Truncated Trigonometric Moment Problems and Determinate Measures

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1. INTRODUCTION

The main problem that we will consider in this paper is the question whether or not a given positive Borel measure \( \mu \) on the \( n \)-dimensional torus \( T^n \) is \( S \)-determinate, which means that \( \mu \) is entirely determined by the subset of its Fourier coefficients \( \{ \hat{\mu}(k), k \in S \} \), where \( S \) is an arbitrary subset of \( \mathbb{Z}^n \) containing 0 and with the property that \( -k \in S \) whenever \( k \in S \). We give necessary and sufficient conditions for \( S \)-determinacy and show that, when \( S \) is a finite set, the validity of this property can be verified algebraically by an algorithmic procedure involving trigonometric polynomials with spectrum in \( S \) vanishing on the support of \( \mu \) and satisfying certain positivity properties.

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one-dimensional situation in which the known moments are those corresponding to the integers from $-N$ to $N$, where $N \geq 0$. In this case, it is well known that a measure is determinate if and only if it consists of finitely many atoms (with possibly different masses associated with different points on the support), the number of these atoms being less than or equal to $N$ (see [2, 3, 9, 12, 13, 15]). In this paper, we consider the determinacy problem for general subsets of $\mathbb{Z}^n$ and obtain various necessary and sufficient conditions for a given measure to be determinate. These results are particularly useful in the case of finite truncated trigonometric moment problems (i.e., problems where only finite many of the moments are known), since the determinacy problem for a given measure can then be reduced to an algebraic problem involving a family of trigonometric polynomials with spectrum in $\mathcal{S}$ and that vanish on the support of the given measure, which has to be a finite set. We obtain along the way a result of independent interest which is a description of the set of points in $\mathbb{T}^n$ that belong to the support of at least one measure solving a given finite truncated trigonometric moment problem (Theorem 4.9). This last problem can be solved through an algorithmic procedure which is particularly efficient in dimension 1 (see Theorem 4.13), but works in arbitrary dimension (see Lemma 4.7). It involves the computation of the common zero set of certain families of trigonometric polynomials with spectrum in $\mathcal{S}$ that vanish on the support of the given measure, which has to be a finite set. We should mention that, although the results established in this paper simplify considerably the problem to decide whether a given measure is determinate or not with respect to a given moment problem, the algebraic conditions that need to be verified are far from being trivial from the computational point of view, because of the positivity restrictions that the trigonometric polynomials involved need to satisfy (see Section 4).

The paper is organized as follows. In Section 2, we review some basic facts about trigonometric moment problems and give a characterization for the extreme points of the convex set consisting of the positive measures on $\mathbb{T}^n$ solving a given moment problem. In Section 3, we consider the general truncated trigonometric moment problem and establish necessary and sufficient conditions for the determinacy of a given measure (Theorem 3.1). We also characterize the subsets of $\mathbb{Z}^n$ having the property that every non-trivial moment problem associated with that subset is indeterminate. Those are in fact the subsets contained in a proper subgroup of $\mathbb{Z}^n$ (Theorem 3.5). The case of finite truncated moment problems is considered in detail in Section 4 and some examples are worked out explicitly.

We briefly review the notation and some basic facts that will be used in this paper. We will identify the $n$-dimensional torus with the set

$$
\mathbb{T}^n = \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n, |z_i| = 1, i = 1, \ldots, n\}.
$$
If $z \in \mathbb{T}^n$ and $k \in \mathbb{Z}^n$, we denote by $z^k$ the complex number of modulus 1 $z_1^{k_1} \cdots z_n^{k_n}$. If $F$ is a set, we denote by $|F|$ its cardinality. If $\mu$ is a Borel measure on $\mathbb{T}^n$, one can associate with it a continuous linear functional acting on the space of complex-valued continuous functions defined on $\mathbb{T}^n$ by the formula

$$
\langle \mu, \phi \rangle = \int_{\mathbb{T}^n} \phi \, d\mu,
$$

(1.1)

for $\phi$ continuous on $\mathbb{T}^n$ and, by the Riesz representation theorem, any such functional can be associated with a Borel measure on $\mathbb{T}^n$ as in (1.1). If $\mu$ is a Borel measure on $\mathbb{T}^n$, we define its norm to be

$$
\|\mu\| = \int_{\mathbb{T}^n} 1 \, d|\mu|,
$$

where $|\mu|$ denotes the total variation of $\mu$ and coincides with $\mu$ if $\mu$ is positive. We will denote by $m$ the normalized Lebesgue measure on $\mathbb{T}^n$. It is the unique translation invariant measure (with respect to the group structure of $\mathbb{T}^n$) that has total mass 1, i.e., such that $\langle m, 1 \rangle = 1$. If $f$ is a function integrable with respect to $m$, we associate with it the measure $f \, dm$ defined by

$$
\langle f \, dm, \phi \rangle = \int_{\mathbb{T}^n} f \phi \, dm.
$$

If $\alpha \in \mathbb{T}^n$, we denote by $\delta_\alpha$ the Dirac mass at $\alpha$, i.e., the positive measure on $\mathbb{T}^n$ defined by

$$
\langle \delta_\alpha, \phi \rangle = \phi(\alpha).
$$

If $\mu$ is a Borel measure on $\mathbb{T}^n$ and $k \in \mathbb{Z}^n$, we define the $k$th Fourier coefficient (or $k$th moment) of $\mu$, to be the number $\hat{\mu}(k) = \langle \mu, z^{-k} \rangle$ and $\mu$ has the Fourier series expansion

$$
\mu = \sum_{k \in \mathbb{Z}^n} \hat{\mu}(k) z^k,
$$

where the series converges in the weak-star topology. Given two Borel measures, $\mu$ and $\nu$ on $\mathbb{T}^n$, one can define their convolution product on $\mathbb{T}^n$, $\mu * \nu$, which is again a Borel measure on $\mathbb{T}^n$ and satisfies $(\mu * \nu)(k) = \hat{\mu}(k) \hat{\nu}(k)$, for all $k \in \mathbb{Z}^n$. Finally, if $\mu$ is a positive Borel measure on $\mathbb{T}^n$ and $1 \leq p \leq \infty$, we will denote by $L^p_\mu$ the classical Lebesgue spaces of functions defined $\mu$-almost everywhere on $\mathbb{T}^n$ whose $p$th power is $\mu$-integrable, when $1 \leq p < \infty$, and that are essentially bounded $\mu$-almost everywhere when $p = \infty$. When $\mu = m$, we will use the notation $L^p(\mathbb{T}^n)$ instead of $L^p_m$. 
2. TRUNCATED TRIGONOMETRIC MOMENT PROBLEMS

Let us consider a positive Borel measure $\mu$ on $\mathbb{T}^n$ and a subset $S$ of $\mathbb{Z}^n$ which is symmetric, i.e., $0 \in S$ and $k \in S$ if and only if $-k \in S$. We say that the measure $\mu$ is $S$-determinate if the equalities

$$\int_{\mathbb{T}^n} z^k \, d\mu = \int_{\mathbb{T}^n} z^k \, d\nu, \quad k \in S, \quad (2.1)$$

for some positive Borel measure $\nu$ on $\mathbb{T}^n$, imply that $\mu = \nu$. If $\mu$ is not $S$-determinate, $\mu$ will be called $S$-indeterminate. We note that the symmetry assumption on the set $S$ is not really restrictive since, if a moment $\hat{\mu}(k)$ is known for a positive measure $\mu$, so is the moment $\hat{\mu}(-k) = \overline{\hat{\mu}(k)}$ and, if $0$ did not belong to $S$, every measure would be $S$-indeterminate since the measure $\mu + C$ would have the same $S$-moments as that of $\mu$ when $C > 0$. Every positive Borel measure is $\mathbb{Z}^n$-determinate, since every measure is completely determined by its Fourier coefficients. However, if $S \neq \mathbb{Z}^n$, there always exist $S$-indeterminate measures. For example, if $k \notin S$, the measures associated with the functions $f(z) = 1$ and $g(z) = 1 + \text{Re}(z^k)/2$, for $z \in \mathbb{T}^n$, have the same $S$-moments. On the other hand, there exist sets $S$ with the property that the only $S$-determinate measure is the 0-measure (of course, the 0-measure is trivially always $S$-determinate since $S$ contains zero). In fact, we will show that this happens exactly when $S$ is contained in a proper subgroup of $\mathbb{Z}^n$ (Theorem 3.5). Given a symmetric set $S \subset \mathbb{Z}^n$, we can define an associated space of trigonometric polynomials $\Pi_S$ that we view as a set of functions on $\mathbb{T}^n$.

**Definition 2.1.** If $S \subset \mathbb{Z}^n$ is a symmetric set, we define

$$\Pi_S = \left\{ P(z) = \sum_{k \in F} a_k z^k, \quad a_k \in \mathbb{C}, \quad F \subset S, \quad |F| < \infty \right\}.$$

$\Pi_S$ is thus the complex vector space consisting of the trigonometric polynomials with spectrum in $S$ and it has dimension $|S|$ if $S$ is a finite set. Note that a polynomial $P(z) = \sum_{k \in S} a_k z^k$ is real if and only if $a_{-k} = \overline{a_k}$, for all $k \in S$, and thus, since $S$ is symmetric, both $\text{Re}(P)$ and $\text{Im}(P)$ are in $\Pi_S$ if $P$ itself belongs to $\Pi_S$.

**Definition 2.2.** A linear functional $L$ on $\Pi_S$ is called positive if $L(P) \geq 0$ for all $P \in \Pi_S$ with $P \geq 0$ on $\mathbb{T}^n$.

It is well known that positive functionals can be associated with positive Borel measures on $\mathbb{T}^n$ (see [16]).
Proposition 2.3. Given a symmetric set $S \subset \mathbb{Z}^n$, a linear functional $L$ on $\Pi_S$ is positive if and only if there exists a positive Borel measure $\mu$ on $\mathbb{T}^n$ such that

$$L(P) = \int_{\mathbb{T}^n} P \, d\mu, \quad \forall P \in \Pi_S. \quad (2.2)$$

Given a positive Borel measure $\mu$ on $\mathbb{T}^n$, we will denote by $\Phi_\mu$ the positive linear functional on $\Pi_S$ defined by the identity (2.2). It is clear that, given two positive Borel measures, $\mu$ and $\nu$, on $\mathbb{T}^n$, we have $\Phi_\mu = \Phi_\nu$ on $\Pi_S$ if and only (2.1) holds.

Definition 2.4. If $S$ is a fixed symmetric set in $\mathbb{Z}^n$ and $\mu$ is a positive Borel measure on $\mathbb{T}^n$, we will denote by $F_\mu$ the positive linear functional on $\Pi_S$ defined by the identity 2.2. It is clear that, given two positive Borel measures, $\mu$ and $\nu$, on $\mathbb{T}^n$, we have $F_\mu = F_\nu$ on $\Pi_S$ if and only if (2.1) holds.

Proposition 2.5. A positive Borel measure $\nu$ is an extreme point of $\mathcal{M}_S(\mu)$ if and only if $\nu$ is dense in $L^1_\mu$.

Proof. If $\nu \in \mathcal{M}_S(\mu)$ and $\Pi_S$ is not dense in $L^1_\mu$, there exists, using the Hahn–Banach theorem, a non-zero function $g \in L^\infty_\mu$ such that

$$\int_{\mathbb{T}^n} P g \, d\nu = 0, \quad \forall P \in \Pi_S.$$

We can assume that $\|g\|_\infty \leq 1/2$ and, since $S$ is symmetric, that $g$ is real. Defining $\nu_1 = (1 + g)\nu$ and $\nu_2 = (1 - g)\nu$, it follows that $\nu \neq \nu_i$ for $i = 1, 2$, while the identities $\nu = (\nu_1 + \nu_2)/2$ and $\Phi_\nu = \Phi_{\nu_1} = \Phi_{\nu_2}$ hold, which shows that $\nu$ is not an extreme point of $\mathcal{M}_S(\mu)$. Conversely, if $\nu \in \mathcal{M}_S(\mu)$ and $\Pi_S$ is dense in $L^1_\nu$, assume that $\nu = (\nu_1 + \nu_2)/2$, where $\nu_i \in \mathcal{M}_S(\mu)$, for $i = 1, 2$. Clearly, both $\nu_1$ and $\nu_2$ are absolutely continuous with respect to $\nu$. There exist two non-negative functions $g_1$ and $g_2$ in $L^1_\nu$ such that $\nu = g_i \nu$, for $i = 1, 2$. Hence, it follows that, for $i = 1, 2$,

$$\int_{\mathbb{T}^n} P (1 - g_i) \, d\nu = 0, \quad \forall P \in \Pi_S.$$

Using the density of $\Pi_S$ in $L^1_\nu$, we deduce that $g_1 = g_2 = 1$ $\nu$-almost everywhere and thus that $\nu = \nu_1 = \nu_2$, i.e., $\nu$ is an extreme point of $\mathcal{M}_S(\mu)$. \hfill \Box
In the case where the set $S$ is finite, the previous result has an important consequence: any extreme point of $\mathcal{M}_S(\mu)$ must consist of finitely many atoms.

**Corollary 2.6.** If $S$ is a finite set and $\mu$ is any positive Borel measure on $\mathbf{T}^n$, any extreme point $\nu$ of $\mathcal{M}_S(\mu)$ has the form

$$\nu = \sum_{\alpha \in F} c_\alpha \delta_\alpha,$$

where $F \subset \mathbf{T}^n$, $|F| \leq |S|$, and $c_\alpha > 0$, for all $\alpha \in F$.

**Proof.** This follows simply from the fact that the identity mapping from $\Pi_S$ to $L_1^\mu$ must be onto and thus the dimension of the space $L_\nu^1$ is at most $|S|$. 

3. **Indeterminate Measures Associated with a General Symmetric Set**

In this section, we consider the case of a general symmetric set $S \subset \mathbf{Z}^n$ and formulate conditions which are necessary and sufficient for a positive Borel measure on $\mathbf{T}^n$ to be $S$-determinate.

**Theorem 3.1.** Let $S \subset \mathbf{Z}^n$ be a symmetric set and let $\mu$ be a positive Borel measure on $\mathbf{T}^n$. Then $\mu$ is $S$-determinate if and only if

(a) $\Pi_S$ is dense in $L_1^\mu$.

(b) Every positive Borel measure $\nu$ on $\mathbf{T}^n$ for which there exists a constant $C > 0$ such that

$$\int_{\mathbf{T}^n} P d\nu \leq C \int_{\mathbf{T}^n} P d\mu, \forall P \in \Pi_S, P \geq 0 \text{ on } \mathbf{T}^n \quad (3.1)$$

has the form $\nu = g \mu$, where $g \in L_\mu^\infty$ and $g \geq 0 \text{ a.e. } d\mu$.

Furthermore, if $S$ is finite, (b) can be replaced by the following weaker condition.

(b') Every $\beta \in \mathbf{T}^n$ for which there exists a constant $C > 0$ such that

$$P(\beta) \leq C \int_{\mathbf{T}^n} P d\mu, \forall P \in \Pi_S, P \geq 0 \text{ on } \mathbf{T}^n \quad (3.2)$$

must satisfy $\mu(\{\beta\}) > 0$. 

Proof. Suppose that $\mu$ is $S$-determinate, then $\mathcal{M}_S(\mu) = \{\mu\}$ and thus $\mu$ is, trivially, an extreme point of $\mathcal{M}_S(\mu)$. Hence, $\Pi_S$ is dense in $L^1_\mu$ by Proposition 2.5. If $\nu$ is a positive Borel measure on $\mathbb{T}^n$ satisfying (3.1) for some constant $C > 0$, the linear form $L$ on $\Pi_S$, defined by

$$L(P) = \int_{\mathbb{T}^n} P d\mu - C^{-1} \int_{\mathbb{T}^n} P d\nu, \quad P \in \Pi_S,$$

is positive. By Proposition 2.3, there exists thus a positive Borel measure $\sigma$ on $\mathbb{T}^n$ such that

$$\int_{\mathbb{T}^n} P d\mu = C^{-1} \int_{\mathbb{T}^n} P d\nu + \int_{\mathbb{T}^n} P d\sigma, \quad P \in \Pi_S.$$

Since $\mu$ is $S$-determinate, it follows that

$$\mu = C^{-1} \nu + \sigma. \quad (3.3)$$

Clearly, $\nu$ is then absolutely continuous with respect to $\mu$ and there exists thus a function $g \geq 0$ a.e. $d\mu$ such that $\nu = g \mu$. Using (3.3), it is easily seen that $g \leq C$ a.e. $d\mu$. Hence, (b) holds (as well as the weaker statement (b')). Conversely, let us assume that $\mu$ is $S$-indeterminate and that $\Pi_S$ is dense in $L^1_\mu$. Let $\nu \in \mathcal{M}_S(\mu)$ with $\nu \neq \mu$. The inequality (3.1) is then trivially satisfied with $C = 1$. If $\nu = g \mu$, for some $g \in L^*_\mu$, it would follow that

$$\int_{\mathbb{T}^n} P(1 - g) d\mu = 0, \quad \forall P \in \Pi_S. \quad (3.4)$$

Using the density of $\Pi_S$ in $L^1_\mu$, we would then deduce from (3.4) that $g = 1$ a.e. $d\mu$, i.e., that $\nu = \mu$, which brings about a contradiction. Hence, (b) fails. Assuming now that $S$ is finite, that $\mu$ is $S$-indeterminate and that $\Pi_S$ is dense in $L^1_\mu$, we need to show that (b') fails. Let $\Lambda$ be the support of $\mu$ (which is then a finite set). We claim that $\mu$ is the only measure in $\mathcal{M}_S(\mu)$ with support contained in $\Lambda$. Indeed, if $\nu$ is such a measure, we can find, for any $\alpha \in \Lambda$, using the density of $\Pi_S$ in $L^1_\mu$, a polynomial $P_\alpha \in \Pi_S$ such that $P_\alpha(\alpha) = 1$ and $P_\beta(\beta) = 0$, for all $\beta \in \Lambda$ with $\beta \neq \alpha$. Hence,

$$\nu(\alpha) = \int_{\mathbb{T}^n} P_\alpha d\nu = \int_{\mathbb{T}^n} P_\alpha d\mu = \mu(\alpha), \quad \alpha \in \Lambda,$$

which shows that $\mu = \nu$. Now let $\sigma$ be an extreme point of $\mathcal{M}_S(\mu)$ which is different from $\mu$. Such an extreme point must exist since $\mu$ is assumed to be $S$-indeterminate and, since $S$ is finite, it must consist of finitely many atoms. Also, by the previous argument, there must exist at least a point $\beta$
in the support of $\sigma$ with $\beta \in \Lambda$. We have thus

$$P(\beta) = \sigma(\{\beta\})^{-1} \int_{\mathbb{T}^n} P d\sigma = \sigma(\{\beta\})^{-1} \int_{\mathbb{T}^n} P d\mu,$$

for all $P \in \Pi_\infty$ with $P \geq 0$ on $\mathbb{T}^n$ while $\mu(\beta) = 0$. Hence, (b') must fail.

**Proposition 3.2.** Let $S \subset \mathbb{Z}^n$ be a symmetric set and let $\mu_1, \mu_2$ be positive Borel measures on $\mathbb{T}^n$.

(a) If $\mu_1$ is $S$-determinate and $\mu_2 \leq C\mu_1$, for some constant $C > 0$ (i.e., $C\mu_1 - \mu_2$ is a positive measure), then $\mu_2$ is $S$-determinate.

(b) If $S$ is finite, the property of $S$-determinacy is a property of the support of a measure; i.e., if $\Lambda$ is the support of some $S$-determinate measure, then every positive Borel measure on $\mathbb{T}^n$ with support $\Lambda$ is also $S$-determinate.

**Proof.** To prove part (a), we can assume, by dividing $\mu_2$ by $C$ if necessary, that $C = 1$. Thus $\mu_1 = \mu_2 + \sigma$, for some positive Borel measure $\sigma$ on $\mathbb{T}^n$. If $\mu_2$ were $S$-indeterminate, we could find a positive Borel measure $\nu_2$, different from $\mu_2$, that belongs to $\mathcal{M}_S(\mu_2)$. The measure $\nu_1 = \nu_2 + \sigma$ would then belong to $\mathcal{M}_S(\mu_1)$, contradicting the fact that $\mu_1$ is determinate. This proves part (a). If $S$ is finite and $\mu_1$ is $S$-determinate, then the support of $\mu$ is a finite set $\Lambda$ (see Corollary 2.6) and thus if $\mu_2$ is another positive measure with same support $\Lambda$, there exists a constant $C > 0$ such that $\mu_2 \leq C\mu_1$. Therefore, $\mu_2$ is determinate by part (a), which proves part (b).

**Examples 3.3.** The following examples show that the conditions (a) and (b), and also (a) and (b'), of Theorem 3.1 are independent of each other while the condition (b') is in general strictly weaker than (b).

Consider the one-dimensional symmetric set $S = \mathbb{Z} \setminus \{\pm 1\}$. Clearly, in this case, a measure $\mu$ is $S$-indeterminate if and only if there exists a non-zero complex number $a$ such that $\mu + \text{Re}(az)$ is a positive measure. If $h \in L^1(\mathbb{T})$ denotes the absolutely continuous part of $\mu$, that condition is then equivalent to $h(z) \geq -\text{Re}(az)$ for a.e. $z \in \mathbb{T}$. The measure $d\mu = (1 + \text{Re}(z)) \, dm$, where $m$ denotes the normalized Lebesgue measure on $\mathbb{T}^n$, is thus $S$-indeterminate since

$$1 + \text{Re}(z) + \text{Re}(az) = 1 + \text{Re}((1 + a)z) \geq 0, \quad z \in \mathbb{T},$$

if $|1 + a| \leq 1$ and, clearly, $\mathcal{M}_S(\mu) = \{(1 + \text{Re}(cz)) \, dm, \ c \in \mathbb{C}, \ |c| \leq 1\}$. We claim that $\Pi_S$ is dense in $L^1_{\mu}$. Indeed if $g \in L^1_{\mu}$, i.e., $g \in L^1(\mathbb{T})$, and

$$\int_{\mathbb{T}} z^k g(z)(1 + \text{Re}(z)) \, dm = 0, \quad \forall k \in S,$
it follows that
\[ g(z)(1 + \Re(z)) = \Re(bz), \quad z \in \mathbb{T} \]
for some constant \( b \in \mathbb{C} \), if we also assume that \( g \) is real, which we can always do since \( S \) is symmetric. If \( g \neq 0 \), we have \( b \neq 0 \) and, for some constant \( C > 0 \), we have the inequality
\[
|\Re(bz)| \leq C|1 + \Re(z)|, \quad z \in \mathbb{T}. \tag{3.5}
\]
Letting \( z = -1 \) in the previous inequality, we deduce that \( b \) is a non-zero imaginary constant. (3.5) then becomes
\[
|\Im(z)| \leq C|1 + \Re(z)|, \quad z \in \mathbb{T}
\]
(with a possibly different constant \( C \)), but this last inequality clearly fails near the point \( z = -1 \). Hence, \( g = 0 \) and, using the Hahn–Banach theorem, we deduce that \( \Pi_S \) is dense in \( L^1_\mu \) and thus the condition (a) of Theorem 3.1 holds for \( \mu \). However, (b) fails since, if \( dv = dm \), the inequality (3.1) actually becomes an equality when \( C = 1 \) but \( dm \neq f d\mu \), for any \( f \in L^1_\mu \). We note also that, in this situation, the weaker condition (b’) holds for \( \mu \). Indeed, no \( \beta \in \mathbb{T} \) can satisfy the inequality (3.2), since, if such \( \beta \) existed, it would follow, by considering the positive linear form \( L \) on \( \Pi_S \) defined by
\[
L(P) = \int_{\mathbb{T}} P d\mu - C^{-1} P(\beta), \quad P \in \Pi_S,
\]
and using Proposition 2.3, that there exists an element \( \nu \) of \( \mathcal{M}_S(\mu) \) satisfying \( \nu(\beta) > 0 \). This contradicts the fact that all the elements of \( \mathcal{M}_S(\mu) \) are absolutely continuous with respect to the Lebesgue measure.

If \( S \) is the one-dimensional symmetric set \( \{0, \pm 1, \pm i\} \) and \( \mu = \delta_1 + \delta_{-1} \), then \( \mu \) is \( S \)-indeterminate since, as can be easily checked, the measure \( \nu = \delta_1 + \delta_{-1} \) belongs to \( \mathcal{M}_S(\mu) \). In this case, it is easily seen that \( \Pi_S \) is dense in \( L^1_\mu \), but the set of points \( \beta \in \mathbb{T} \) for which (3.2) holds is the set \( \{\pm 1, \pm i\} \). To see this, we note that the validity of the inequality (3.2) for some \( \beta \) is equivalent to the existence of a measure \( \sigma \in \mathcal{M}_S(\mu) \) with \( \sigma(\{\beta\}) > 0 \) (as shown in the proof of Theorem 3.1). Since in this special case, we have
\[
\int_{\mathbb{T}} 1 d\sigma = 2, \quad \int_{\mathbb{T}} z^4 d\sigma = 2
\]
if \( \sigma \in \mathcal{M}_S(\mu) \), it follows that the support of \( \sigma \) must be contained in the set \( \{\pm 1, \pm i\} \), and conversely, every point in that set belongs to the support of
some measure in $\mathcal{M}_S(\mu)$. Since the set $\{\pm 1, \pm i\}$ is strictly larger than the support of $\mu$, the condition (b') fails for $\mu$. For the same set $S$, the measure $\mu_2 = \delta_{1} + \delta_{-1} + \delta_{1} + \delta_{-1} = \mu + \nu$ is also $S$-indeterminate, since $2\mu \in \mathcal{M}_S(\mu_2)$. Note that (b') of Theorem 3.1 holds for $\mu_2$ since, by an argument similar to that given for $\mu$, any measure in $\mathcal{M}_S(\mu_2)$ must be supported in the set $\{\pm 1, \pm i\}$. However, the condition (a) of Proposition (3.1) fails for $\mu_2$. Indeed, it is easily checked that the subspace of $\Pi_S$ consisting of the polynomials in $\Pi_S$ vanishing on the set $\{\pm 1, \pm i\}$ is the two-dimensional subspace spanned by the polynomials $P_1(z) = z^4 - 1$ and $P_2(z) = z^4 - 1$. Therefore, since the kernel of the natural map from $\Pi_S$ to $L^1_{\mu_2}$ has dimension 2, the dimension of the image is $|S| - 2 = 3$ while $\dim L^1_{\mu_1} = 4$.

Our next goal is to characterize the symmetric sets $S$ with the property that every non-zero positive Borel measure on $\mathbf{T}^n$ is $S$-indeterminate.

**Definition 3.4.** If $S$ is a subset of $\mathbf{Z}^n$, we denote by $\mathcal{G}(S)$ the subgroup of $\mathbf{Z}^n$ generated by $S$, i.e., the smallest subgroup of $\mathbf{Z}^n$ containing $S$.

**Theorem 3.5.** Given a symmetric set $S \subset \mathbf{Z}^n$, the following are equivalent:

(a) Every non-zero positive Borel measure on $\mathbf{T}^n$ is $S$-indeterminate.

(b) The measure $\delta_1$ is $S$-indeterminate, where 1 denotes the identity element of the group $\mathbf{T}^n$.

(c) $\mathcal{G}(S) \neq \mathbf{Z}^n$.

**Proof.** The statement (a) obviously implies (b). If (b) holds, let $\nu$ be a positive Borel measure in $\mathcal{M}_S(\delta_1)$ with $\nu \neq \delta_1$. Then, for every $P \in \Pi_S$, we have the identity

$$P(1) = \int_{\mathbf{T}^n} P \, d\nu. \quad (3.6)$$

Let $S_j$, $j \geq 1$, be an increasing sequence of finite symmetric sets such that $\bigcup_{j \geq 1} S_j = S$. Noticing that the polynomial $P_j(z) = \sum_{k \in S_j} |z^k - 1|^2$ belongs to $\Pi_{S_j}$ is non-negative on $\mathbf{T}^n$ and vanishes at $z = 1$, we obtain, replacing $P$ by $P_j$ in (3.6), that the support of $\nu$ is contained in the set $\{z \in \mathbf{T}^n, z^k = 1, \forall k \in S\}$. Since $\nu \neq \delta_1$, there exists thus at least a point $\alpha \neq 1$ in the support of $\nu$ such that $\alpha^k = 1$, for all $k \in S$. This shows that $\mathcal{G}(S) \neq \mathbf{Z}^n$.

Finally, if (c) holds, there exists some $\alpha \neq 1$ in $\mathbf{T}^n$ such that $S \subset \{k \in \mathbf{Z}^n, \alpha^k = 1\}$. Let $\mu$ be any non-zero positive Borel measure on $\mathbf{T}^n$. The measure $\mu_\alpha = \delta_\alpha * \mu$, where $*$ denotes the convolution product on $\mathbf{T}^n$, belongs to $\mathcal{M}_S(\mu)$ since $\hat{\mu}_\alpha(k) = \alpha^{-k} \hat{\mu}(k) = \hat{\mu}(k)$ if $k \in S$. Thus $\mu$ is $S$-indeterminate if $\mu_\alpha \neq \mu$. On the other hand, if $\mu_\alpha = \mu$, then $\hat{\mu}(k) = 0$.
unless $\alpha^k = 1$. Therefore, if $m \in \mathbb{Z}^n$ and $\alpha^m \neq 1$, we have

$$\int_{\mathbb{T}^n} z^k \overline{z}^m \, d\mu = \int_{\mathbb{T}^n} z^{k+m} \, d\mu = 0, \quad k \in S.$$
DEFINITION 4.3. Given a finite set $\Lambda \subset \mathbb{T}^n$ and another set $\Omega$ with $\Lambda \subset \Omega \subset \mathbb{T}^n$, we say that $\Omega$ has the property $P$ if every polynomial $P \in \Pi_S$ such that $P = 0$ on $\Lambda$ and $P \geq 0$ on $\Omega$ satisfies $P = 0$ on $\Omega$.

Clearly, $\Lambda$ itself satisfies property $P$ and the class of sets that satisfy property $P$ is closed under arbitrary unions. Furthermore, if $\Omega$ satisfies $P$, so does its closure $\overline{\Omega}$. This motivates the following definition.

DEFINITION 4.4. Given a finite set $\Lambda \subset \mathbb{T}^n$, we define the set $\mathcal{B}(\Lambda)$ to be the largest subset of $\mathbb{T}^n$ that satisfies $P$. It is simply the union of all subsets of $\mathbb{T}^n$ that satisfies $P$ and it is a closed subset since its closure must also satisfy $P$.

The following lemma provides a different characterization of sets that satisfy property $P$.

LEMMA 4.5. Suppose that $S \subset \mathbb{Z}^n$ is a finite symmetric set and that $\Lambda \subset \mathbb{T}^n$ is an arbitrary finite set. Consider a set $\Omega$ such that $\Lambda \subset \Omega \subset \mathbb{T}^n$. Then, the following are equivalent,

(a) $\Omega$ satisfies property $P$.

(b) For every $\beta \in \Omega$ and for every positive measure $\mu$ with support $\Lambda$, there exists a positive constant $C = C(\beta, \mu)$ such that

$$P(\beta) \leq C \int_{\mathbb{T}^n} P \, d\mu, \quad \forall P \in \Pi_S, P \geq 0 \text{ on } \Omega. \quad (4.2)$$

Proof. The implication (b) $\Rightarrow$ (a) is easy. Let $\beta \in \Omega$ and assume that (b) holds. If $P \in \Pi_S$ is such that $P = 0$ on $\Lambda$ and $P \geq 0$ on $\Omega$, then it follows from (4.2) that $P(\beta) \leq 0$, and thus that $P(\beta) = 0$, since $\beta \in \Omega$. Hence (a) holds. Conversely, if (a) holds, consider the subspace of $\Pi_S$ defined by $N(\Omega) = \{P \in \Pi_S, P = 0 \text{ on } \Omega\}$, and the quotient space $\Pi_S/N(\Omega)$ with norm defined by

$$\|\tilde{P}\| = \sup_{\beta \in \Omega} |P(\beta)|,$$

for $\tilde{P} = P + N(\Omega) \in \Pi_S/N(\Omega)$. Let

$$K = \{\tilde{P} \in \Pi_S/N(\Omega), P \geq 0 \text{ on } \Omega, \|\tilde{P}\| = 1\}.$$

Note that the definition of $K$ does not depend on the representative $P$ chosen; i.e., if $\tilde{P} = Q$ and $P \geq 0$ on $\Omega$, then $Q \geq 0$ on $\Omega$. We claim that $K$ is compact. Indeed, $K$ is clearly bounded in $\Pi_S/N(\Omega)$ and it is also closed since if $\tilde{P}_n \in K$ and $\tilde{P}_n \to \tilde{P}$, then $P_n \to P$ uniformly on $\Omega$ and thus $P \geq 0$ on $\Omega$. Hence, since we have also $\|\tilde{P}\| = \lim \|\tilde{P}_n\| = 1$, it follows that
\[ \tilde{P} \in K. \] Being a closed and bounded subset of a finite dimensional space, \( K \) is therefore compact. Let us consider the mapping \( \Phi: K \to \mathbb{R} \) defined by

\[ \Phi(\tilde{P}) = \int_{\mathbb{T}} P d\mu, \]

for \( \tilde{P} = P + N(\Omega) \in \Pi_\delta/N(\Omega) \). Note that \( \Phi \) is well defined since \( \Lambda \subset \Omega \) and it is also continuous since

\[ |\Phi(\tilde{P})| \leq \|\mu\| \sup_{\beta \in \Omega} |P(\beta)| = \|\mu\| \|\tilde{P}\|. \]

Let \( t = \inf_{\tilde{P} \in K} \Phi(\tilde{P}) \). We claim that \( t > 0 \). Indeed, if \( t = 0 \), there would exist, by compactness, an element \( \tilde{P}_0 \in K \) such that \( \Phi(\tilde{P}_0) = 0 \). Since \( P_0 \geq 0 \) on \( \Omega \), this implies that \( P_0 = 0 \) on \( \Lambda \) and, thus, that \( P_0 = 0 \) on \( \Omega \) by (a), but this contradicts the fact that \( \|P_0\| = 1 \). We have thus shown that, if \( P \in \Pi_\delta \) and \( P \geq 0 \) on \( \Omega \), then

\[ \int_{\mathbb{T}} P d\mu \geq t \sup_{\beta \in \Omega} |P(\beta)|, \]

and, in particular, that (b) holds with \( C = 1/t \).

Before getting to the main result of this section, we need to introduce another family of sets.

**Definition 4.6.** If \( S \subset \mathbb{Z}^n \) is a finite symmetric set and \( \Lambda \subset \mathbb{T}^n \) is finite, we define

\[ \mathcal{E}_1(\Lambda) = \{\beta \in \mathbb{T}^n, P(\beta) = 0, \forall P \in \Pi_\delta, \text{ with } P \geq 0 \text{ on } \mathbb{T}^n, P = 0 \text{ on } \Lambda\} \]

and, for every integer \( k \geq 1 \),

\[ \mathcal{E}_{k+1}(\Lambda) = \{\beta \in \mathbb{T}^n, P(\beta) = 0, \forall P \in \Pi_\delta, \text{ with } P \geq 0 \text{ on } \mathcal{E}_k(\Lambda), P = 0 \text{ on } \Lambda\}. \]

We also define \( \mathcal{E}(\Lambda) = \bigcap_{k \geq 1} \mathcal{E}_k(\Lambda) \).

An easy induction argument shows that \( \mathcal{E}_{k+1}(\Lambda) \subset \mathcal{E}_k(\Lambda) \), for every \( k \geq 1 \). The following lemma shows, in particular, that the sequence of sets \( \mathcal{E}_1(\Lambda), \mathcal{E}_2(\Lambda), \ldots \) becomes constant after a finite number of steps.
Lemma 4.7. If the sets $\mathcal{C}(k)(\Lambda)$, $k \geq 1$, and $\mathcal{C}(\Lambda)$ are as in Definition 4.6 and $N = \dim(P \in \Pi_k, P = 0$ on $\Lambda$), then there exists an integer $k_0$ with $1 \leq k_0 \leq N + 1$ such that $\mathcal{C}(k)(\Lambda) = \mathcal{C}(k)(\Lambda)$ for $k \geq k_0$. Furthermore,

$$\mathcal{C}(\Lambda) = \{ \beta \in \mathbb{T}^n, P(\beta) = 0, \forall P \in \Pi_k, \text{ with } P \geq 0 \text{ on } \mathcal{C}(\Lambda), P = 0 \text{ on } \Lambda \}.$$  \hfill (4.3)

Proof. We will argue by contradiction. Suppose that $\mathcal{C}(k+1)(\Lambda) \subsetneq \mathcal{C}(k)(\Lambda)$ for $1 \leq k \leq N + 1$. This means that, for each $k = 1, \ldots, N + 1$, we can find a polynomial $P_k \in \Pi_k$, with the properties that $P_k = 0$ on $\Lambda$, $P_k \geq 0$ on $\mathcal{C}(k)(\Lambda)$, and $P_k(\beta_k) \neq 0$ for some $\beta_k \in \mathcal{C}(k)(\Lambda)$. Note that, by definition of $\mathcal{C}(k+1)(\Lambda)$, $P_k = 0$ on $\mathcal{C}(k+1)(\Lambda)$. Suppose now that

$$\sum_{k=1}^{N+1} a_k P_k = 0,$$ \hfill (4.4)

for some constants $a_1, \ldots, a_{N+1}$. Evaluating (4.4) at the point $\beta_{N+1} \in \mathcal{C}(N+1)(\Lambda)$ and using the fact that $P_k = 0$ on $\mathcal{C}(N+1)(\Lambda)$ if $1 \leq k \leq N$, we obtain that $a_{N+1} P_{N+1}(\beta_{N+1}) = 0$ and thus that $a_{N+1} = 0$. A repeated use of this argument, using successively the points $\beta_N, \ldots, \beta_1$, shows that $a_k = 0$, for each $k = 1, \ldots, N + 1$. We conclude therefore that the $N + 1$ polynomials $P_1, \ldots, P_{N+1}$ are linearly independent, but this contradicts the definition of $N$. Thus an integer $k_0$ with the properties claimed above must exist. Finally, the identity (4.3) follows from the fact that $\mathcal{C}(\Lambda) = \mathcal{C}(k_0)(\Lambda) = \mathcal{C}(k_0+1)(\Lambda)$.

The importance of the set $\mathcal{C}(\Lambda)$ stems from the following observation.

Lemma 4.8. If $S$ and $\Lambda$ are as in Definition 4.6 and $\mu$ is any positive measure whose support is the set $\Lambda$, then any measure $\nu \in \mathcal{M}(\mu)$ is supported in $\mathcal{C}(\Lambda)$.

Proof. Suppose that $P \in \Pi_k$ is a polynomial that satisfies $P = 0$ on $\Lambda$, $P \geq 0$ on $\mathbb{T}^n$, and $P(\beta) > 0$ at some point $\beta \in \mathbb{T}^n$. Then, it follows immediately from the identities

$$0 = \int_{\mathbb{T}^n} P d\mu = \int_{\mathbb{T}^n} P d\nu$$ \hfill (4.5)

that $\beta$ does not belong to the support of $\nu$. Thus $\text{supp}(\nu) \subset \mathcal{C}(\Lambda)$. Moreover, if we know that $\text{supp}(\nu) \subset \mathcal{C}(k)(\Lambda)$, for some $k \geq 1$, we deduce again from (4.5) that, if $P$ is a polynomial in $\Pi_k$ such that $P = 0$ on $\Lambda$, $P \geq 0$ on $\mathcal{C}(k)(\Lambda)$, and $P(\beta_k) > 0$, at some point $\beta_k \in \mathbb{T}^n$, then $\beta_k$ cannot belong to the support of $\nu$. Hence, $\text{supp}(\nu) \subset \mathcal{C}(k+1)(\Lambda)$, and, since this is true for all $k \geq 1$, we conclude that $\text{supp}(\nu) \subset \mathcal{C}(\Lambda)$.
We can now prove the main result of this section.

**Theorem 4.9.** Suppose that $S \subseteq \mathbb{Z}^n$ is a finite symmetric set and that $\Lambda \subseteq T^n$ is an arbitrary finite set. Then $\mathcal{A}(\Lambda) = \mathcal{B}(\Lambda) = \mathcal{C}(\Lambda)$. Furthermore, if $\mu$ is any positive measure with support $\Lambda$, then any measure in $\mathcal{M}_s(\mu)$ has its support contained in $\mathcal{C}(\Lambda)$ and conversely, given any point $\beta \in \mathcal{C}(\Lambda)$, there exists a measure $\nu \in \mathcal{M}_s(\mu)$ such that $\nu(\{\beta\}) > 0$.

**Proof.** We will first deduce the inclusions

$$\mathcal{B}(\Lambda) \subseteq \mathcal{A}(\Lambda) \subseteq \mathcal{C}(\Lambda) \subseteq \mathcal{B}(\Lambda),$$

which will prove the first statement of the theorem. Since $\mathcal{B}(\Lambda)$ satisfies property (P), there exists, by Lemma 4.5, for any $\beta \in \mathcal{B}(\Lambda)$ and any positive measure $\mu$ whose support is $\Lambda$, a constant $C > 0$ such that

$$P(\beta) \leq C \int_{T^n} P \, d\mu, \quad \forall P \in \Pi_S, \ P \geq 0 \text{ on } \mathcal{B}(\Lambda).$$

Therefore, the inequality (4.1) is also satisfied at the point $\beta$, since it involves a smaller collection of polynomials. Thus $\mathcal{B}(\Lambda) \subseteq \mathcal{A}(\Lambda)$. If $\beta \in \mathcal{A}(\Lambda)$, there exists a constant $C > 0$, such that the linear form $L$, defined by

$$L(P) = \int_{T^n} P \, d\mu - C^{-1}P(\beta), \quad P \in \Pi_S,$$

is positive on $\Pi_S$. By Proposition 2.3, there exists thus a positive Borel measure $\sigma$ on $T^n$ such that

$$\int_{T^n} P \, d\mu = C^{-1}P(\beta) + \int_{T^n} P \, d\sigma, \quad P \in \Pi_S.$$

Hence the measure $\nu = C^{-1}\delta_\beta + \sigma$ belongs to $\mathcal{M}_s(\mu)$ and its support is thus contained in $\mathcal{C}(\Lambda)$ by Lemma 4.8. Hence, $\beta \in \mathcal{C}(\Lambda)$ and $\mathcal{A}(\Lambda) \subset \mathcal{C}(\Lambda)$. Finally, $\mathcal{C}(\Lambda) \subset \mathcal{B}(\Lambda)$, since, by (4.3), $\mathcal{C}(\Lambda)$ satisfies property (P) and, by definition, $\mathcal{B}(\Lambda)$ is the largest subset of $T^n$ satisfying that property. The first part of the last statement made in the theorem follows from Lemma 4.8, while the second part follows from the inclusion $\mathcal{C}(\Lambda) \subset \mathcal{A}(\Lambda)$, which, as seen in the first part of the proof, allows for the construction of a measure $\nu$ with the stated property.

We remark that, of the definitions of the three sets $\mathcal{A}(\Lambda)$, $\mathcal{B}(\Lambda)$, and $\mathcal{C}(\Lambda)$, the only one which is constructible is that of $\mathcal{C}(\Lambda)$. It is given in terms of an algorithm that provides an effective construction of $\mathcal{C}(\Lambda)$, using the sets $\mathcal{C}_1(\Lambda), \mathcal{C}_2(\Lambda), \ldots$ which are obtained by considering the
common zero set of the polynomials in \( \Pi_S \) vanishing on \( \Lambda \) and satisfying certain positivity properties. The previous result, combined with our previous characterization of \( S \)-determinate measures, will allow us to prove the following theorem characterizing the \( S \)-determinate measures.

**Theorem 4.10.** Let \( S \subset \mathbb{Z}^n \) be a finite symmetric set and let \( \Lambda \subset \mathbb{T}^n \) be a finite set. If \( \alpha \in \mathbb{T}^n \), let \( \Phi_{\alpha} \) denote the linear functional \( P \mapsto P(\alpha) \) defined on \( \Pi_S \). Then, the following are equivalent:

(a) \( \Lambda \) is the support of an \( S \)-determinate measure \( \mu \).

(b) The collection \( \{\Phi_{\alpha}\}_{\alpha \in \Lambda} \) of linear functions on \( \Pi_S \) is linearly independent and \( \mathcal{B}(\Lambda) = \Lambda \).

(c) For every set \( \Lambda_1 \subset \Lambda \), \( \mathcal{B}(\Lambda_1) = \Lambda_1 \).

**Proof.** The fact that the collection \( \{\Phi_{\alpha}\}_{\alpha \in \Lambda} \) of linear functionals on \( \Pi_S \) is linearly independent is equivalent to density of \( \Pi_S \) in \( L^1_\mu \). The equivalence of (a) and (b) follows then immediately from Theorem 3.1 and Theorem 4.9. If \( \Lambda \) is the support of an \( S \)-determinate measure, then so is every subset \( \Lambda_1 \) of \( \Lambda \), by Proposition 3.2, and in particular, \( \mathcal{B}(\Lambda_1) = \Lambda_1 \). Hence (c) holds. Conversely, if (c) holds, then, clearly, \( \mathcal{B}(\Lambda) = \Lambda \). If the collection \( \{\Phi_{\alpha}\}_{\alpha \in \Lambda} \) of linear functional on \( \Pi_S \) was not linearly independent, we could find numbers \( a_\alpha \), \( \alpha \in \Lambda \), not all zero, such that

\[
\sum_{\alpha \in \Lambda} a_\alpha P(\alpha) = 0, \quad \forall P \in \Pi_S.
\]

Since \( P \in \Pi_S \) whenever \( P \in \Pi_S \), we can assume that all the \( a_\alpha \) are real numbers. Letting \( \Lambda_1 = \{\alpha \in \Lambda, a_\alpha \geq 0\} \) and \( \Lambda_2 = \Lambda \setminus \Lambda_1 \), we find thus that

\[
\sum_{\lambda \in \Lambda_1} |a_\lambda| P(\lambda) = \sum_{\lambda \in \Lambda_2} |a_\lambda| P(\lambda).
\]

Applying the previous identity to the polynomial \( P = 1 \), we deduce the strict inclusions \( \emptyset \subsetneq \Lambda_1 \subsetneq \Lambda \). That same identity shows then that \( \Lambda_2 \) is a set disjoint from \( \Lambda_1 \), but contained in \( \mathcal{B}(\Lambda_1) \), which would contradict (c). Hence, the collection \( \{\Phi_{\alpha}\}_{\alpha \in \Lambda} \) is linearly independent on \( \Pi_S \) and thus (b) holds, which concludes the proof.

**Remark 4.11.** We note that the identity \( \mathcal{B}(\Lambda) = \Lambda \) is equivalent to the following property: for any set \( \Omega \) satisfying \( \Lambda \subset \Omega \subset \mathbb{T}^n \), there must exist a polynomial \( P_0 \in \Pi_S \) such that \( P_0 = 0 \) on \( \Lambda \), \( P_0 \geq 0 \) on \( \Omega \), and \( P_0(\beta) > 0 \), for at least one point \( \beta \in \Omega \setminus \Lambda \). This follows from the identity \( \mathcal{B}(\Lambda) = \mathcal{B}(\Lambda) \).
In what follows, we investigate more thoroughly the algorithm for the construction of the set \( \mathcal{E}(\Lambda) \) via the decreasing sequence of sets \( \{ \mathcal{E}_k(\Lambda) \}_{k \geq 1} \). We will show, in particular, that in dimension \( n = 1 \), that sequence becomes constant after the first step, i.e., \( \mathcal{E}_k(\Lambda) = \mathcal{E}_1(\Lambda) \), for \( k \geq 2 \), while in higher dimensions, more steps might be needed. We need the following lemma.

**Lemma 4.12.** Let \( S \subset \mathbb{Z}^n \) be a finite symmetric set, let \( \Lambda \subset \mathbb{T}^n \) be a finite set, and consider the associated sequence \( \{ \mathcal{E}_k(\Lambda) \}_{k \geq 1} \) as defined in Definition 4.6. Let us define \( \mathcal{E}_0(\Lambda) = \mathbb{T}^n \). Suppose that for some \( k \geq 0 \), \( \mathcal{E}_{k+1}(\Lambda) \subset \mathcal{E}_k(\Lambda) \). Then, there exists a polynomial \( P_k \in \Pi_S \) such that \( P_k = 0 \) on \( \mathcal{E}_{k+1}(\Lambda) \) and \( P_k > 0 \) on \( \mathcal{E}_k(\Lambda) \setminus \mathcal{E}_{k+1}(\Lambda) \).

**Proof.** Consider the set \( M_k \) consisting of the non-identically zero polynomials in \( \Pi_S \) that vanish on \( \mathcal{E}_{k+1}(\Lambda) \) and are non-negative on \( \mathcal{E}_k(\Lambda) \). Choose a countable subset \( N_k = \{ Q_i, i = 1, 2, \ldots \} \) of \( M_k \), which is dense in \( M_k \) for the topology induced by the sup-norm. Define

\[
P_k = \sum_{i=1}^{\infty} 2^{-i} \frac{Q_i}{\|Q_i\|_{\infty}},
\]

Clearly, \( P_k \) belongs to \( M_k \) and vanishes on \( \mathcal{E}_{k+1}(\Lambda) \). Furthermore, if \( \beta \) belongs to \( \mathcal{E}_k(\Lambda) \setminus \mathcal{E}_{k+1}(\Lambda) \), there exists a polynomial \( P \in \Pi_S \) such that \( P = 0 \) on \( \Lambda \), \( P \geq 0 \) on \( \mathcal{E}_k(\Lambda) \), and \( P(\beta) > 0 \). If \( Q_i \in N_k \) is chosen so that \( \|P - Q_i\|_{\infty} < P(\beta)/2 \), it then follows that \( Q_i(\beta) > P(\beta)/2 \) and, in particular, that \( P_k(\beta) > 0 \), which proves the claim.

The following result shows that, in the one-dimensional case, the algorithm given above to construct \( \mathcal{E}(\Lambda) \) is very efficient: one simply needs to compute \( \mathcal{E}_1(\Lambda) \) and \( \mathcal{E}_2(\Lambda) \). Note that the proof given depends in a very essential way on the simple structure of the zeros of a non-zero one-dimensional trigonometric polynomial, which must be isolated.

**Theorem 4.13.** Let \( S \subset \mathbb{Z} \) be a finite one-dimensional symmetric set, let \( \Lambda \subset \mathbb{T} \) be a finite set, and consider the associated decreasing sequence \( \{ \mathcal{E}_k(\Lambda) \}_{k \geq 1} \) together with the set \( \mathcal{E}(\Lambda) = \cap_{k=1}^{\infty} \mathcal{E}_k(\Lambda) \). Then, either \( \mathcal{E}(\Lambda) = \mathcal{E}_1(\Lambda) \) or \( \mathcal{E}(\Lambda) = \mathcal{E}_2(\Lambda) \).

**Proof.** Let \( k_0 \geq 1 \) be the smallest integer such that \( \mathcal{E}_1(\Lambda) = \mathcal{E}_{k_0}(\Lambda) \), for \( k \geq k_0 \) (see Lemma 4.7). If \( k_0 = 1 \), then \( \mathcal{E}(\Lambda) = \mathcal{E}_1(\Lambda) \) and we are done. If \( k_0 \geq 2 \), then there exists at least one polynomial in \( \Pi_S \) that vanishes on \( \Lambda \), is non-negative on \( \mathbb{T} \), and is non-identically zero. Since we are in dimension 1 and \( \mathcal{E}_2(\Lambda) \) is contained in the zero set of that polynomial, it follows that \( \mathcal{E}_2(\Lambda) \) and, thus all the sets \( \mathcal{E}_k(\Lambda) \), \( k \geq 1 \), are finite sets. By Lemma 4.12, we can find polynomials \( P_1, \ldots, P_N \) in \( \Pi_S \).
where \( N = k_0 - 1 \), such that \( P_k > 0 \) on \( \mathcal{C}_{k_0 - k + 1}(\Lambda) \) and \( P_k = 0 \) on \( \mathcal{C}_{k_0 - k + 1}(L) \), for \( k = 1, \ldots, N \). We will now construct, by induction, polynomials \( Q_1, \ldots, Q_N \) in \( \Pi_S \) with the properties that \( Q_k = 0 \) on \( \mathcal{C}_{k_0}(L) \) and \( Q_k > 0 \) on \( \mathcal{C}_{k_0 - k}(L) \). We simply take \( Q_1 = P_1 \) and if \( Q_k \), with \( 1 \leq k \leq N - 1 \), has been constructed, we choose \( Q_{k+1} \) of the form \( Q_{k+1} = Q_k + \lambda P_{k+1} \), where \( \lambda > 0 \) is chosen large enough so that \( \lambda P_{k+1} > -Q_k \) on the set \( \mathcal{C}_{k_0 - k - 1}(L) \). Note that this is possible, since \( \mathcal{C}_{k_0 - k - 1}(L) \) is a finite set and \( P_{k+1} > 0 \) on \( \mathcal{C}_{k_0 - k}(L) \) while \( Q_k > 0 \) on \( \mathcal{C}_{k_0 - k}(L) \). The polynomial \( Q_N \) satisfies \( Q_N > 0 \) on \( \mathcal{C}_{2}(L) \) and \( Q_N = 0 \) on \( \mathcal{C}_{k_0}(L) \). This shows that \( \mathcal{C}_{2}(L) \subset \mathcal{C}_{k_0}(L) \) and, thus, that \( \mathcal{C}_{2}(L) = \mathcal{C}_{k_0}(L) \), which concludes the proof.

**Corollary 4.14.** Let \( S \subset \mathbb{Z} \) be a finite one-dimensional symmetric set. Then the finite set \( \Lambda \subset T \) is the support of an \( S \)-determinate measure \( \mu \) if and only if \( \Pi_S \) is dense in \( L^1_\mu \) and there exists a polynomial \( P_0 \in \Pi_S \) such that

\[
P_0 = 0 \quad \text{on} \; \Lambda, \quad P_0 > 0 \quad \text{on} \; \mathcal{C}_{2}(L) \setminus \Lambda.
\]

**Proof.** By Theorems 4.9 and 4.10, the only thing left to show is that the existence of the polynomial \( P_0 \) above is equivalent to the identity \( \mathcal{C}(L) = \Lambda \). If (4.6) holds for some polynomial \( P_0 \in \Pi_S \), then either \( \mathcal{C}_1(\Lambda) = \Lambda \) or \( \mathcal{C}_{2}(\Lambda) = \Lambda \), which implies, in either case, that \( \mathcal{C}(\Lambda) = \Lambda \). Conversely, if \( \mathcal{C}(\Lambda) = \Lambda \), it follows from Theorem 4.13 that either \( \mathcal{C}_1(\Lambda) = \Lambda \) or \( \mathcal{C}_{2}(\Lambda) = \Lambda \). In the first case, we can take \( P_0 = 0 \), while in the second case, the existence of \( P_0 \) follows from Lemma 4.12.

**Example 4.15.** As an application of our results, we consider the classical situation in which \( S \) is the one-dimensional symmetric set \( S = \{ k \in \mathbb{Z}, |k| \leq N \} \), where \( N \) is a positive integer (cf. [2, 3, 9, 12, 15]). Let \( \Lambda \) be a finite subset of \( T \) and let \( \mu \) be any positive measure on \( T \) whose support is \( \Lambda \). A routine application of the Lagrange interpolation formula shows that the condition that \( \Pi_S \) be dense in \( L^1_\mu \) is equivalent to the inequality \( |\Lambda| \leq 2N + 1 \). For the computation of \( \mathcal{C}(\Lambda) \), we use the well-known result of Riesz stating that a polynomial \( P \in \Pi_S \) satisfying \( P \geq 0 \) on \( T \) has the form \( P = Q^2 \), where \( Q \) is a polynomial in \( \Pi_k \), where the set \( K = \{ k \in \mathbb{Z}, 0 \leq k \leq N \} \). Using the fact that if \( |\Lambda| \leq N \), there exists a polynomial \( Q \) in \( \Pi_k \) whose zero set is exactly \( \Lambda \), while if \( \Lambda \geq N + 1 \), the 0-polynomial is the only polynomial in \( \Pi_k \) that vanishes on \( \Lambda \), it is easily seen that \( \mathcal{C}_2(\Lambda) = \Lambda = \mathcal{C}(\Lambda) \) if \( |\Lambda| \leq N \), while \( \mathcal{C}_2(\Lambda) = T = \mathcal{C}(\Lambda) \) if \( |\Lambda| \geq N + 1 \). Using Theorem 4.10, we recover thus the classical result that a measure \( \mu \) whose support is \( \Lambda \) is \( S \)-determinate if and only if \( |\Lambda| \leq N \).
The following example illustrates the fact that, in dimension \( n = 1 \), the case where \( \mathcal{E}(\Lambda) = \mathcal{E}_2(\Lambda) \subsetneq \mathcal{E}(\Lambda) \) can occur.

**Proposition 4.16.** Let \( S \) be the one-dimensional set \( \{0, \pm 1, \pm 3\} \). Then, a non-empty finite set \( \Lambda \subset T \) is the support of an \( S \)-determinate measure if and only if one of the following three (mutually exclusive) conditions is satisfied:

(a) \( \Lambda = \{\alpha\}, \alpha \in T \);

(b) \( \Lambda = \{\alpha, \beta\}, \alpha, \beta \in T, \alpha \neq \beta, \) and \( \text{Re}(\alpha \bar{\beta}) \geq -\frac{1}{2} \);  

(c) \( \Lambda = \{\alpha, \alpha e^{2\pi i/3}, \alpha e^{4\pi i/3}\}, \alpha \in T \).

**Proof.** We can assume, without loss of generality, that \( 1 \) belongs to \( \Lambda \), since if a measure \( \mu \) is \( S \)-determinate on \( T^n \), so is every measure of the form \( \delta_\alpha * \mu \), where \( \alpha \in T^n \). If \( |\Lambda| = 1 \), then \( \Lambda = \{1\} \). Since \( \mathcal{E}(S) = \mathbb{Z} \), it follows from Theorem 3.5 that \( \mu \) is \( S \)-determinate. If \( |\Lambda| = 2 \), then \( \Lambda = \{1, \alpha\} \), for some \( \alpha \in T \), with \( \alpha \neq 1 \). To compute \( \mathcal{E}_i(\Lambda) \), we consider a polynomial \( P \in \Pi_s \) such that \( P \geq 0 \) on \( T \) and satisfying \( P(1) = P(\alpha) = 0 \).

It follows from the result of Riesz that was mentioned in Example 4.15 that \( P(z) = C(z - 1)(z - \alpha)(z - \beta)^2 \), where \( C \geq 0 \) and \( \beta \in \mathbb{C} \) has to be chosen so that \( P \in \Pi_s \) when \( \beta \neq 0 \). It is easily checked that this last requirement reduces to the identity

\[
\alpha(1 + |\beta|^2) + 2(1 + \alpha) \beta = 0. \tag{4.7}
\]

Letting \( r = |\beta| \), we deduce from (4.7) that \( 2|1 + \alpha|r = 1 + r^2 \). This last equation has two real solutions given by \( r = |1 + \alpha| \pm \sqrt{|1 + \alpha|^2 - 1} \) when \( \text{Re}(\alpha) > -\frac{1}{2} \), in which case \( r \neq 1 \), one solution given by \( r = 1 \) when \( \text{Re}(\alpha) = -\frac{1}{2} \) and no real solution when \( \text{Re}(\alpha) < -\frac{1}{2} \). We conclude therefore that \( \mathcal{E}(\Lambda) = \Lambda = \mathcal{E}(\Lambda) \) when \( \text{Re}(\alpha) > -\frac{1}{2} \), that \( \mathcal{E}_i(\Lambda) = T = \mathcal{E}(\Lambda) \) when \( \text{Re}(\alpha) < -\frac{1}{2} \), and that \( \mathcal{E}_s(\Lambda) = \{1, \omega, \omega^2\} \), where \( \omega = \exp(2\pi i/3) \), if \( \text{Re}(\alpha) = -\frac{1}{2} \), i.e., in the case where \( \alpha = \omega \) or \( \omega^2 \). In this last situation, one has to compute \( \mathcal{E}_s(\Lambda) \). If \( \alpha = \omega \), the polynomial

\[
P(z) = \frac{(z - 1)(1 - \omega z)}{(\omega^2 - 1)(1 - \omega^2)}, \quad z \in T,
\]

belongs to \( \Pi_s \) and satisfies \( P(1) = P(\omega) = 0 \) and \( P(\omega^2) = 1 > 0 \). Hence, it follows that \( \mathcal{E}_s(\Lambda) = \{1, \omega\} = \mathcal{E}(\Lambda) \). A similar computation shows that, when \( \alpha = \omega^2 \), \( \mathcal{E}(\Lambda) = \{1, \omega^2\} = \Lambda \).

If \( |\Lambda| = 3 \), i.e., \( \Lambda = \{\alpha_1, \alpha_2, \alpha_3\} \), where \( \alpha_1, \alpha_2, \alpha_3 \in T \) are distinct and not equal to 1, it follows, exactly as above, that if a polynomial \( P \in \Pi_s \) satisfies \( P \geq 0 \) on \( T \) and \( P = 0 \) on \( \Lambda \), then \( P \) vanishes identically on \( T \).
unless \( \Lambda = \{1, \omega, \omega^2\} \), in which case \( P(z) = C|z^2 - 1|^2 \) for some constant \( C \geq 0 \) and thus \( \mathcal{O}(\Lambda) = \Lambda \). If \( |\Lambda| \geq 4 \), any non-negative polynomial that vanishes on \( \Lambda \) has to be identically zero and, therefore \( \mathcal{O}_3(\Lambda) = T = \mathcal{O}(\Lambda) \) in that case. Since it is easily checked that \( \Pi_5 \) is dense in \( L^1_\mu \), if \( \mu \) has \( \Lambda \) for support and \( |\Lambda| \leq 3 \), the result follows from the previous considerations and Theorem 4.10.

We conclude this paper with a two-dimensional example which illustrates the fact that Theorem 4.13 does not hold in dimensions higher than 1.

**Proposition 4.17.** Let \( \alpha \in T \) be such that \( \alpha^8 \neq 1 \) and consider the subsets of \( T^2 \) and \( Z^2 \) defined respectively by \( \Lambda = \{(1,1),(-1,1),(\alpha, \alpha^2)\} \) and

\[
S = \{(0,0),(1,0),(-1,0),(0,1),(0,-1),\]
\[
(2,1),(-2,-1),(2,-1),(-2,1)\}.
\]

Then, we have the inclusions \( \mathcal{O}(\Lambda) \subseteq \mathcal{O}_3(\Lambda) \subseteq \mathcal{O}_2(\Lambda) \subseteq \mathcal{O}(\Lambda) \) and any positive measure \( \mu \) with support \( \Lambda \) is \( S \)-determinate.

**Proof.** We first note that the polynomials that appear in the definition of the sets \( \mathcal{O}_i(\Lambda) \) can be assumed to be real-valued polynomials, by considering their real parts if necessary. Such a polynomial in \( \Pi_5 \) has the form

\[
P(z_1, z_2) = a_0 + \text{Re}(a_1 z_1) + \text{Re}(a_2 z_2) + \text{Re}(a_3 z_1^2 z_2) + \text{Re}(a_4 z_1^2 z_2^2),
\]

(4.8)

where \( a_0 \in \mathbb{R} \) and \( a_1, a_2, a_3, a_4 \in \mathbb{C} \). We will start by showing that a polynomial \( P \in \Pi_5 \) which satisfies \( P \geq 0 \) on \( T^2 \) and \( P = 0 \) on \( \Lambda \) must necessarily have the form \( P(z_1, z_2) = a_0(1 - \text{Re}(z_1^2 z_2^2)) \), where \( a_0 \geq 0 \). Indeed, if \( P \geq 0 \) on \( T^2 \) and is of the form (4.8), then, clearly, \( a_0 \geq 0 \). The fact that \( P \) vanishes on \( \Lambda \) implies that the real parts of the numbers \( a_1, a_2, a_3, a_4 \), and \( a_0 + a_1 \alpha + a_2 \alpha^2 + a_3 \alpha^4 + a_4 \) are all equal to 0, while the fact that \( P \) must also have local minima at each point of \( \Lambda \) implies the vanishing of the imaginary parts of the numbers \( a_1, a_2, a_3, a_4, a_2 + a_3 - a_4, a_1 \alpha + 2a_3 \alpha^4 + 2a_4, \) and \( a_2 \alpha^2 + a_3 \alpha^4 - a_4 \). From this, we deduce immediately that \( a_1 = 0 \) and, after some computations using this last set of identities, that

\[
a_2 = \frac{2i \overline{\alpha}}{\text{Re}(\alpha)} \text{Im}(a_4), \quad a_3 = \frac{-i(\overline{\alpha})^2}{\text{Re}(\alpha^2)} \text{Im}(a_4).
\]
Using the first set of identities and the fact that $a_4 = 0$, we arrive at the equations

$$a_0 + \Re(a_4) = 0, \quad \Im(a_4) \left\{ \frac{2\Im(\alpha)}{\Re(\alpha)} - \frac{\Im(a^2)}{\Re(a^2)} \right\} = 0,$$

which, after taking into account our hypothesis on $\alpha$, show that $a_4 = -a_0$ and $a_2 = a_3 = 0$. This gives the required form of the polynomial $P$. Hence, it follows immediately that $\mathcal{E}_2(\Lambda) = \{(z_1, z_2) \in \mathbb{T}^2, \ z_2 = z_1^2\}$. To compute $\mathcal{E}_3(\Lambda)$, we consider a real-valued polynomial $P$ of the form (4.8) which satisfies $P \geq 0$ on $\mathcal{E}_2(\Lambda)$ and $P = 0$ on $\Lambda$. Defining the one-variable polynomial $Q(z)$ by

$$Q(z) = P(z, z^2) = a_0 + \Re(a_4) + \Re(a_2 z) + \Re(a_2 z^2) + \Re(a_3 z^4), \quad z \in \mathbb{T}, \quad (4.9)$$

we see that the requirements imposed on $P$ are equivalent to $Q \geq 0$ on $\mathbb{T}$ and $Q(1) = Q(-1) = Q(\alpha) = 0$. Using again the result of Riesz (see Example 4.15), we find that $Q(z) = C[(z^2 - 1)(z - \alpha)(z - \beta)]^2$ where $C \geq 0$, and $\beta \in \mathbb{C}$ has to be chosen so that this factorized expression for $Q$ is compatible with (4.9). It is easily checked that this last condition is equivalent to $2\beta = -\alpha(1 + |\beta|^2)$. Letting $\beta = r\zeta$, where $r \geq 0$ and $|\zeta| = 1$ and equating the absolute value of both sides of the previous identity, we obtain that $r = 1$, and, thus, that $\beta = -\alpha$, i.e.,

$$Q(z) = C[(z^2 - 1)(z^2 - \alpha^2)]^2.$$

This shows that

$$P(z_1, z_2) = C \left\{ (2 + |\alpha|^2) + \Re(-4(1 + \alpha z_2)) + \Re(2\alpha^2 z_2 z_1^3) \right\}$$

$$+ \Re(a_4(z_1^2 z_2^2 - 1)),$$

where $C \geq 0$ and $\alpha_4 \in \mathbb{C}$ is arbitrary. Hence, we conclude that

$$\mathcal{E}_2(\Lambda) = \{(1, 1), (-1, 1), (\alpha, \alpha^2), (-\alpha, \alpha^2)\}.$$

To show that $\mathcal{E}_3(\Lambda) = \Lambda = \mathcal{E}(\Lambda)$, we note that the polynomial

$$R(z_1, z_2) = \frac{z_2 - 1 + \alpha(z_1 - z_1)}{2(\alpha^2 - 1)}, \quad (z_1, z_2) \in \mathbb{T}^2,$$
belongs to \( \Pi_S \), vanishes on \( \Lambda \), and satisfies \( R(-\alpha, \alpha^2) = 1 > 0 \). This proves the first part of the claim. Finally, since the polynomials

\[
P_1(z_1, z_2) = z_2 - 1, \quad P_2(z_1, z_2) = (z_1 + 1)(\overline{z_1} - \overline{\alpha}), \quad P_3(z_1, z_2) = (z_1 - 1)(\overline{z_1} - \overline{\alpha})
\]

belong to \( \Pi_S \) and each one of them vanishes on a different subset of \( \Lambda \) with two elements, it follows that \( \Pi_S \) is dense in \( L^1_\mu \), if \( \mu \geq 0 \) and \( \text{supp } \mu = \Lambda \). Corollary 4.14 shows that \( \mu \) is thus \( S \)-determinate, which concludes the proof. \( \blacksquare \)

Remark 4.18. Of course, the previous construction can be extended to dimensions higher than 2 by considering the natural embeddings of \( \mathbb{Z}^2 \) and \( \mathbb{T}^2 \) in \( \mathbb{Z}^n \) and \( \mathbb{T}^n \), respectively, when \( n \geq 3 \).

REFERENCES

3. A. Arimoto and T. Ito, Singularly positive definite sequences and parametrization of extreme points, Linear Algebra Appl. 239 (1996), 127–149.