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On the low regularity of the fifth order Kadomtsev–Petviashvili I equation

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Abstract

We consider the fifth order Kadomtsev–Petviashvili I (KP-I) equation as $\partial_t u + \alpha\partial_x^3 u + \partial_x^5 u + \partial_x^{-1}\partial_y^2 u + uu_x = 0$, while $\alpha \in \mathbb{R}$. We introduce an interpolated energy space E_s to consider the well-posedness of the initial value problem (IVP) of the fifth order KP-I equation. We obtain the local well-posedness of IVP of the fifth order KP-I equation in E_s for $0 < s \leqslant 1$. To obtain the local well-posedness, we present a bilinear estimate in the Bourgain space in the framework of the interpolated energy space. It crucially depends on the dyadic decomposed Strichartz estimate, the fifth order dispersive smoothing effect and maximal estimate.
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1. Introduction

We consider the initial value problem (IVP) of the fifth order Kadomtsev–Petviashvili (KP) equation

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$$\begin{cases} \partial_t u + \alpha \partial_x^3 u + \beta \partial_x^5 u + \partial_x^{-1} \partial_y^2 u + u \partial_x u = 0, \\ u(0, x, y) = u_0(x, y), \quad (x, y) \in \mathbb{R}^2. \end{cases} \tag{1}$$

Here $\alpha, \beta \in \mathbb{R}$ and u_0 is a real valued function. If $\beta > 0$ Eq. (1) is called the fifth order KP-I and if $\beta < 0$ it takes the name the fifth order KP-II. This equation occurs naturally in the modeling of a long dispersive wave. Kawahara [15] introduced the fifth order Korteweg–de Vries equation

$$\partial_t u + \alpha \partial_x^3 u + \beta \partial_x^5 u + u \partial_x u = 0, \tag{2}$$

which models the wave propagation in one direction. While the KP equation models the propagation along the x-axis of a nonlinear dispersive long wave on the surface of a fluid with a slow variation along the y-axis (see [14,21,22] and the references therein).

We begin with a few facts about KP equations. The Fourier transform of a Schwarz function $f(x, y)$ is defined by

$$\hat{f}(\xi, \mu) = \frac{1}{2\pi} \int_{\mathbb{R}^2} f(x, y) e^{-i(x\xi + y\mu)} \, dx \, dy.$$

The dispersive function of the KP equation is

$$\omega(\xi, \mu) = \beta \xi^5 - \alpha \xi^3 + \frac{\mu^2}{\xi}. \tag{3}$$

The analysis of the IVP of the KP equation depends crucially on the sign of α and β. We take a glance on the case $\beta = 0$. In this case, Eq. (1) turns out to be the third order KP equation. Without loss of generality, we assume $|\alpha| = 1$. If $\alpha = -1$, the equation is called the third order KP-I equation. While if $\alpha = 1$, the equation is called the third order KP-II equation. By computing the gradient of ω, we get that for the third order KP-I

$$\left|\nabla \omega(\xi, \mu)\right| = \left| \left(3\xi^2 - \frac{\mu^2}{\xi^2}, 2\frac{\mu}{\xi} \right) \right| \gtrsim |\xi|. \tag{4}$$

For the third order KP-II equation, we have

$$\left|\nabla \omega(\xi, \mu)\right| = \left| \left(-3\xi^2 - \frac{\mu^2}{\xi^2}, 2\frac{\mu}{\xi} \right) \right| \gtrsim |\xi|^2. \tag{5}$$

One can easily recover a full derivative smoothness along the x direction by (5), but only a half derivative smoothness by (4). Since the nonlinear term in the third order KP equation involves a full derivative along the x direction, this explains partially to get the well-posedness for the IVP of KP-I is much more difficult than that of KP-II.

Another important concept in the analysis of dispersive equation is the resonance function. Still considering the third order KP equation, the resonance function is defined by

$$\begin{aligned} R(\xi_1, \xi_2, \mu_1, \mu_2) &= \omega(\xi_1 + \xi_2, \mu_1 + \mu_2) - \omega(\xi_1, \mu_1) - \omega(\xi_2, \mu_2) \\ &= -\frac{\xi_1 \xi_2}{(\xi_1 + \xi_2)} \left(3\alpha(\xi_1 + \xi_2)^2 + \left(\frac{\mu_1}{\xi_1} - \frac{\mu_2}{\xi_2} \right)^2 \right). \end{aligned}$$

Thus for the third order KP-II equation, we always have the following inequality

$$\left|R(\xi_1,\xi_2,\mu_1,\mu_2)\right| \geqslant C|\xi_1||\xi_2||\xi_1+\xi_2|. \tag{6}$$

However, for the third order KP-I equation, the inequality (6) is not true all the time. In this case, resonant interaction happens frequently. The resonant interaction means the resonance function is zero or close to zero. Generally, we use (6) to recover the derivative on x by the regularity on t. Thus, the simpler the corresponding zero set, the easier it is to deal with the problem. This facts also implies that the well-posedness problem of KP-II is easier than that of KP-I.

A natural function space to consider the well-posedness of the IVP of the KP equation is the non-isotropic Sobolev space:

$$H^{s_1,s_2}\left(\mathbb{R}^2\right) := \left\{f \in \mathcal{S}'\left(\mathbb{R}^2\right);\ \left\|\langle\xi\rangle^{s_1}\langle\mu\rangle^{s_2}\hat{f}\right\|_{L^2_{\xi,\mu}} < \infty\right\}, \tag{7}$$

where $\langle\xi\rangle = (1+|\xi|)$. Keep in mind that we are still in the case of $\beta = 0$. A scaling argument (e.g. see [21]) shows that $s_1 + 2s_2 > -\frac{1}{2}$ is expected for the local well-posedness of the IVP of the KP equations in H^{s_1,s_2}. As we pointed out, the third order KP-II has better dispersive effect than the third order KP-I. The results about the third order KP-II are very close to the expected indices. In [2], Bourgain showed the global well-posedness of the third order KP-II in L^2, i.e. $s_1 = s_2 = 0$. This result had been improved by Takaoka and Tzvetkov [24] and Isaza and Mejía [13] to $s_1 > -\frac{1}{3}$, $s_2 \geqslant 0$. In [23], Takaoka obtained the local well-posedness of the IVP of the third order KP-II for $s_1 > -\frac{1}{2}$, $s_2 = 0$ and an additional low frequency condition $|D_x|^{-\frac{1}{2}+\varepsilon}u_0 \in L^2$. Recently, Hadac [9] removed the additional condition on the initial value above. This means in the case $s_2 = 0$, the result on the third order KP-II equation is sharp. While for the third order KP-I equation, the situation is far from the expected. By compactness method, Iório and Nunes [12] obtained the local well-posedness of the IVP of the third KP-I equation for data in the normal Sobolev space $H^s(\mathbb{R}^2)$, $s > 2$, and satisfying a "zero-mass" condition. They used only the divergence form of the nonlinearity and the skew-adjointness of the (linear) dispersion operator. The condition on s is needed to control the gradient of the solution in the L^∞. In [7], Colliander, Kenig and Staffilani obtained well-posedness for small data in a weighted Sobolev space with essentially H^2 regularity.

Another natural space to consider the well-posedness of the IVP of the KP-I equation is the energy space. We first notice that the KP equation (1) satisfies the following two conversations.

Mass

$$\|u\|_{L^2} = \|u_0\|_{L^2}. \tag{8}$$

Hamiltonian

$$\begin{aligned} H(u) = \frac{\beta}{2}\int\left(\partial_x^2 u\right)^2 dx\,dy - \frac{\alpha}{2}\int(\partial_x u)^2\,dx\,dy \\ + \frac{1}{2}\int\left(\partial_x^{-1}\partial_y u\right)^2 dx\,dy + \frac{1}{6}\int u^3\,dx\,dy = H(u_0). \end{aligned} \tag{9}$$

Combining the above two conversations together, we can define the energy space for the fifth order KP-I equation ($\beta = 1$) by

$$E(5\text{th}) = \left\{u \in \mathcal{S}'\left(\mathbb{R}^2\right);\ \|u\|_{E(5\text{th})} = \left\|\left(1+|\xi|^2+|\xi|^{-1}|\mu|\right)\hat{u}(\xi,\eta)\right\|_{L^2} < \infty\right\}. \tag{10}$$

For the third order KP-I equation ($\beta = 0, \alpha = -1$), the energy space can be defined by

$$E(3\text{th}) = \left\{u \in \mathcal{S}'(\mathbb{R}^2);\ \|u\|_{E(3\text{th})} = \left\|\left(1 + |\xi| + |\xi|^{-1}|\mu|\right)\hat{u}(\xi, \eta)\right\|_{L^2} < \infty\right\}. \tag{11}$$

On these function spaces, we can prove that for $\beta = 1$,

$$\left\|u(t)\right\|_{E(5\text{th})} \leqslant C\|u_0\|_{E(5\text{th})}, \tag{12}$$

and for $\beta = 0, \alpha = -1$

$$\left\|u(t)\right\|_{E(3\text{th})} \leqslant C\|u_0\|_{E(3\text{th})} \tag{13}$$

for any sufficiently smooth solution u of KP-I equation, uniformly in time (see also [5,22]). Thus it would be expected to obtain local well-posedness in this kind of spaces. But the recent results of Molinet, Saut and Tzvetkov [19,20] showed that, for the third order KP-I ($\beta = 0, \alpha < 0$), one cannot prove local well-posedness in any type of non-isotropic L^2-based Sobolev space H^{s_1,s_2}, or in the energy space (see also [18]), by applying Picard iteration to the integral equation formulation of the third order KP-I equation. To avoid the difficulty, one must abandon Picard iteration or find out an alternative space with similar regularity with H^{s_1,s_2} or energy space. Recently, Colliander, Ionescu, Kenig and Staffilani [6] set up the local well-posedness of the IVP of the third order KP-I equation with small data in the intersection of energy space E and weighted space P defined by

$$E = \left\{f\colon\ f \in L^2,\ \partial_x f \in L^2,\ \partial_x^{-1}\partial_y f \in L^2\right\} \quad \text{and} \quad P = \left\{f\colon\ (y + i)f \in L^2\right\}. \tag{14}$$

Kenig [16] established the global well-posedness of the IVP of the third order KP-I equation in the following function space

$$Z_0 = \left\{u \in L^2(\mathbb{R}^2)\colon\ \|u\|_{L^2} + \left\|\partial_x^{-1}\partial_y u\right\|_{L^2} + \left\|\partial_x^2 u\right\|_{L^2} + \left\|\partial_x^{-2}\partial_y^2 u\right\|_{L^2} < \infty\right\}.$$

As far as we know, the best well-posedness result of the third KP-I equation is due to Ionescu, Kenig and Tataru [11]. They set up the global well-posedness of the third order KP-I equation in the $E(3\text{th})$ space. Thus a more interesting question is to set up the global well-posedness of the third order KP-I equation in L^2. It is still open.

We now turn our attention back to the fifth order KP-I equation. Without loss of the generality, we may assume that $\beta = 1$ from now on. The fifth order equation has a higher dispersive term than a third order KP equation, which helps us to obtain some better results than the third order KP equation. As before, we first consider the dispersive function of the fifth order KP equation. Since there is an interaction between the third order dispersive term and the fifth order dispersive term, we cannot get a dispersive smoothing effect as (4) or (5) for all $(\xi, \mu) \in \mathbb{R}^2$, but we still have

$$\left|\nabla\omega(\xi, \mu)\right| = \left|\left(5\xi^4 + \alpha 3\xi^2 - \frac{\mu^2}{\xi^2}, 2\frac{\mu}{\xi}\right)\right| \gtrsim |\xi|^2, \quad \text{if } |\xi|^2 > |\alpha|. \tag{15}$$

This inequality can help us to recover a full derivative which is important in the analysis of the fifth order KP-I equation. We also consider the resonance function

$$
\begin{aligned}
&R(\xi_1,\xi_2,\mu_1,\mu_2)\\
&\quad=\omega(\xi_1+\xi_2,\mu_1+\mu_2)-\omega(\xi_1,\mu_1)-\omega(\xi_2,\mu_2)\\
&\quad=\frac{\xi_1\xi_2}{(\xi_1+\xi_2)}\left((\xi_1+\xi_2)^2\left[5\left(\xi_1^2+\xi_1\xi_2+\xi_2^2\right)-3\alpha\right]-\left(\frac{\mu_1}{\xi_1}-\frac{\mu_2}{\xi_2}\right)^2\right).
\end{aligned}
\tag{16}
$$

The first result of the fifth order KP-I equation in the context of energy space is due to Saut and Tzvetkov [22]. They obtained the local well-posedness for the fifth order KP-I equation with data satisfying

$$
\|u_0\|_{L^2}+\big\||D_x|^s u_0\big\|_{L^2}+\big\||D_y|^k u_0\big\|<\infty,\quad s\geqslant 1,\ k\geqslant 0,\ |\xi|^{-1}\hat{u}_0(\xi,\mu)\in\mathcal{S}'\big(\mathbb{R}^2\big).
$$

Here $|D_x|^s u_0=(|\xi|^s\hat{u}_0)^\vee$. They also set up the global well-posedness for the data satisfies $u_0\in L^2$ and $H(u_0)<\infty$. Recently, Ionescu and Kenig [10] got the global well-posedness for the IVP of the fifth order periodic KP-I equation absenting the third order dispersive term with the initial data in $E(5\text{th})$. For the IVP of the fifth order KP-II equation, Saut and Tzvetkov [22] also obtained the global well-posedness for the initial data in L^2. And they put forward an open problem whether one can get the local and global well-posedness of the IVP of the fifth order KP-I equation with the initial data in L^2.

To connect the known results with the L^2 conjecture, we introduce the function space E_s consisting of all the functions satisfying

$$
\|f\|_s=:\|f\|_{E_s}=\left\|\left(1+|\xi|^2+\frac{|\mu|}{|\xi|}\right)^s\hat{f}(\xi,\mu)\right\|_{L^2}<\infty,\quad \forall s\in\mathbb{R}.
$$

It is easy to see when $s=0$, $E_0=L^2$, and when $s=1$, $E_1=E(5\text{th})$. To get the low regularity of the KP equation, we need a careful analysis on the time-spatial spaces. In this case, Bourgain type space is needed. Below, we may abuse $\hat{f}$ as the Fourier transform of a function in (x,y) or (x,y,t). One may figure it out in the context.

Definition 1. Let $\chi_0(\tau-\omega(\xi,\mu))=\chi_{[0,1]}(|\tau-\omega(\xi,\mu)|)$, $\chi_j(\tau-\omega(\xi,\mu))=\chi_{[2^{j-1},2^j]}(|\tau-\omega(\xi,\mu)|)$ for $j\in\mathbb{N}$. For $s,b\in\mathbb{R}$, we define the space $X_{s,b}$ through the following norm:

$$
\|f\|_{X_{s,b}}=\sum_{j\geqslant 0}2^{jb}\left\|\chi_j\big(\tau-\omega(\xi,\mu)\big)\left(1+|\xi|^2+\frac{|\mu|}{|\xi|}\right)^s\hat{f}(\xi,\mu,\tau)\right\|_{L^2}.
\tag{17}
$$

Furthermore, for an interval $I\subset\mathbb{R}$ the localized Bourgain space $X_{s,b}(I)$ can be defined via requiring

$$
\|u\|_{X_{s,b}(I)}=\inf_{w\in X_{s,b}}\big\{\|w\|_{X_{s,b}}\colon\ w(t)=u(t)\text{ on interval }I\big\}.
$$

We now state the well-posedness result in $X_{s,b}$ with initial data in E_s.

Theorem 1.1. *Assume that $\beta=1,\alpha\in\mathbb{R}$, and $1\geqslant s>0$. For any real valued function $u_0\in E_s$, there exist $T=T(\|u_0\|_{E_s})$ and a unique solution u of* (1) *in $X_{s,\frac{1}{2}+}(I)$ with $I=[-T,T]$.*

Moreover the map $u_0 \to u$ *is smooth from* E_s *to* $X_{s,\frac{1}{2}+}(I)$. *By Sobolev embedding, we have* $u \in C([-T,T]; E_s)$. *Here* $\frac{1}{2}+ > \frac{1}{2}$ *and is as close as possible to* $\frac{1}{2}$.

By (12) and Theorem 1.1, we can recover the global well-posedness of the IVP of the fifth order KP-I equation in the energy space:

Theorem 1.2. *(See also [22].) Assume that* $\beta = 1$, $\alpha \in \mathbb{R}$, $s = 1$. *For any real valued* $u_0 \in E_1$, *there exists a unique solution of the IVP of the fifth order KP-I equation*

$$u \in C(\mathbb{R}, E_1).$$

Remark 1. Even though the conjecture that the global well-posedness for the IVP of the fifth order KP-I equation with data in L^2 is still open, it seems the function space E_s will be expected to consider this open problem. Since E_s contains the specific feature $(1+|\xi|^2+|\mu||\xi|^{-1})$ of KP-I equation, and is different from the Sobolev space H^{s_1,s_2} or H^s, we have independent interest in obtaining the global or local well-posedness of the IVP of the fifth order KP-I equations in E_s for $s \in \mathbb{R}$.

Remark 2. In our argument, dyadic Strichartz estimates are essential. Especially, when we dispose the "high–low" interaction in the bilinear estimate, a low order derivative on the low frequency part is needed. In this case, $s > 0$ is necessary.

Our main argument to prove Theorem 1.1 is to set up a bilinear estimate as in Section 3 below. Recently, Colliander, Ionescu, Kenig and Staffilani [6] discovered a counterexample which showed that one could not set up a similar bilinear estimate in the Bourgain type space in the third KP-I case. But we find their counterexample does not work in our case, since the fifth order dispersive function can help us to recover a full derivative. Also, we do not recourse to the weighted space. In [6], a weighted space is also used to dispose the case when the very high and very low frequency interaction happens. In our paper, we can overcome this difficulty by the fifth order smoothing effect and the dyadic decomposed Strichartz estimate.

In the rest of the paper we would like to use the notation $A \lesssim B$ if there exists a constant $C > 0$ which does not depend on B such that $A \leqslant CB$. If $C < \frac{1}{100}$, we would like to use $A \ll B$. If there exist c and C which are $\frac{1}{100} < c < C < 100$, such that $cA \leqslant B \leqslant CA$, the notation $A \sim B$ will be used. And the constants c and C will be possibly different from line to line.

This paper is organized as follows. In Section 2, we present some results on linear KP equation and some useful estimates. In Section 3, we present the bilinear estimate which is crucial to set up our local well-posedness. In Section 4, we finish the proof of Theorem 1.1.

2. The linear estimates

We begin with the IVP of linear KP equation

$$\begin{cases} \partial_t u + \alpha\partial_x^3 u + \partial_x^5 u + \partial_x^{-1}\partial_y^2 u = 0, \\ u(0,x,y) = u_0(x,y), \quad (x,y) \in \mathbb{R}^2. \end{cases} \tag{18}$$

By Fourier transform, the solution of (18) can be defined as

$$u = S(t)u_0(x, y) = \int_{\mathbb{R}^2} e^{i(x\xi + y\mu + t\omega(\xi,\mu))}\hat{u}_0(\xi, \mu)\, d\xi\, d\mu.$$

By Duhamel's formula, (1) can be reduced to the integral formulation:

$$u(t) = S(t)u_0 - \frac{1}{2}\int_0^t S(t-t')\partial_x\big(u^2(t')\big)\, dt'. \tag{19}$$

Indeed, to get the local existence result, we apply the fixed point argument to the nonlinear map defined as the right-hand side of the following integral equation:

$$u(t) = \psi(t)\left[S(t)u_0 - \frac{1}{2}\int_0^t S(t-t')\partial_x\big(\psi_T^2(t')u^2(t')\big)\, dt'\right], \tag{20}$$

where $t \in \mathbb{R}$ and, ψ is a time cut-off function satisfying

$$\psi \in C_0^\infty(\mathbb{R}), \quad \operatorname{supp}\psi \subset [-2, 2], \quad \psi \equiv 1 \quad \text{on } [-1, 1], \tag{21}$$

and $\psi_T(\cdot) = \psi(\cdot/T)$.

To run the fixed point argument, we first set up the following homogeneous and inhomogeneous linear estimates.

Proposition 2.1. *Assume $\psi \in C^\infty$ as above and $s \in \mathbb{R}$, $\frac{1}{2} \leqslant b < 1$, then*

$$\big\|\psi(t)S(t)u_0\big\|_{X_{s,b}} \leqslant C\|u_0\|_{E_s}, \tag{22}$$

$$\left\|\psi(t)\int_0^t S(t-t')h(t')\, dt'\right\|_{X_{s,b}} \leqslant C\|h\|_{X_{s,b-1}}. \tag{23}$$

Proof. We observe that

$$\big(\psi(t)S(t)u_0\big)\hat{\ }(\xi, \mu, \tau) = \hat{\psi}\big(\tau - \omega(\xi, \mu)\big)\hat{u}_0(\xi, \mu). \tag{24}$$

To prove (22), we need to estimate the following integral expression:

$$\sum_{j\geqslant 0} 2^{jb}\left(\int_{\mathbb{R}^3} w(\xi, \mu)^{2s}\chi_j(\tau - \omega)\big|\hat{\psi}(\tau - \omega)\big|^2|\hat{u}_0|^2\, d\xi\, d\mu\, d\tau\right)^{\frac{1}{2}}, \tag{25}$$

where $w(\xi,\mu)=(1+|\xi|^2+\frac{|\mu|}{|\xi|})$. We observe that for $j=0$

$$\int_{\mathbb{R}} \left|\hat{\psi}(\lambda)\right|^2 \chi_j(\lambda)\,d\lambda \lesssim \|\hat{\psi}\|_{L^\infty}^2, \tag{26}$$

and for $j \geqslant 1$

$$\int_{\mathbb{R}} \left|\hat{\psi}(\lambda)\right|^2 \chi_j(\lambda)\,d\lambda \lesssim 2^j \frac{1}{(1+2^j)^{2N}} \left\|\left(1+|s|\right)^N \hat{\psi}(s)\right\|_{L^\infty}^2 \tag{27}$$

for any $N \in \mathbb{N}$. When we insert (26) and (27) into (25) we obtain the bound

$$\|u_0\|_{E_s}\left(\|\hat{\psi}\|_{L^\infty} + \sum_{j\geqslant 1} \frac{2^{(\frac{1}{2}+b)j}}{(1+2^j)^N} \left\|\left(1+|s|\right)^N \hat{\psi}(s)\right\|_{L^\infty}\right). \tag{28}$$

It is easy to see that for $N>2$, $\sum_{j\geqslant 1} \frac{2^{(\frac{1}{2}+b)j}}{(1+2^j)^N} \leqslant C$, then (22) is proved.

To prove (23), we write

$$\psi(t)\int_0^t S(t-t')h(t')\,dt' = I + II,$$

where

$$I = \psi(t)\int_{-\infty}^{\infty}\int_{\mathbb{R}^2} e^{i(x\xi+y\mu)}\hat{h}(\xi,\mu,\tau)\psi(\tau-\omega)\frac{e^{it\tau}-e^{it\omega}}{\tau-\omega(\xi,\mu)}\,d\xi\,d\mu\,d\tau$$

and

$$II = \psi(t)\int_{-\infty}^{\infty}\int_{\mathbb{R}^2} e^{i(x\xi+y\mu)}\hat{h}(\xi,\mu,\tau)\left[1-\psi(\tau-\omega)\right]\frac{e^{it\tau}-e^{it\omega}}{\tau-\omega(\xi,\mu)}\,d\xi\,d\mu\,d\tau.$$

By Taylor expansion we can write I as

$$I = \sum_{k=1}^{\infty}\frac{i^k}{k!}t^k\psi(t)\int_{\mathbb{R}^2} e^{i(x\xi+y\mu+t\omega)}\left(\int_{-\infty}^{\infty}\hat{h}(\xi,\mu,\tau)(\tau-\omega)^{k-1}\psi(\tau-\omega)\,d\tau\right)d\xi\,d\tau. \tag{29}$$

For $k \geqslant 1$, we write

$$t^k\psi(t) = \psi_k(t).$$

It is easy to show for $s \in \mathbb{R}$,

$$\left|\hat{\psi}_k(s)\right| \leqslant C,$$

and for any $|s| > 1$,

$$\left|\hat{\psi}_k(s)\right| \leqslant C \frac{(1+k)^2}{(1+|s|)^2}.$$

From (29) it is easy to see

$$I = \sum_{k=1}^{\infty} \frac{i^k}{k!} \psi_k(t) S(t) h_k(x, y),$$

where

$$\hat{h}_k(\xi, \mu) = \int_{-\infty}^{\infty} \hat{h}(\xi, \mu, \tau)(\tau - \omega)^{k-1} \psi(\tau - \omega)\, d\tau.$$

Then by (22), we obtain

$$\|I\|_{X_{s,b}} \lesssim \sum_{k \geqslant 1} \frac{(1+k)^2}{k!} \|h_k\|_{E_s}.$$

On the other hand, from the definition of E_s and $X_{s,b}$, it is easy to see that

$$\|h_k\|_{E_s} \lesssim \|h\|_{X_{s,b-1}}.$$

We now pass to II. We write $II = II_1 + II_2$, where

$$II_1 = \psi(t) \int_{-\infty}^{\infty} \int_{\mathbb{R}^2} e^{i(x\xi + y\mu)} \hat{h}(\xi, \mu, \tau)\big[1 - \psi(\tau - \omega)\big] \frac{e^{it\tau}}{\tau - \omega(\xi, \mu)}\, d\xi\, d\mu\, d\tau,$$

$$II_2 = -\psi(t) \int_{\mathbb{R}^2} e^{i(x\xi + y\mu)} \int_{-\infty}^{\infty} \hat{h}(\xi, \mu, \tau)\big[1 - \psi(\tau - \omega)\big] \frac{e^{it\omega}}{\tau - \omega(\xi, \mu)}\, d\tau\, d\xi\, d\mu.$$

Again by the definition of $X_{s,b}$, we obtain

$$\|II_1\|_{X_{s,b}} \lesssim \|h\|_{X_{s,b-1}}.$$

By (22), we get

$$\|II_2\|_{X_{s,b}} \lesssim \|\tilde{h}\|_{E_s},$$

where

$$\hat{\tilde{h}}(\xi,\mu)=\int_{-\infty}^{\infty}\left[1-\psi(\tau-\omega)\right]\frac{\hat{h}(\xi,\mu,\tau)}{\tau-\omega}\,d\tau.$$

By the following estimate

$$\|\tilde{h}\|_{E_s}\lesssim\|h\|_{X_{s,b-1}},$$

we finish the proof of Proposition 2.1. □

Proposition 2.2. *(See [1].) Let $\delta(r)=2(\frac{1}{2}-\frac{1}{r})$, $2\leqslant r<\infty$. For any $0<T<1$, there exists C independent of T such that*

$$\left\||D_x|^{\frac{\delta(r)}{2}}S(t)u_0(x,y)\right\|_{L_T^q(L^r_{(x,y)})}\leqslant C\|u_0\|_{L^2_{(x,y)}},\qquad \frac{2}{q}=\delta(r). \tag{30}$$

Here

$$\|f\|_{L_T^q(L^r_{(x,y)})}=\left(\int_{-T}^{T}\left(\iint\left|f(x,y,t)\right|^r dx\,dy\right)^{\frac{q}{r}}dt\right)^{\frac{1}{q}}.$$

The following dyadic decomposed Strichartz estimates are crucial in our bilinear estimates.

Proposition 2.3. *Let $\chi_j(\xi,\mu,\tau)=\chi_j(\tau-\omega(\xi,\mu))$, $j\geqslant 0$, and (q,r) as in Proposition 2.2. Denote $f_j=(\chi_j(\xi,\mu,\tau)|\hat{f}|(\xi,\mu,\tau))^{\vee}$. For any $0<T<1$, we have*

$$\left\||D_x|^{\frac{\delta(r)}{2}}f_j\right\|_{L_T^q(L^r_{(x,y)})}\lesssim 2^{\frac{j}{2}}\|f_j\|_{L^2}. \tag{31}$$

Here

$$\|f\|_{L^2}=\left(\iiint\left|f(\xi,\mu,\tau)\right|^2 d\xi\,d\mu\,d\tau\right)^{\frac{1}{2}}.$$

For the sake of convenience, we would like to state the following special cases:

$$\|f_j\|_{L_T^\infty(L^2_{(x,y)})}\lesssim 2^{j/2}\|f_j\|_{L^2(x,y,t)}, \tag{32}$$

$$\left\||D_x|^{\frac{1}{4}}f_j\right\|_{L_T^4(L^4_{(x,y)})}\lesssim 2^{j/2}\|f_j\|_{L^2(x,y,t)}. \tag{33}$$

For $0<\delta<\frac{1}{2}$

$$\left\||D_x|^{\delta}f_j\right\|_{L_T^{\frac{1}{\delta}}(L^{\frac{2}{1-2\delta}}_{(x,y)})}\lesssim 2^{j/2}\|f_j\|_{L^2_{(x,y,t)}} \tag{34}$$

and

$$\big\||D_x|^{\frac{1}{2}-\delta} f_j\big\|_{L_T^{\frac{2}{1-2\delta}}(L_{(x,y)}^{\frac{1}{\delta}})} \lesssim 2^{j/2}\|f_j\|_{L^2_{(x,y,t)}}. \tag{35}$$

Proof of Proposition 2.3. We first note that

$$f_j(x,y,t) = \int_{\mathbb{R}^3} e^{i(x\xi+y\mu+t\tau)}|\hat{f}|\chi_j(\xi,\mu,\tau)\,d\xi\,d\mu\,d\tau.$$

By a simple change of variables we can write

$$\begin{aligned} f_j(x,y,t) &= \int_{\mathbb{R}^3} e^{i(x\xi+y\mu+t(\lambda+\omega))}|\hat{f}|(\xi,\mu,\lambda+\omega)\chi_j(\lambda)\,d\xi\,d\mu\,d\lambda \\ &= \int_{\mathbb{R}} e^{it\lambda}\chi_j(\lambda)\bigg[\int_{\mathbb{R}^2} e^{i(x\xi+y\mu+t\omega)}|\hat{f}|(\xi,\mu,\lambda+\omega)\,d\xi\,d\mu\bigg]d\lambda \\ &= \int_{\mathbb{R}} e^{it\lambda}\chi_j(\lambda)S(t)f_\lambda(x,y)\,d\lambda. \end{aligned}$$

Here $\hat{f}_\lambda(\xi,\mu) = |\hat{f}|(\xi,\mu,\lambda+\omega)$. Then (31) follows from Minkowski's inequality, Strichartz estimate (30) and Cauchy–Schwarz inequality. □

To set up the bilinear estimate in the next section, we will encounter the interaction between high frequency and very low frequency. Then the following maximal estimate will be useful when we dispose the very low frequency.

Proposition 2.4 *(Maximal estimate). Let T_m be the operator such that $\hat{T}_m f(\xi,\mu,\tau) = m(\xi,\mu)\hat{f}(\xi,\mu,\tau)$. Then*

$$\big\|T_m(f)\big\|_{L_t^2(L^\infty_{x,y})} \lesssim \|m\|_{L^2_{\xi,\mu}}\|f\|_{L^2}. \tag{36}$$

Proof. We first notice that

$$T_m f(x,y,t) = \int_{\mathbb{R}^2} \check{m}(x-x',y-y')f(x',y',t)\,dx'\,dy'.$$

Here and below, we use $\check{m}$ to denote the inverse Fourier transform of a function m. Then

$$\big|T_m f(x,y,t)\big| \lesssim \|m\|_{L^2}\big\|f(\cdot,\cdot,t)\big\|_{L^2_{x,y}}.$$

To end the proof one only take the L^2 norm in the t variable. □

At the end of this section, we would like to set up the following proposition, whose idea comes from Lemma 3.1 of [8].

Proposition 2.5. *Let f be a function with compact support (in time) in $[-T,T]$ and $b \geqslant 0$. For any $a>0$, there exists $\sigma=\sigma(a)>0$, such that*

$$\|f\|_{X_{0,(b-a)}} \lesssim T^{\sigma}\|f\|_{X_{0,b}}. \tag{37}$$

Proof. We first show that

$$\big\|\langle\tau-\omega\rangle^{-a}\hat{f}\big\|_{L^2} \lesssim T^{\sigma}\|f\|_{L^2}. \tag{38}$$

We rewrite

$$\big\|\langle\tau-\omega\rangle^{-a}\hat{f}\big\|_{L^2} = \big\|S(t)\langle\partial_t\rangle^{-a}S(-t)f\big\|_{L^2}.$$

Since $S(t)$ is a unit operator in L^2 space and preserves the support properties in time, we have

$$\begin{aligned}\big\|\langle\tau-\omega\rangle^{-a}\hat{f}\big\|_{L^2} = \big\|\langle\partial_t\rangle^{-a}S(-t)f\big\|_{L^2} &\lesssim T^{\frac{1}{2}-\frac{1}{q'}}\big\|\langle\partial_t\rangle^{-a}S(-t)f\big\|_{L^2_{(x,y)}(L_t^{q'})}\\ &\lesssim T^{\frac{1}{2}-\frac{1}{q'}}\big\|S(-t)f\big\|_{L^2},\end{aligned} \tag{39}$$

where $\frac{1}{2}-\frac{1}{q'}=a<\frac{1}{2}$ or $q'=\infty$, if $a>\frac{1}{2}$. We now turn to show (37) by (38). By the definition of $X_{0,b}$, we have

$$\begin{aligned}\|f\|_{X_{0,b-a}} &= \sum_{j\geqslant 0}2^{j(b-a)}\big\|\chi_j\big(\tau-\omega(\xi,\mu)\big)\hat{f}\big\|_{L^2}\\ &\lesssim \sum_{j\geqslant 0}2^{-aj/2}\big\|\langle\tau-\omega\rangle^{-a/2}\langle\tau-\omega\rangle^{b}\chi_j\big(\tau-\omega(\xi,\mu)\big)\hat{f}\big\|_{L^2}\\ &\lesssim \sum_{j\geqslant 0}2^{-aj/2}T^{\sigma}\big\|\langle\tau-\omega\rangle^{b}\chi_j\big(\tau-\omega(\xi,\mu)\big)\hat{f}\big\|_{L^2}\\ &\lesssim T^{\sigma}\|f\|_{X_{0,b}}. \qquad \square\end{aligned}$$

3. The bilinear estimates

Theorem 3.1. *Assume $0<s\leqslant 1$, and u,v with compact time support on $[-T,T]$, $0<T<1$. There exists $\sigma>0$ such that*

$$\big\|\partial_x(uv)\big\|_{X_{s,-\frac{1}{2}+}} \lesssim T^{\sigma}\|u\|_{X_{s,\frac{1}{2}+}}\|v\|_{X_{s,\frac{1}{2}+}}. \tag{40}$$

Here $-\frac{1}{2}+=(\frac{1}{2}+)-1$.

Remark 3. The bilinear estimate above plays a key role in the method of Picard iteration. There are many literatures considering the multilinear estimates. Among them we prefer to pay more attention on [17] and [25]. In [17], Kenig, Ponce and Vega present a bilinear estimate in the studying of the IVP of KdV. It mainly depends on the estimate of the resonance function. Since in the KdV case, the resonant set is very simple, the decomposition of frequency method can bring us enough benefit. Recently, the first two authors [3] obtained the low regularity of modified KdV–Burgers equation by this method. In [25], Tao presented another program to obtain the multilinear estimates. He used the dual argument and dyadic decomposition to transform the multilinear estimate into the estimates of some multipliers on some basic boxes. This method can be used to study some more complicated cases. We also applied this method in a recent paper [4] to set up the well-posedness of the IVP of the modified KdV equation with a dissipative term. As pointed out in [25], the estimate in the box for the KP equation is much complicated. In this paper, we would like to use the dyadic decomposition, the Strichartz estimates and the dispersive smoothing effect to exhaust the structure of the zero set of KP-I resonance function.

We use the duality to prove the bilinear estimate (40). To make our argument more clear, we would like to divide our estimates into two catalogs according to the main term in $(1+|\xi|^2+|\mu||\xi|^{-1})$. It means that we need to estimate, for $g_j \geqslant 0$,

$$\sum_{j\geqslant 0} 2^{j(-\frac{1}{2}+)} \int_{A*} g_j(\xi,\mu,\tau)\chi_j\big(\tau-\omega(\xi,\mu)\big)\chi_1(\xi,\mu) \times |\xi|\big(1+|\xi|^2\big)^s |\hat{u}|(\xi_1,\mu_1,\tau_1)|\hat{v}|(\xi_2,\mu_2,\tau_2)\,d\xi_1\,d\mu_1\,d\tau_1\,d\xi_2\,d\mu_2\,d\tau_2 \tag{41}$$

and

$$\sum_{j\geqslant 0} 2^{j(-\frac{1}{2}+)} \int_{A*} g_j(\xi,\mu,\tau)\chi_j\big(\tau-\omega(\xi,\mu)\big)\chi_2(\xi,\mu) \times |\xi|\left(1+\frac{|\mu|}{|\xi|}\right)^s |\hat{u}|(\xi_1,\mu_1,\tau_1)|\hat{v}|(\xi_2,\mu_2,\tau_2)\,d\xi_1\,d\mu_1\,d\tau_1\,d\xi_2\,d\mu_2\,d\tau_2, \tag{42}$$

where $A*$ is the set $\{\xi_1+\xi_2=\xi,\ \mu_1+\mu_2=\mu,\ \tau_1+\tau_2=\tau\}$, $\chi_1(\xi,\mu)$ is the characteristic function of the set $\{|\xi|^2 \geqslant \frac{|\mu|}{|\xi|}\}$, $\chi_2(\xi,\mu)$ is the characteristic function of the set $\{|\xi|^2 < \frac{|\mu|}{|\xi|}\}$ and $\|g_j\chi_1\chi_j\|_{L^2} \leqslant 1$ and $\|g_j\chi_2\chi_j\|_{L^2} \leqslant 1$. It is clear that by symmetry one can always assume that $|\xi_1| \geqslant |\xi_2|$. The KP-I problem is difficult since resonant set is complicated. We will decompose $A*$ into several domains. For each domain, we decompose it into some tiny sets, and use the estimates in Section 2 on these tiny sets. For instance, when the resonant happens, we will consult to the maximum estimates and the dyadic decomposed Strichartz estimates. We start by subdividing $A*$ into three domains of integration by

Low–Low interaction domain

$$A_1 = \big\{|\xi_1| \geqslant |\xi_2|;\ |\xi_1| \leqslant 100\max\big(1,\sqrt{|\alpha|}\big)\big\};$$

High–High interaction domain

$$A_2 = \big\{|\xi_1| \geqslant |\xi_2|;\ |\xi_2| \sim |\xi_1| \geqslant 100\max\big(1,\sqrt{|\alpha|}\big)\big\};$$

High–Low interaction domain

$$A_3 = \big\{|\xi_1| \gg |\xi_2|;\ |\xi_1| \geqslant 100\max\big(1,\sqrt{|\alpha|}\big)\big\}.$$

Proof of Theorem 3.1. Denote

$$\hat{\phi}_1(\xi,\mu,\tau)=\big(1+|\xi|^2+|\mu|/|\xi|\big)^s|\hat{u}|(\xi,\mu,\tau)$$

and

$$\hat{\phi}_2(\xi,\mu,\tau)=\big(1+|\xi|^2+|\mu|/|\xi|\big)^s|\hat{v}|(\xi,\mu,\tau).$$

Then we need to prove, there exists $\sigma>0$ such that

$$\big\|\partial_x(uv)\big\|_{X_{s,-\frac{1}{2}+}}\lesssim T^\sigma\|\phi_1\|_{X_{0,\frac{1}{2}+}}\|\phi_2\|_{X_{0,\frac{1}{2}+}}.$$

By Proposition 2.5, it suffices to show that

$$\big\|\partial_x(uv)\big\|_{X_{s,-\frac{1}{2}+}}\lesssim\|\phi_1\|_{X_{0,\frac{1}{2}+}}\|\phi_2\|_{X_{0,\frac{1}{2}}}+\|\phi_1\|_{X_{0,\frac{1}{2}}}\|\phi_2\|_{X_{0,\frac{1}{2}+}}. \tag{43}$$

We now control the following two terms by the right-hand side of (43):

$$\sum_{j\geqslant 0}2^{j(-\frac{1}{2}+)}\int_{A*}g_j(\xi,\mu,\tau)\chi_j\big(\tau-\omega(\xi,\mu)\big)\chi_1(\xi,\mu) \times|\xi|\big(1+|\xi|^2\big)^s\frac{\hat{\phi}_1(\xi_1,\mu_1,\tau_1)}{(1+|\xi_1|^2+\frac{|\mu_1|}{|\xi_1|})^s}\frac{\hat{\phi}_2(\xi_2,\mu_2,\tau_2)}{(1+|\xi_2|^2+\frac{|\mu_2|}{|\xi_2|})^s} \tag{44}$$

and

$$\sum_{j\geqslant 0}2^{j(-\frac{1}{2}+)}\int_{A*}g_j(\xi,\mu,\tau)\chi_j\big(\tau-\omega(\xi,\mu)\big)\chi_2(\xi,\mu) \times|\xi|\left(1+\frac{|\mu|}{|\xi|}\right)^s\frac{\hat{\phi}_1(\xi_1,\mu_1,\tau_1)}{(1+|\xi_1|^2+\frac{|\mu_1|}{|\xi_1|})^s}\frac{\hat{\phi}_2(\xi_2,\mu_2,\tau_2)}{(1+|\xi_2|^2+\frac{|\mu_2|}{|\xi_2|})^s}. \tag{45}$$

Another assumption is that function

$$G_{i,j}(x,y,t)=\mathcal{F}^{-1}\left(|\xi|\left(1+|\xi|^2+\frac{|\mu|}{|\xi|}\right)^s g_j(\xi,\mu,\tau)\chi_j\big(\tau-\omega(\xi,\mu)\big)\chi_i(\xi,\mu)\right)(x,y,t)$$

has compact support in time (supporting in the set $[-T,T]$) for $i=1,2$, $j\in\mathbb{N}$. In fact, if we denote

$$\Phi_i(x,y,t)=\mathcal{F}^{-1}\left(\frac{\hat{\phi}_i(\xi_i,\mu_i,\tau_i)}{(1+|\xi_i|^2+\frac{|\mu_i|}{|\xi_i|})^s}\right)(x,y,t),\quad \text{for } i=1,2,$$

the integral in (44) and (45) can be written as an inner product $\langle G_{i,j},\Phi_1\Phi_2\rangle$. Since u and v have compact support with respect to $t\in[-T,T]$, then $\Phi_1\Phi_2$ has the same compact support in time with u and v. Thus the inner product $\langle G_{i,j},\Phi_1\Phi_2\rangle$ can be restricted on the interval $[-T,T]$

according to the time axis. It means we can assume that $G_{i,j}$ has the same compact support in time. We also need some other notations:

$$\hat{\phi}_{i,j_i} = \hat{\phi}_i \chi_{j_i}\big(\tau_i - \omega(\xi_i, \mu_i)\big), \quad i = 1, 2,$$

$$\hat{\phi}_{i,j_i,m_i} = \hat{\phi}_i \chi_{j_i}\big(\tau_i - \omega(\xi_i, \mu_i)\big)\theta_{m_i}(\xi_i), \quad i = 1, 2,$$

$$\hat{\phi}_{i,j_i,n_i} = \hat{\phi}_i \chi_{j_i}\big(\tau_i - \omega(\xi_i, \mu_i)\big)\theta_{n_i}(\mu_i), \quad i = 1, 2,$$

and

$$\hat{\phi}_{i,j_i,m_i,n_i} = \hat{\phi}_i \chi_{j_i}\big(\tau_i - \omega(\xi_i, \mu_i)\big)\theta_{m_i}(\xi_i)\theta_{n_i}(\mu_i), \quad i = 1, 2.$$

Here we used the notation $\theta_0(\eta) = \chi_{[0,1]}(|\eta|)$, $\theta_m(\eta) = \chi_{[2^{m-1},2^m]}(|\eta|)$, $m \in \mathbb{N}$. Some times, we may use g_j instead of $g_j(\xi, \mu, \tau)\chi_j(\tau - \omega(\xi, \mu))$, one can figure out it in the context. Then we can decompose (44) and (45) by

$$\sum_{j_1,j_2 \geqslant 0} \sum_{j \geqslant 0} 2^{j(-\frac{1}{2}+)} \int_{A*} g_j(\xi, \mu, \tau)\chi_j\big(\tau - \omega(\xi, \mu)\big)\chi_1(\xi, \mu) \times |\xi|\big(1 + |\xi|^2\big)^s \frac{\hat{\phi}_{1,j_1}(\xi_1, \mu_1, \tau_1)}{(1 + |\xi_1|^2 + \frac{|\mu_1|}{|\xi_1|})^s} \frac{\hat{\phi}_{2,j_2}(\xi_2, \mu_2, \tau_2)}{(1 + |\xi_2|^2 + \frac{|\mu_2|}{|\xi_2|})^s} \tag{46}$$

and

$$\sum_{j_1,j_2 \geqslant 0} \sum_{j \geqslant 0} 2^{j(-\frac{1}{2}+)} \int_{A*} g_j(\xi, \mu, \tau)\chi_j\big(\tau - \omega(\xi, \mu)\big)\chi_2(\xi, \mu) \times |\xi|\left(1 + \frac{|\mu|}{|\xi|}\right)^s \frac{\hat{\phi}_{1,j_1}(\xi_1, \mu_1, \tau_1)}{(1 + |\xi_1|^2 + \frac{|\mu_1|}{|\xi_1|})^s} \frac{\hat{\phi}_{2,j_2}(\xi_2, \mu_2, \tau_2)}{(1 + |\xi_2|^2 + \frac{|\mu_2|}{|\xi_2|})^s}. \tag{47}$$

Low–Low interaction

Case A. $|\xi_1 + \xi_2|^2 \geqslant \frac{|\mu_1+\mu_2|}{|\xi_1+\xi_2|}$.

In this case, we have $|\xi_1 + \xi_2| \lesssim |\xi_1| \lesssim \max(1, \sqrt{|\alpha|})$. And we also have $|\mu_1 + \mu_2| \leqslant |\xi_1 + \xi_2|^3 \lesssim \max(1, |\alpha|^{\frac{3}{2}})$. Thus we have

$$\begin{aligned}(46) &\lesssim \sum_{j,j_1,j_2 \geqslant 0} 2^{j(-\frac{1}{2}+)} \big\| \big(m(\xi, \mu)g_j\big)^\vee \big\|_{L^2_T(L^\infty_{x,y})} \|\phi_{1,j_1}\|_{L^2(\xi,\mu,\tau)} \|\phi_{2,j_2}\|_{L^\infty_T(L^2_{x,y})} \\ &\lesssim \sum_{j,j_1,j_2 \geqslant 0} 2^{j(-\frac{1}{2}+)} \|g_j\|_{L^2} \|\phi_{1,j_1}\|_{L^2} 2^{j_2/2} \|\phi_{2,j_2}\|_{L^2} \\ &\lesssim \|\phi_1\|_{X_{0,\frac{1}{2}}} \|\phi_2\|_{X_{0,\frac{1}{2}}}. \end{aligned} \tag{48}$$

Here $m(\xi, \mu) = \chi_{|\xi| \lesssim \max(1,|\alpha|^{\frac{1}{2}}),\, |\mu| \lesssim \max(1,|\alpha|^{\frac{3}{2}})}$, which belongs to $L^2(\mathbb{R} \times \mathbb{R})$.

Case B. $|\xi_1+\xi_2|^2 \leqslant \frac{|\mu_1+\mu_2|}{|\xi_1+\xi_2|}$.

We first note that if $\frac{|\mu_1+\mu_2|}{|\xi_1+\xi_2|} \leqslant 1$, then argument above can also bring us the same estimate. We need only to consider the case $\frac{|\mu_1+\mu_2|}{|\xi_1+\xi_2|} \geqslant 1$.

$$(47) \lesssim \sum_{j_1,j_2\geqslant 0} \sum_{j\geqslant 0} 2^{j(-\frac{1}{2}+)} \int_{A_1} g_j(\xi,\mu,\tau)\chi_j\big(\tau-\omega(\xi,\mu)\big)\chi_2(\xi,\mu)$$
$$\times |\xi|^{1-s}|\mu|^s \frac{\hat{\phi}_{1,j_1}(\xi_1,\mu_1,\tau_1)}{(1+|\xi_1|^2+\frac{|\mu_1|}{|\xi_1|})^s} \frac{\hat{\phi}_{2,j_2}(\xi_2,\mu_2,\tau_2)}{(1+|\xi_2|^2+\frac{|\mu_2|}{|\xi_2|})^s}.$$

We then consider two subcases.

Subcase B1. $|\mu_1| \leqslant |\mu_2|$.

If $\frac{|\mu_2|}{|\xi_2|} \leqslant |\xi_2|^2$, then $|\mu_2| \lesssim \max(1,|\alpha|^{\frac{3}{2}})$. Since $|\xi_1+\xi_2| \lesssim \max(1,|\alpha|^{1/2})$, we have

$$\begin{aligned}(47) &\lesssim \sum_{j,j_1,j_2\geqslant 0} 2^{j(-\frac{1}{2}+)} \big\|\big(m(\xi,\mu)g_j\big)^\vee\big\|_{L_T^2(L_{x,y}^\infty)} \|\phi_{1,j_1}\|_{L^2} \|\phi_{2,j_2}\|_{L_T^\infty(L_{x,y}^2)} \\ &\lesssim \sum_{j,j_1,j_2\geqslant 0} 2^{j(-\frac{1}{2}+)} \|g_j\|_{L^2} \|\phi_{1,j_1}\|_{L^2} 2^{j_2/2} \|\phi_{2,j_2}\|_{L^2} \\ &\lesssim \|\phi_1\|_{X_{0,\frac{1}{2}}} \|\phi_2\|_{X_{0,\frac{1}{2}}}. \end{aligned} \tag{49}$$

Here $m(\xi,\mu) = \chi_{|\xi|\lesssim \max(1,|\alpha|^{\frac{1}{2}}),|\mu|\lesssim \max(1,|\alpha|^{\frac{3}{2}})}$.

If $\frac{|\mu_2|}{|\xi_2|} \geqslant |\xi_2|^2$. We first consider the case $\frac{|\mu|}{|\xi|} \lesssim \frac{|\mu_2|}{|\xi_2|}$. Thus we can choose $\min(\frac{1}{2},s) > \delta > 0$ as small as possible such that

$$(47) \lesssim \sum_{j_1,j_2\geqslant 0} \sum_{j\geqslant 0} 2^{j(-\frac{1}{2}+)} \int_{A_1} g_j(\xi,\mu,\tau)\chi_j\big(\tau-\omega(\xi,\mu)\big)\chi_2(\xi,\mu)$$
$$\times |\xi|^{1-\delta}\left(\frac{|\mu|}{|\xi|}\right)^{s-\delta} |\mu|^\delta \hat{\phi}_{1,j_1}(\xi_1,\mu_1,\tau_1) \frac{\hat{\phi}_{2,j_2}(\xi_2,\mu_2,\tau_2)}{(\frac{|\mu_2|}{|\xi_2|})^{s-\delta}(\frac{|\mu_2|}{|\xi_2|})^\delta}$$
$$\lesssim \sum_{j_1,j_2,j\geqslant 0} 2^{j(-\frac{1}{2}+)} \int_{A_1} g_j(\xi,\mu,\tau)\chi_j\big(\tau-\omega(\xi,\mu)\big)\chi_2(\xi,\mu)$$
$$\times |\xi|^{1-\delta} \hat{\phi}_{1,j_1}(\xi_1,\mu_1,\tau_1)|\xi_2|^\delta \hat{\phi}_{2,j_2}(\xi_2,\mu_2,\tau_2).$$

If $j \leqslant j_2$, by Hölder's inequality and (33), we get

$$(47) \lesssim \sum_{j_1, j_2 \geqslant 0} \sum_{j_2 \geqslant j \geqslant 0} 2^{j(-\frac{1}{2}+)} \big\| |D_x|^{\frac{1}{4}} g_j^{\vee} \big\|_{L_T^4(L_{x,y}^4)} \big\| |D_x|^{\frac{1}{4}} \phi_{1,j_1} \big\|_{L_T^4(L_{x,y}^4)} \|\phi_{2,j_2}\|_{L^2}$$

$$\lesssim \sum_{j_1, j_2 \geqslant 0} \sum_{j_2 \geqslant j \geqslant 0} 2^{j(-\frac{1}{2}+)} 2^{j/2} 2^{j_1/2} \|g_j\|_{L^2} \|\phi_{1,j_1}\|_{L^2} \|\phi_{2,j_2}\|_{L^2}$$

$$\lesssim \|\phi_1\|_{X_{0,\frac{1}{2}}} \|\phi_2\|_{X_{0,\frac{1}{2}}}.$$

If $j \geqslant j_2$, by Hölder's inequality and (34) and (35), we obtain

$$(47) \lesssim \sum_{j, j_1 \geqslant 0} \sum_{j \geqslant j_2 \geqslant 0} 2^{j(-\frac{1}{2}+)} \|g_j\|_{L^2} \big\| |D_x|^{\frac{1}{2}-\delta} \phi_{1,j_1} \big\|_{L_T^{\frac{2}{1-2\delta}}(L_{x,y}^{\frac{1}{\delta}})} \big\| |D_x|^{\delta} \phi_{2,j_2} \big\|_{L_T^{\frac{1}{\delta}}(L_{x,y}^{\frac{2}{1-2\delta}})}$$

$$\lesssim \sum_{j, j_1 \geqslant 0} \sum_{j \geqslant j_2 \geqslant 0} 2^{j(-\frac{1}{2}+)} 2^{j_1/2} 2^{j_2/2} \|g_j\|_{L^2} \|\phi_{1,j_1}\|_{L^2} \|\phi_{2,j_2}\|_{L^2}$$

$$\lesssim \|\phi_1\|_{X_{0,\frac{1}{2}}} \|\phi_2\|_{X_{0,\frac{1}{2}}}.$$

If $\frac{|\mu|}{|\xi|} \gg \frac{|\mu_2|}{|\xi_2|}$ and $0 < s \leqslant \frac{1}{2}$, the proof above can also work. We only need to estimate the case $\frac{1}{2} < s \leqslant 1$.

$$(47) \lesssim \sum_{j_1, j_2, j \geqslant 0} 2^{j(-\frac{1}{2}+)} \int_{A_1} g_j(\xi, \mu, \tau) \chi_j\big(\tau - \omega(\xi, \mu)\big) \chi_2(\xi, \mu)$$
$$\times |\xi|^{1-s} |\mu|^s \frac{\hat{\phi}_{1,j_1}(\xi_1, \mu_1, \tau_1)}{(1+|\mu_1|)^s} \frac{\hat{\phi}_{2,j_2}(\xi_2, \mu_2, \tau_2)}{|\mu_2|^s}.$$

In addition, we decompose $|\mu_1| \sim 2^{n_1}$ for $n_1 \geqslant 0$. Thus

$$(47) \lesssim \sum_{j_1, j_2 \geqslant 0} \sum_{j \geqslant 0} \sum_{n_1 \geqslant 0} 2^{j(-\frac{1}{2}+)} 2^{-n_1 s} \|g_j\|_{L^2} \|\phi_{1,j_1,n_1}\|_{L_T^2(L_{x,y}^{\infty})} \|\phi_{2,j_2}\|_{L_T^{\infty}(L_{x,y}^2)}$$

$$\lesssim \sum_{j_1, j_2 \geqslant 0} \sum_{j \geqslant 0} \sum_{n_1 \geqslant 0} 2^{j(-\frac{1}{2}+)} 2^{j_2/2} 2^{-n_1(s-\frac{1}{2})} \|g_j\|_{L^2} \|\phi_{1,j_1,n_1}\|_{L^2} \|\phi_{2,j_2}\|_{L^2}$$

$$\lesssim \|\phi_1\|_{X_{0,\frac{1}{2}}} \|\phi_2\|_{X_{0,\frac{1}{2}}}.$$

Here we used the fact that $|\xi_1| \lesssim \max(1, \sqrt{|\alpha|})$ and $|\mu_1| \lesssim 2^{n_1}$ with Proposition 2.4.

Subcase B2. $|\mu_1| \geqslant |\mu_2|$.

If $|\mu_2| < 1$, we obtain

$$(47) \lesssim \sum_{j, j_1, j_2 \geqslant 0} 2^{j(-\frac{1}{2}+)} \|g_j\|_{L^2} \|\phi_{1,j_1}\|_{L_T^{\infty}(L_{x,y}^2)} \big\| \big(m(\xi_2, \mu_2)\hat{\phi}_{2,j_2}\big)^{\vee} \big\|_{L_T^2(L_{x,y}^{\infty})}$$

$$\lesssim \sum_{j, j_1, j_2 \geqslant 0} 2^{j(-\frac{1}{2}+)} \|g_j\|_{L^2} 2^{j_1/2} \|\phi_{1,j_1}\|_{L^2} \|\phi_{2,j_2}\|_{L^2}$$

$$\lesssim \|\phi_1\|_{X_{0,\frac{1}{2}}} \|\phi_2\|_{X_{0,\frac{1}{2}}}.$$

Here $m(\xi_2,\mu_2)$ denotes the characteristic function of the set $\{(\xi_2,\mu_2); |\xi_2| \lesssim \max(1,|\alpha|^{\frac{1}{2}}), |\mu_2|<1\}$. Thus, we need only to consider the case $|\mu_2| \geqslant 1$. In this case, we can run the same argument with Subcase B1 by interchanging the positions of $|\mu_1|$ and $|\mu_2|$. We omit the details.

High–High interaction

Case A. $|\xi_1+\xi_2|^2 \geqslant \frac{|\mu_1+\mu_2|}{|\xi_1+\xi_2|}$.

We can also assume that $|\xi_1+\xi_2| \gtrsim \max(1,|\alpha|^{1/2})$. Otherwise we go back to (48). We now run dyadic decomposition with respect to $|\xi_1| \sim 2^{m_1}$ (hence $|\xi_2| \sim 2^{m_1}$) and $|\xi| \sim 2^m$ with $m_1+1 \geqslant m \geqslant 0$.

$$\begin{aligned}(46) \lesssim & \sum_{j_1,j_2\geqslant 0}\sum_{j\geqslant 0}\sum_{m_1+1\geqslant m\geqslant 0} 2^{j(-\frac{1}{2}+)}\int_{A_2} g_j(\xi,\mu,\tau)\chi_j\big(\tau-\omega(\xi,\mu)\big)\chi_1(\xi,\mu)\\ &\times 2^{m(1+2s)}2^{m_1(-4s)}\hat{\phi}_{1,j_1,m_1}(\xi_1,\mu_1,\tau_1)\hat{\phi}_{2,j_2,m_1}(\xi_2,\mu_2,\tau_2).\end{aligned}$$

We now consider two subcases.

Subcase A1. $\max(j,j_2) \geqslant 2m_1$.

If $j \leqslant j_2$, we obtain

$$\begin{aligned}(46) &\lesssim \sum_{j_1,j_2\geqslant 0}\sum_{j_2\geqslant j\geqslant 0}\sum_{\frac{j_2}{2}\geqslant m_1>0} 2^{j(-\frac{1}{2}+)}2^{m_1(\frac{1}{2}-2s)}\big\||D_x|^{\frac{1}{4}}g_j^{\vee}\big\|_{L_T^4(L_{x,y}^4)}\\ &\quad\times \big\||D_x|^{\frac{1}{4}}\phi_{1,j_1,m_1}\big\|_{L_T^4(L_{x,y}^4)}\|\phi_{2,j_2,m_1}\|_{L^2}\\ &\lesssim \sum_{j_1,j_2\geqslant 0}\sum_{j_2\geqslant j\geqslant 0} 2^{j(-\frac{1}{2}+)}2^{j/2}2^{j_1/2}2^{j_2(\frac{1}{4}-s)}\|g_j\|_{L^2}\|\phi_{1,j_1}\|_{L^2}\|\phi_{2,j_2}\|_{L^2}\\ &\lesssim \sum_{j_1,j_2\geqslant 0}\sum_{j_2\geqslant j\geqslant 0} 2^{j(-\frac{1}{2}+)}2^{j/2}2^{j_1/2}2^{j_2(\frac{1}{2}+)}2^{-j_2(s+\frac{1}{4}+)}\|g_j\|_{L^2}\|\phi_{1,j_1}\|_{L^2}\|\phi_{2,j_2}\|_{L^2}\\ &\lesssim \|\phi_1\|_{X_{0,\frac{1}{2}}}\|\phi_2\|_{X_{0,\frac{1}{2}+}}.\end{aligned}$$

If $j > j_2$, we also have

$$\begin{aligned}(46) &\lesssim \sum_{j_1\geqslant 0}\sum_{j>j_2\geqslant 0}\sum_{\frac{j}{2}\geqslant m_1>0} 2^{j(-\frac{1}{2}+)}2^{m_1(\frac{1}{2}-2s)}\big\||D_x|^{\frac{1}{4}}\phi_{2,j_2,m_1}\big\|_{L_T^4(L_{x,y}^4)}\\ &\quad\times \big\||D_x|^{\frac{1}{4}}\phi_{1,j_1,m_1}\big\|_{L_T^4(L_{x,y}^4)}\|g_j\|_{L^2}\\ &\lesssim \sum_{j_1\geqslant 0}\sum_{j>j_2\geqslant 0} 2^{j(-\frac{1}{2}+)}2^{j\max(0,\frac{1}{4}-s)}2^{j_1/2}2^{j_2/2}\|g_j\|_{L^2}\|\phi_{1,j_1}\|_{L^2}\|\phi_{2,j_2}\|_{L^2}\\ &\lesssim \|\phi_1\|_{X_{0,\frac{1}{2}}}\|\phi_2\|_{X_{0,\frac{1}{2}}}.\end{aligned}$$

Subcase A2. $\max(j, j_2) \leqslant 2m_1$.

Subsubcase 1. $|\frac{\mu_1}{\xi_1} - \frac{\mu_2}{\xi_2}|^2 \leqslant \frac{1}{2}|\xi_1+\xi_2|^2|5(\xi_1^2+\xi_1\xi_2+\xi_2^2)-3\alpha|$.

In this case, the resonant interaction does not happen. By the definition of resonance function, we can get a useful estimate. Writing

$$\begin{aligned}
&\tau - \omega(\xi_1+\xi_2, \mu_1+\mu_2) - \tau_1 + \omega(\xi_1, \mu_1) - \tau_2 + \omega(\xi_2, \mu_2)\\
&\quad = -\frac{\xi_1\xi_2}{(\xi_1+\xi_2)}\left((\xi_1+\xi_2)^2\big[5\big(\xi_1^2+\xi_1\xi_2+\xi_2^2\big)-3\alpha\big] - \left(\frac{\mu_1}{\xi_1} - \frac{\mu_2}{\xi_2}\right)^2\right),
\end{aligned} \tag{50}$$

since $\langle \tau - \omega(\xi,\mu)\rangle \sim 2^j$, $\langle \tau_1 - \omega(\xi_1,\mu_1)\rangle \sim 2^{j_1}$ and $\langle \tau_2 - \omega(\xi_2,\mu_2)\rangle \sim 2^{j_2}$, we have $2^{\max(j,j_1,j_2)} \geqslant |\xi_1|^4|\xi_1+\xi_2| \geqslant |\xi_1|^4 \sim 2^{4m_1}$. It is clear that we have $j_1 = \max(j, j_1, j_2) \geqslant 4m_1$. Thus $|\xi_1+\xi_2| \lesssim 2^{j_1-4m_1}$. We now choose $\delta > 0$ such that $\min(\frac{1}{4}, s) > \delta > 0$ and $1 - 4\delta + \frac{1}{2} > \frac{1}{2}+$. Therefore

$$\begin{aligned}
(46) &\lesssim \sum_{j_1\geqslant 0}\ \sum_{\frac{j_1}{4}\geqslant m_1\geqslant 0}\ \sum_{2m_1\geqslant\max(j_2,j)\geqslant 0} 2^{j(-\frac{1}{2}+)}2^{(j_1-4m_1)(\frac{1}{2}-2\delta)}\big\||D_x|^{\frac{1}{4}}g_j^{\vee}\big\|_{L_T^4(L_{x,y}^4)}\\
&\quad \times \|\phi_{1,j_1,m_1}\|_{L_t^2(L_{x,y}^2)}\big\||D_x|^{\frac{1}{4}}\phi_{2,j_2,m_1}\big\|_{L_T^4(L_{x,y}^4)}\\
&\lesssim \sum_{j_1\geqslant 0}\ \sum_{m_1\geqslant 0}\ \sum_{2m_1\geqslant\max(j_2,j)\geqslant 0} 2^{j(-\frac{1}{2}+)}2^{j/2}2^{(j_1-4m_1)(\frac{1}{2}-2\delta)}2^{j_2/2}\\
&\quad \times\|g_j\|_{L^2}\|\phi_{1,j_1,m_1}\|_{L^2}\|\phi_{2,j_2,m_1}\|_{L^2}\\
&\lesssim \|\phi_1\|_{X_{0,\frac{1}{2}}}\|\phi_2\|_{X_{0,\frac{1}{2}}}.
\end{aligned}$$

Subsubcase 2. $|\frac{\mu_1}{\xi_1} - \frac{\mu_2}{\xi_2}|^2 > \frac{1}{2}|\xi_1+\xi_2|^2|5(\xi_1^2+\xi_1\xi_2+\xi_2^2)-3\alpha|$.

In this case, the resonant interaction may happen. We have to do some delicate estimates. Let $\theta_1 = \tau_1 - \omega(\xi_1,\mu_1)$ and $\theta_2 = \tau_2 - \omega(\xi_2,\mu_2)$, we can control (46) by

$$\begin{aligned}
&\sum_{j_1,m_1\geqslant 0}\ \sum_{2m_1>\max(j,j_2)\geqslant 0} 2^{j(-\frac{1}{2}+)}2^{m_1(1-2s)}\int g_j\big(\xi,\mu,\theta_1+\omega(\xi_1,\mu_1)+\theta_2+\omega(\xi_2+\mu_2)\big)\\
&\quad \times \chi_j\big(\theta_1+\theta_2+\omega(\xi_1,\mu_2)+\omega(\xi_2+\mu_2)-\omega(\xi_1+\xi_2,\mu_1+\mu_2)\big)\\
&\quad \times \hat{\phi}_{1,j_1,m_1}\big(\xi_1,\mu_1,\theta_1+\omega(\xi_1,\mu_1)\big)\hat{\phi}_{2,j_2,m_1}\big(\xi_2,\mu_2,\theta_2+\omega(\xi_2,\tau_2)\big)\,d\xi_1\,d\mu_1\,d\xi_2\,d\mu_2\,d\theta_1\,d\theta_2.
\end{aligned} \tag{51}$$

We divide the above quantity into two cases.

Subsubsubcase 2a. $|5(\xi_1^4-\xi_2^4) - 3\alpha(\xi_1^2-\xi_2^2) - [(\frac{\mu_1}{\xi_1})^2 - (\frac{\mu_2}{\xi_2})^2]| > 1$.

We change the variables by

$$\begin{cases} u = \xi_1 + \xi_2, \\ v = \mu_1 + \mu_2, \\ w = \theta_1 + \omega(\xi_1, \mu_1) + \theta_2 + \omega(\xi_2 + \mu_2), \\ \mu_2 = \mu_2. \end{cases} \tag{52}$$

The determinant of the Jacobian associating to this change of variables is

$$\begin{aligned} J_\mu &= \begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 5\xi_1^4 - 3\alpha\xi_1^2 - \frac{\mu_1^2}{\xi_1^2} & 5\xi_2^4 - 3\alpha\xi_2^2 - \frac{\mu_2^2}{\xi_2^2} & 2\frac{\mu_1}{\xi_1} & 2\frac{\mu_2}{\xi_2} \\ 0 & 0 & 0 & 1 \end{vmatrix} \\ &= 5(\xi_1^4 - \xi_2^4) - 3\alpha(\xi_1^2 - \xi_2^2) - \left[\left(\frac{\mu_1}{\xi_1}\right)^2 - \left(\frac{\mu_2}{\xi_2}\right)^2\right]. \end{aligned} \tag{53}$$

Thus $|J_\mu| > 1$. We have

$$\begin{aligned} (46) \lesssim \sum_{j_1, m_1 \geqslant 0} \sum_{2m_1 > \max(j, j_2) \geqslant 0} & 2^{j(-\frac{1}{2}+)} 2^{m_1(1-2s)} \int g_j \chi_j(u, v, w) \\ & \times |J_\mu|^{-1} H(u, v, w, \mu_2, \theta_1, \theta_2)\, du\, dv\, dw\, d\mu_2\, d\theta_1\, d\theta_2. \end{aligned} \tag{54}$$

Here $H(u, v, w, \mu_2, \theta_1, \theta_2)$ denotes the transformation of $\hat{\phi}_{1,j_1,m_1}\hat{\phi}_{2,j_2,m_1}$. For fixed θ_1, θ_2, ξ_1, ξ_2, μ_1, we calculate the set length where the free variable μ_2 can range. More precisely, we denote this length by Δ_{μ_2}. Let

$$f(\mu) = \theta_1 + \theta_2 - \frac{\xi_1\xi_2}{(\xi_1 + \xi_2)}\left((\xi_1 + \xi_2)^2\left[5(\xi_1^2 + \xi_1\xi_2 + \xi_2^2) - 3\alpha\right] - \left(\frac{\mu_1}{\xi_1} - \frac{\mu}{\xi_2}\right)^2\right),$$

we have $|f'(\mu_2)| > |\xi_1|^2$. Since

$$\begin{aligned} &\left|\theta_1 + \theta_2 + \omega(\xi_1, \mu_2) + \omega(\xi_2 + \mu_2) - \omega(\xi_1 + \xi_2, \mu_1 + \mu_2)\right| \\ &\quad = \left|\theta_1 + \theta_2 - \frac{\xi_1\xi_2}{(\xi_1 + \xi_2)}\left((\xi_1 + \xi_2)^2\left[5(\xi_1^2 + \xi_1\xi_2 + \xi_2^2) - 3\alpha\right] - \left(\frac{\mu_1}{\xi_1} - \frac{\mu_2}{\xi_2}\right)^2\right)\right| \sim 2^j. \end{aligned} \tag{55}$$

This means that we have $\Delta_{\mu_2} \leqslant 2^{j-2m_1}$. By Cauchy–Schwarz and the inverse change of variables we have

$$\begin{aligned} &\int g_j \chi_j(u, v, w) |J_\mu|^{-1} H(u, v, w, \mu_2, \theta_1, \theta_2)\, du\, dv\, dw\, d\mu_2\, d\theta_1\, d\theta_2 \\ &\quad \lesssim 2^{j/2 - m_1} \int g_j \chi_j(u, v, w) \left(\int |J_\mu|^{-2} H^2(u, v, w, \mu_2, \theta_1, \theta_2)\, d\mu_2\right)^{1/2} du\, dv\, dw\, d\mu_2\, d\theta_1\, d\theta_2 \end{aligned}$$

$$
\begin{aligned}
&\lesssim 2^{j/2-m_1}\|g_j\chi_j\|_{L^2}\int\left(\int |J_\mu|^{-2}H^2(u,v,w,\mu_2,\theta_1,\theta_2)\,du\,dv\,dw\,d\mu_2\right)^{1/2}d\theta_1\,d\theta_2\\
&\lesssim 2^{j/2-m_1}\|g_j\chi_j\|_{L^2}\int\left(\int |J_\mu|^{-1}H^2(u,v,w,\mu_2,\theta_1,\theta_2)\,du\,dv\,dw\,d\mu_2\right)^{1/2}d\theta_1\,d\theta_2\\
&=2^{j/2-m_1}\|g_j\chi_j\|_{L^2}\int\left(\int \prod_{i=1,2}\hat{\phi}^2_{i,j_i,m_1}\big(\xi_i,\mu_i,\theta_i+\omega(\xi_i,\mu_i)\big)\,d\xi_1\,d\mu_1\,d\xi_2\,d\mu_2\right)^{1/2}d\theta_1\,d\theta_2\\
&\lesssim 2^{j/2-m_1}2^{j_1/2}2^{j_2/2}\|g_j\|_{L^2}\|\phi_{1,j_1,m_1}\|_{L^2}\|\phi_{2,j_2,m_1}\|_{L^2}.
\end{aligned}
$$

It follows from (54) that

$$
\begin{aligned}
(46)\lesssim \sum_{m_1,j_1\geqslant 0}\ \sum_{2m_1>\max(j_2,j)\geqslant 0} & 2^{j(-\frac{1}{2}+)}2^{j/2-m_1}2^{m_1(1-2s)}2^{j_1/2}2^{j_2/2}\\
&\times \|g_j\|_{L^2}\|\phi_{1,j_1,m_1}\|_{L^2}\|\phi_{2,j_2,m_1}\|_{L^2}\\
\lesssim \|\phi_1\|_{X_{0,\frac{1}{2}}}\|\phi_2\|_{X_{0,\frac{1}{2}}}.
\end{aligned}
$$

Subsubsubcase 2b. $|5(\xi_1^4-\xi_2^4)-3\alpha(\xi_1^2-\xi_2^2)-[(\frac{\mu_1}{\xi_1})^2-(\frac{\mu_2}{\xi_2})^2]|\leqslant 1$.

In this case the change of variables above cannot be used because the determinant of Jacobian may become zero. We consider the change of variables instead:

$$
\begin{cases}
u=\xi_1+\xi_2,\\
v=\mu_1+\mu_2,\\
w=\theta_1+\omega(\xi_1,\mu_1)+\theta_2+\omega(\xi_2+\mu_2),\\
\xi_1=\xi_1.
\end{cases}
\tag{56}
$$

In this case the determinant of Jacobian J_ξ is given by

$$
\begin{aligned}
J_\xi&=\begin{vmatrix}
1 & 1 & 0 & 0\\
0 & 0 & 1 & 1\\
5\xi_1^4-3\alpha\xi_1^2-\frac{\mu_1^2}{\xi_1^2} & 5\xi_2^4-3\alpha\xi_2^2-\frac{\mu_2^2}{\xi_2^2} & 2\frac{\mu_1}{\xi_1} & 2\frac{\mu_2}{\xi_2}\\
1 & 0 & 0 & 0
\end{vmatrix}\\
&=2\left(\frac{\mu_1}{\xi_1}-\frac{\mu_2}{\xi_2}\right).
\end{aligned}
\tag{57}
$$

An easy calculation shows that $|J_\xi|\gtrsim|\xi_1|$. In this time, we fixed $\theta_1,\theta_2,\xi_2,\mu_1,\mu_2$, and calculate the interval length Δ_{ξ_1} of the free variable ξ_1. Set

$$
h(\xi)=5\big(\xi^4-\xi_2^4\big)-3\alpha\big(\xi^2-\xi_2^2\big)-\left[\left(\frac{\mu_1}{\xi}\right)^2-\left(\frac{\mu_2}{\xi_2}\right)^2\right].
\tag{58}
$$

We compute

$$
h'(\xi)=20\xi^3-6\alpha\xi+2(\mu_1/\xi)^2\xi^{-1}.
\tag{59}
$$

Since now $h'(\xi_1)$ has the same sign as ξ_1, we have $|h'(\xi_1)| \gtrsim |\xi_1|^3$. Thus $\Delta_{\xi_1} \lesssim 2^{-3m_1}$. Remind

$$(46) \lesssim \sum_{j_1,m_1\geqslant 0}\ \sum_{2m_1>\max(j,j_2)\geqslant 0} 2^{j(-\frac{1}{2}+)} 2^{m_1(1-2s)} \int g_j\chi_j(u,v,w) \times |J_\xi|^{-1} H(u,v,w,\xi_1,\theta_1,\theta_2)\,du\,dv\,dw\,d\xi_1\,d\theta_1 d\theta_2. \tag{60}$$

Again denote by $H(u,v,w,\xi_1,\theta_1,\theta_2)$ the transformation of $\prod_{i=1,2}\hat{\phi}_{i,j_i,m_1}$ under the change of variables (56).

$$\begin{aligned}
&\int g_j\chi_j(u,v,w)|J_\xi|^{-1} H(u,v,w,\xi_1,\theta_1,\theta_2)\,du\,dv\,dw\,d\xi_1\,d\theta_1\,d\theta_2\\
&\lesssim 2^{-\frac{3}{2}m_1}\int g_j\chi_j(u,v,w)\left(\int |J_\xi|^{-2}H^2(u,v,w,\xi_1,\theta_1,\theta_2)\,d\xi_1\right)^{1/2} du\,dv\,dw\,d\theta_1\,d\theta_2\\
&\lesssim 2^{-\frac{3}{2}m_1}\|g_j\chi_j\|_{L^2}\int\left(\int |J_\xi|^{-2}H^2(u,v,w,\xi_1,\theta_1,\theta_2)\,du\,dv\,dw\,d\xi_1\right)^{1/2} d\theta_1\,d\theta_2\\
&\lesssim 2^{-2m_1}\|g_j\chi_j\|_{L^2}\int\left(\int |J_\xi|^{-1}H^2(u,v,w,\xi_1,\theta_1,\theta_2)\,du\,dv\,dw\,d\xi_1\right)^{1/2} d\theta_1\,d\theta_2\\
&= 2^{-2m_1}\|g_j\chi_j\|_{L^2}\int\left(\int \prod_{i=1,2}\hat{\phi}^2_{i,j_i,m_1}\big(\xi_i,\mu_i,\theta_i+\omega(\xi_i,\mu_i)\big)\,d\xi_i\,d\mu_i\right)^{1/2} d\theta_1\,d\theta_2\\
&\lesssim 2^{-2m_1}2^{j_1/2}2^{j_2/2}\|g_j\|_{L^2}\|\phi_{1,j_1,m_1}\|_{L^2}\|\phi_{2,j_2,m_1}\|_{L^2}.
\end{aligned}$$

Thus

$$\begin{aligned}
(46) &\lesssim \sum_{m_1,j_1\geqslant 0}\ \sum_{2m_1>\max(j_2,j)\geqslant 0} 2^{j(-\frac{1}{2}+)}2^{-2m_1}2^{m_1(1-2s)}2^{j_1/2}2^{j_2/2}\|g_j\|_{L^2}\|\phi_{1,j_1,m_1}\|_{L^2}\|\phi_{2,j_2,m_1}\|_{L^2}\\
&\lesssim \|\phi_1\|_{X_{0,\frac{1}{2}}}\|\phi_2\|_{X_{0,\frac{1}{2}}}.
\end{aligned}$$

Case B. $|\xi_1+\xi_2|^2 \leqslant \frac{|\mu_1+\mu_2|}{|\xi_1+\xi_2|}$.

If $\frac{|\mu_1+\mu_2|}{|\xi_1+\xi_2|} \lesssim 1$, this case can also be proved by (48). Thus we need only to consider the case $\frac{|\mu_1+\mu_2|}{|\xi_1+\xi_2|} \gtrsim 1$.

Subcase B1. $|\mu_1| \leqslant |\mu_2|$.

Subsubcase B1a. $\frac{|\mu_2|}{|\xi_2|} \leqslant |\xi_2|^2$.

In this case, $|\mu_2| \leqslant |\xi_2|^3$ and $|\mu_1+\mu_2| \leqslant 2|\xi_2|^3$. We now decompose $|\xi_1| \sim |\xi_2| \sim 2^{m_2}$. Then in this case we bound (47) by

$$\sum_{j_1,j_2,j\geqslant 0}\sum_{m_2\geqslant 0}2^{j(-\frac{1}{2}+)}\int\limits_{A_2} g_j(\xi,\mu,\tau)\chi_j\big(\tau-\omega(\xi,\mu)\big)\chi_2(\xi,\mu)$$
$$\times|\xi_1+\xi_2|^{1-s}2^{-m_2 s}\hat{\phi}_{1,j_1,m_2}(\xi_1,\mu_1,\tau_1)\hat{\phi}_{2,j_2,m_2}(\xi_2,\mu_2,\tau_2). \tag{61}$$

We first consider the case that two high frequency waves interaction forms a very low wave, i.e. $|\xi_1+\xi_2|<1$.

$$\begin{aligned}(47)&\lesssim\sum_{j,m_2\geqslant 0}\sum_{j_1,j_2\geqslant 0}2^{j(-\frac{1}{2}+)}2^{m_2(-\frac{1}{2}-s)}\big\||D_x|^{\frac{1}{4}}\phi_{2,j_2,m_2}\big\|_{L^4_T(L^4_{x,y})}\\&\quad\times\big\||D_x|^{\frac{1}{4}}\phi_{1,j_1,m_2}\big\|_{L^4_T(L^4_{x,y})}\|g_j\|_{L^2}\\&\lesssim\sum_{j,m_2\geqslant 0}\sum_{j_1,j_2\geqslant 0}2^{j(-\frac{1}{2}+)}2^{m_2(-\frac{1}{2}-s)}2^{j_1/2}2^{j_2/2}\|g_j\|_{L^2}\|\phi_{1,j_1,m_2}\|_{L^2}\|\phi_{2,j_2,m_2}\|_{L^2}\\&\lesssim\|\phi_1\|_{X_{0,\frac{1}{2}}}\|\phi_2\|_{X_{0,\frac{1}{2}}}.\end{aligned}$$

For the case $|\xi_1+\xi_2|>1$, one can use the argument in Case A again to obtain

$$(47)\lesssim\|\phi_1\|_{X_{0,\frac{1}{2}+}}\|\phi_2\|_{X_{0,\frac{1}{2}}}+\|\phi_1\|_{X_{0,\frac{1}{2}}}\|\phi_2\|_{X_{0,\frac{1}{2}+}}.$$

Subsubcase B1b. $\frac{|\mu_2|}{|\xi_2|}\geqslant|\xi_2|^2$.

We bound (47) by

$$\sum_{j_1,j_2,j\geqslant 0}2^{j(-\frac{1}{2}+)}\int\limits_{A_2} g_j(\xi,\mu,\tau)\chi_j\big(\tau-\omega(\xi,\mu)\big)\chi_2(\xi,\mu)$$
$$\times|\xi_1+\xi_2|^{1-s}|\mu_1+\mu_2|^s\frac{\hat{\phi}_{1,j_1}(\xi_1,\mu_1,\tau_1)}{|\xi_1|^{2s}}\frac{\hat{\phi}_{2,j_2}(\xi_2,\mu_2,\tau_2)}{(\frac{|\mu_2|}{|\xi_2|})^s}.$$

Of course a dyadic decomposition with respect to ξ_1 is also needed. Let $|\xi_1|\sim 2^{m_1}$, we bound (47) by

$$\sum_{j_1,j_2,j\geqslant 0}\sum_{m_1\geqslant 0}2^{j(-\frac{1}{2}+)}\int\limits_{A_2} g_j(\xi,\mu,\tau)\chi_j\big(\tau-\omega(\xi,\mu)\big)\chi_2(\xi,\mu)$$
$$\times|\xi_1+\xi_2|^{1-s}2^{-m_1 s}\hat{\phi}_{1,j_1,m_1}(\xi_1,\mu_1,\tau_1)\hat{\phi}_{2,j_2}(\xi_2,\mu_2,\tau_2).$$

Then one can also run the above argument by considering two cases: $|\xi_1+\xi_2|\leqslant 1$ and $|\xi_1+\xi_2|\geqslant 1$. We now give some details in the case $|\xi_1+\xi_2|\geqslant 1$.

Subsubsubcase 1. $\max(j,j_2)\geqslant 2m_1$.

If $j\leqslant j_2$ and $0<s\leqslant\frac{1}{4}$, we choose $0<\delta<\frac{1}{2}$

$$
\begin{aligned}
(47) &\lesssim \sum_{j_1,j_2\geqslant 0}\ \sum_{j_2\geqslant \max(j,2m_1)\geqslant 0} 2^{j(-\frac12+)}2^{m_1(\frac12-2s)}\big\||D_x|^{\delta} g_j^{\vee}\big\|_{L_T^{\frac1\delta}(L_{x,y}^{\frac{2}{1-2\delta}})}\\
&\quad\times \big\||D_x|^{\frac12-\delta}\phi_{1,j_1,m_1}\big\|_{L_T^{\frac{2}{1-2\delta}}(L_{x,y}^{\frac1\delta})}\|\phi_{2,j_2}\|_{L^2}\\
&\lesssim \sum_{j_1,j_2\geqslant 0}\ \sum_{j_2\geqslant \max(j,2m_1)\geqslant 0} 2^{j(-\frac12+)}2^{m_1(\frac12-2s)}2^{j/2}2^{j_1/2}\|g_j\|_{L^2}\|\phi_{1,j_1,m_1}\|_{L^2}\|\phi_{2,j_2}\|_{L^2}\\
&\lesssim \|\phi_1\|_{X_{0,\frac12}}\|\phi_2\|_{X_{0,\frac12}}.
\end{aligned}
$$

If $j\leqslant j_2$ and $\frac14<s\leqslant 1$, we bound (47) by

$$
\begin{aligned}
&\sum_{j_1,j_2\geqslant 0}\ \sum_{j_2\geqslant \max(j,2m_1)\geqslant 0} 2^{j(-\frac12+)}2^{m_1(\frac12-2s)}\|g_j\|_{L^2}\big\||D_x|^{\frac14}\phi_{1,j_1,m_1}\big\|_{L_T^4(L_{x,y}^4)}\big\||D_x|^{\frac14}\phi_{2,j_2}\big\|_{L_T^4(L_{x,y}^4)}\\
&\quad\lesssim \sum_{j_1,j_2\geqslant 0}\ \sum_{j_2\geqslant \max(j,2m_1)\geqslant 0} 2^{j(-\frac12+)}2^{m_1(\frac12-2s)}2^{j_2/2}2^{j_1/2}\|g_j\|_{L^2}\|\phi_{1,j_1,m_1}\|_{L^2}\|\phi_{2,j_2}\|_{L^2}\\
&\quad\lesssim \|\phi_1\|_{X_{0,\frac12}}\|\phi_2\|_{X_{0,\frac12}}.
\end{aligned}
$$

If $j>j_2$, we also have

$$
\begin{aligned}
(47) &\lesssim \sum_{j,j_1\geqslant 0}\ \sum_{j>\max(j_2,2m_1)\geqslant 0} 2^{j(-\frac12+)}2^{m_1(\frac12-2s)}\big\||D_x|^{\frac14}\phi_{2,j_2}\big\|_{L_T^4(L_{x,y}^4)}\\
&\quad\times \big\||D_x|^{\frac14}\phi_{1,j_1,m_1}\big\|_{L_T^4(L_{x,y}^4)}\|g_j\|_{L^2}\\
&\lesssim \sum_{j,j_1\geqslant 0}\ \sum_{j>\max(j_2,2m_1)\geqslant 0} 2^{j(-\frac12+)}2^{j\max(0,\frac14-s)}2^{j_1/2}2^{j_2/2}\|g_j\|_{L^2}\|\phi_{1,j_1,m_1}\|_{L^2}\|\phi_{2,j_2}\|_{L^2}\\
&\lesssim \|\phi_1\|_{X_{0,\frac12}}\|\phi_2\|_{X_{0,\frac12}}.
\end{aligned}
$$

Subsubsubcase 2. $\max(j,j_2)<2m_1$.

In this case, the argument in Case A can still work by replacing the $\frac14$ derivative on g_j by $\frac14$ derivative on ϕ_1 when $\frac14<s\leqslant 1$, and $|\frac{\mu_1}{\xi_1}-\frac{\mu_2}{\xi_2}|^2>\frac12|\xi_1+\xi_2|^2[5(\xi_1^2+\xi_1\xi_2+\xi_2^2)-3\alpha]$. We omit the rest details.

Subcase B2. $|\mu_1|\geqslant|\mu_2|$. One can use the same argument presented in Subcase B1 by inverting the role of (ξ_1,μ_1) and (ξ_2,μ_2).

High–Low interaction In this domain, the estimates will be more complicated. Roughly speaking, we will consider the term $\frac{|\mu_2|}{|\xi_2|}$ in two regions, $\frac{|\mu_2|}{|\xi_2|}\gtrsim\max(|\xi_1|^2,\frac{|\mu_1|}{|\xi_1|})$ and $\frac{|\mu_2|}{|\xi_2|}\ll\max(|\xi_1|^2,\frac{|\mu_1|}{|\xi_1|})$.

Region I. $\frac{|\mu_2|}{|\xi_2|}\gtrsim\max(|\xi_1|^2,\frac{|\mu_1|}{|\xi_1|})$.

Case A. $|\xi_1+\xi_2|^2 \gtrsim \frac{|\mu_1+\mu_2|}{|\xi_1+\xi_2|}$.

Subcase A1. $|\xi_1|^2 \gtrsim \frac{|\mu_1|}{|\xi_1|}$.

We apply the dyadic decomposition with respect to $|\xi| \sim |\xi_1| \sim 2^{m_1}$ to bound (46) by

$$\sum_{j_1,j_2,j\geqslant 0}\sum_{m_1\geqslant 0} 2^{j(-\frac{1}{2}+)}\int_{A_3} g_j(\xi,\mu,\tau)\chi_j\big(\tau-\omega(\xi,\mu)\big)\chi_1(\xi,\mu) \times 2^{m_1}\hat{\phi}_{1,j_1,m_1}(\xi_1,\mu_1,\tau_1)\frac{\hat{\phi}_{2,j_2}(\xi_2,\mu_2,\tau_2)}{(\frac{|\mu_2|}{|\xi_2|})^s}. \tag{62}$$

Subsubcase A1a. $|\xi_2|\geqslant 1$ and $\max(j,j_2)\geqslant \frac{3}{2}m_1$.

We first notice that

$$(46) \lesssim \sum_{j_1,m_1\geqslant 0}\sum_{\max(j,j_2)\geqslant\frac{3}{2}m_1\geqslant 0} 2^{j(-\frac{1}{2}+)}\int_{A_3} g_j(\xi,\mu,\tau)\chi_j\big(\tau-\omega(\xi,\mu)\big)\chi_1(\xi,\mu) \times 2^{m_1(1-2s)}\hat{\phi}_{1,j_1,m_1}(\xi_1,\mu_1,\tau_1)\hat{\phi}_{2,j_2}(\xi_2,\mu_2,\tau_2).$$

If $j\geqslant j_2$,

$$\begin{aligned}(46) &\lesssim \sum_{j_1,j\geqslant 0}\sum_{j\geqslant\max(j_2,\frac{3}{2}m_1)\geqslant 0} 2^{j(-\frac{1}{2}+)}2^{m_1(\frac{3}{4}-2s)}\|g_j\|_{L^2} \\ &\quad\times \big\||D_x|^{\frac{1}{4}}\phi_{1,j_1,m_1}\big\|_{L^4_T(L^4_{x,y})}\big\||D_x|^{\frac{1}{4}}\phi_{2,j_2}\big\|_{L^4_T(L^4_{x,y})} \\ &\lesssim \sum_{j_1,j\geqslant 0}\sum_{j\geqslant\max(j_2,\frac{3}{2}m_1)\geqslant 0} 2^{j(-\frac{1}{2}+)}2^{m_1(\frac{3}{4}-2s)}\|g_j\|_{L^2}2^{j_1/2}2^{j_2/2}\|\phi_{1,j_1,m_1}\|_{L^2}\|\phi_{2,j_2}\|_{L^2} \\ &\lesssim \|\phi_1\|_{X_{0,\frac{1}{2}}}\|\phi_2\|_{X_{0,\frac{1}{2}}}.\end{aligned}$$

If $j<j_2$,

$$\begin{aligned}(46) &\lesssim \sum_{j_1,j_2\geqslant 0}\sum_{j_2\geqslant\max(\frac{3}{2}m_1,j)\geqslant 0} 2^{j(-\frac{1}{2}+)}2^{m_1(\frac{1}{2}-2s)}\big\||D_x|^{\frac{1}{4}}g_j\big\|_{L^4_T(L^4_{x,y})} \\ &\quad\times \big\||D_x|^{\frac{1}{4}}\phi_{1,j_1,m_1}\big\|_{L^4_T(L^4_{x,y})}\|\phi_{2,j_2}\|_{L^2} \\ &\lesssim \sum_{j_1,j_2\geqslant 0}\sum_{j_2\geqslant\max(j,\frac{3}{2}m_1)\geqslant 0} 2^{j(-\frac{1}{2}+)}2^{j/2}2^{m_1(\frac{1}{2}-2s)}2^{j_1/2}\|g_j\|_{L^2}\|\phi_{1,j_1,m_1}\|_{L^2}\|\phi_{2,j_2}\|_{L^2} \\ &\lesssim \|\phi_1\|_{X_{0,\frac{1}{2}}}\|\phi_2\|_{X_{0,\frac{1}{2}+}}.\end{aligned}$$

Subsubcase A1b. $|\xi_2| \geqslant 1$ and $\max(j, j_2) \leqslant \frac{3}{2}m_1$.

As in the estimates in the high frequency interaction domain, it is necessary to consider more cases.

Subsubsubcase 1. $|\frac{\mu_1}{\xi_1} - \frac{\mu_2}{\xi_2}|^2 < \frac{1}{2}|\xi_1 + \xi_2|^2 |5(\xi_1^2 + \xi_1\xi_2 + \xi_2^2) - 3\alpha|$.

In this case, the resonant interaction does not happen. By inequality (50) and $|\xi_2| > 1$, we get that $j_1 = \max(j, j_1, j_2) \geqslant 4m_1$. We now bound (46) by

$$\sum_{j_1\geqslant 0} \sum_{j_1\geqslant 4m_1\geqslant 0} \sum_{\frac{3}{2}m_1\geqslant \max(j,j_2)\geqslant 0} 2^{j(-\frac{1}{2}+)} 2^{m_1(\frac{3}{4}-2s)} \big\| |D_x|^{\frac{1}{4}} g_j^{\vee} \big\|_{L_T^4(L_{x,y}^4)}$$
$$\times \|\phi_{1,j_1,m_1}\|_{L^2} \big\| |D_x|^{\frac{1}{4}} \phi_{2,j_2} \big\|_{L_T^4(L_{x,y}^4)}$$
$$\lesssim \sum_{j_1\geqslant 0} \sum_{j_1\geqslant 4m_1\geqslant 0} \sum_{\frac{3}{2}m_1\geqslant \max(j,j_2)\geqslant 0} 2^{j(-\frac{1}{2}+)} 2^{j/2} 2^{m_1(\frac{3}{4}-2s)} 2^{j_2/2} \|g_j\|_{L^2} \|\phi_{1,j_1,m_1}\|_{L^2} \|\phi_{2,j_2}\|_{L^2}$$
$$\lesssim \|\phi_1\|_{X_{0,\frac{1}{2}+}} \|\phi_2\|_{X_{0,\frac{1}{2}}}.$$

Subsubsubcase 2. $|\frac{\mu_1}{\xi_1} - \frac{\mu_2}{\xi_2}|^2 \geqslant \frac{1}{2}|\xi_1 + \xi_2|^2 |5(\xi_1^2 + \xi_1\xi_2 + \xi_2^2) - 3\alpha|$.

We need to divide the estimate into two cases:

$$\left|5\left(\xi_1^4 - \xi_2^4\right) - 3\alpha\left(\xi_1^2 - \xi_2^2\right) - \left[\left(\frac{\mu_1}{\xi_1}\right)^2 - \left(\frac{\mu_2}{\xi_2}\right)^2\right]\right| \geqslant 1$$

and

$$\left|5\left(\xi_1^4 - \xi_2^4\right) - 3\alpha\left(\xi_1^2 - \xi_2^2\right) - \left[\left(\frac{\mu_1}{\xi_1}\right)^2 - \left(\frac{\mu_2}{\xi_2}\right)^2\right]\right| < 1.$$

As we known, the first inequality means the determinant of the Jacobian of the change of variables (52) $|J_\mu| \geqslant 1$. So we get

$$(46) \lesssim \sum_{m_1, j_1\geqslant 0} \sum_{\frac{3}{2}m_1 > \max(j_2, j)\geqslant 0} 2^{j(-\frac{1}{2}+)} 2^{j/2-m_1} 2^{m_1(1-2s)} 2^{j_1/2} 2^{j_2/2} \|g_j\|_{L^2} \|\phi_{1,j_1,m_1}\|_{L^2} \|\phi_{2,j_2}\|_{L^2}$$
$$\lesssim \|\phi_1\|_{X_{0,\frac{1}{2}}} \|\phi_2\|_{X_{0,\frac{1}{2}}}.$$

For the second inequality, we recur to the change of variables (56). In the same way, we get

$$(46) \lesssim \sum_{m_1, j_1\geqslant 0} \sum_{2m_1 > \max(j_2, j)\geqslant 0} 2^{j(-\frac{1}{2}+)} 2^{-2m_1} 2^{m_1(1-2s)} 2^{j_1/2} 2^{j_2/2} \|g_j\|_{L^2} \|\phi_{1,j_1,m_1}\|_{L^2} \|\phi_{2,j_2}\|_{L^2}$$
$$\lesssim \|\phi_1\|_{X_{0,\frac{1}{2}}} \|\phi_2\|_{X_{0,\frac{1}{2}}}.$$

Subsubcase A1c. $|\xi_2| < 1$.

If $|\mu_2| \lesssim 1$, since $\frac{|\mu_2|}{|\xi_2|} \gtrsim |\xi_1|^2$, we have that $|\xi_2| \lesssim |\xi_1|^{-2}$. Thus we bound (46) by

$$\sum_{j,j_1,j_2\geqslant 0}\sum_{m_1\geqslant 0} 2^{j(-\frac{1}{2}+)}2^{m_1(1-2s)}\|g_j\|_{L^2}\|\phi_{1,j_1,m_1}\|_{L_T^\infty(L_{x,y}^2)}\left\|\left(m(\xi_2,\mu_2)\hat{\phi}_{2,j_2}\right)^\vee\right\|_{L_T^2(L_{x,y}^\infty)}$$
$$\lesssim \sum_{j,j_1,j_2\geqslant 0}\sum_{m_1\geqslant 0} 2^{j(-\frac{1}{2}+)}2^{m_1(1-2s)}\|g_j\|_{L^2}2^{j_1/2}\|\phi_{1,j_1,m_1}\|_{L^2}2^{-m_1}\|\phi_{2,j_2}\|_{L^2}$$
$$\lesssim \|\phi_1\|_{X_{0,\frac{1}{2}}}\|\phi_2\|_{X_{0,\frac{1}{2}}}.$$

Here $m(\xi_2,\mu_2)$ denotes the characteristic function of set $\{|\xi_2| \lesssim 2^{-2m_1}, |\mu_2| < 1\}$.

If $|\mu_2| \gtrsim 1$ and $\max(j, j_2) \geqslant m_1$, when $j = \max(j, j_2)$, we choose $\min(\frac{1}{2}, s) > \delta > 0$ such that $\frac{1}{2} - 2s + \delta < |-\frac{1}{2}+|$ and bound (46) by

$$\sum_{j_1,j\geqslant 0}\sum_{0\leqslant\max(j_2,m_1)\leqslant j} 2^{j(-\frac{1}{2}+)}2^{m_1(1/2-2s+\delta)}\|g_j\|_{L^2}$$
$$\times\left\||D_x|^{\frac{1}{2}-\delta}\phi_{1,j_1,m_1}\right\|_{L_T^{\frac{2}{1-2\delta}}(L_{x,y}^{\frac{1}{\delta}})}\left\||D_x|^{\delta}\phi_{2,j_2}\right\|_{L_T^{\frac{1}{\delta}}(L_{x,y}^{\frac{2}{1-2\delta}})}$$
$$\lesssim \sum_{j_1,j\geqslant 0}\sum_{0\leqslant\max(j_2,m_1)\leqslant j} 2^{j(-\frac{1}{2}+)}2^{m_1(1/2-2s+\delta)}2^{j_1/2}2^{j_2/2}\|g_j\|_{L^2}\|\phi_{1,j_1,m_1}\|_{L^2}\|\phi_{2,j_2}\|_{L^2}$$
$$\lesssim \|\phi_1\|_{X_{0,\frac{1}{2}}}\|\phi_2\|_{X_{0,\frac{1}{2}}}.$$

While for the case $j_2 = \max(j, j_2)$, we bound (46) by

$$\sum_{j_1,j_2\geqslant 0}\sum_{0\leqslant\max(j,m_1)\leqslant j_2} 2^{j(-\frac{1}{2}+)}2^{m_1(1/2-2s)}\left\||D_x|^{\delta}g_j^\vee\right\|_{L_T^{\frac{1}{\delta}}(L_{x,y}^{\frac{2}{1-2\delta}})}$$
$$\times\left\||D_x|^{\frac{1}{2}-\delta}\phi_{1,j_1,m_1}\right\|_{L_T^{\frac{2}{1-2\delta}}(L_{x,y}^{\frac{1}{\delta}})}\|\phi_{2,j_2}\|_{L^2}$$
$$\lesssim \sum_{j_1,j_2\geqslant 0}\sum_{0\leqslant\max(j,m_1)\leqslant j_2} 2^{j(-\frac{1}{2}+)}2^{m_1(1/2-2s)}2^{j/2}2^{j_1/2}\|g_j\|_{L^2}\|\phi_{1,j_1,m_1}\|_{L^2}\|\phi_{2,j_2}\|_{L^2}$$
$$\lesssim \|\phi_1\|_{X_{0,\frac{1}{2}}}\|\phi_2\|_{X_{0,\frac{1}{2}+}}.$$

If $|\mu_2| \gtrsim 1$ and $\max(j, j_2) < m_1$, we have to divided two subcases to estimate (46).

Subsubsubcase a. $|\frac{\mu_1}{\xi_1} - \frac{\mu_2}{\xi_2}|^2 < \frac{1}{2}|\xi_1+\xi_2|^2|5[\xi_1^2+\xi_1\xi_2+\xi_2^2]-3\alpha|$.

As we know, the estimate on the resonance function can be used now. We have $|\xi_1|^4|\xi_2| \lesssim 2^{\max(j,j_1,j_2)}$. Unfortunately, since $|\xi_2| < 1$, the element inequality is not as good as we have used. We claim that $|\xi_2| \geqslant |\xi_1|^{-2}$. Otherwise, if $|\frac{\mu_1}{\xi_1}| \sim |\frac{\mu_2}{\xi_2}|$, then $|\mu_2| \lesssim |\xi_1|^2|\xi_2| \lesssim 1$. And if $|\frac{\mu_1}{\xi_1}| \ll |\frac{\mu_2}{\xi_2}|$, since we are in Subsubsubcase a: $|\frac{\mu_1}{\xi_1} - \frac{\mu_2}{\xi_2}|^2 < \frac{1}{2}|\xi_1+\xi_2|^2|5[\xi_1^2+\xi_1\xi_2+\xi_2^2]-3\alpha|$, we have $|\mu_2| \lesssim |\xi_1|^2|\xi_2| \lesssim 1$. These conflict with the assumption $|\mu_2| \gtrsim 1$. Thus we have $2^{2m_1} \leqslant 2^{\max(j, j_1, j_2)}$. It is clear that $j_1 = \max(j, j_1, j_2)$. We bound (46) with

$$\sum_{j_1 \geqslant j,\, j_2 \geqslant 0} \sum_{0 \leqslant 2m_1 \leqslant j_1} 2^{j(-\frac{1}{2}+)} 2^{m_1(1/2-2s+\delta)} \big\| |D_x|^{\frac{1}{2}-\delta} g_j^{\vee} \big\|_{L_T^{\frac{2}{1-2\delta}}(L_{x,y}^{\frac{1}{\delta}})}$$
$$\times \|\phi_{1,j_1,m_1}\|_{L^2} \big\| |D_x|^{\delta} \phi_{2,j_2} \big\|_{L_T^{\frac{1}{\delta}}(L_{x,y}^{\frac{2}{1-2\delta}})}$$
$$\lesssim \sum_{j_1 \geqslant j,\, j_2 \geqslant 0} \sum_{0 \leqslant 2m_1 \leqslant j_1} 2^{j(-\frac{1}{2}+)} 2^{m_1(1/2-2s+\delta)} 2^{j/2} 2^{j_2/2} \|g_j\|_{L^2} \|\phi_{1,j_1,m_1}\|_{L^2} \|\phi_{2,j_2}\|_{L^2}$$
$$\lesssim \|\phi_1\|_{X_{0,\frac{1}{2}+}} \|\phi_2\|_{X_{0,\frac{1}{2}}}.$$

Subsubsubcase b. $|\frac{\mu_1}{\xi_1} - \frac{\mu_2}{\xi_2}|^2 \geqslant \frac{1}{2}|\xi_1 + \xi_2|^2 |5[\xi_1^2 + \xi_1\xi_2 + \xi_2^2] - 3\alpha|$. In this case, one can run the same argument in Subsubcase A1b.

Subcase A2. $|\xi_1|^2 \ll \frac{|\mu_1|}{|\xi_1|}$.

The argument in Subcase A1 above can also help us to get the same estimates. We would like to show the different point when we encounter the case $|\mu_2| \gtrsim 1$, $|\frac{\mu_1}{\xi_1} - \frac{\mu_2}{\xi_2}|^2 < \frac{1}{2}|\xi_1 + \xi_2|^2 |5[\xi_1^2 + \xi_1\xi_2 + \xi_2^2] - 3\alpha|$. Here we still have $|\xi_1|^4|\xi_2| \lesssim 2^{\max(j, j_1, j_2)}$. If $|\mu_1| \lesssim |\mu_2|$, we have $\frac{|\mu_2|}{|\xi_2|} \gg \frac{|\mu_1|}{|\xi_1|}$. It means that we also have $|\xi_2| > |\xi_1|^{-2}$. If $|\mu_1| \gg |\mu_2|$, then we have $|\mu_1 + \mu_2| \sim |\mu_1|$, thus $\frac{|\mu_1+\mu_2|}{|\xi_1+\xi_2|} \gg |\xi_1 + \xi_2|^2$. This does not appear since we are in case $|\xi_1 + \xi_2|^2 \gtrsim \frac{|\mu_1+\mu_2|}{|\xi_1+\xi_2|}$. Then we can run the argument in Subcase A1.

Case B. $|\xi_1 + \xi_2|^2 \ll \frac{|\mu_1+\mu_2|}{|\xi_1+\xi_2|}$.

Subcase B1. $|\mu_1| \lesssim |\mu_2|$.

In this region, we also have $\frac{|\mu_2|}{|\xi_2|} \gg |\xi_1|^2$. Similar to the argument presented in the second part of Subcase B1 of domain A_2, we can bound (47) with

$$\sum_{j_1, j_2, j \geqslant 0} \sum_{m_1 \geqslant 0} 2^{j(-\frac{1}{2}+)} \int_{A_3} g_j(\xi, \mu, \tau) \chi_j\big(\tau - \omega(\xi, \mu)\big) \chi_2(\xi, \mu)$$
$$\times |\xi_1 + \xi_2|^{1-s} 2^{-m_1 s} \hat{\phi}_{1,j_1,m_1}(\xi_1, \mu_1, \tau_1) \hat{\phi}_{2,j_2}(\xi_2, \mu_2, \tau_2).$$

Then the estimate in Case A above works.

Subcase B2. $|\mu_1| \gg |\mu_2|$.

It is clear that $\frac{|\mu_1|}{|\xi_1|} \sim \frac{|\mu_1+\mu_2|}{|\xi_1+\xi_2|}$. Thus (47) by

$$\sum_{j_1,j_2,j\geqslant 0}\sum_{m_1\geqslant 0}2^{j(-\frac{1}{2}+)}\int_{A_3} g_j(\xi,\mu,\tau)\chi_j\big(\tau-\omega(\xi,\mu)\big)\chi_2(\xi,\mu)$$
$$\times 2^{m_1(1-2s)}\hat{\phi}_{1,j_1,m_1}(\xi_1,\mu_1,\tau_1)\hat{\phi}_{2,j_2}(\xi_2,\mu_2,\tau_2).$$

If $|\mu_2|<1$, then we also have $|\xi_2|\leqslant|\xi_1|^{-2}$. By the same argument in Subsubcase A1c, we bound (47) by $\|\phi_1\|_{X_{0,\frac{1}{2}}}\|\phi_2\|_{X_{0,\frac{1}{2}+}}$.

If $|\mu_2|\geqslant 1$, the estimates in Case A above can also work until we come to the case $|\xi_2|<|\xi_1|^{-2}$, $\max(j,j_2)<2m_1$ and $|\frac{\mu_1}{\xi_1}-\frac{\mu_2}{\xi_2}|^2<\frac{1}{2}|\xi_1+\xi_2|^2|5[\xi_1^2+\xi_1\xi_2+\xi_2^2]-3\alpha|$. Of course, in this case, the estimate on resonance function can also bring us

$$|\xi_1|^4|\xi_2|\lesssim 2^{\max(j,j_1,j_2)}.$$

But this estimate cannot help us to get any benefit since $|\xi_2|<|\xi_1|^{-2}$. Fortunately, in this case, for fixed μ_1,ξ_1,ξ_2, the variable μ_2 can range in two symmetry intervals with length $\Delta_{\mu_2}\lesssim|\xi_1|^2|\xi_2|\lesssim 1$. Represent the change of variables (51) here,

$$\sum_{j_1,m_1\geqslant 0}\sum_{2m_1>\max(j,j_2)\geqslant 0}2^{j(-\frac{1}{2}+)}2^{m_1(1-2s)}\int g_j\big(\xi,\mu,\theta_1+\omega(\xi_1,\mu_1)+\theta_2+\omega(\xi_2+\mu_2)\big)$$
$$\times\chi_j\big(\theta_1+\theta_2+\omega(\xi_1,\mu_2)+\omega(\xi_2+\mu_2)-\omega(\xi_1+\xi_2,\mu_1+\mu_2)\big)$$
$$\times\hat{\phi}_{1,j_1,m_1}\big(\xi_1,\mu_1,\theta_1+\omega(\xi_1,\mu_1)\big)\hat{\phi}_{2,j_2}\big(\xi_2,\mu_2,\theta_2+\omega(\xi_2,\tau_2)\big)\,d\xi_1\,d\mu_1\,d\xi_2\,d\mu_2\,d\theta_1\,d\theta_2.$$

By Cauchy–Schwarz inequality, we control the integral (51) by

$$\|\phi_{1,j_1,m_1}\|_{L^2}\bigg(\int\bigg|\int H(\xi_1,\xi_2,\mu_1,\mu_2,\theta_1,\theta_2)\,d\xi_2\,d\mu_2\,d\theta_2\bigg|^2 d\xi_1\,d\mu_1\,d\theta_1\bigg)^{\frac{1}{2}}$$
$$\lesssim 2^{j_2/2}2^{-m_1}\|\phi_{1,j_1,m_1}\|_{L^2}\bigg(\iint\big|H(\xi_1,\xi_2,\mu_1,\mu_2,\theta_1,\theta_2)\big|^2\,d\xi_2\,d\mu_2\,d\theta_2\,d\xi_1\,d\mu_1\,d\theta_1\bigg)^{\frac{1}{2}}$$
$$\lesssim 2^{j_2/2}2^{-m_1}\|\phi_{1,j_1,m_1}\|_{L^2}\|g_j\|_{L^2}\|\phi_{2,j_2}\|.$$

Here $H(\xi_1,\xi_2,\mu_1,\mu_2,\theta_1,\theta_2)$ denotes $g_j(\xi,\mu,\theta_1+\omega(\xi_1,\mu_1)+\theta_2+\omega(\xi_2+\mu_2))\chi_j(\theta_1+\theta_2+\omega(\xi_1,\mu_2)+\omega(\xi_2+\mu_2)-\omega(\xi_1+\xi_2,\mu_1+\mu_2))\hat{\phi}_{2,j_2}(\xi_2,\mu_2,\theta_2+\omega(\xi_2,\tau_2))$. Now we put this estimate into the summation above to obtain

$$(47)\lesssim\|\phi_1\|_{X_{0,\frac{1}{2}+}}\|\phi_2\|_{X_{0,\frac{1}{2}}}.$$

Region II. $\frac{|\mu_2|}{|\xi_2|}\ll\max(|\xi_1|^2,\frac{|\mu_1|}{|\xi_1|})$.

Case A. $|\xi_1|^2\gg\frac{|\mu_1|}{|\xi_1|}$.

Since $|\xi_1+\xi_2|^3\sim|\xi_1|^3\gg|\mu_1|$ and $|\xi_1|^3\gg|\xi_1|^2|\xi_2|\gg|\mu_2|$, the resonant interaction does not happen, so $2^{\max(j,j_1,j_2)}\geqslant|\xi_1|^4|\xi_2|$.

If $|\mu_2| < 1$ and $j = \max(j, j_1, j_2)$, then $2^j \geqslant 2^{4m_1+m_2}$. In the same way, we bound (46) by

$$\sum_{j,j_1,j_2\geqslant 0} \sum_{j\geqslant 4m_1+m_2} \sum_{m_2<m_1} 2^{j(-\frac{1}{2}+)} 2^{m_1} \|g_j\|_{L^2} \|\phi_{1,j_1,m_1}\|_{L^\infty_T(L^2_{x,y})} \times \big\| \big(m_{m_2}(\xi_2,\mu_2)\hat{\phi}_{2,j_2}\big)^\vee \big\|_{L^2_T(L^\infty_{x,y})}$$
$$\lesssim \sum_{j,j_1,j_2\geqslant 0} \sum_{j\geqslant 4m_1+m_2} \sum_{m_2<m_1} 2^{j(-\frac{1}{2}+)} 2^{m_1} \|g_j\|_{L^2} 2^{j_1/2} \|\phi_{1,j_1,m_1}\|_{L^2} 2^{m_2/2} \|\phi_{2,j_2}\|_{L^2}$$
$$\lesssim \|\phi_1\|_{X_{0,\frac{1}{2}}} \|\phi_2\|_{X_{0,\frac{1}{2}+}}.$$

Here we used Proposition 2.4 with m_{m_2} denoting a class of multipliers which are the characteristic functions of the sets $\{|\xi_2| \sim 2^{m_2}, |\mu_2| < 1\}$.

If $|\mu_2| < 1$ and $j_1 = \max(j, j_1, j_2)$ or $j_2 = \max(j, j_1, j_2)$ is the maximal value, similarly we have

$$(46) \lesssim \sum_{j,j_1,j_2\geqslant 0} \sum_{j_1\geqslant 4m_1+m_2} \sum_{m_2<m_1} 2^{j(-\frac{1}{2}+)} 2^{m_1} \|\phi_{1,j_1,m_1}\|_{L^2} \|g_j^\vee\|_{L^\infty_T(L^2_{x,y})} \times \big\| \big(m_{m_2}(\xi_2,\mu_2)\hat{\phi}_{2,j_2}\big)^\vee \big\|_{L^2_T(L^\infty_{x,y})}$$
$$\lesssim \sum_{j,j_1,j_2\geqslant 0} \sum_{j_1\geqslant 4m_1+m_2} \sum_{m_2<m_1} 2^{j(-\frac{1}{2}+)} 2^{m_1} 2^{j/2} 2^{m_2/2} \|g_j\|_{L^2} \|\phi_{1,j_1,m_1}\|_{L^2} \|\phi_{2,j_2}\|_{L^2}$$
$$\lesssim \|\phi_1\|_{X_{0,\frac{1}{2}}} \|\phi_2\|_{X_{0,\frac{1}{2}+}}.$$

If $|\mu_2| \geqslant 1$ and $\max(j, j_2) \geqslant 2m_1$, let $j = \max(j, j_2)$, there exists $\min(\frac{1}{2}, s) > \delta > 0$ and $|-\frac{1}{2}+| > \frac{1}{4} + \frac{1}{2}\delta > 0$ such that

$$(46) \lesssim \sum_{j_1,j\geqslant 0} \sum_{0\leqslant \max(j_2,2m_1)\leqslant j} 2^{j(-\frac{1}{2}+)} 2^{m_1(1/2+\delta)} \|g_j\|_{L^2} \times \big\| |D_x|^{\frac{1}{2}-\delta} \phi_{1,j_1,m_1} \big\|_{L_T^{\frac{2}{1-2\delta}}(L_{x,y}^{\frac{1}{\delta}})} \big\| |D_x|^{\delta} \phi_{2,j_2} \big\|_{L_T^{\frac{1}{\delta}}(L_{x,y}^{\frac{2}{1-2\delta}})}$$
$$\lesssim \sum_{j_1,j\geqslant 0} \sum_{0\leqslant \max(j_2,2m_1)\leqslant j} 2^{j(-\frac{1}{2}+)} 2^{m_1(1/2+\delta)} 2^{j_1/2} 2^{j_2/2} \|g_j\|_{L^2} \|\phi_{1,j_1,m_1}\|_{L^2} \|\phi_{2,j_2}\|_{L^2}$$
$$\lesssim \|\phi_1\|_{X_{0,\frac{1}{2}}} \|\phi_2\|_{X_{0,\frac{1}{2}}}.$$

For the case $j_2 = \max(j, j_2)$, we bound (46) by

$$\sum_{j_1,j_2\geqslant 0} \sum_{0\leqslant \max(j,2m_1)\leqslant j_2} 2^{j(-\frac{1}{2}+)} 2^{m_1/2} \big\| |D_x|^{\frac{1}{4}} g_j^\vee \big\|_{L^4_T(L^4_{x,y})} \big\| |D_x|^{\frac{1}{4}} \phi_{1,j_1,m_1} \big\|_{L^4_T(L^4_{x,y})} \|\phi_{2,j_2}\|_{L^2}$$
$$\lesssim \sum_{j_1,j_2\geqslant 0} \sum_{0\leqslant \max(j,2m_1)\leqslant j_2} 2^{j(-\frac{1}{2}+)} 2^{m_1/2} 2^{j/2} 2^{j_1/2} \|g_j\|_{L^2} \|\phi_{1,j_1,m_1}\|_{L^2} \|\phi_{2,j_2}\|_{L^2}$$
$$\lesssim \|\phi_1\|_{X_{0,\frac{1}{2}}} \|\phi_2\|_{X_{0,\frac{1}{2}}}.$$

If $|\mu_2| \geqslant 1$ and $\max(j, j_2) < 2m_1$, we would like to perform a dyadic decomposition by setting $|\xi_i| \sim 2^{m_i}$ with $i = 1, 2$ and $m_1 \geqslant 0$, $m_2 \in \mathbb{Z}$. The dyadic decomposition with respect to $|\mu_2| \sim 2^{n_2}$, $n_2 \geqslant 0$ will be useful. Another useful note is that $m_2^* = \max(n_2 - m_2, 2m_2)$.

We perform the change of variables (52). It is easy to see that $|J_\mu| > |\xi_1|^4$, so

$$\begin{aligned}(46) \lesssim & \sum_{j_1, m_1, n_2 \geqslant 0} \sum_{m_2} \sum_{2m_1 > \max(j, j_2) \geqslant 0} 2^{j(-\frac{1}{2}+)} 2^{m_1} \int g_j(u, v, w) \chi_j(u, v, w) \\ & \times |J_\mu|^{-1} H(u, v, w, \mu_2, \theta_1, \theta_2)\, du\, dv\, dw\, d\mu_2\, d\theta_1\, d\theta_2 \\ \lesssim & \sum_{j_1, m_1, n_2 \geqslant 0} \sum_{m_2} \sum_{2m_1 > \max(j, j_2) \geqslant 0} 2^{j(-\frac{1}{2}+)} 2^{-m_1} 2^{n_2/2} 2^{j_1/2} 2^{j_2/2} \\ & \times \|g_j\|_{L^2} \|\phi_{1, j_1, m_1}\|_{L^2} \frac{\|\phi_{2, j_2, m_2, n_2}\|_{L^2}}{\max(1, 2^{m_2^*})^s}.\end{aligned}$$

If $m_2 \geqslant 0$ and $n_2 - m_2 < 0$, we bound (46) with

$$\begin{aligned}& \sum_{j, j_1, j_2, m_1 \geqslant 0} \sum_{0 \leqslant n_2 \leqslant m_2 < m_1} 2^{j(-\frac{1}{2}+)} 2^{-m_1} 2^{n_2/2} 2^{(-2m_2)s} \\ & \quad \times 2^{j_1/2} 2^{j_2/2} \|g_j\|_{L^2} \|\phi_{1, j_1, m_1}\|_{L^2} \|\phi_{2, j_2, m_2, n_2}\|_{L^2} \\ & \quad \lesssim \|\phi_1\|_{X_{0, \frac{1}{2}+}} \|\phi_2\|_{X_{0, \frac{1}{2}}}.\end{aligned}$$

If $2m_2 \geqslant n_2 - m_2 \geqslant 0$ and $j > 2m_2$, we have

$$\begin{aligned}(46) \lesssim & \sum_{j_1, j_2 \geqslant 0} \sum_{0 \leqslant n_2 - m_2 < 2m_2} \sum_{j \geqslant 2m_2 \geqslant 0} \sum_{m_1 \geqslant m_2 \geqslant 0} 2^{j(-\frac{1}{2}+)} 2^{-m_1} 2^{(n_2 - m_2)/2} 2^{m_2/2} 2^{-2m_2 s} \\ & \times 2^{j_1/2} 2^{j_2/2} \|g_j\|_{L^2} \|\phi_{1, j_1, m_1}\|_{L^2} \|\phi_{2, j_2, m_2, n_2}\|_{L^2} \\ \lesssim & \|\phi_1\|_{X_{0, \frac{1}{2}+}} \|\phi_2\|_{X_{0, \frac{1}{2}}}.\end{aligned}$$

If $n_2 - m_2 \geqslant 2m_2 \geqslant 0$ and $j > 2m_2$, since $\frac{|\mu_2|}{|\xi_2|} \ll \max(|\xi_1|^2, \frac{|\mu_1|}{|\xi_1|})$ and $|\xi_1|^2 \gg \frac{|\mu_1|}{|\xi_1|}$, one can get $(n_2 - m_2) \leqslant 2m_1$. Recalling that $s > 0$, we obtain

$$\begin{aligned}(46) \lesssim & \sum_{j_1, j_2 \geqslant 0} \sum_{0 < 2m_2 \leqslant n_2 - m_2} \sum_{j \geqslant 2m_2 \geqslant 0} \sum_{m_1 \geqslant m_2 \geqslant 0} 2^{j(-\frac{1}{2}+)} 2^{-m_1} 2^{(n_2 - m_2)/2} 2^{m_2/2} 2^{-(n_2 - m_2)s} \\ & \times 2^{j_1/2} 2^{j_2/2} \|g_j\|_{L^2} \|\phi_{1, j_1, m_1}\|_{L^2} \|\phi_{2, j_2, m_2, n_2}\|_{L^2} \\ \lesssim & \|\phi_1\|_{X_{0, \frac{1}{2}+}} \|\phi_2\|_{X_{0, \frac{1}{2}}}.\end{aligned}$$

If $m_2 < 0$, we have

$$
\begin{aligned}
(46) \lesssim & \sum_{j_1,m_1\geqslant 0}\sum_{m_2<0}\sum_{2m_1\geqslant n_2-m_2\geqslant 0}\sum_{2m_1\geqslant \max(j_2,j)\geqslant 0} 2^{j(-\frac{1}{2}+)}2^{-m_1}2^{(n_2-m_2)(1/2-s)}2^{j_1/2}2^{j_2/2}2^{m_2/2}\\
&\times \|g_j\|_{L^2}\|\phi_{1,j_1,m_1}\|_{L^2}\|\phi_{2,j_2,m_2,n_2}\|_{L^2}\\
\lesssim & \sum_{j_1,m_1\geqslant 0}\sum_{2m_1\geqslant \max(j_2,j)\geqslant 0} 2^{j(-\frac{1}{2}+)}2^{-2m_1 s}2^{j_1/2}2^{j_2/2}\|g_j\|_{L^2}\|\phi_{1,j_1,m_1}\|_{L^2}\|\phi_{2,j_2}\|_{L^2}\\
\lesssim & \|\phi_1\|_{X_{0,\frac{1}{2}}}\|\phi_2\|_{X_{0,\frac{1}{2}}}.
\end{aligned}
$$

We now consider the case $0\leqslant \max(j,j_2)\leqslant m_1$, $0\leqslant j<2m_2$ and $n_2-m_2>0$. It is clear that $j_1=\max(j,j_1,j_2)$ and $|\xi_1|^4\lesssim 2^{j_1}$, since $2^{\max(j,j_1,j_2)}\gtrsim |\xi_1|^4|\xi_2|$. We bound (46) by

$$
\begin{aligned}
&\sum_{j_1,j_2\geqslant 0}\sum_{j\geqslant 0} 2^{j(-\frac{1}{2}+)}\int_{A_3} g_j(\xi,\mu,\tau)\chi_j(\xi,\mu,\tau)\\
&\qquad\times |\xi_1+\xi_2|\hat{\phi}_{1,j_1}(\xi_1,\mu_1,\tau_1)\frac{\hat{\phi}_{2,j_2}(\xi_2,\mu_2,\tau_2)}{(1+|\xi_2|^2+\frac{|\mu_2|}{|\xi_2|})^s}.
\end{aligned}
$$

There exists $\min(\frac{1}{2},s)>\delta>0$ small enough such that

$$
\begin{aligned}
(46) \lesssim & \sum_{m_1\geqslant 0}\sum_{j_1\geqslant 4m_1\geqslant 0}\sum_{j_1\geqslant 4m_1\geqslant 0}\sum_{2m_1\geqslant \max(j_2,j)\geqslant 0} 2^{j(-\frac{1}{2}+)}2^{m_1(1/2+\delta)}\\
&\times \big\||D_x|^{1/2-\delta}g_j^{\vee}\big\|_{L_T^{\frac{2}{1-2\delta}}(L_{x,y}^{\frac{1}{\delta}})}\|\phi_{1,j_1,m_1}\|_{L^2}\big\||D_x|^{\delta}\phi_{2,j_2}\big\|_{L_T^{\frac{1}{\delta}}(L_{x,y}^{\frac{2}{1-2\delta}})}\\
\lesssim & \sum_{m_1\geqslant 0}\sum_{j_1\geqslant 4m_1\geqslant 0}\sum_{2m_1\geqslant \max(j_2,j)\geqslant 0} 2^{j(-\frac{1}{2}+)}2^{m_1(1/2+\delta)}2^{j/2}2^{j_2/2}\|g_j\|_{L^2}\|\phi_{1,j_1,m_1}\|_{L^2}\|\phi_{2,j_2}\|_{L^2}\\
\lesssim & \|\phi_1\|_{X_{0,\frac{1}{2}+}}\|\phi_2\|_{X_{0,\frac{1}{2}}}.
\end{aligned}
$$

Case B. $|\xi_1|^2\ll \frac{|\mu_1|}{|\xi_1|}$.

We first note that $|\mu_2|\ll|\mu_1|$, otherwise we have $\frac{|\mu_2|}{|\xi_2|}\gtrsim\frac{|\mu_1|}{|\xi_1|}$, which is contradiction with the assumption $\frac{|\mu_2|}{|\xi_2|}\ll\frac{|\mu_1|}{|\xi_1|}$ and $|\xi_2|\ll|\xi_1|$. Thus we have $|\mu_1+\mu_2|\sim|\mu_1|$. The argument in Case A can be run smoothly until we come to the case $|\mu_2|\geqslant 1$ and $\max(j,j_2)<2m_1$. We perform the change of variables (52). It is easy to see that $|J_\mu|\gtrsim|\xi_1|^4$. By the same estimate in (55), for fixed $\theta_1,\theta_2,\xi_1,\xi_2,\mu_1$, the length of the symmetric intervals where free variable μ_2 can range is $\Delta_{\mu_2}<2^{j-2m_1}$. Then we have

$$
\begin{aligned}
(47) \lesssim & \sum_{j_1,m_1\geqslant 0}\sum_{2m_1>\max(j_2,j)\geqslant 0} 2^{j(-\frac{1}{2}+)}2^{m_1}\int g_j(u,v,w)\chi_j(u,v,w)\\
&\times |J_\mu|^{-1}H(u,v,w,\mu_2,\theta_1,\theta_2)\,du\,dv\,dw\,d\mu_2\,d\theta_1\,d\theta_2
\end{aligned}
$$

$$\lesssim \sum_{j_1,m_1\geqslant 0}\ \sum_{2m_1>\max(j_2,j)\geqslant 0} 2^{j(-\frac{1}{2}+)}2^{j/2-2m_1}2^{j_1/2}2^{j_2/2}\|g_j\|_{L^2}\|\phi_{1,j_1,m_1}\|_{L^2}\|\phi_{2,j_2}\|_{L^2}$$

$$\lesssim \|\phi_1\|_{X_{0,\frac{1}{2}}}\|\phi_2\|_{X_{0,\frac{1}{2}}}.$$

Case C. $|\xi_1|^2 \sim \frac{|\mu_1|}{|\xi_1|}$.

Since $\frac{|\mu_2|}{|\xi_2|} \ll |\xi_1|^2 \sim \frac{|\mu_1|}{|\xi_1|}$, we also have $|\mu_1+\mu_2| \sim |\mu_1|$. In this case, the resonant interaction will happen. We bound (46) and (47) by

$$\sum_{j_1,j_2\geqslant 0}\sum_{j\geqslant 0}\int_{A_3} g_j(\xi,\mu,\tau)\chi_j(\xi,\mu,\tau)|\xi_1+\xi_2|\hat{\phi}_{1,j_1}(\xi_1,\mu_1,\tau_1)\frac{\hat{\phi}_{2,j_2}(\xi_2,\mu_2,\tau_2)}{(1+|\xi_2|^2+\frac{|\mu_2|}{|\xi_2|})^s}.$$

We decompose $|\xi_1| \sim 2^{m_1}$, $m_1 \geqslant 0$, and first consider a special case $|\mu_2| < 1$ and $|\xi_2| \leqslant |\xi_1|^{-2-\varepsilon}$ for some $\varepsilon > 0$ small enough. In this case, we can use Proposition 2.4. (46) and (47) can be bounded by

$$\sum_{j,j_1,j_2\geqslant 0}\ \sum_{m_1\geqslant 0} 2^{j(-\frac{1}{2}+)}2^{m_1}\|g_j\|_{L^2}\|\phi_{1,j_1,m_1}\|_{L_T^\infty(L^2_{x,y})}\big\|\big(m(\xi_2,\mu_2)\hat{\phi}_{2,j_2}\big)^\vee\big\|_{L_T^2(L^\infty_{x,y})}$$

$$\lesssim \sum_{j,j_1,j_2\geqslant 0}\ \sum_{m_1\geqslant 0} 2^{j(-\frac{1}{2}+)}2^{-\frac{\varepsilon}{2}m_1}\|g_j\|_{L^2}2^{j_1/2}\|\phi_{1,j_1,m_1}\|_{L^2}\|\phi_{2,j_2}\|_{L^2}$$

$$\lesssim \|\phi_1\|_{X_{0,\frac{1}{2}}}\|\phi_2\|_{X_{0,\frac{1}{2}}}.$$

In the remaining estimates, we always have $|\xi_2| > |\xi_1|^{-2-\varepsilon}$ for the same ε as above. In fact, $|\mu_2| > 1$ implies $|\xi_2| > |\xi_1|^{-2} > |\xi_1|^{-2-\varepsilon}$, since $|\mu_2| \ll |\xi_1|^2|\xi_2|$.

Now we consider the case $\max(j,j_2) \geqslant (2-\varepsilon)m_1$ for the same ε as above. When $j=\max(j,j_2)$, there exists $\min(\frac{1}{6},s) > \delta > 0$ small enough such that

$$\begin{aligned}(46), (47) &\lesssim \sum_{j_1,j\geqslant 0}\ \sum_{0\leqslant \max(j_2,(2-\varepsilon)m_1)\leqslant j} 2^{j(-\frac{1}{2}+)}2^{m_1(1/2+\delta)}2^{(2+\varepsilon)m_1\delta}\|g_j\|_{L^2}\\
&\quad\times \big\||D_x|^{\frac{1}{2}-\delta}\phi_{1,j_1,m_1}\big\|_{L_T^{\frac{2}{1-2\delta}}(L_{x,y}^{\frac{1}{\delta}})}\big\||D_x|^{\delta}\phi_{2,j_2}\big\|_{L_T^{\frac{1}{\delta}}(L_{x,y}^{\frac{2}{1-2\delta}})}\\
&\lesssim \sum_{j_1,j\geqslant 0}\ \sum_{0\leqslant \max(j_2,(2-\varepsilon)m_1)\leqslant j} 2^{j(-\frac{1}{2}+)}2^{m_1(1/2+(3+\varepsilon)\delta)}2^{j_1/2}2^{j_2/2}\\
&\quad\times \|g_j\|_{L^2}\|\phi_{1,j_1,m_1}\|_{L^2}\|\phi_{2,j_2}\|_{L^2}\\
&\lesssim \|\phi_1\|_{X_{0,\frac{1}{2}}}\|\phi_2\|_{X_{0,\frac{1}{2}}}.\end{aligned}$$

When $j_2=\max(j,j_2)$,

$$\begin{aligned}
(46), (47) &\lesssim \sum_{j_1, j_2 \geqslant 0} \sum_{j_2 \geqslant \max((2-\varepsilon)m_1, j) \geqslant 0} 2^{j(-\frac{1}{2}+)} 2^{m_1/2} \big\| |D_x|^{\frac{1}{4}} g_j^\vee \big\|_{L_T^4(L_{x,y}^4)} \\
&\quad \times \big\| |D_x|^{\frac{1}{4}} \phi_{1,j_1,m_1} \big\|_{L_T^4(L_{x,y}^4)} \|\phi_{2,j_2}\|_{L^2} \\
&\lesssim \sum_{j_1, j_2 \geqslant 0} \sum_{j_2 \geqslant \max(j,(2-\varepsilon)m_1) \geqslant 0} 2^{j(-\frac{1}{2}+)} 2^{j/2} 2^{m_1/2} 2^{j_1/2} \|g_j\|_{L^2} \|\phi_{1,j_1,m_1}\|_{L^2} \|\phi_{2,j_2}\|_{L^2} \\
&\lesssim \|\phi_1\|_{X_{0,\frac{1}{2}}} \|\phi_2\|_{X_{0,\frac{1}{2}+}}.
\end{aligned}$$

At last, we consider the case $\max(j, j_2) < (2-\varepsilon)m_1$ for the same ε as above.

Subcase 1. $|\frac{\mu_1}{\xi_1} - \frac{\mu_2}{\xi_2}|^2 < \frac{1}{2}|\xi_1 + \xi_2|^2 |5(\xi_1^2 + \xi_1\xi_2 + \xi_2^2) - 3\alpha|$.

Since now the resonant interaction does not happen, we have $|\xi_1|^4|\xi_2| \leqslant 2^{\max(j,j_1,j_2)}$. And because $|\xi_2| > |\xi_1|^{-2-\varepsilon}$, we get that $j_1 = \max(j, j_1, j_2) \geqslant (2-\varepsilon)m_1$. By choosing $\min(\frac{1}{6}, s) > \delta > 0$ small enough, we have

$$\begin{aligned}
(46), (47) &\lesssim \sum_{j_1 \geqslant 0} \sum_{0 \leqslant \max(j,j_2) \leqslant (2-\varepsilon)m_1 \leqslant j_1} 2^{j(-\frac{1}{2}+)} 2^{m_1(1/2+\delta)} 2^{(2+\varepsilon)m_1\delta} \|\phi_{1,j_1,m_1}\|_{L^2} \\
&\quad \times \big\| |D_x|^{\frac{1}{2}-\delta} g_j^\vee \big\|_{L_T^{\frac{2}{1-2\delta}}(L_{x,y}^{\frac{1}{\delta}})} \big\| |D_x|^{\delta} \phi_{2,j_2} \big\|_{L_T^{\frac{1}{\delta}}(L_{x,y}^{\frac{2}{1-2\delta}})} \\
&\lesssim \sum_{j_1 \geqslant 0} \sum_{0 \leqslant \max(j,j_2) \leqslant (2-\varepsilon)m_1 \leqslant j_1} 2^{j(-\frac{1}{2}+)} 2^{m_1(1/2+(3+\varepsilon)\delta)} 2^{j/2} 2^{j_2/2} \\
&\quad \times \|g_j\|_{L^2} \|\phi_{1,j_1,m_1}\|_{L^2} \|\phi_{2,j_2}\|_{L^2} \\
&\lesssim \|\phi_1\|_{X_{0,\frac{1}{2}+}} \|\phi_2\|_{X_{0,\frac{1}{2}}}.
\end{aligned}$$

Subcase 2. $|\frac{\mu_1}{\xi_1} - \frac{\mu_2}{\xi_2}|^2 \geqslant \frac{1}{2}|\xi_1 + \xi_2|^2 |5(\xi_1^2 + \xi_1\xi_2 + \xi_2^2) - 3\alpha|$.

As we know, we also need to consider two cases:

$$\left| 5(\xi_1^4 - \xi_2^4) - 3\alpha(\xi_1^2 - \xi_2^2) - \left[\left(\frac{\mu_1}{\xi_1}\right)^2 - \left(\frac{\mu_2}{\xi_2}\right)^2 \right] \right| \geqslant |\xi_1|^{\frac{1}{2}} \tag{63}$$

and

$$\left| 5(\xi_1^4 - \xi_2^4) - 3\alpha(\xi_1^2 - \xi_2^2) - \left[\left(\frac{\mu_1}{\xi_1}\right)^2 - \left(\frac{\mu_2}{\xi_2}\right)^2 \right] \right| < |\xi_1|^{\frac{1}{2}}. \tag{64}$$

(63) means the determinant of Jacobian of the change of variables (52), $|J_\mu| \geqslant |\xi_1|^{\frac{1}{2}}$. Thus we have

$$\begin{aligned}(46),(47) &\lesssim \sum_{m_1,j_1\geqslant 0}\ \sum_{(2-\varepsilon)m_1>\max(j_2,j)\geqslant 0} 2^{j(-\frac{1}{2}+)}2^{j/2-m_1}2^{m_1(1-\frac{1}{4})}2^{j_1/2}2^{j_2/2}\\ &\quad\times \|g_j\|_{L^2}\|\phi_{1,j_1,m_1}\|_{L^2}\|\phi_{2,j_2}\|_{L^2}\\ &\lesssim \|\phi_1\|_{X_{0,\frac{1}{2}}}\|\phi_2\|_{X_{0,\frac{1}{2}}}.\end{aligned}$$

When (64) occurs, we recur to the change of variables (56). By the argument in (59) and (58), for fixed $\theta_1, \theta_2, \xi_2, \mu_1, \mu_2$, the length of the interval where ξ_1 ranges is $|\xi_1| < 2^{(\frac{1}{2}-3)m_1}$. Thus we obtain

$$\begin{aligned}(46),(47) &\lesssim \sum_{m_1,j_1\geqslant 0}\ \sum_{(2-\varepsilon)m_1>\max(j_2,j)\geqslant 0} 2^{j(-\frac{1}{2}+)}2^{-(2-\frac{1}{2})m_1}2^{m_1}2^{j_1/2}2^{j_2/2}\\ &\quad\times \|g_j\|_{L^2}\|\phi_{1,j_1,m_1}\|_{L^2}\|\phi_{2,j_2}\|_{L^2}\\ &\lesssim \|\phi_1\|_{X_{0,\frac{1}{2}}}\|\phi_2\|_{X_{0,\frac{1}{2}}}.\end{aligned}$$

We now finish the proof of Theorem 3.1. □

4. Proof of main theorem

We now state the proof of Theorem 1.1.

Proof. Considering the integral equation according to (1)

$$u(t) = \psi(t)\left[S(t)u_0 - \frac{1}{2}\int_0^t S(t-t')\partial_x\big(\psi_T^2(t')u^2(t')\big)\,dt'\right], \tag{65}$$

where $0 < T < 1$, and $\psi_T(t)$ is the same bump function with (21). It is clear that a solution for (65) is a fixed point of the nonlinear operator

$$L(u) = \psi(t)S(t)u_0 - \frac{1}{2}\psi(t)\int_0^t S(t-t')\partial_x\big(\psi_T^2(t')u^2(t')\big)\,dt'. \tag{66}$$

Thus we need to prove the operator L is a contractive mapping from the following closed set to itself

$$B_a = \big\{u \in X_{s,b},\ \|u\|_{X_{s,b}} \leqslant a\big\}, \tag{67}$$

where $a = 4C\|u_0\|_{E_s}$. By Proposition 2.1 and Theorem 3.1, there exist $\sigma > 0$ such that

$$\big\|L(u)\big\|_{X_{s,\frac{1}{2}+}} \leqslant C\|u_0\|_{E_s} + CT^\sigma\|u\|^2_{X_{s,\frac{1}{2}+}}. \tag{68}$$

Next, since $\partial_x(u^2) - \partial_x(v^2) = \partial_x[(u-v)(u+v)]$, we get in the same way that

$$\left\| L(u) - L(v) \right\|_{X_{s,\frac{1}{2}+}} \leqslant CT^{\sigma} \|u - v\|_{X_{s,\frac{1}{2}+}} \left(\|u\|_{X_{s,\frac{1}{2}+}} + \|v\|_{X_{s,\frac{1}{2}+}} \right). \tag{69}$$

By choosing $T = T(\|u_0\|_{E_s})$ such that $8CT^{\sigma}\|u_0\|_{E_s} < 1$, we deduce that from (68) and (69) that L is strictly contractive on the ball B_a. Thus, there exists unique solution to the IVP of the fifth order KP-I equation $u \in X_{s,\frac{1}{2}+}$ on the interval $[-T, T]$. The smoothness of the mapping from E_s to $X_{s,\frac{1}{2}+}$ follows from the fixed point argument. Since $X_{s,\frac{1}{2}+} \subset C([-T, T]; E_s)$, we finish the proof of Theorem 1.1. $\square$

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