THE APPROXIMATE EVALUATION OF CERTAIN INTEGRALS

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Abstract—An adaptation of the decomposition method allows one to calculate an integral not expressible in terms of elementary functions nor adequately tabulated. It is shown that only two terms of the series solution lead to an excellent agreement with the actual solution of the problem.

INTRODUCTION

The decomposition method [1,2] can be an effective procedure for the evaluation of certain difficult integrals by changing them to easily solved differential equations and consequently evaluating the integrals in an easily computed convergent series. In general, differential equations are treated as the problems of integrations. However, integrations are neither always trivial, nor conveniently tabulated for ready reference.

ANALYSIS

To demonstrate the idea that the decomposition method may offer advantages in carrying out integrations, we consider reducing integrations to differential equations since the latter problems are solved easily and quickly by the said method [2,3].

Consider the simple first order differential equation

\[ \frac{dy}{dx} + P(x)y = Q(x), \quad y(0) = 0. \]

In the standard form [2], equation (1) can be rewritten as

\[ Ly + Py = Q, \quad y(0) = 0, \]

where \( L = \frac{d}{dx} \).

It follows immediately that the solution of (1) is given by

\[ y(x) = \int Qf(x) \, dx, \quad f(x) = e^{\int P \, dx}. \]

It may happen that the integral in r.h.s of (4) is of great difficulty and not computable in terms of known functions, nor tabulated in standard form. We may, however, consider the decomposition solution of this equation very easily as [2]

\[ y = \sum y_n = y_0 + y_1 + y_2 + \cdots \]

where \( y_0 = y(0) + L^{-1}Q = L^{-1}Q, \quad y_{k+1} = -L^{-1}(Py_k), \quad k = 0, 1, 2, \ldots \)
The $L^{-1}$ operator is, of course, an integration, and we assume the necessary integrability, i.e., 

$L^{-1} = \int f(x) \, dx$.

The expression $\phi_n = \sum_{i=0}^{n-1} y_i$ is the $n$ terms of approximation of the solution $y$. Convergence is well established [2], and it is also seen in [2] when numerical computation of the analytic approximation is carried out, a rapid stabilization to an acceptable accuracy is evident. Hence, we obtain from (4)

$$\int Qf(x) \, dx = \sum y_n f(x). \quad (6)$$

For example, let $P = \sin x, Q = \text{Constant} = m$ (say). We easily get from (4)

$$y = me^\cos x \left[ \int e^{-\cos x} \, dx \right]. \quad (7)$$

The integral in (7) is not easy to calculate. Decomposition immediately yields

$$y_0 = mx,$$

$$y_1 = mx \cos x - m \sin x,$$

similarly for $y_2, y_3$, etc.

Considering a two-term approximation, we have

$$y \approx y_0 + y_1 = mx + mx \cos x - m \sin x.$$

Needless to say, one can go further with little effort [2]. It is interesting to note that for small $x$, the decomposition solution gives $y = mx - mx + mx = mx$, and from (7) we have $y = me e^{-1} dx = mx$.

For comparison with the known results, and to stress the point that the decomposition series is very rapidly convergent and only a few terms of the series solution are sufficient for most purposes, we take $P = -2x, Q = 1$, whence it follows that

$$\int e^{-x^2} \, dx = y(x)e^{-x^2} = \sum y_n e^{-x^2}, \quad (8)$$

where $y$ is given by

$$\frac{dy}{dx} - 2xy = 1,$$

$$y(0) = 0. \quad (9)$$

We easily find the components of decomposition as

$$y_0 = y(0) + L^{-1}(1) = x, \quad y_1 = L^{-1}(2xy_0) = \frac{2}{3} x^3,$$

and similarly for $y_2, y_3$, etc. Considering only two terms, we get

$$\int e^{-x^2} \, dx = (x + \frac{2}{3} x^3)e^{-x^2} = R(x) \, (\text{say}). \quad (10)$$

The integral in the left is known and equal to $\frac{\sqrt{x}}{2} \text{erf}(x)$. The comparison is recorded in the following table.

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<th>$x$</th>
<th>$\text{erf}(x)$</th>
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For large values of \( z \), the above computation requires some more components of decomposition to obtain a fair accuracy. We may conclude, however, that in many cases, complicated integrals may be handled more easily, quickly, and elegantly by decomposition method than the traditional numerical method. The number of terms required to obtain a computable and accurate solution is generally small [2,3].

REFERENCES