n)| = O(\sqrt{n})$. Then we consider the problem of an arbitrary fixed group Γ acting on a large set. We prove that if for any action of Γ on a set Y, for each proper normal subgroup H of Γ, $D_H(Y) \le 2$, then there is an integer n such that for any set X with $|X| \ge n$, for any action of Γ on X with no fixed points, $D_\Gamma(X) \le 2$.

Keywords: Distinguishing number; Distinguishing set of group actions; Symmetric groups; Group actions; Graphs

1. Introduction

Distinguishing labeling was first defined by Albertson and Collins [1] for graphs. A labeling of a graph G, $f : V(G) \to \{1, 2, \ldots, r\}$, is said to be *$r$-distinguishing* if no non-trivial automorphism of G preserves all the vertex labels. In other words, f is r-distinguishing if for any $\sigma \in \mathrm{Aut}(G)$, $\sigma \neq 1$, there is a vertex x such that $f(x) \neq f(\sigma(x))$. The *distinguishing number* of a graph G is defined as

$$D(G) = \min\{r : \text{ there exists an } r\text{-distinguishing labeling of } G\}.$$

Distinguishing labeling can be naturally extended to general group actions [16]. Let Γ be a group acting on a set X. For a positive integer r, an r-labeling $f : X \to \{1, 2, \ldots, r\}$ of X is said to be *r-distinguishing with respect to the action of Γ* if for any $\sigma \in \Gamma$, $f(x) = f(\sigma(x))$ for all x only if $\sigma x = x$ for all x. If the action of Γ on X is faithful, then $1 \in \Gamma$ is the only group element σ for which $\sigma x = x$ for all x. In this case, a labeling $f : X \to \{1, 2, \ldots, r\}$ of X is r-distinguishing with respect to the action of Γ if for any $\sigma \in \Gamma$, if $\sigma \neq 1$, then there exists an $x \in X$ for which $f(x) \neq f(\sigma(x))$. (We use 1 to denote the unit of a group.) We shall be mainly interested in faithful group actions.

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However, for the purpose of using inductions, we need to consider group actions that are not faithful. The *distinguishing number* $D_\Gamma(X)$ of the action of Γ on X is defined as

$$D_\Gamma(X) = \min\{r : \text{ there exists an } r\text{-distinguishing labeling of } X\}.$$

The distinguishing number of graphs and group actions have been studied in [1–11,14,16]. It was proven in [16] that if Γ is a non-trivial abelian group then $D_\Gamma(X) = 2$ for any action of Γ on a set X, and if Γ is a dihedral group, then $D_\Gamma(X) \le 3$ for any action of Γ on a set X. The result was generalized in [5], where it was proved that if Γ is nilpotent of class c or supersolvable of length c then $D_\Gamma(X) \le c + 1$ for any action of Γ on a set X. It was proved in [16] that for any group Γ, if $|\Gamma| < (k+1)!$ then $D_\Gamma(X) \le k$ for any action of Γ on a set X. It was conjectured in [17] that for any action of S_n on a set X, $D_{S_n}(X) = \lceil n^{1/k} \rceil$ or $\lceil (n-1)^{1/k} \rceil$ for some positive integer k, and the conjecture was proved to be true [17] for almost all n.

All the results above explore the relation between the distinguishing number $D_\Gamma(X)$ and the structure of the group Γ. In this paper, we are interested in a different problem: How can the cardinality of X affect the distinguishing number? Assume that $X = [n] = \{1, 2, \dots, n\}$. It is obvious that $D_\Gamma([n]) \le n$, as the labeling which assigns to each element of $[n]$ a distinct label is certainly distinguishing. So for a non-trivial group Γ action on $[n]$, we have $2 \le D_\Gamma([n]) \le n$ (an action of Γ on X is trivial if $\sigma x = x$ for all $\sigma \in \Gamma$ and for all $x \in X$). Now given any positive integer $2 \le k \le n$, is it possible to find an action of a group Γ on $[n]$ with $D_\Gamma([n]) = k$? The answer is yes. For any $2 \le k \le n$, the subgroup of S_n which fixes each of $k+1, k+2, \dots, n$ and whose action on $\{1, 2, \dots, k\}$ is isomorphic to S_k has distinguishing number k. However, this answer is not very convincing, because such an action is basically an action on the set $\{1, 2, \dots, k\}$. Is there an action Γ on $[n]$ which is transitive on $[n]$ and has $D_\Gamma([n]) = k$? More precisely, for a positive integer n, let

$$D(n) = \{D_\Gamma([n]) : \Gamma \text{ is a transitive subgroup of } S_n\}.$$

The question, first asked by Chan [4,5], is to determine $D(n)$. In this paper, we shall prove that $|D(n)| = O(\sqrt{n})$, which means that most of the integers in the interval $[2, n]$ do not belong to $D(n)$. On the other hand, for infinitely many n, we have $|D(n)| \ge \log_2 n$. The next question that we are interested in also concerns the relation between the distinguishing number and the cardinality of the set X. If Γ is a group, is there an integer $n = n(\Gamma)$ such that for any set X with $|X| \ge n$ and for any action of Γ on X, we have $D_\Gamma(X) \le 2$?

If there is an action of Γ on a set X with $D_\Gamma(X) > 2$, then the answer is trivially 'no' since we can add to X arbitrarily many elements x that are fixed by all the group elements, i.e., let X' be a large superset of X, and extend the action of Γ to X' by letting $\sigma(x) = x$ for all $\sigma \in \Gamma$ and all $x \in X' \setminus X$. The addition of such elements does not change the distinguishing number. To make the problem interesting, one needs to avoid such 'trivial' actions. Suppose O is an orbit of the action of Γ on X. Let K be the pointwise stabilizer of O. Then K is a normal subgroup of Γ. We say O is a restrictive orbit if for any action of Γ on a set Y, we have $D_K(Y) \le 2$. We shall prove that for any group Γ, there is an integer $n = n(\Gamma)$ such that for any action of Γ on X, if the union of the restrictive orbits has cardinality at least n, then $D_\Gamma(X) = 2$.

For $x \in X$, let $\text{stab}_x = \{\sigma \in \Gamma : \sigma x = x\}$ be the point-stabilizer of x. To each orbit O, one associates the conjugacy class $\pi(O) = \{\text{stab}_x : x \in O\}$ of subgroups of Γ. Let $\|O\|$ be the number of orbits O' with $\pi(O') = \pi(O)$. We shall prove that if for each orbit O, the cardinality satisfies $2^{\|O\|} \ge |O|$, then $D_\Gamma(X) = 2$.

2. Bounding the size of $D(n)$

This section discusses the distinguishing number of transitive actions on $[n]$. Assume Γ acts transitively on $[n]$. Then Γ is either a primitive subgroup of S_n or an imprimitive subgroup of S_n. In the latter case, it is known (see [13]) that we can write $[n]$ as the Cartesian product of $[k]$ and $[m]$, and $\Gamma = K \wr H$ is the wreath product of K and H, where K is a primitive subgroup of S_k and H is a transitive subgroup of S_m. Recall that the wreath product $K \wr H$ has elements

$$\{(f, h) : h \in H \text{ and } f \text{ is a map from } [m] \text{ to } K\}.$$

We denote the image of $j \in [m]$ by $f_j \in K$. The action on $[k] \times [m]$ is defined for $(i, j) \in [k] \times [m]$ by $(f, h)(i, j) = (f_j(i), h(j))$. The sets $B_j = \{(i, j) : i \in [k]\}$ are called blocks. Thus the element $(f, h) \in K \wr H$ maps the block B_j to the block $B_{h(j)}$.

Lemma 2.1. *Suppose $n = km$ and $k, m \geq 2$. View $[n]$ as the Cartesian product $[k] \times [m]$. Suppose K is a primitive subgroup of S_k and H is a transitive subgroup of S_m, and consider the action of $K \wr H$ on $[n]$ defined above.*

1. *If $k > m$ then the wreath products $S_k \wr H$ and $A_k \wr H$ have distinguishing numbers $D_{S_k \wr H}([n]) = k + 1$ and $D_{A_k \wr H}([n]) = k - 1$.*
2. *If $D_K([k]) = t$ and r is an integer with $\binom{r}{t} \geq D_H([m])$, then $D_{K \wr H}([n]) \leq r$.*

Proof. The distinguishing number of the action of the wreath product of groups is studied in [4]. Lemma 2.1 follows from results in [4]. However, we include here a short direct proof.

Proof of (1). Assume $k > m$. If $K = S_k$, then let ϕ be the $(k + 1)$-labeling of $[k] \times [m]$ which labels the k elements of block B_j with the k distinct labels in $\{1, 2, \ldots, k + 1\} \setminus \{j\}$. We claim that ϕ is a distinguishing labeling. Indeed, if $(f, h) \in S_k \wr H$ preserves the labeling ϕ, then since $\phi(B_j) = \phi(B_{h(j)})$ and distinct blocks use different label sets, we conclude that $h(j) = j$. Thus $(f, h)(i, j) = (f_j(i), j)$. But distinct elements in B_j are labeled by distinct labels. So $f_j(i) = i$, and hence $(f, h) = id_{[k]\times[m]}$. On the other hand, since $K = S_k$, a distinguishing labeling of $[k] \times [m]$ must label each block with k distinct labels, and there must be two blocks that use different label sets. Therefore $D_{S_k \wr H}([n]) = k + 1$.

If $K = A_k$, then let ϕ be the $(k - 1)$-labeling of $[k] \times [m]$ which labels the k elements of block B_j with the $k - 1$ labels in $\{1, 2, \ldots, k - 1\}$ such that label j is used twice, and every other label is used once. Now we show that ϕ is a distinguishing labeling. If $(f, h) \in A_k \wr H$ preserves the labeling, then since $\phi(B_j) = \phi(B_{h(j)})$ and colour j is the only colour used twice in B_j, colour $h(j)$ is the only colour used twice in $B_{h(j)}$, we conclude that $h(j) = j$. In B_j distinct elements are labeled by distinct labels, except that two elements, say q, q', are both labeled by label j. Thus f_j fixes all elements of B_j, except possibly interchanging q and q'. However, since $f_j \in A_k$ is an even permutation, we know that $f_j \neq (qq')$. Therefore f_j fixes all elements of B_j. On the other hand, a distinguishing labeling of a single block already needs $k - 1$ labels. Indeed, if ϕ is a labeling of $[k] \times [m]$ which uses less than $k - 1$ labels, then for each $j \in [k]$, either there are three elements, say q, q', q'', of B_j labeled by the same label, or there are four elements, say q, q' and p, p', such that q, q' are labeled by the same label, and p, p' are labeled by the same label. Let $h(j) = j$ for all j, and let

$$f_j = \begin{cases} (qq'q''), & \text{if there exist } q, q', q'' \in [k] \text{ with } \phi(q) = \phi(q') = \phi(q''), \\ (qq')(pp'), & \text{if there exist } q, q', p, p' \in [k] \text{ with } \phi(q) = \phi(q') \text{ and } \phi(p) = \phi(p'). \end{cases}$$

Then $(f, h) \in A_k \wr H$ preserves the labeling ϕ. Thus we have proven that $D_{A_k \wr H}([n]) = k - 1$.

Proof of (2). Suppose K is a subgroup of S_k with $D_K([k]) = t$ and r is an integer with $\binom{r}{t} \geq D_H([m])$. We define an r-labeling of $[k] \times [m]$ as follows: Let $\binom{[r]}{t}$ be the family of t-subsets of the r labels. Let $\phi : [m] \to \binom{[r]}{t}$ be a distinguishing labeling of the action of H on $[m]$. Now for each block B_j, use the t labels in the t-subset $\phi(j)$ to label the elements of B_j in such a way that it is a distinguishing labeling of $[k]$ with respect to the action of K. Now if $(f, h) \in K \wr H$ preserves the labeling ϕ, then since $\phi(B_j) = \phi(B_{h(j)})$, we conclude that $h(j) = j$. Within the block B_j, since the labeling is distinguishing with respect to the action of K, we have that f_j is the identity mapping. So ϕ is a distinguishing labeling and hence $D_{K \wr H}([n]) \leq r$. □

For a group action of K on $[k]$, two labelings l_1 and l_2 of $[k]$ are equivalent if there exists $\sigma \in K$ such that $l_1(\sigma x) = l_2(x)$ for all $x \in [k]$. In the second half of Lemma 2.1, the condition that $\binom{r}{t} \geq D_H([m])$ can be replaced by the condition that "the number of non-equivalent r-labelings of $[k]$ with respect to K is greater than or equal to $D_H([m])$". If $D_K([k]) = t$, then of course $D_{K \wr H}([n]) \geq t$. If the number of non-equivalent t-labelings of $[k]$ with respect to K is greater than or equal to $D_H([m])$, then we have $D_{K \wr H}([n]) = t$.

Lemma 2.2. *Suppose k, n are positive integers and $k > 5\sqrt{n}$ and $k \neq n$. Then $k \in D(n)$ if and only if $k + 1$ or $k - 1$ is a factor of n.*

Proof. If $k \neq n$, $k > 5\sqrt{n}$ and $k + 1$ or $k - 1$ is a factor of n, then it follows from Lemma 2.1 that $k \in D(n)$. Conversely, assume $k \in D(n)$ and $k > 5\sqrt{n}$. We shall show that $k + 1$ or $k - 1$ is a factor of n. Assume to the contrary that neither $k + 1$ nor $k - 1$ is a factor of n. Let Γ be a group acting transitively on $[n]$ with $D_\Gamma([n]) = k$.

As $D_\Gamma([n]) = k \neq n$ and $k \neq n-1$, we conclude that $\Gamma \neq S_n, A_n$. If Γ is a transitive, primitive subgroup of S_n, then it follows from a result of [12] that $|\Gamma| \leq 50n^{\sqrt{n}}$. It is proved in [16] that if $|H| \leq k!$ then $D_H(Z) \leq k$ for any action of H on any set Z. Therefore $k = D_\Gamma([n]) \leq 5\sqrt{n}$, which is a contradiction.

Assume Γ is an imprimitive subgroup of S_n. Then there are positive integers t, m such that $t \times m = n$ and $\Gamma = K \wr H$, where K is a primitive subgroup of S_t and H is a transitive subgroup of S_m.

We claim $D_K([t]) \leq 5\sqrt{n/2}$. If $t \leq \sqrt{n}$, then $D_K([t]) \leq t < 5\sqrt{n/2}$. Assume $t > \sqrt{n}$. If $K = S_t$, then by Lemma 2.1, $k = D_\Gamma([n]) = t+1$. Hence $k-1 = t$ is a factor of n, contrary to our assumption. If $K = A_t$, then $k = D_\Gamma(X) = t-1$. Hence $k+1 = t$ is a factor of n, again contrary to our assumption. Hence $K \neq S_t, A_t$, and so $|K| \leq 50t^{\sqrt{t}}$ and hence $D_K([t]) \leq 5\sqrt{t} \leq 5\sqrt{n/2}$.

Suppose $D_K([t]) = s$. Then for $r = 5\sqrt{n/2} + 2$, we have $\binom{r}{s} \geq n > D_H([m])$. By Lemma 2.1, $D_\Gamma([n]) \leq r \leq 5\sqrt{n}$, again contrary to our assumption. This completes the proof of Lemma 2.2. □

Theorem 2.3. *Let $D(n) = \{D_\Gamma([n]) : \Gamma$ is a transitive subgroup of $S_n\}$. Then*

$$|D(n)| = O(\sqrt{n}).$$

Proof. By Lemma 2.2, if $k \in D(n)$, then either $k \leq 5\sqrt{n}$ or $k = n$ or $k-1$ is a factor of n or $k+1$ is a factor of n. As n has at most $O(\ln n)$ factors, it follows that $|D(n)| = O(\sqrt{n})$. □

We do not know whether the upper bound given in Theorem 2.3 is optimal. It follows from Lemma 2.1 that if k is a factor of n such that $k < n < k^2$, then $k-1, k+1 \in D(n)$. If $n = 2^m$ and m is odd, then n has $\frac{m-1}{2}$ factors k with $k < n$ and $k^2 > n$ and hence $|D(n)| \geq m-1 = \log_2 n - 1$. As $2 \in D(n)$, if $m \geq 5$, we have $|D(n)| \geq m = \log_2 n$.

3. A group acting on a large set

Tucker and Conder [15] have shown that there are only a finite number of 3-connected planar graphs G with $D(G) > 2$. Thus if G is a sufficiently large 3-connected planar graph then $D(G) \leq 2$. This leads to the following question:

Suppose Γ is a group which acts on a sufficiently large set X. Under what conditions is $D_\Gamma(X) \leq 2$?

If $x \in X$ is a fixed point of Γ, then let $X' = X \setminus \{x\}$. It is obvious that $D_\Gamma(X) = D_\Gamma(X')$. Thus for the question to be interesting, we assume that Γ has no fixed points. However, for some groups Γ, having no fixed points and X large enough still does not imply that $D_\Gamma(X) \leq 2$. For example, consider the action of S_n on $[n] = \{1, 2, \ldots, n\}$. We extend the action to a large set $X = [n] \cup_{i=1}^{k} \{a_{i,0}, a_{i,1}\}$ (where k is a large integer) as follows: If σ is even, then $\sigma(a_{i,j}) = a_{i,j}$; if σ is odd, then $\sigma(a_{i,j}) = a_{i,1-j}$. If $\sigma \in S_n$ is even, then $\sigma x = x$ for every $x \in \cup_{i=1}^{k} \{a_{i,0}, a_{i,1}\}$. Thus if f is a labeling which labels $a_{1,0}$ and $a_{1,1}$ with distinct labels, then the label is preserved by $\sigma \in S_n$ if and only if $\sigma \in A_n$. Therefore $D_{S_n}(X) = D_{A_n}([n]) = n-1$. The problem in this example is that the "large part" of the set X is pointwise fixed by the "large subgroup" A_n of S_n.

An action of Γ on X and an action of Γ on Y are isomorphic if there is a one-to-one correspondence $\phi : X \to Y$ such that for any $\sigma \in \Gamma$, $\phi(\sigma x) = \sigma\phi(x)$. Let H be a subgroup of Γ. There is a natural action of Γ on cosets of H defined as $\sigma(\tau H) = \sigma\tau H$ for $\sigma \in \Gamma$. For subgroups H_1, H_2 of Γ, we notice first that the action of Γ on cosets of H_1 is isomorphic to the action of Γ on cosets of H_2 if H_1 is conjugate to H_2 (i.e. $H_1 = \sigma H_2 \sigma^{-1}$ for some $\sigma \in \Gamma$). Suppose O is an orbit of the action of Γ on X. Then the action of Γ on O is isomorphic to the action of Γ on cosets of stab_x for any $x \in O$.

We denote the conjugacy class $\{\mathrm{stab}_x : x \in O\}$ by $\pi(O)$ and denote the set of all conjugacy classes of Γ by $\mathcal{S}$. Given a conjugacy class C in $\mathcal{S}$, let v_C be the number of orbits O in X with $\pi(O) = C$ and let $\|O\| = v_{\pi(O)}$. Then, up to isomorphism, the action of Γ on a set X is characterized by the integer vector $\vec{v} = \{v_C : C \in \mathcal{S}\}$.

Lemma 3.1. *Suppose Γ is a group acting on a set X. For a positive integer n, define an action of Γ on $X \times [n]$ as $\sigma(x, i) = (\sigma x, i)$. Then $D_\Gamma(X \times [n]) = \lceil (D_\Gamma(X))^{1/n} \rceil$.*

Proof. Suppose $D_\Gamma(X \times [n]) = d$ and f is a d-distinguishing labeling of $X \times [n]$ with respect to Γ. Let $g : X \to [d]^n$ be defined as $g(x) = (f(x,1), f(x,2), \ldots, f(x,n))$. If $\sigma \in \Gamma$ and $g(\sigma x) = g(x)$ for all x, then $f(\sigma(x,i)) = f(x,i)$ for all $(x,i) \in X \times [n]$. Since f is a distinguishing labeling, we conclude that $\sigma x = x$ for all x. So g is a d^n-distinguishing labeling of X with respect to Γ. Hence $D_\Gamma(X) \leq (D_\Gamma(X \times [n]))^n$. Conversely, suppose s is an integer and $D_\Gamma(X) \le s^n$. Let $g : X \to [s]^n$ be a distinguishing labeling of X with respect to Γ. Let $f : X \times [n] \to [s]$ be defined as $f(x, i) = j$ if the i-th component of $g(x)$ is j. If $\sigma \in \Gamma$ and $f(\sigma(x, i)) = f(x, i)$ for all x, i, then $g(\sigma x) = g(x)$ for all x. Since g is distinguishing, we conclude that $\sigma x = x$ for all x. Hence f is a distinguishing labeling and so $D_\Gamma(X \times [n]) \le s$. Therefore $D_\Gamma(X \times [n]) \le \lceil (D_\Gamma(X))^{1/n} \rceil$. □

For an orbit O of the action of Γ on X, let $K_O = \cap_{H \in \pi(O)} H$. Then K_O is a normal subgroup of Γ.

Lemma 3.2. *Suppose Γ is a group acting on a set X and $O_1, O_2, \ldots, O_k$ are orbits of the action. Let $Y = \cup_{j=1}^k O_j$ and $Z = X \setminus Y$. Let $K = \cap_{j=1}^k K_{O_j}$. Then*

$$D_K(Z) \le D_\Gamma(X) \le \max\{D_\Gamma(Y), D_K(Z)\}.$$

Proof. Assume f is a distinguishing labeling of X with respect to Γ. If $\sigma \in K$ and $f(\sigma x) = f(x)$ for all $x \in Z$, then $f(\sigma x) = f(x)$ for all $x \in X$ (because for $x \notin Z$, $\sigma x = x$). Therefore $\sigma x = x$ for all $x \in Z$, i.e., the restriction of f to Z is a distinguishing labeling of Z with respect to K. This proves the first inequality.

Let $r = \max\{D_\Gamma(Y), D_K(Z)\}$, let $f_1 : Y \to \{1, 2, \ldots, r\}$ be an r-distinguishing labeling of Y with respect to Γ and let $f_2 : Z \to \{1, 2, \ldots, r\}$ be an r-distinguishing labeling of Z with respect to K. Let

$$g(x) = \begin{cases} f_1(x), & \text{if } x \in Y, \\ f_2(x), & \text{if } x \in Z. \end{cases}$$

If $\sigma \in \Gamma$ and $g(\sigma x) = g(x)$ for all x, then $f_1(\sigma x) = f_1(x)$ for all $x \in Y$. Since f_1 is a distinguishing labeling of Y with respect to Γ (note that $\sigma Y = Y$ and $\sigma Z = Z$), we have $\sigma x = x$ for all $x \in Y$. So $\sigma \in K$. Because $f_2(\sigma x) = f_2(x)$ for all $x \in Z$ and f_2 is a distinguishing labeling of Z with respect to K, we conclude that $\sigma x = x$ for all $x \in Z$. Therefore $\sigma x = x$ for all $x \in X$ and g is a distinguishing labeling of X with respect to Γ. This proves the second inequality. □

Theorem 3.3. *Suppose Γ acts on X and O is an orbit of the action. Then*

$$D_\Gamma(X) \le \max\{(D_\Gamma(O))^{1/\|O\|}, D_{K_O}(X)\}.$$

Proof. Let $O_1, O_2, \ldots, O_{\|O\|}$ be the set of orbits with $\pi(O_j) = \pi(O)$. Let $Y = \cup_{j=1}^{\|O\|} O_j$ and $Z = X \setminus Y$. Since $K_{O_j} = K_O$ for all j, and $D_{K_O}(X) = D_{K_O}(Z)$, the conclusion follows from Lemma 3.2. □

Suppose Γ acts on X and O is an orbit. We say O is a *restrictive orbit* if for any action of Γ on a set Y, we have $D_{K_O}(Y) \le 2$.

Corollary 3.4. *For an action of Γ on X, if there is a restrictive orbit O with $2^{\|O\|} \ge D_\Gamma(O)$, then $D_\Gamma(X) \le 2$.*

Proof. Assume O is a restrictive orbit with $2^{\|O\|} \ge D_\Gamma(O)$. Apply Theorem 3.3, we conclude that $D_\Gamma(X) \le 2$. □

Corollary 3.5. *Suppose Γ is a group such that for any proper normal subgroup H of Γ, for any action of Γ on a set Y, we have $D_H(Y) \le 2$. Then there is an integer $n = n(\Gamma)$ such that for any set X with $|X| \ge n$, for any action of Γ on X with no fixed points, $D_\Gamma(X) \le 2$.*

Proof. The assumption implies that all the orbits are restrictive. Thus the conclusion follows from Corollary 3.4. □

The following is a special case of Corollary 3.5.

Corollary 3.6. *If Γ is a simple group and X is a sufficiently large set, then for any action of Γ on X without fixed points, $D_\Gamma(X) \le 2$.*

Suppose Γ acts on X. For $\sigma \in \Gamma$, let $m(\sigma) = |\{x \in X : \sigma x \ne x\}|$, and let $m(\Gamma) = \min\{m(\sigma) : \sigma \in \Gamma, m(\sigma) \ne 0\}$. It was shown in [14] that if $m(\Gamma) > 2\log_2(|\Gamma|)$, then $D_\Gamma(X) = 2$. Corollary 3.6 also follows from this result, because if Γ is simple, then for each non-trivial orbit O (i.e., orbits containing at least two elements), for each $\sigma \in \Gamma$ with $\sigma \ne 1$, there is an element $x \in O$ for which $\sigma x \ne x$. Hence $m(\Gamma)$ is at least as large as the number of non-trivial orbits.

If X is sufficiently large and there is no fixed points, then there are sufficiently many non-trivial orbits, and hence $D_\Gamma(X) = 2$.

The condition in Corollary 3.4 is necessary in the following sense: If X is large but the union of restrictive orbits is small, then it is possible that $D_\Gamma(X) > 2$. Let O be an orbit of the action of Γ on X. Assume there is an action of Γ on Y with $D_{K_O}(Y) \geq 3$. Then for any positive integer m, the action of Γ to $X = Y \cup (O \times [m])$ has $D_\Gamma(X) \geq D_{K_O}(A) \geq 3$ by Lemma 3.2.

Theorem 3.7. *Suppose Γ acts on X. If $(D_\Gamma(O))^{1/\|O\|} \leq d$ for each orbit O, then $D_\Gamma(X) \leq d$.*

Proof. For each conjugacy class C of subgroups of Γ, let Y_C be the union of orbits O with $\pi(O) = C$. By Lemma 3.1, $D_\Gamma(Y_C) \leq d$ for each C. Since $D_\Gamma(X) \leq \max_{C \in \mathcal{S}}\{D_\Gamma(Y_C)\}$, we have $D_\Gamma(X) \leq d$. □

Corollary 3.8. *If $(D_\Gamma(O))^{1/\|O\|} \leq 2$ for each orbit O, then $D_\Gamma(X) = 2$.*

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