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A Characterization of Certain Weak*-Closed Subalgebras of $L_\infty(G)$

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Let G be an arbitrary locally compact Hausdorff group, and let $L_\infty(G)$ be the set of essentially bounded measurable functions on G with respect to left invariant Haar measure. Let S be a linear subspace of $L_\infty(G)$ which is (i) left and right translation invariant; (ii) weak*-closed; (iii) self-adjoint, i.e., $f \in S$ implies $\bar{f} \in S$; and (iv) an algebra containing the constant functions. Then there exists a unique closed normal subgroup H of G such that $S = \{f \in L_\infty(G): {}_af = f_a = f, \forall a \in H\}$. This extends to arbitrary locally compact groups, a result known only for Abelian groups.

1. INTRODUCTION

In their study of the structure of sets invariant under a probability measure μ, Pathak and Shapiro [3] were lead in a natural way to characterize subalgebras of $L_\infty(\mathbb{R})$ which are in a sense periodic. They extended their result to locally compact Abelian (LCA) groups. Their proof, however, is completely based on the Fourier analysis of an LCA group, and as such cannot be used in the non-Abelian case. In this paper we prove their result for general locally compact Hausdorff groups. Lemmas 1 and 2 leading to this result are also of interest in their own right.

Let G be an arbitrary locally compact Hausdorff group, and $L_1 \equiv L_1(G)$ and $L_\infty \equiv L_\infty(G)$ the usual corresponding spaces with respect to left Haar measure dx or μ. A subset S of L_∞ is called self-adjoint if $f \in S$ implies $\bar{f} \in S$, where $\bar{f}$ is the complex conjugate of f. Left and right translation of a function f are defined by $({}_af)(x) = f(a^{-1}x)$, $f_a(x) = f(xa^{-1})$, and we put $\tilde{f}(x) = f(x^{-1})$ · cl_{w*} denotes the closure in the w^*-topology. For two functions f and g on G, the convolution $f * g$ is defined by

$$(f*g)(x) = \int_G f(xy)\, g(y^{-1})\, dy = \int_G f(y)\, g(y^{-1}x)\, dy.$$

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Given a normal subgroup H of G, and putting $S = \{f \in L_\infty : {}_af = f_a = f, \forall a \in H\}$, we can verify that S is a linear subspace of L_∞ which is (i) left and right translation invariant, (ii) w^*-closed, (iii) self-adjoint, and (iv) an algebra containing the constant functions.

We prove the following converse theorem:

THEOREM. *Let S be a linear subspace of L_∞ with the properties* (i), (ii), (iii), *and* (iv). *Then there exists a unique closed normal subgroup H in G such that $S = \{f \in L_\infty : {}_af = f_a = f, \forall a \in H\}$.*

2. PROOF OF THE THEOREM

We will obtain the proof of the theorem as a result of the following lemmas, of which Lemmas 1 and 2 are of a more general nature.

LEMMA 1. *For $g \in L_\infty$ we have $g \in \mathrm{cl}_{w^*}(L_1 * g)$.*

Proof. Given n functions k_i ($i = 1,..., n$) in L_1 and $\epsilon > 0$, we have to find a function f_1 in L_1 such that

$$\left| \int_G k_i(x)\,(f_1 * g)\,(x)\,dx - \int_G k_i(x)\,g(x)\,dx \right| < \epsilon. \tag{1}$$

Since each $k_i * \tilde{g}$ is continuous there exists a neighborhood U of the identity e of G such that $|(k_i * \tilde{g})(x) - (k_i * \tilde{g})(e)| < \epsilon$ for each x in U, and we may assume that $0 < \mu(U) < \infty$. Put $f_1 = (1/\mu(U))\,\chi_U$, where χ_U is the characteristic function of U. Using the identity

$$\int_G k(x)\,(f * g)\,(x)\,dx = \int_G f(x)\,(k * \tilde{g})\,(x)\,dx \tag{2}$$

for any f, k in L_1 and g in L_∞, it is easily verified that the constructed function f_1 indeed fulfills condition (1). ∎

COROLLARY 1. *If S is a w^*-closed L_1-submodule of L_∞ (so $L_1 * S \subset S$), then $S = \mathrm{cl}_{w^*}(L_1 * S)$.*

Indeed, from the foregoing lemma each g in S belongs to $\mathrm{cl}_{w^*}(L_1 * S)$, and so $S \subset \mathrm{cl}_{w^*}(L_1 * S)$. The reversed inclusion is trivial by the w^*-closedness of S. ∎

LEMMA 2. *Let S be a left translation invariant linear subspace of L_∞. Then $f * g \in \mathrm{cl}_{w^*}S$, for all f in L_1 and g in S.*

Proof. Suppose first that f has the form $f = \chi_K$, where K is a compact subset of G. Given $k_1, \ldots, k_n$ in L_1 we have, using (2),

$$\int_G k_i(y)\,(\chi_K * g)(y)\,dy = \int_G \chi_K(x)\,(k_i * \tilde{g})(x)\,dx = \int_K (k_i * \tilde{g})(x)\,dx. \tag{3}$$

Since each function $k_i * \tilde{g}$ is continuous on the compact set K there exists, given $\epsilon > 0$, a partition $K = \bigcup_{j=1}^m K_j$ of K such that $|(k_i * \tilde{g})(x) - (k_i * \tilde{g})(y)| \leqslant \epsilon/\mu(K)$ for x and y belonging to the same set K_j in the partition. For each j $(1 \leqslant j \leqslant n)$ we choose a point a_j in K_j and we put $\alpha_{ij} = (k_i * \tilde{g})(a_j)$ $(1 \leqslant i \leqslant n)$. If $f_i = \sum_{j=1}^m \alpha_{ij}\chi_{K_j}$, then $|f_i(x) - (k_i * \tilde{g})(x)| \leqslant \epsilon/\mu(K)$ (x in K), and so we obtain, in view of (3),

$$\left| \int_G k_i(y)\,(\chi_K * g)(y)\,dy - \int_K f_i(y)\,dy \right| \leqslant \epsilon.$$

But

$$\int_K f_i(y)\,dy = \sum_{j=1}^m \mu(K_j)\,(k_i * g)(a_j) = \int_G k_i(x) \left(\sum_{j=1}^m \mu(K_j)_{a_j} g \right)(x)\,dx.$$

Since $\sum_{j=1}^m \mu(K_j)_{a_j} g$ belongs to S, the lemma is proved for such particular functions f in L_1.

Now let f be an arbitrary function in L_1. Using the inner regularity of Haar measure there exists, given $\epsilon > 0$, a simple function $h = \sum_{k=1}^p \beta_k \chi_{U_k}$, where the U_k are compact, such that $\| f - h \|_1 < \epsilon/\| g \|_\infty$, and so $\| f * g - h * g \|_\infty < \epsilon$. Since by the first part $h * g$ belongs to $\mathrm{cl}_{w^*}(S)$, the same is true of $f * g$. ▮

COROLLARY 2. *Let S be a w^*-closed linear subspace of L_∞. Then S is left translation invariant iff $L_1 * S \subset S$.*

Proof. That S is an L_1-submodule if S is left invariant is the contents of lemma 2. Conversely, if $L_1 * S \subset S$ it follows from corollary 1 that $S = \mathrm{cl}_{w^*}(L_1 * S)$. Hence each w^*-neighborhood of $g \in S$ contains a function of the form $f * h$, where f and h belong to L_1 and S, respectively. It is then easily verified that, given any point a in G, ${}_a g$ can be w^*-approximated by functions of type ${}_a(f * h) = {}_a f * h$. So S is left invariant. ▮

From now on we suppose that S is a linear subspace of L_∞ having the properties (i), (ii), (iii), and (iv) as in the statement of the theorem. From Corollary 2 together with Corollary 1 it follows that such an S also has property (v): $S = \mathrm{cl}_{w^*}(L_1 * S)$.

Putting $H = \{a \in G : {}_a g = g_a = g, \forall g \in S\}$, it is easily verified that H is a normal subgroup of G. We first give another description of H.

LEMMA 3. $H = \{a \in G : {}_a g = g_a = g, \forall g \in L_1 * S\}$.

This is more or less clear from (v); we omit the easy verification. ▮

In view of Lemma 3, any function g in $L_1 * S$ is constant on each coset aH ($=Ha$) corresponding to each point a in G. If a and b are points in G belonging to different cosets, then there exists a function g in $L_1 * S$ such that $g(a) \neq g(b)$; for since $L_1 * S$ is left and right translation invariant, the assumption $g(a) = g(b)$ for all $g \in L_1 * S$ leads to ${}_{ab^{-1}}g = g = g_{ab^{-1}}$ and so, by Lemma 3, $ab^{-1} \in H$ or $Ha = Hb$. From these remarks we conclude that $H = \bigcap_{g \in L_1 * S} \{a \in G\colon g(a) = g(e)\}$. Since each g in $L_1 * S$ is continuous, H is closed.

As in [1, 19.23(b)] we define $C_{ru} \equiv C_{ru}(G)$ to be the linear space of all right uniformly continuous, bounded, complex-valued functions on G; i.e., for $g \in C_{ru}$ and $\epsilon > 0$, there exists a neighborhood U of e such that

$$| g(x) - g(y)| < \epsilon \qquad \text{if} \quad xy^{-1} \in U. \tag{4}$$

LEMMA 4. $L_1 * S = C_{ru} \cap S$.

Proof. Since $L_1 * S \subset L_1 * L_\infty = C_{ru}$ [2, 32.45(b)], $L_1 * S \subset C_{ru} \cap S$ and the set on the right-hand side is norm closed in L_∞, since this is so for C_{ru} and S. On the other hand, since $L_1 * S \subset S$, we deduce from the factorization theorem [2, 32.22] that $L_1 * S$ is also norm closed. Hence to prove the lemma it suffices to show that $L_1 * S$ is norm dense in $C_{ru} \cap S$.

Given g in $C_{ru} \cap S$ and $\epsilon > 0$, put $f_1 = (1/\mu(U))\, \chi_U$, where U is a neighborhood as in (4) with $0 < \mu(U) < \infty$. Then, for x in G,

$$|(f_1 * g)(x) - g(x)| = \frac{1}{\mu(U)} \left| \int_{x^{-1}U} (g(y^{-1}) - g(x))\, dy \right|,$$

and so $\| f_1 * g - g \|_\infty < \epsilon$. Hence the result. ∎

From Lemma 4 we conclude that $L_1 * S$ is a closed self-adjoint algebra containing the constant functions.

LEMMA 5. *Let g be a function in C_{ru} which is constant on each coset of H. Then g belongs to S.*

Proof. Given $k_1, \ldots, k_n$ in L_1 and $\epsilon > 0$, there exists a compact set K in G such that $\int_{G \setminus K} | k_i(x)|\, dx < \epsilon/2 \| g \|_\infty$ $(1 \leqslant i \leqslant n)$.

From the remark following Lemma 4 and an application of the Stone–Weierstrass theorem we conclude that there exists a function h in $L_1 * S$ such that g and h are equal on K. Since all functions in $L_1 * S$ are bounded, another application of the Stone–Weierstrass theorem on suitable compact subsets of $\mathbb{C}$ tells us that $L_1 * S$ is closed under all continuous operations. So if φ is that complex function defined on the complex numbers such that $\varphi(z) = z$ for $| z | \leqslant \| g \|_\infty$ and $\varphi(z) = (z/\| z \|)\, \| g \|_\infty$ for $| z | > \| g \|_\infty$, then the composition $\varphi \circ h$ belongs to $L_1 * S$, $\| \varphi \circ h \|_\infty \leqslant \| g \|_\infty$, and $(\varphi \circ h)(z) = h(z) = g(z)$ for z

in K. Since $\int_G = \int_{G\setminus K} + \int_K$, we obtain $| \int_G k_i(x)((\varphi \circ h)(x) - g(x))\, dx | < \epsilon$, from which the lemma follows. ∎

LEMMA 6. *Let g be a function in L_∞ such that ${}_ag = g_a = g$, for each a in H. Then g belongs to S.*

Proof. For each f in L_1 and a in H we have $(f * g)_a = f * g_a = f * g$. Then let b and c be two points in G belonging to a same coset of H. Then $c^{-1}b \in H$ and so $(f * g)_{c^{-1}b} = f * g$, or $(f * g)_{c^{-1}} = (f * g)_{b^{-1}}$. Since $f * g$ is everywhere defined, we have $(f * g)(c) = (f * g)(b)$. By Lemma 5, $f * g$ belongs to S, i.e., $L_1 * g \subset S$. Using Lemma 1 we conclude that g is in S. ∎

The proof of the theorem, except the uniqueness condition, now follows quickly. Indeed, we have constructed a closed normal subgroup H of G, and by Lemma 6 we have $\{g \in L_\infty: {}_ag = g_a = g, \forall a \in H\} \subset S$. By definition of H the reversed inclusion is clear. Hence $S = \{g \in L_\infty: {}_ag = g_a = g, \forall a \in H\}$.

Finally we have to settle the uniqueness question. Suppose H_1 is also a closed normal subgroup of G such that $S = \{g \in L_\infty: {}_ag = g_a = g, \forall a \in H_1\}$. Clearly, then, H_1 is a subset of H. Suppose there exists an element a in $H \setminus H_1$. Let G/H_1 be the quotient group, π the natural homomorphism of G onto G/H_1; then G/H_1 is locally compact and Hausdorff, and $\pi(a) \neq \pi(e)$. So there exists a continuous bounded function φ on G/H_1, such that $(\varphi \circ \pi)(a) \neq (\varphi \circ \pi)(e)$. For each c in H_1 we have ${}_c(\varphi \circ \pi) = (\varphi \circ \pi)_c = \varphi \circ \pi$, and so $\varphi \circ \pi$ would belong to S. This is impossible, however, since ${}_{a^{-1}}(\varphi \circ \pi) \neq \varphi \circ \pi$, although a^{-1} belongs to H.

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