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The topology of fluid flow past a sequence of cylinders

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Abstract

This paper analyzes conditions under which dynamical systems in the plane have indecomposable continua or even infinite nested families of indecomposable continua. Our hypotheses are patterned after a numerical study of a fluid flow example, but should hold in a wide variety of physical processes. The basic fluid flow model is a differential equation in $\mathbb{R}^2$ which is periodic in time, and so its solutions can be represented by a time-1 map $F:\mathbb{R}^2 \to \mathbb{R}^2$. We represent a version of this system "with noise" by considering any sequence of maps $F_n:\mathbb{R}^2 \to \mathbb{R}^2$, each of which is ε-close to F in the C^1 norm, so that if p is a point in the fluid flow at time n, then $F_n(p)$ is its position at time $n+1$. We show that indecomposable continua still exist for small ε.

Keywords: Indecomposable continua; Horseshoes; Fluid flow; Noisy dynamical system; Lagrangian dynamics; Area-preserving

AMS classification: Primary 58F12; 54F20, Secondary 54F50; 58F13; 58F15; 58F20

The motivation and origins of this paper lie in a computer investigation of a model of fluid flow, and that makes it quite different from most topology papers. The hypotheses for our rigorous results are based on the numerical observations in that investigation. Our purpose is to study the topology present in the model (and ideally in the actual fluid), the topology that is forced to occur given that our observations hold. Although our observations

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were made by studying this particular model of a nondiffusive, incompressible fluid, we feel that they often hold for actual fluid flows, and thus our results have applications to a variety of problems.

We prove that when our hypotheses hold, indecomposable continua must be associated with our flow and its Poincaré return map. An important aspect of our work is that the topological structure we find (which involves indecomposable continua), persists in the presence of small random fluctuations in the flow. When random fluctuations are introduced, it is no longer possible to talk about invariant points, periodic orbits, and invariant sets (except for the cylinders themselves), but the indecomposability remains, forever tangled around downstream cylinders and downstream continua and floating in the stream.

This is the second paper we have written concerning this fluid flow model. The first, "Indecomposable continua in dynamical systems with noise: fluid flow past an array of cylinders" [9], was written for physicists, and contains extensive information about the model and the observations which are only summarized here. This paper, on the other hand, contains a careful discussion of the rigorous results coming out of that fluid flow investigation. In particular, proofs left out of the physics paper are included here. Two related papers, also written for physicists, are [10,7].

1. Brief discussion of the model and the observations

Rather than study fluid flow via the usual Navier–Stokes equations, we used a model based on Lagrangian dynamics. The rationale here is much the same as a cancer researcher who studies cancer in rats in order to understand the more complicated problems of cancer in humans. It is currently impossible to obtain sufficiently accurate numerical solutions to Navier–Stokes equations for the topological investigation presented here, since we require an accurate time-1 Poincaré return map. The flow associated with Navier–Stokes has a very thin boundary layer with a large derivative too close to the fixed cylinders in our flow for meaningful observations and study. Thus, as in [7] which had one cylinder, our team physicists created a plausible 2-dimensional stream function that is an area-preserving flow (formally identical to Hamilton's equations), with terms to give the background stream flow, vortices, and the cylinder obstacles. The model yields a flow whose velocity goes to 0 near the cylinders with the expected vortices being spun off downstream from each cylinder in a periodic fashion, flowing downstream, and then dying.

A schematic diagram of the experiment is provided in Fig. 1. (The "cylinders" are actually invariant simple closed curves that are nearly circular. We call the curves cylinders because we are thinking of the fluid model representing a layer of a 3-dimensional fluid flow. Thus, our "cylinders" stick out of the page.) The initial investigation involved extensive numerical studies of the model by Miguel Sanjuan using the software *Dynamics* [6]. Because of the periodicity in time involved in the models, it was natural to simplify our problem by investigating the Poincaré return map associated with the flow. Thus, our differentiable flow ψ is associated with a plane diffeomorphism F defined

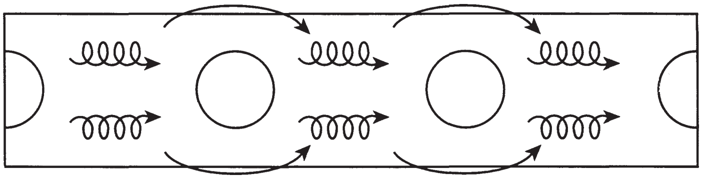

Fig. 1. The figure shows an array of cylinders, with the fluid flowing downstream. Vortices are shed periodically behind each cylinder, they move along the channel S and they die out. In most of our pictures the vertical scale is changed so that the cylinders appear highly elliptical. The horizontal lines show the range of y ($-2 \leqslant y \leqslant 2$) used in all the figures.

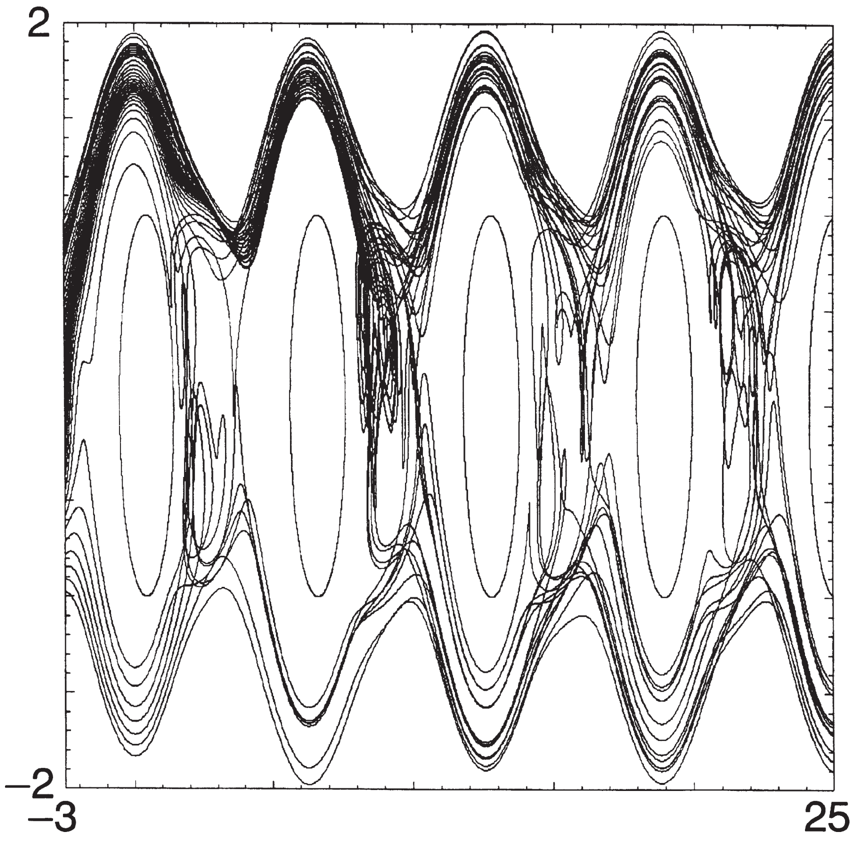

Fig. 2. Several continuous time trajectories are shown, illustrating the chaos between cylinders.

by $F(x, y) = \psi(x, y, 1)$, since we have periodicity 1 in time built into the flow. The flow is thus viewed stroboscopically. While the second figure shows actual trajectories in the flow, the remaining figures are associated with the Poincaré return map of the flow.

The flow trajectories plotted in Fig. 2 illustrate the chaos between the cylinder obstacles causing the complications in the flow. (Recall that the "cylinders" are actually simple

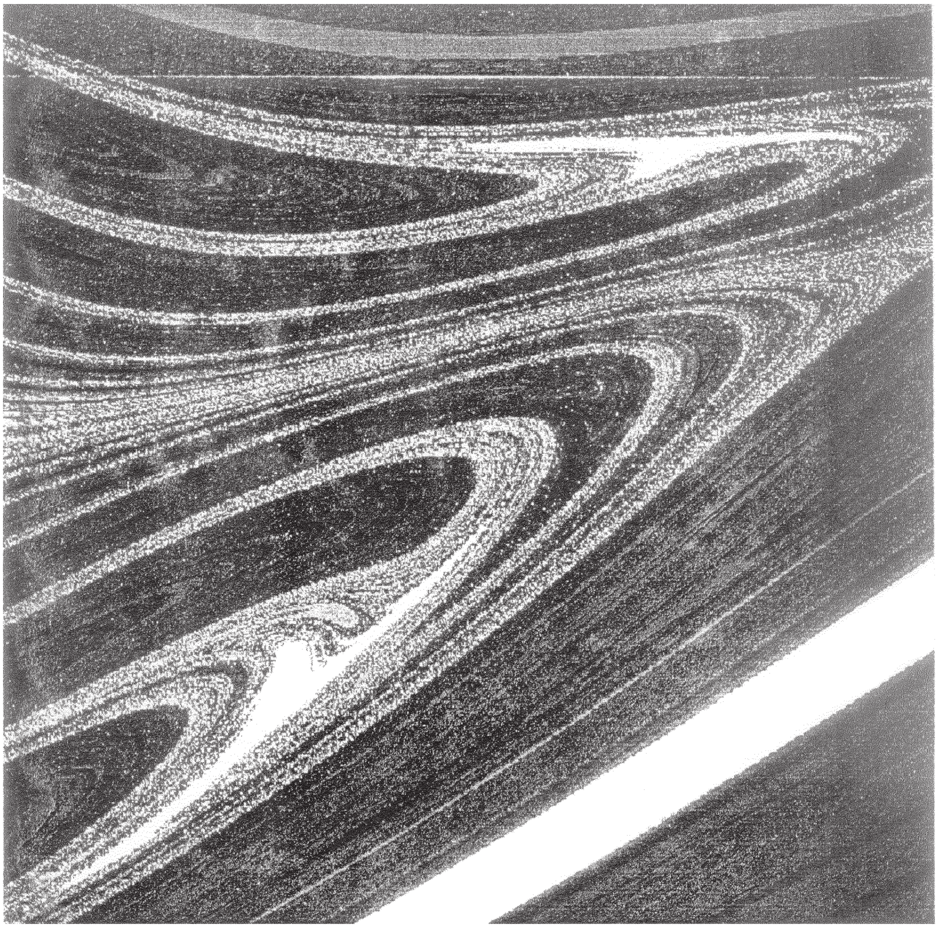

Fig. 3. The figure shows the exit times, the time required to pass to the next cylinder, in different shades of black-to-gray in the region $1.25 < x < 1.7$ and $-0.6 < y < 0.5$. This region is to the immediate right of a cylinder, and the two invariant bubbles (white) are clearly visible in this region. Solid color regions have small exit times, while the speckled region has long exit times.

closed curves that are nearly circular. They appear elliptical here because of the choice of the y-scale.) In particular, note that a number of the trajectories actually cross from above the cylinders to below, and vice versa. In Fig. 3 we show a study of exit times (relative to the Poincaré map) for points. The different bands represent different numbers of iterations for the shaded points to exit the screen. The speckled region reveals the existence of a chaotic region, because those points requiring long exit times (where the "pile up" of layers occurs) are those near the bounded trajectories, since in those regions some points never leave the screen (the region viewed). Fluid flows downstream, from left to right in the figures, but points inside and on the boundaries of the cylinders are fixed. Far away from the cylinders, above and below, the flow is almost perfectly horizontal (due to the background flow component function *bflow*). *We studied the sets* $S^+(x_0)$ *and*

$S^-(x_0)$. *The set* $S^+(x_0)$ *is defined to be the set of points* (x, y) *at time* $t_0 = 0$ *with the property that the trajectory* $(x(t), y(t))$ *satisfies* $x(t) \geqslant x_0$ *for all time* (*positive and negative*). *The points in* $S^-(x_0)$ *have the trajectories satisfying* $x(t) \leqslant x_0$ *for all time*. Notice that $S^+(x_0)$ includes all cylinders to the right of x_0. We add the point at ∞ in the plane to the sets $S^+(x_0)$ and $S^-(x_0)$ purely for convenience: it means we can discuss our work in the setting of compact sets. Most trajectories flow from $x = -\infty$ to $x = +\infty$. *An important aspect of our work is that the topology of the sets* $S^+(x_0)$ *and* $S^-(x_0)$ *of semi-bounded trajectories persists in the presence of small random fluctuations in the flow.*

Suppose that $\overline{\mathbb{R}^2} = \mathbb{R}^2 \cup \{\overline{\infty}\}$ denotes our compactified plane, and that $F : \overline{\mathbb{R}^2} \to \overline{\mathbb{R}^2}$ ($F(\overline{\infty}) = \overline{\infty}$) denotes the Poincaré time-1 return map of our flow. Note that since the vector field is periodic in x with period 2π, if $(\overline{x}, \overline{y}) = F(x, y)$, then $F(x + 2\pi, y) = (\overline{x} + 2\pi, \overline{y})$. (The distance between the centers of consecutive cylinders is 2π.) There is a non-empty, connected, invariant set S, i.e., $F(S) = S$, such that there is a uniform bound $\widetilde{\delta}$ on $|y|$ for all (x, y) in S. Also, $(x, y) \in S$ implies $(x + 2\pi i, y) \in S$ for all integers i. There is a uniform $\vartheta > 0$ such that if $(x, y) \notin S$, then the x-coordinate of $F(x, y)$ is greater than $x + \vartheta$, that is, everywhere outside the band S, the fluid moves uniformly to the right. (The set S is the nearly horizontal band in the plane outside of which the flow is laminar. This is built into the model via the *bflow* term.) We made the following observations based on the numerical studies.

Observation 1. Let $L(x_0)$ be the vertical line with x-coordinate x_0. There is a value x_0 such that F maps each point on $L(x_0)$ strictly to the right of $L(x_0)$. (This indicates the flow is generally from left to right even inside the band S. Actually we found several such lines between each pair of cylinders. The first we found was to the right of a cylinder, but to the left of the associated quadrilateral Q_0 (discussed below), and was difficult to find. Later we discovered that many easy-to-find such lines occur between Q_0 and the next cylinder.) See Fig. 4.

Observation 2. There is a quadrilateral Q_0 that satisfies the *lockout property*. That is, if $q \in Q_0$ and for some $k > 0$, $F^k(q) \notin Q_0$, then further iterates of q remain outside Q_0; i.e., $F^n(q) \notin Q_0$ if $n \geqslant k$. We observe that Q_0 lies between $L(x_0)$ and $L(x_0 + 2\pi)$.

Observation 3. The map F is a hyperbolic horseshoe map on Q_0 in the sense of Smale [11]. Write A, B, C, D for the vertices of Q_0 as in Fig. 5. In particular, if $G : \mathbb{R}^2 \to \mathbb{R}^2$ is a horseshoe map on Q_0, then the top DC and bottom AB have images $G(AB)$ and $G(DC)$ that lie outside Q_0 and the image $G(Q_0)$ of Q_0 stretches at least twice across Q_0 as shown in Fig. 6, without intersecting sides AD or BC. Also, for almost every $q \in Q_0$ there is an $n > 0$ depending on q for which $G^n(q) \notin Q_0$. Now every horseshoe map (associated with a diffeomorphism) must contain at least two saddle fixed points. We let p_0 denote any one of these. Our map F is of the form G^2—again this is inherent in the model. Thus, our observations indicate that $G = \sqrt{F}$ is itself a horseshoe map, so that $F(Q_0)$ stretches four times across Q_0. (See Fig. 7.)

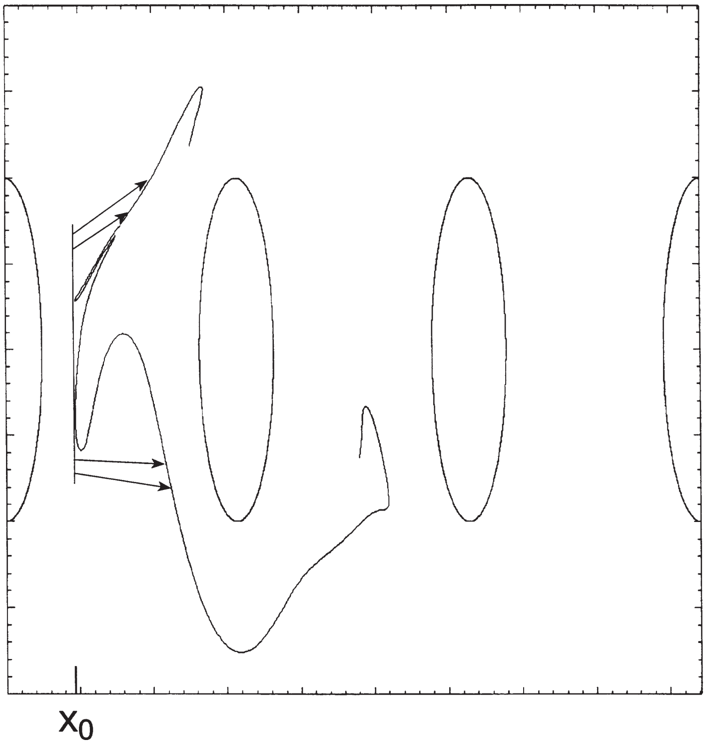

Fig. 4. The value x_0 was chosen carefully in the figure so that all points on it are mapped to the right by the time-1 map F. The curve shown is the image of this segment shown at $x = x_0$. While it is hard to see, there is a gap between this line segment and its image, so that the segment maps strictly to the right.

Now define Q_i to be the horizontal translate by $x = 2\pi i$ of Q_0. Note that because of the periodicity assumption we can assume that Q_i has properties analogous to those of Q_0. Also, let Cyl_0 denote the cylinder just to the left of Q_0, and let Cyl_i denote the ith cylinder, i.e., the one which a horizontal translate of Cyl_0 by $2\pi i$.

Suppose that $\widetilde{F}:\mathbb{R}^2 \to \mathbb{R}^2$ is a C^1-diffeomorphism of the plane, and p is a hyperbolic fixed point of $\widetilde{F}$ (i.e., none of the eigenvalues associated with $D\widetilde{F}(p)$ has norm 1). Then p is either an attracting fixed point, a repelling fixed point, or a saddle. If p is a saddle point, then there are sets of points attracted to and repelled from p. The set of points attracted to p is called the *stable manifold* of p, is denoted $W^s(p)$, and is a continuous, differentiable, one-to-one image of the real line $\mathbb{R}$. The set of points repelled from p is called the *unstable manifold* of p, is denoted $W^u(p)$, and is also a continuous, differentiable, one-to-one image of the real line $\mathbb{R}$. Now our fluid flow map $F:\mathbb{R}^2 \to \mathbb{R}^2$ is an *area-preserving* C^1-diffeomorphism of the plane, and it cannot have attracting or repelling points. All hyperbolic fixed points for F must be saddle points. The fact that our observations indicate that F is a hyperbolic horseshoe on Q_0 (and thus on each Q_i) means that there is an invariant Cantor set C_0 (C_i, respectively) in Q_0 (Q_i) which contains a dense set of periodic

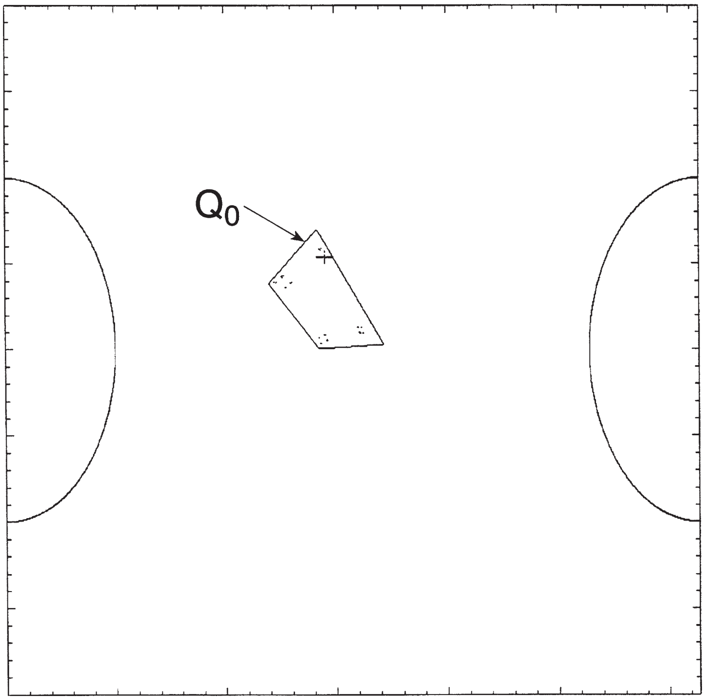

Fig. 5. The points shown inside the quadrilateral Q_0 constitute the Cantor set, whose trajectories of points remain inside Q_0 for all times $t = 0, \pm 1, \pm 2, \ldots$.

points, and at least two hyperbolic fixed points. Thus, both must be saddle points, and we were able to locate these (approximately) numerically, and also to compute numerically the approximate eigenvalues associated with these. (*Dynamics* [6] has this capability.)

We observed that:

Observation 4. If p_i denotes one of the saddle fixed points contained in Q_i, the unstable manifold of p_i intersects transversally both the stable manifold of p_i and the stable manifold of p_{i+1}. See Figs. 8 and 9.

Observation 5. There is a connected segment U_i of the unstable manifold of p_i and a connected segment S_{i+1} of the stable manifold of p_{i+1} which have the same end points and together bound a region J that contains the cylinder Cyl_{i+1} in its interior but excludes the cylinder Cyl_{i+2}. See Fig. 10.

Observation 6. For each i, there is an $\overline{x}_i \in (x_i, x_i + 2\pi)$ such that $F(L(\overline{x}_i))$ is to the right of $L(\overline{x}_i)$, $L(\overline{x}_i)$ lies to the right of Q_i, and there is an integer N with the property that if $q \in Q_i$ and $F(q)$ is not in Q_i, then $F^N(q)$ is to the right of $L(\overline{x}_i)$. (The integer N is independent of the choice of q. In our example we found such a line with $N = 2$. See

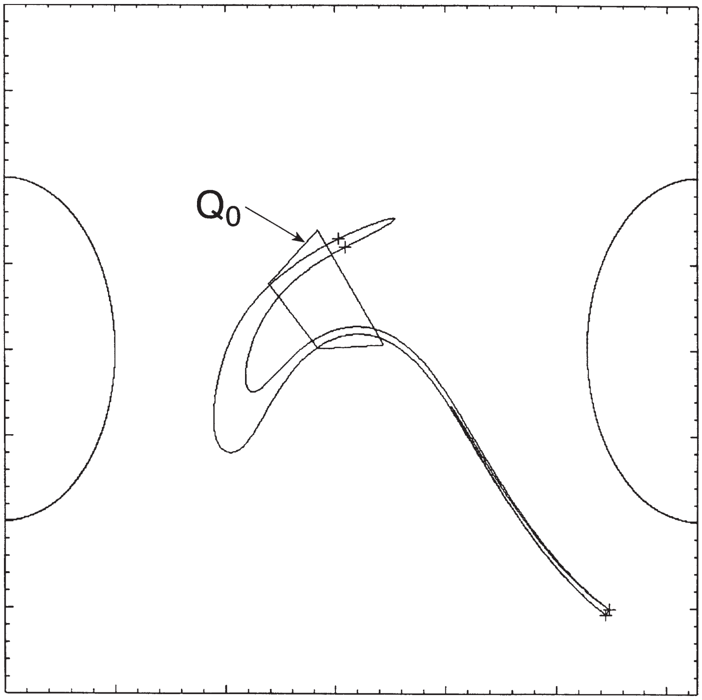

Fig. 6. The figure shows a horseshoe. The crosses are the images of the vertices of the quadrilateral Q_0 under the action of G. This map G is a special map that is the "square root" of F, that is, $G(G(x, y)) = F(x, y)$ for all $(x, y) \in Q_0$. See the discussion of the model for more details.

Fig. 11. Observation 6 is not assumed as a hypothesis until we introduce noise into the flow in Section 4.)

Observation 7. Finally, we observed the presence of several invariant open sets ("bubbles") between the C_0 and C_1 cylinders. These open sets do not intersect the quadrilaterals Q_i, are typical features of Hamiltonian systems, and represent points that neither flow downstream nor come from upstream. They would thus be contained in our sets $S^+(x_0)$ and $S^-(x_0)$ of semi-bounded trajectories. The interior of all bubbles between cylinders Cyl_0 and Cyl_1 must be mapped to itself under F—they are trapped for all time between the Cyl_0 and Cyl_1 cylinders. By the periodicity inherent in the model, such trapped bubbles occur between each consecutive pair of cylinders. (See Fig. 3.)

Remark. How "good" are our observations? In particular, are they based on questionable numerics? *No, they are not*. Our observations (except possibly for Observation 7) entailed making only short computations, using well understood algorithms. Finding the quadrilateral Q_0 and vertical lines $L(x_0)$ was difficult, but once located, the computations were not. While the computation of a large part of the unstable manifold (such as that seen in Fig. 9) involves more computation and is perhaps somewhat less certain, it was not

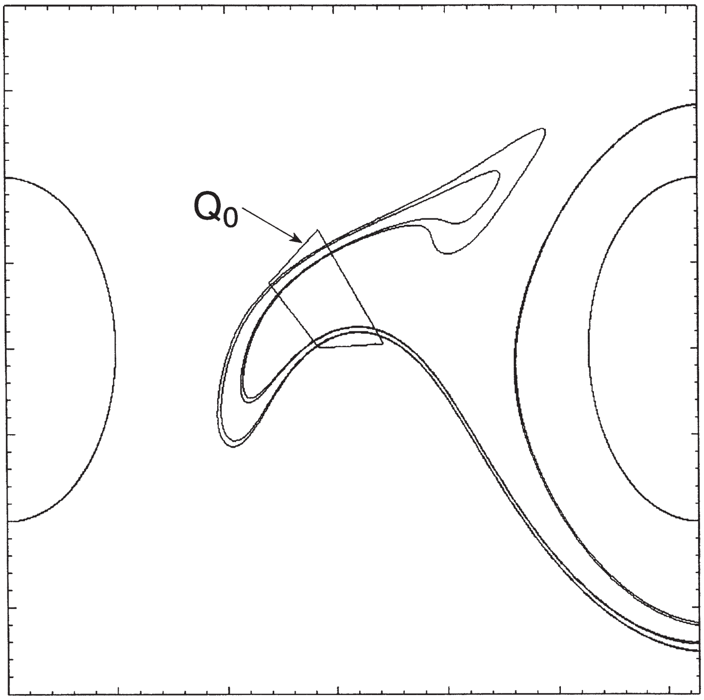

Fig. 7. The picture shows the first iterate $F(Q_0)$ of the quadrilateral, which is also $G^2(x, y)$.

used to make our basic observations about the crossings of the various stable and unstable manifolds, nor the images of the vertical lines $L(x_0)$ and quadrilateral Q_0. The pictures involving more extensive computations are included because they *show* the objects that must be there, given our basic observations and the mathematics they imply. These figures behave as expected and predicted, and they offer additional numerical evidence that our conclusions are correct.

2. Background and notation for the rigorous results

A *continuum* is a compact, connected metric space. A subset of a continuum which is itself a continuum is a *subcontinuum*. A continuum is *indecomposable* if it is not the union of two (necessarily overlapping) proper subcontinua. Equivalently, a continuum is indecomposable if every proper subcontinuum has empty interior (relative to the continuum). If x is a point in the continuum X, then the *composant* $Com(x)$ in X containing x is the set of all points y in X such that there is a proper subcontinuum in X that contains both x and y. The collection $\mathcal{C}(X)$ of all composants of an indecomposable continuum X partitions X into $\mathfrak{c}$ (the cardinality of the real numbers) many mutually disjoint, first category, F_σ-set connected sets. (For more information and references concerning indecomposable continua, see [4].)

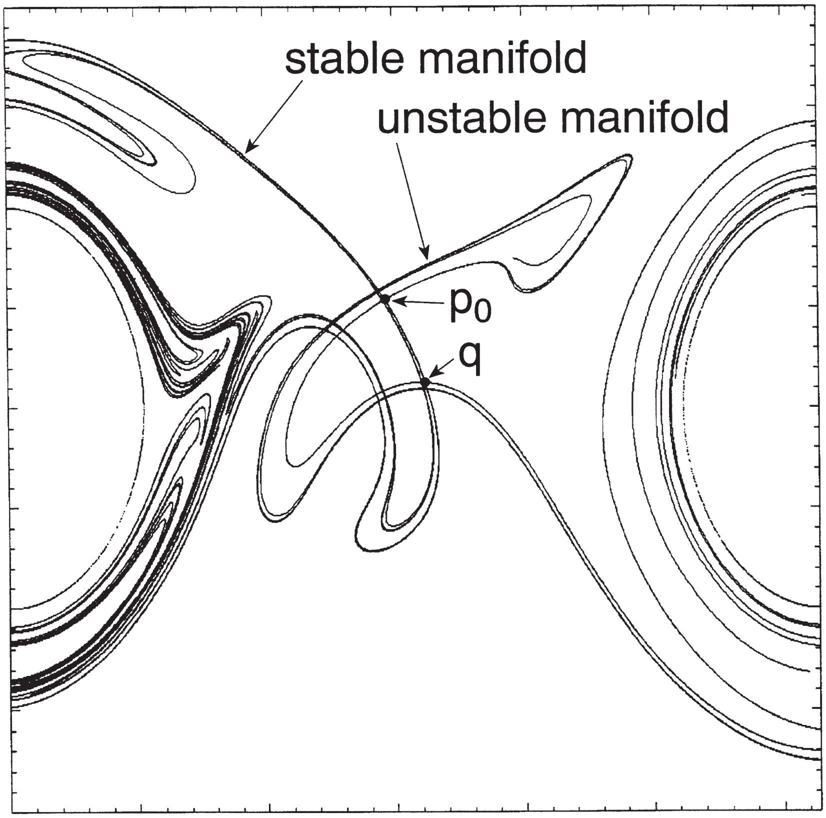

Fig. 8. The horseshoe F on Q_0 in Figs. 5–7 has some fixed points. One of these, p_0, is shown here. The stable and unstable manifolds of p_0 intersect at a point $q \neq p_0$. The closure of such a set of intersection points is the Cantor set shown in Fig. 5, and it was created by plotting the intersections of the stable and unstable manifolds.

If X is a space and A is a subset of X, then we use the notation A°, $\overline{A}$, and ∂A to denote the interior, closure, and boundary of A in X, respectively. If Y is a subspace of X (with the inherited topology), $A \subseteq Y$, and we wish to discuss the interior, closure, or boundary of A in the subspace Y, we use the notation $Int_Y(A)$, $Cl_Y(A)$, and $Bdy_Y(A)$, respectively, to avoid confusion. The symbols $\mathbb{Z}$, $\mathbb{N}$, and $\widetilde{\mathbb{N}}$ are used to denote the integers, the positive integers, and the nonnegative integers, respectively. We use the usual metric in the plane, i.e., for $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathbb{R}^2$,

$$|x - y| = d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}.$$

If $\varepsilon > 0$, $A, B \subset \mathbb{R}^2$, let $D_\varepsilon(A) = \{y \in \mathbb{R}^2 \mid |y - x| < \varepsilon$ for some $x \in A\}$, and let $d(A, B) = \inf\{|x - y| \mid x \in A,\ y \in B\}$. For $x = (x_1, x_2) \in \mathbb{R}^2$, we denote the projections of x, to the x- and y-axes as $\pi_1(x) = x_1$, and $\pi_2(x) = x_2$, respectively.

Suppose that (X, d) is a metric space, and $f : X \to X$ is a homeomorphism. If $\delta > 0$, $\{x_j\}_{j=j_1}^{j_2}$ is a sequence of points in X, then $\{x_j\}_{j=j_1}^{j_2}$ is a δ-*chain* (*of length* $n = j_2 - j_1$

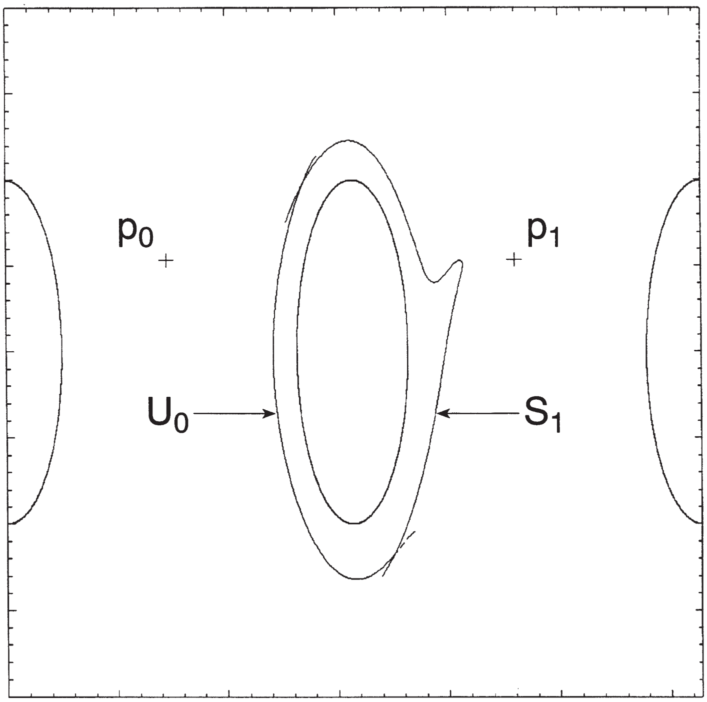

Fig. 9. The figure shows that Observation 5 is satisfied by our example. The cylinder is encapsulated by the segments of the stable and unstable manifolds of the fixed points p_0 and p_1, respectively. The fixed point z_0 in Theorem 3.15 (and then Corollary 3.16) can be any point of the cylinder and z_1 is any point of any other cylinder. Of course, all cylinder points are automatically fixed points. The fixed points p_0 and p_1 are *not* in the segments used for encapsulation.

from x_{j_1} *to* x_{j_2}), if $d(f(x_j), x_{j+1}) < \delta$ for each $j \in \{j_1, \ldots, j_2\}$. We also allow $j_1 = -\infty$ and $j_2 = \infty$, in which case it may not be possible to speak of the length of the chain, nor the points at which it begins and ends. A point y in X *ε-shadows* $\{x_j\}_{j=j_1}^{j_2}$ provided $d(f^j(y), x_j) < \varepsilon$ for $j_1 \leqslant j \leqslant j_2$. The homeomorphism f is *expansive* on X if there exists some positive constant c such that for each pair x, y of distinct points of X, there is some integer $n_{x,y}$ such that $d(f^{n_{x,y}}(x), f^{n_{x,y}}(y)) \geqslant c$. If Λ is a closed subset of X such that $f(\Lambda) = \Lambda$, then Λ is *invariant* under f. If there is some neighborhood U of X containing the closed invariant set Λ in its interior and $\bigcap_{n\in\mathbb{Z}} f^n(U) = \Lambda$, then Λ is an *isolated set* for f, and U is an *isolating neighborhood* for Λ under f.

Suppose that $\widetilde{F}: X \to X$ is a homeomorphism on a compact metric space X, and p is a point in X. Then the *forward limit set of* p, denoted $\omega(p)$, is the set of all accumulation points of the sequence $\widetilde{F}(p), \widetilde{F}^2(p), \ldots,$ and the *backward limit set of* p, denoted $\alpha(p)$, is the set of all accumulation points of the sequence $\widetilde{F}^{-1}(p), \widetilde{F}^{-2}(p), \ldots.$ The *orbit of* p, denoted $O(p)$ or $O_{\widetilde{F}}(p)$, is the set $\{\widetilde{F}^n(p) \mid n \in \mathbb{Z}\}$. The *forward orbit of* p, denoted $O^+(p)$ or $O^+_{\widetilde{F}}(p)$, is the set $\{\widetilde{F}^n(p) \mid n \in \widetilde{\mathbb{N}}\}$, and the *backward orbit of* p, denoted $O^-(p)$ or $O^-_{\widetilde{F}}(p)$, is the set $\{\widetilde{F}^{-n}(p) \mid n \in \mathbb{N}\}$. The point p is *wandering* if there is an open set o

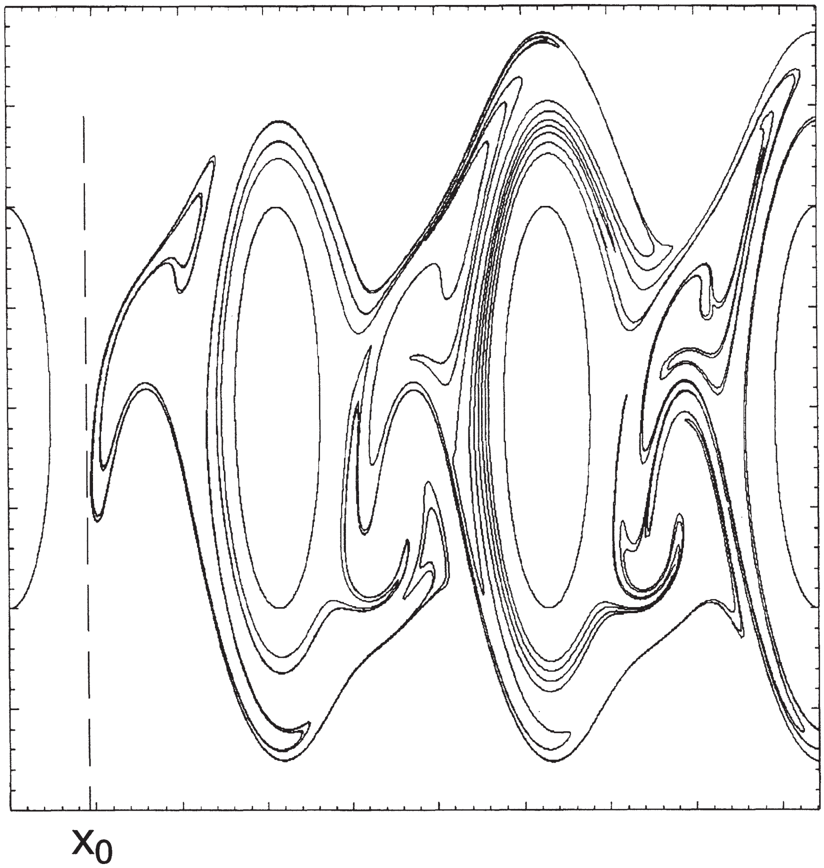

Fig. 10. The set $S^+(x_0)$ shown is the set of points whose trajectories remain for all time ($t = 0, \pm 1, \pm 2, \ldots$) to the right of the dashed line at $x = x_0$.

in X such that the collection $\{\widetilde{F}^n(o) \mid n \in \mathbb{Z}\}$ is mutually disjoint; and we say that the open set o is a *wandering open set*.

Suppose that $f:\mathbb{R}^n \to \mathbb{R}^n$ is a C^1-diffeomorphism of $\mathbb{R}^n$. Fix a point p in $\mathbb{R}^n$. A *tangent vector at* p is a pair $(p, \boldsymbol{v})$ where $\boldsymbol{v} \in \mathbb{R}^n$, and is written $\boldsymbol{v}_p$. The collection of all vectors at the point p is the *tangent space at* p, and is denoted by $T_p\mathbb{R}^n$. The tangent space $T_p\mathbb{R}^n$ is a vector space with $(p, \boldsymbol{v}) + (p, \boldsymbol{w}) = (p, \boldsymbol{v} + \boldsymbol{w})$. The disjoint union of the tangent vectors at different points is called the *tangent bundle* or the *tangent space* of $\mathbb{R}^n$, is denoted $T\mathbb{R}^n$, and is isomorphic to $\mathbb{R}^n \times \mathbb{R}^n$. An invariant set Λ has a *hyperbolic structure* for a diffeomorphism f on the differentiable manifold X if

(1) at each point p in Λ, the tangent space $T_p = T_p\mathbb{R}^n$ to $\mathbb{R}^n$ splits as the direct sum of E^u_p and E^s_p, i.e., $T_p = E^u_p \oplus E^s_p$,
(2) the splitting is invariant under the action of the derivative map DF in the sense that $Df(p)(E^u_p) = E^u_{f(p)}$ and $Df(p)(E^s_p) = E^s_{f(p)}$, and
(3) there exist $0 < \lambda < 1$ and $C \geqslant 1$ such that for all $n \in \widetilde{\mathbb{N}}$,

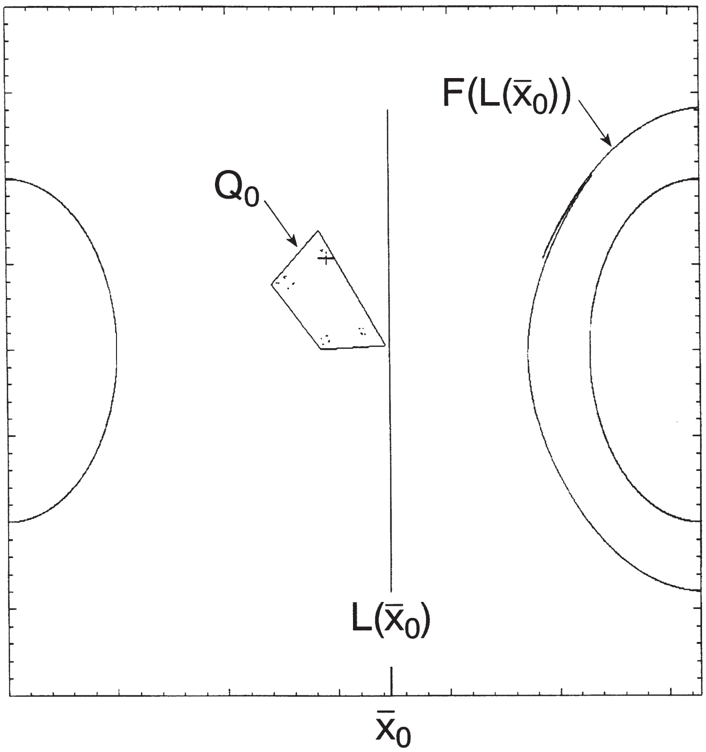

Fig. 11. In our example, the vertical line $L(\overline{x}_0)$ (shown above with x-coordinate $\overline{x}_0$) is mapped to the right of itself (as is the line $L(x_0)$); in addition, Observation 6 is satisfied because for each point $q \in Q_0$ for which $F(q)$ is not in Q_0, $F^2(q)$ is to the right of $L(\overline{x}_0)$, and the same is true for all $F^n(q)$ for $n \leqslant 2$, since once a point is to the right of $L(\overline{x}_0)$, it must stay to the right. The curve shown is the image of this segment shown at $x = \overline{x}_0$.

$$\begin{aligned} \left|Df^n(p)(\boldsymbol{v}^s)\right| &\leqslant C\lambda^n|\boldsymbol{v}^s| \quad \text{for } \boldsymbol{v}^s \in E^s_p, \text{ and} \\ \left|Df^{-n}(p)(\boldsymbol{v}^u)\right| &\leqslant C\lambda^n|\boldsymbol{v}^u| \quad \text{for } \boldsymbol{v}^u \in E^u_p. \end{aligned}$$

If an invariant set Λ has hyperbolic structure for f, we also say that Λ is a *hyperbolic invariant set.* If Q is a closed subset of $\mathbb{R}^n$, then f is *hyperbolic* on Q if f is hyperbolic on $\bigcap_{n\in\mathbb{Z}} f^n(Q)$ (which is an invariant set for f).

We need the following facts pertaining to invariant hyperbolic sets. (See [8] for proofs and more details.)

Corollary to the Stable Manifold Theorem for Hyperbolic Sets. *Suppose that X is a differentiable manifold, d denotes a (compatible) metric on X, $f: X \to X$ is a diffeomorphism, and Λ is a closed, invariant, hyperbolic subset of X under f. If Λ is an isolated set for f, and U is an isolating neighborhood for Λ, then Λ is totally disconnected.*

Shadowing Theorem. *Suppose that X is a differentiable manifold, d denotes a (compatible) metric on X, $f: X \to X$ is a diffeomorphism, and Λ is a closed, invariant, hyperbolic* *subset of X under f. If* $\varepsilon > 0$*, then there exist positive numbers* δ *and* η *such that if* $\{x_j\}_{j=j_1}^{j_2}$ *is a* δ*-chain for* f *with* $d(\{x_j\}, \Lambda) < \eta$ *for* $j_1 \leqslant j \leqslant j_2$*, then there is some* y *in* X *which* ε*-shadows* $\{x_j\}_{j=j_1}^{j_2}$*. If* $j_1 = -\infty$ *and* $j_2 = \infty$ *for the* δ*-chain, then* y *is unique. If* $j_1 = -\infty$*,* $j_2 = \infty$*, and* Λ *is an isolated invariant set, then the unique point* y *is in* Λ*.*

Expansiveness Theorem. *Suppose that* X *is a differentiable manifold,* d *denotes a* (*compatible*) *metric on* X*,* $f : X \to X$ *is a diffeomorphism, and* Λ *is a closed, invariant, hyperbolic subset of* X *under* f*. Then* f *is expansive on* Λ*, i.e., the homeomorphism* $f|\Lambda : \Lambda \to \Lambda$ *is expansive.*

Suppose that $\widetilde{F} : X \to X$ is a C^1-diffeomorphism of the differentiable n-manifold X, and p is a hyperbolic saddle fixed point in X under the action of $\widetilde{F}$. Suppose further that $W^s(p)$, and $W^u(p)$ denote the stable and unstable manifolds, respectively, of the point p, and suppose that $W^u(p)$ is one-dimensional. Denote the two branches of $W^u(p)$ by $W^{u+}(p)$ and $W^{u-}(p)$. If o is an open set that contains p, then $W^s_{loc}(p)$ is the component of $W^s(p)$ that contains p in o and is called the local stable manifold of p. Similarly, we can define $W^u_{loc}(p)$, the local unstable manifold of p, and its two local branches $W^{u+}_{loc}(p)$ and $W^{u-}_{loc}(p)$. (If X is the plane, then both $W^s(p)$ and $W^u(p)$ are continuous, one-to-one images of the real line $\mathbb{R}$, and each is differentiable.) (See [8] or [1] for more information.)

We need the following theorem (see [1]).

Horseshoe Theorem. *Let* $f : \mathbb{R}^2 \to \mathbb{R}^2$ *be a diffeomorphism and let* p *be a hyperbolic saddle fixed point for* f*. If the stable and unstable manifolds of* p *cross transversally, then there is a hyperbolic horseshoe for some iterate* f^n *of* f*.*

Marcy Barge [2] proved the following theorem. We state a modified version (the version we need) of his theorem for a 2-manifold. The original theorem is more general.

Barge's Theorem. *Suppose that* X *is a differentiable* 2*-manifold,* $\widetilde{F} : X \to X$ *is a* C^1*-diffeomorphism with hyperbolic saddle fixed point* p*, and* $\widetilde{F}$ *satisfies the following properties*:

(B1) $\overline{W^{u+}(p)}$ *is compact and nowhere dense in* X*, and*

(B2) $(W^{u+}(p) \cap W^s(p)) \setminus \{p\}$ *is not empty, but* $W^{u+}(p)$ *is not a subset of* $W^s(p)$*.*

Then $\overline{W^{u+}(p)}$ *is an indecomposable continuum.*

Suppose that G is a plane homeomorphism (not necessarily a diffeomorphism). Then we say that the map G is a (*topological*) *horseshoe map on the quadrilateral* Q with vertices A, B, C, and D and sides AB, BC, CD, and DA if

(1) the sides CD and AB have images $G(AB)$ and $G(CD)$ that lie outside Q;
(2) if K is an arc in Q that intersects both sides CD and AB, then $Q \cap G(K)$ consists at least two components each component of which intersects both CD and AB; and
(3) $G(Q)$ does not intersect DA or BC.

If the map G is, in addition, a diffeomorphism, then G is a *hyperbolic horseshoe map on a quadrilateral* Q with vertices A, B, C, and D and sides AB, BC, CD, and DA if it is a horseshoe map on Q (i.e., it satisfies the conditions above), and if Γ denotes the invariant set $\bigcap_{n\in\mathbb{Z}} G^n(Q)$, then G is hyperbolic on Γ.

We say that the closed neighborhood B satisfies the *lockout property* if when $q \in B$ and for some $k > 0$, $G^k(q) \notin B$, then further iterates of q remain outside B; i.e., $G^n(q) \notin B$ if $n \geqslant k$.

In the material that follows, we give $\overline{\mathbb{R}^2}$, S, $L(x_0)$, $S^+(x_0)$, and $S^-(x_0)$ the meanings they had in the previous sections. Likewise, *when we speak of a fluid flow diffeomorphism* F, then for each integer i, Cyl_i denotes the ith cylinder, Q_i denotes the ith quadrilateral (between Cyl_i and Cyl_{i+1}), and p_i denotes a saddle fixed point in Q_i, with $p_{i+1} = (p_{i1} + 2\pi, p_{i2})$ where $p_i = (p_{i1}, p_{i2})$.

Now since $\overline{\mathbb{R}^2}$ is homeomorphic to the 2-sphere, there is a metric $\widetilde{d}$ on $\overline{\mathbb{R}^2}$ which is compatible with its topology. Since $\overline{\mathbb{R}^2}$ is a compact metric space, so is the space $\mathcal{F}(\overline{\mathbb{R}^2})$ consisting of all closed subsets of $\overline{\mathbb{R}^2}$ with the topology induced by the Hausdorff metric ν (relative to the metric $\widetilde{d}$). Thus, if H and K are in $\mathcal{F}(\overline{\mathbb{R}^2})$, then $\nu(H, K) = \inf\{\varepsilon > 0 \mid$ each point of H is within ε (under the metric $\widetilde{d}$) of some point of K and each point of K is within ε (under the metric $\widetilde{d}$) of some point of $H\}$.

Another topology on collections of closed subsets of a compact subset D of $\mathbb{R}^2$ that we need is the quotient topology. Suppose that D is a compact subset of $\mathbb{R}^2$, and $\mathcal{D}$ denotes a decomposition of D into disjoint closed sets which is upper semicontinuous. The collection $\mathcal{D}$, when endowed with the quotient topology, is a compact metric space (with the points of $\mathcal{D}$ (considered as space) being the sets in the collection $\mathcal{D}$ (considered as collection in $\mathbb{R}^2$)). Let $P: D \to \mathcal{D}$ denote the projection map associated with the decomposition. The map P is continuous and onto. We say that the set D is a *quotient Cantor set* if there is an upper semicontinuous decomposition $\mathcal{D}$ of D such that $\mathcal{D}$ endowed with the quotient topology is a Cantor set. Note that $\mathcal{D} \subset \mathcal{F}(\overline{\mathbb{R}^2})$. Every set that is open in the quotient topology on $\mathcal{D}$ is also open in the topology induced by the Hausdorff metric on $\mathcal{D}$. If $D \in \mathcal{D}$, then D is a *point of continuity* of $\mathcal{D}$ if every sequence $D_1, D_2, \ldots$ in $\mathcal{D}$ which converges to D in the quotient topology also converges to D in the topology induced by the Hausdorff metric on $\mathcal{D}$. The points of continuity of an upper semicontinuous decomposition on a compact metric space contain a dense G_δ-subset of the decomposition space. Whenever X is a compact metric space and $\mathcal{D}$ is an upper semicontinuous decomposition of X, then the quotient space $\mathcal{D}$ is itself (i.e., endowed with the quotient topology) a compact metric space. Whenever X is a compact metric space, then $\mathcal{D} = \{C\colon C$ is a component of $X\}$ is an upper semicontinuous decomposition of X, and the quotient space is a totally disconnected compact metric space. (See [5] for more information.) We say that the set D is a *quotient Cantor set of continua* if there is an upper semicontinuous decomposition $\mathcal{D}$ of D such that $\mathcal{D}$ endowed with the quotient topology is a Cantor set and each point of this Cantor set is a continuum in the space X.

Another tool from dynamics that we need is the Lambda Lemma:

The Lambda Lemma. *Suppose that $\widetilde{F}$ is a plane diffeomorphism, and p is a hyperbolic saddle fixed point of $\widetilde{F}$. Suppose that a differentiable curve $\widetilde{L}$ crosses the stable manifold $W^s(p)$ of p transversally* (*that is, there is a point q in $W^s(p) \cap \widetilde{L}$ such that the angle* *between the tangent lines to $W^s(p)$ and $W^u(p)$ at the point q is neither* 0 *nor* π*). Then each point in the unstable manifold of p is a limit point of* $\bigcup_{n>0} \widetilde{F}(L)$.

3. Observations become hypotheses: the fluid flow theorems with and without noise

Now we turn our observations into the hypotheses of the theorems. In the statements that follow, F can be taken to be the Poincaré return map associated with our model, and, unless explicitly stated otherwise, it can also be taken to be any plane diffeomorphism satisfying the hypotheses given. Whenever F is a homeomorphism on $\mathbb{R}^2$, it is always possible to extend F to a homeomorphism on $\overline{\mathbb{R}^2} = \mathbb{R}^2 \cup \overline{\infty}$ by defining $F(\overline{\infty}) = \overline{\infty}$. In the results and statements that follow, when a homeomorphism or a diffeomorphism F on $\mathbb{R}^2$ is discussed, we assume that F is defined on $\overline{\mathbb{R}^2}$ in this way. We call the Poincaré return map associated with our model the fluid flow diffeomorphism F. To make this precise, we call F the *fluid flow diffeomorphism* if it has the following properties:

- (F1) F is area preserving on $\mathbb{R}^2$, and for each $i \in \mathbb{Z}$,
- (F2) there is a number $x_i \in [2\pi i, 2\pi(i+1)]$ such that $F(L(x_i))$ is completely to the right of $L(x_i)$,
- (F3) $F(S) = S$, and Q_i is the quadrilateral in $\mathbb{R}^2$ located in the almost horizontal band S between the ith cylinder Cyl_i and $(i+1)$st cylinder Cyl_{i+1} with $Q_i \subseteq V_i$, where $V_i = [x_i - 2\pi i, x_i + 2\pi i] \times [-\widetilde{\delta}, \widetilde{\delta}] \subset S$,
- (F4) F is a hyperbolic horseshoe on each quadrilateral Q_i, and p_i is a saddle fixed point in Q_i,
- (F5) F satisfies the lockout property on Q_i, and $Q_i \subseteq S^+(x_i)$,
- (F6) $F(x) = x$ for each point x in $\bigcup_{i\in\mathbb{Z}} Cyl_i$,
- (F7) $(W^s(p_i) \cap W^u(p_i)) \backslash \{p_i\} \neq \emptyset$,
- (F8) there is a point q in $(W^u(p_i) \cap W^s(p_{i+1})) \backslash \{p_i\}$ such that the angle between $W^u(p_i)$ and $W^s(p_{i+1})$ at q is not equal to 0 or π, but $W^u(p_{i+1}) \cap W^s(p_i) = \emptyset$,
- (F9) there are connected segments U_i of the unstable manifold of p_i and S_{i+1} of the stable manifold of p_{i+1} that have the same end points and bound a region J_i in S that contains the cylinder Cyl_{i+1}, while other cylinders Cyl_j $(j \neq i)$ do not intersect $\overline{J_i}$,
- (F10) if $F(x, y) = (\overline{x}, \overline{y})$, then $F(x + 2\pi, y) = (\overline{x} + 2\pi, \overline{y})$, and
- (F11) there are positive numbers ϑ and λ such that if (x, y) is not in S, then $\pi_1 F(x, y) - x > \vartheta$, and $|\pi_2 F(x, y) - y| < \lambda < 1$.

Now consider a new assumption.

Adding noise. Let $\varepsilon > 0$. Instead of applying the fluid flow diffeomorphism F at each time i, we instead assume that for each i, a diffeomorphism F_i which is close to F in the sense that $|F(q) - F_i(q)| < \varepsilon$ and $|DF(q) - DF_i(q)| < \varepsilon$ for each i and q, is applied. Further, assume that each point x which is in one of the cylinders is fixed. We refer to ε as the "noise level".

With this assumption we can still talk about the trajectory of a point if we replace $F(q_0)$ with $F_0(q_0)$, $F^2(q_0)$ with $F_1 \circ F_0(q_0)$, and so forth. In general, the trajectory of q_0 is the bisequence $\ldots q_{-2}, q_{-1}, q_0, q_1, \ldots$, where $q_i = F_{i-1} \circ F_{i-2} \circ \cdots \circ F_0(q_0)$ and $q_{-i} = (F_{-1} \circ F_{-2} \circ \cdots \circ F_{-i})^{-1}(q_0) = F_{-i}^{-1} \circ \cdots \circ F_{-2}^{-1} \circ F_{-1}^{-1}(q_0)$ for $i > 0$. It no longer makes sense to talk about *invariant* Cantor sets, *invariant* points, or *invariant* continua. However, we *can* talk about those points the x-coordinate of whose forward trajectory does not go to $+\infty$, and those points the x-coordinate of whose backward trajectory does not go to $-\infty$. Define then the *entrainment set* $\widetilde{S}^+(x_0)$ to be *the set of points* (x, y) *whose entire trajectory* (*under a sequence of noisy maps*) *is to the right of* x_0, define $\widetilde{S}^-(x_0)$ to be *the set of points* (x, y) *whose entire trajectory* (*under a sequence of noisy maps*) *is to the left of* x_0, and define Z_0 to be the *set of all points* (x, y) *whose entire trajectory* (*under a sequence of noisy maps*) *is inside the quadrilateral* Q_0. Similarly, and as before, we can define the sets $\widetilde{S}^+(x_i)$, $\widetilde{S}^-(x_i)$, and Z_i for each integer i. We also need an additional assumption for the fluid flow diffeomorphism F, which corresponds to Observation 6 (not necessary if for each i, $F_i = F$). Thus we assume that F, in addition to satisfying properties (F1)–(F11), also satisfies (F12):

(F12) *Strong lockout property.* For each i, there is an $\overline{x}_i \in (x_i, x_i + 2\pi)$ such that $F(L(\overline{x}_i))$ is to the right of $L(\overline{x}_i)$, $L(\overline{x}_i)$ lies to the right of Q_i, and there is an integer N_F with the property that if $q \in Q_i$ and $F(q)$ is not in Q_i, then $F^{N_F}(q)$ is to the right of $L(\overline{x}_i)$. (The integer N_F is independent of the choice of q or of i. It is also the case, because of the background flow (property (F11)) that there is some $\Upsilon > 0$ such that $d(L(\overline{x}_i), F(L(\overline{x}_i)) > \Upsilon$.)

To avoid cumbersome notation, define for integers $m \leqslant n$, $\widetilde{F}_{n,m} = F_n \circ F_{n-1} \circ \cdots \circ F_m$. Thus, the trajectory of the point q_0 is the bisequence $\ldots q_{-2}, q_{-1}, q_0, q_1, \ldots$, where $q_i = F_{i-1} \circ F_{i-2} \circ \cdots \circ F_0(q_0) = \widetilde{F}_{i-1,0}(q_0)$ and $q_{-i} = (F_{-1} \circ F_{-2} \circ \cdots \circ F_{-i})^{-1}(q_0) = (\widetilde{F}_{-1,-i})^{-1}(q_0) = F_{-i}^{-1} \circ \cdots \circ F_{-2}^{-1} \circ F_{-1}^{-1}(q_0)$ for $i > 0$. Suppose then that the diffeomorphism F_i is applied at integer time i. Note that for a sufficiently small choice of ε, each F_i has the following properties:

(Fi1) for each $i' \in \mathbb{Z}$, there is a number $x_{i'} \in [2\pi i', 2\pi(i'+1)]$ such that $F_i(L(x_{i'}))$ is completely to the right of $L(x_{i'})$,

(Fi2) F_i is a hyperbolic horseshoe on each quadrilateral $Q_{i'}$,

(Fi3) $F_i(x) = x$ for each point x in $\bigcup_{i' \in \mathbb{Z}} Cyl_{i'}$, and

(Fi4) for each i', there is an $\overline{x}_{i'} \in (x_{i'}, x_{i'} + 2\pi)$ such that $F_i(L(\overline{x}_{i'}))$ is to the right of $L(\overline{x}_{i'})$, $L(\overline{x}_{i'})$ lies to the right of $Q_{i'}$, and if $q \in Q_{i'}$ and $\widetilde{F}_i(q)$ is not in $Q_{i'}^{\circ}$, then $F_i^{N_F}(q)$ is to the right of $L(\overline{x}_{i'})$,

(Fi5) if (x, y) is not in S, then $\pi_1 F_i(x, y) - x > \vartheta/2$, and $|\pi_2 F(x, y) - \pi_2 F_i(x, y)\,| < \varepsilon < 2\lambda$.

Proposition 3.1. *Suppose that F is an area preserving plane homeomorphism, Q is a quadrilateral in the plane, for almost every point q in Q there is some positive integer n_q such that $F^{n_q}(q) \notin Q$, and F satisfies the lockout property on Q. Then for almost every $q \in Q$, $\limsup_{n\to\infty} F^n(q) = \overline{\infty}$.*

Proof. Let B denote the set $F(Q) - Q$. Since F has the lockout property on Q, so does each $F^n(Q)$, and each finite union $\bigcup_{j=k}^{l} F^j(Q)$ $(0 \leqslant k \leqslant l)$. Also, $F(B)$ does not intersect B, so that $F^k(B)$ does not intersect $F^{k-1}(B)$. By the lockout property, $F(B)$ does not intersect Q, and so $F(B)$ does not intersect $Q \cup F(Q)$. Then $F^k(B)$ does not intersect $\bigcup_{j=0}^{k} F^j(Q)$, from which it follows that the members of the finite sequence $B, F(B), \ldots, F^{k-1}(B)$ are mutually disjoint. But each of these disjoint sets has the same nonzero area as B, because F is area-preserving. Let V be a bounded open set in $\mathbb{R}^2$. Let $B_V \subseteq B$ be the set such that $b \in B_V$ implies that $F^n(b) \in V$ for all $n > 0$. We claim B_V has area 0. The sets $B_V, F(B_V), F^2(B_V), \ldots,$ are disjoint because $F^k(B_V) \subseteq F^k(B)$. Furthermore all $F^k(B_V)$ have area equal to the area of B_V. Since all $F^k(B_V)$ lie in V,

$$\mathit{area}(V) \geqslant \sum_{k=1}^{\infty} \mathit{area}\big(F^k(B_V)\big) = \infty \times \mathit{area}(B_V).$$

Hence, the area of B_V is 0, since otherwise the union of all $F^k(B_V)$ would have infinite area, proving the claim. It follows that for almost every $q \in Q$ there is some $n > 0$ such that $F^n(q)$ is not in V. Thus, for almost every $q \in Q$, $\limsup_{n\to\infty} F^n(q) = \overline{\infty}$. □

The situation for the noisy case is more complicated, and we need to know more about F, namely that points that leave Q are eventually mapped outside a set containing Q in its interior which has the property points mapped into that set cannot leave it. For us, that means assuming that the homeomorphism F satisfies property (F12), which is really a stronger version of the lockout property. For $x \in \mathbb{R}$, let

$$L(x) = \big\{z = (z_1, z_2) \in \mathbb{R}^2 \colon\ z_1 = x\big\}$$

and let

$$L^+(x) = \big\{z = (z_1, z_2) \in \mathbb{R}^2 \colon\ z_1 > x\big\},$$
$$L^-(x) = \big\{z = (z_1, z_2) \in \mathbb{R}^2 \colon\ z_1 < x\big\}.$$

Proposition 3.2. *Suppose that F is an area preserving plane homeomorphism; Q is a quadrilateral in the plane such that F has the lockout property on Q; for almost every point q in Q there is some positive integer n_q such that $F^{n_q}(q) \notin Q$; and there are some positive integer N_F and some $\overline{x} \in \mathbb{R}$ such that $F(L(\overline{x}))$ is to the right of $L(\overline{x})$, $\overline{L^+(\overline{x})} \cap Q = \emptyset$, and if $q \in Q$ and $F(q)$ is not in Q, then $F^m(q)$ is in $L^+(\overline{x})$, for $m \geqslant N_F$. There is $\varepsilon > 0$ such that if for each nonnegative integer j, F_j is an area-preserving homeomorphism on $\overline{\mathbb{R}^2}$ such that for each $q \in \mathbb{R}^2$, $|F(q) - F_j(q)| < \varepsilon$, then for almost every $q \in Q$,* $\limsup_{n\to\infty} \widetilde{F}_{n,0}(q) = \overline{\infty}$.

Proof. Let $N = N_F$, and let

$$\tau = \inf\big\{|x - y| \colon\ x \in Q,\ \ y \in L^+(\overline{x})\big\} > 0, \quad \text{and}$$
$$0 < \tau' < \inf\big\{|F(x) - F(y)| \colon\ x \in Q,\ y \in L^+(\overline{x})\big\}, \quad \tau' < \tau/2.$$

Let $B = F(Q)\backslash Q$. Choose $\varepsilon' > 0$ so that if $\widehat{F}_1, \ldots, \widehat{F}_N$ is a collection of N area preserving homeomorphisms on $\overline{\mathbb{R}^2}$ such that

(i) $\widehat{F}_i(L(\overline{x}))$ is in $L^+(\overline{x})$ for each $1 \leqslant i \leqslant N$;

(ii) for each $q \in \mathbb{R}^2$, $|F(q) - \widehat{F}_i(q)| < \varepsilon'$ and if $i_1, \ldots, i_N$ is a permutation of the finite sequence $1, \ldots, N$, then the composition $\widehat{F}_{i_1} \circ \cdots \circ \widehat{F}_{i_N}$ has the property that if $q \in D_{\varepsilon'}(F(Q) - Q)$, then $\widehat{F}_{i_1} \circ \cdots \circ \widehat{F}_{i_N}(q)$ is in $L^+(\overline{x})$ (in other words, each $\widehat{F}_i$ is chosen so close to F that the resulting composition of N homeomorphisms satisfies an appropriately modified version of (Fi4)); and

(iii) $\varepsilon' < \tau'$.

There is $\varepsilon' > \varepsilon > 0$ such that if $\widehat{F}$ is a plane homeomorphism and $|F(q) - \widehat{F}(q)| < \varepsilon$ for each $q \in Q$, then

$$\widehat{F}\big(F^{-1}\big(D_\varepsilon(Q)\big)\big) \subset D_{\varepsilon'}(Q) \quad \text{and} \quad \widehat{F}^{-1}(Q) \subset D_\varepsilon\big(F^{-1}(Q)\big).$$

Then suppose that for each nonnegative integer j, F_j is an area-preserving homeomorphism on $\overline{\mathbb{R}^2}$, and for each $q \in \mathbb{R}^2$, $|F(q) - F_j(q)| < \varepsilon$.

For each $n \in \mathbb{N}$, let B_n denote the set $F_n(Q) - Q$, and let $B^+ = \bigcup_{n\in\widetilde{\mathbb{N}}} B_n$. Then $B^+ \subseteq D_\varepsilon(F(Q) - Q)$, $\widetilde{F}_{N,1}(B^+)$ lies completely to the right of $L(x)$, and $area(\widetilde{F}_{N,1}(B^+)) = area(B^+)$. Let V be a bounded open set in $\mathbb{R}^2$ that contains B^+. Let $B_V \subseteq B^+$ be the set such that $b_0 \in B_V$ implies that $b_n = \widetilde{F}_{n,0}(b_0) \in V$ for all $n > 0$. We claim B_V has area 0. Since $\widetilde{F}_{N,1}(B^+) \subset L^+(x)$, $\widetilde{F}_{N,1}(B^+) \cap Q = \emptyset$, and moreover, $d(\widetilde{F}_{N,1}(B^+), Q) \geqslant \tau$, so that $d(\widetilde{F}_{N+1,1}(B^+), B^+) \geqslant \tau'/2$, and $\widetilde{F}_{N+1,1}(B^+) \cap B^+ = \emptyset$. Since no F_i moves a point back across the line $L(x)$, $\widetilde{F}_{2N+1,1}(B^+) \subseteq L(x)$, so that $d(\widetilde{F}_{2N+1,1}(B^+), Q) \geqslant \tau$, $d(\widetilde{F}_{2N+2,1}(B^+), B^+) \geqslant \tau'/2$, and $\widetilde{F}_{2N+2,1}(B^+)$ does not intersect B^+. Because $\widetilde{F}_{2N+2,N+2}$ is a homeomorphism, $\widetilde{F}_{2N+2,1}(B^+)$ does not intersect $\widetilde{F}_{2N+2,N+1}(B^+)$. Continuing, it follows that if k is a positive integer, the members of the finite sequence $B^+, \widetilde{F}_{kN+k,1}(B^+), \widetilde{F}_{kN+k,N+1}(B^+), \ldots, \widetilde{F}_{kN+k,(k-1)N+(k-1)}(B^+)$ are mutually disjoint. But each of these disjoint sets has the same nonzero area as B^+, because each F_i is area preserving. Thus, the sets $B_V, \widetilde{F}_{kN+k,1}(B_V), \widetilde{F}_{kN+k,N+1}(B_V), \ldots, \widetilde{F}_{kN+k,(k-1)N+(k-1)}(B_V)$ are all disjoint, since each $\widetilde{F}_{kN+k,iN+i}(B_V) \subseteq \widetilde{F}_{kN+k,iN+i}(B^+)$. It follows that

$$area(V) \geqslant \lim_{k\to\infty} \sum_{i=1}^{k-1} area\big(\widetilde{F}_{kN+k,iN+i}(B_V)\big) = \infty \times area(B_V).$$

Hence, the area of B_V is 0, proving the claim. It follows that for almost every $q \in Q$ the trajectory of q eventually leaves V. Thus, any point that leaves V has a trajectory that is eventually to the right of the line $L(x)$, and for almost every $q \in Q$, $\limsup_{n\to\infty} \widetilde{F}_{n,0}(q) = \overline{\infty}$. □

Proposition 3.3. *Suppose that F is the fluid flow diffeomorphism and $i' \in \mathbb{Z}$. There is $\varepsilon > 0$ such that if for each nonnegative integer j, F_j is an area preserving homeomorphism on $\overline{\mathbb{R}^2}$ such that for each $q \in \mathbb{R}^2$, $|F(q) - F_j(q)| < \varepsilon$, then for almost every $q \in Q_{i'}$, $\lim_{n\to\infty} \widetilde{F}_{n,0}(q) = \overline{\infty}$ and, in particular, the x-coordinate of $\widetilde{F}_{n,0}(q)$ tends to $+\infty$ as*

$n \to \infty$. Furthermore, almost every $q \in Q_{i'}$ is a wandering point for F, and for almost every $q \in Q_{i'}$,

$$\lim_{n\to\infty} F^n(q) = \overline{\infty}, \quad \textit{and} \quad \lim_{n\to\infty} \pi_1(F^n(q)) = \infty.$$

Proof. We use the notation and results of the previous proposition. Thus, we define

$$L^+(\overline{x}_{i'}) = \{(z_1, z_2) \in \mathbb{R}^2 \mid z_1 > \overline{x}_{i'}\}, \quad N = N_F,$$
$$\tau = \inf\{|x - y| \mid x \in Q_{i'},\ y \in L^+(\overline{x}_{i'})\} > 0, \quad \text{and}$$
$$0 < \tau' < \inf\{|F(x) - F(y)| \mid x \in Q_{i'},\ y \in L^+(\overline{x}_{i'})\}, \quad \tau' < \tau/2.$$

(Note that τ is independent of the choice of i'.) Choose $\varepsilon' > 0$ so that if $\widehat{F}_1, \ldots, \widehat{F}_N$ is a collection of N area preserving homeomorphisms on $\overline{\mathbb{R}^2}$ such that

(i) $\widehat{F}_i(L(\overline{x}_{i'}))$ is in $L^+(\overline{x}_{i'})$ for each $1 \leqslant i \leqslant N$;

(ii) for each $q \in \mathbb{R}^2$, $|F(q) - \widehat{F}_i(q)| < \varepsilon'$ and if $i_1, \ldots, i_N$ is a permutation of the finite sequence $1, \ldots, N$, then the composition $\widehat{F}_{i_1} \circ \cdots \circ \widehat{F}_{i_N}$ has the property that if $q \in D_{\varepsilon'}(F(Q) - Q)$, then $\widehat{F}_{i_1} \circ \cdots \circ \widehat{F}_{i_N}(q)$ is in $L^+(x)$ (in other words, each $\widehat{F}_i$ is chosen so close to F that the resulting composition of N homeomorphisms satisfies an appropriately modified version of (Fi4));

(iii) if (x, y) is not in S, then $\pi_1\widehat{F}_i(x, y) - x > \vartheta/2$, and $|\pi_2 F(x, y) - \pi_2\widehat{F}_i(x, y)\ | < \varepsilon' < 2\lambda$ for each $0 \leqslant i \leqslant N$; and

(iv) $\varepsilon' < \tau'$.

There is $\varepsilon' > \varepsilon > 0$ such that if $\widehat{F}$ is a plane homeomorphism and $|F(q) - \widehat{F}(q)| < \varepsilon$ for each $q \in Q_{i'}$, then

$$\widehat{F}\bigl(F^{-1}\bigl(D_\varepsilon(Q_0)\bigr)\bigr) \subset D_{\varepsilon'}(Q_0) \quad \text{and} \quad \widehat{F}^{-1}(Q_0) \subset D_\varepsilon\bigl(F^{-1}(Q_0)\bigr).$$

Then suppose that for each nonnegative integer j, F_j is an area-preserving homeomorphism on $\overline{\mathbb{R}^2}$, and for each $q \in \mathbb{R}^2$, $|F(q) - F_j(q)| < \varepsilon$.

Without loss of generality, let $i' = 0$. For each $n \in \mathbb{N}$, let B_n denote the set $F_n(Q_0) - Q_0$, and let $B^+ = \bigcup_{n\in\widetilde{\mathbb{N}}} B_n$. Then $B^+ \subseteq D_\varepsilon(F(Q_0) - Q_0)$, and if V is a bounded open set in $\mathbb{R}^2$ that contains B^+, and $B_V \subseteq B^+$ is the set such that $b_0 \in B_V$ implies that $\widetilde{F}_{n,0}(b_0) \in V$ for all $n \geqslant 0$, then, applying the previous proposition, B_V has area 0. It follows that for almost every $q_0 \in Q_0$ the trajectory of q eventually leaves V. There is some $M > 0$ such that $M\vartheta/2 > 2\pi$. Now if $V = [\overline{x}_0 - 2m\pi, \overline{x}_0 + 2m\pi] \times [-\widetilde{\delta} - 4mM(\lambda), \widetilde{\delta} + 4mM(\lambda)]$ for some $m > 0$, then a trajectory of a point which is leaving V must leave it through either the (right) line $L(\overline{x}_0 + 2m\pi)$, or the top or bottom lines $y = \pm(\widetilde{\delta} + 2mM(\lambda))$. If a trajectory leaves V through the top or the bottom line, then after $2mM$ applications of the appropriate maps F_i, the point cannot have re-entered the strip S, and the x-coordinate will have moved to the right of the line $L(\overline{x}_0 + 2m\pi)$. Thus, any point that leaves V has a trajectory that is eventually to the right of the line $L(\overline{x}_0 + 2m\pi)$, and for almost every $q \in Q_0$,

$$\lim_{n\to\infty} \widetilde{F}_{n,0}(q_0) = \lim_{n\to\infty} q_n = \overline{\infty}, \quad \text{and} \quad \lim_{n\to\infty} \pi_1\bigl(\widetilde{F}_{n,0}(q_0)\bigr) = \lim_{n\to\infty} \pi_1(q_n) = \infty.$$

That almost every $q \in Q_{i'}$ is a wandering point for F follows from the observation that if for each i, we choose $F_i = F$, then for almost every $q \in Q_0$,

$$\lim_{n\to\infty} F^n(q_0) = \overline{\infty}, \quad \text{and} \quad \lim_{n\to\infty} \pi_1\bigl(F^n(q_0)\bigr) = \infty. \qquad \square$$

Theorem 3.4. *Suppose that F is a plane homeomorphism, Q is a quadrilateral in the plane such that F is a horseshoe on Q, for almost every point q in Q there is some positive integer n_q such that $F^{n_q}(q) \notin Q$, and s and s' denote the opposite sides of Q that $F(Q)$ intersects. Then*

(1) *$Z = \{x\colon\ F^n(x) \in Q$ for all $n \in \mathbb{Z}\}$ is a nowhere dense, invariant, closed set contained in the interior of Q, and $\mathcal{Z} = \{C\colon\ C$ is a component of $Z\}$ endowed with the quotient topology is a totally disconnected, compact metric space*;

(2) *$Z' = \bigcup\{C\colon\ C \in \mathcal{Z}$ and C is not an isolated point of $\mathcal{Z}\} \subseteq Z$ is an invariant closed set, and $\mathcal{Z}' = \{C\colon\ C \in \mathcal{Z}$ and $C \subset Z'\}$ endowed with the quotient topology is a Cantor set, and Z' is a quotient Cantor set of continua*;

(3) *$Z'' = \bigcup\{C\colon\ C \in \mathcal{Z}'$ such that, for some sequence $\{F^{-i}(E_i)\}_{i=0}^{\infty}$, each member of which is a component of $\bigcap_{i\geqslant 0} F^i(Q)$ that intersects both s and s', $C = \lim_{i\to\infty} F^{-i}(E_i)$ (in the quotient topology)$\} \subseteq Z'$ is an invariant closed set, $\mathcal{Z}'' = \{C\colon\ C$ is a component of $Z''\}$ endowed with the quotient topology is a Cantor set, and Z'' is a quotient Cantor set of continua*; *and*

(4) *$Z''' = \overline{\bigcup\{C\colon\ C \text{ is a point of continuity of } \mathcal{Z}''\}} \subseteq Z''$ is an invariant closed set, $\mathcal{Z}''' = \{C\colon\ C$ is a component of $Z'''\}$ endowed with the quotient topology is a Cantor set, Z''' is a quotient Cantor set of continua, and if $C \in \mathcal{Z}'''$, $x \in C$, then $x \in \overline{Z'''\setminus C}$.*

Proof. Suppose that $M (\geqslant 2)$ is the fold number of F. Consider $Z = \{q \in Q \mid F^n(q) \in Q$ for all $n \in \mathbb{Z}\}$. If K is an arc in Q that intersects both opposite sides s and s' of Q, then $Q \cap F(K)$ contains at least M components each of which intersects both s and s'. For each n, $\bigcap_{j=0}^{n} F^j(Q)$ contains at least M^n components each of which intersects both s and s', and no component of $\bigcap_{j=0}^{n} F^j(Q)$ intersects either of the other two sides. Further,

$$\bigcap_{j=0}^{n+1} F^j(Q) \subset Int_Q\left(\bigcap_{j=0}^{n} F^j(Q)\right),$$

and applying the properties of a horseshoe map, $\bigcap_{j=0}^{\infty} F^j(Q) = \widetilde{B}_0$ consists of a collection of continua each of which intersects either s or s'. Since for almost every point q in Q, there is some positive integer n_q such that $F^{n_q}(q) \notin Q$, we know that each component of $\widetilde{B}_0$ is nowhere dense in Q. Now for each $n \in \mathbb{N}$,

$$F^n\left(\bigcap_{j=0}^{\infty} F^j(Q)\right) = \bigcap_{j=n}^{\infty} F^j(Q) \supset \bigcap_{j=n-1}^{\infty} F^j(Q) \supset \cdots \supset \bigcap_{j=0}^{\infty} F^j(Q),$$

so that

$$F^{-n}\left(\bigcap_{j=0}^{\infty} F^j(Q)\right) \subset \cdots \subset \bigcap_{j=0}^{\infty} F^j(Q).$$

Thus, $Z = \bigcap_{j=-\infty}^{\infty} F^j(Q)$ is a nowhere dense, invariant set of continua, none of which intersects ∂Q, and, applying the properties of the quotient map, $\mathcal{Z} = \{C\colon C$ is a component of $Z\}$ endowed with the quotient topology is a totally disconnected, compact metric space. Furthermore, $Z' = \bigcup\{C\colon C \in \mathcal{Z}$ and C is not an isolated point of $\mathcal{Z}\} \subseteq Z$ is an invariant closed set, and $\mathcal{Z}' = \{C\colon C \in \mathcal{Z}$ and $C \subset Z'\}$ endowed with the quotient topology is a Cantor set so that Z' is a quotient Cantor set of continua.

Consider then $Z'' = \bigcup\{C\colon C \in \mathcal{Z}'$ such that, for some sequences $\{n_i\}_{i=0}^{\infty}$ of positive integers and $\{E_i\}_{i=0}^{\infty}$ of continua each member of which is a component of $\bigcap_{i\geqslant 0} F^i(Q)$ that intersects both s and s', $C = \lim_{i\to\infty} F^{n_i}(E_i)$ (in the quotient topology)$\}$. By construction, Z'' is closed. Note that if for each n, $\mathcal{G}_n = \{G_{n,1}, \ldots, G_{n,\beta_n}\} = \{G\colon G$ is a component of $F^n(Q) \cap F^{-n}(Q)$ with nonempty interior$\}$, then $\mathcal{G}_n$ is a mutually disjoint cover of $F^{-n}(\widetilde{B}_0)$ by closed neighborhoods. (If a component C of $F^n(Q) \cap F^{-n}(Q)$ has empty interior, then $F^{n+1}(C) \cap Q = \emptyset$. Thus, we can assume without loss of generality that each component of $F^n(Q) \cap F^{-n}(Q)$ has nonempty interior.) Suppose further that $\mathcal{G}'_n = \{G_{n,1}, \ldots, G_{n,\alpha_n}\} \subseteq \mathcal{G}_n = \{G_{n,1}, \ldots, G_{n,\beta_n}\}$, where $\mathcal{G}'_n$ contains all those members of $\mathcal{G}_n$ which contain some $F^{-n}(C)$, where C is a component of $\widetilde{B}_0$ that intersects both s and s' (and the listing of members of $\mathcal{G}_n$ reflects this property). Then $\mathcal{G}' = \bigcup_{n\geqslant 0} \mathcal{G}'_n$ is a countable base for the set Z'', where each $G_{n,i}$ in $\mathcal{G}'_n$ contains at least M^n members of $\mathcal{G}'_{n+1}$, but $\mathcal{G}'_{n+1}$ may contain neighborhoods not contained in $\bigcup \mathcal{G}'_n$. For each $C \in \mathcal{Z}''$, there are a positive integer N_C and a unique nested sequence $\{G_{n,i_n}\}_{n\geqslant N_C}$ of members of $\mathcal{G}'$ such that $C = \bigcap_{n\geqslant N_C} G_{n,i_n} \in \mathcal{G}'_n$. From this, it follows that the set $Z'' \subseteq Z'$, that $\mathcal{Z}''$ is an upper semicontinuous decomposition of Z'', and that Z'' is a quotient Cantor set.

Finally, consider $Z''' = \overline{\bigcup\{C\colon C \text{ is a point of continuity of } \mathcal{Z}''\}} \subseteq Z''$, and $\mathcal{Z}''' = \{C\colon C$ is a component of $Z'''\}$ endowed with the quotient topology. Then, applying the properties of upper semicontinuous decompositions of compact metric spaces, $\underline{Z'''}$ is a quotient Cantor set of continua, Z''' is invariant, and if $C \in \mathcal{Z}'''$, $x \in C$, then $x \in \overline{Z'''\setminus C}$. $\square$

Theorem 3.5. *Suppose that F is a plane homeomorphism; Q is a quadrilateral in the plane such that F is a horseshoe on Q; for almost every point q in Q there is some positive integer n_q such that $F^{n_q}(q) \notin Q$; F satisfies the strong lockout property on Q (i.e., there are a positive integer integer N_F, $\tau > 0$, and a point $\overline{x}$ in $\mathbb{R}$ such that $F(L(\overline{x})) \subset L^+(\overline{x})$, $Q \subset L^-(\overline{x})$, $d(F(L(\overline{x})), L(\overline{x})) > \tau$); if $q \in Q$ and $F(q)$ is not in Q, then $F^{N_F}(q)$ is in $L^+(\overline{x})$; and s and s' denote the sides of Q that intersect $F(Q)$. There is $\varepsilon > 0$ such that if for each integer j, F_j is a homeomorphism on $\mathbb{R}^2$ such that $|F(x) - F_j(x)| < \varepsilon$ for each x in $\mathbb{R}^2$, then*

(1) *$Z_0 = \{q_0 \in Q \mid$ the trajectory $\{q_j\}_{j=-\infty}^{\infty} \subset Q\}$, is a nowhere dense, invariant, closed set contained in the interior of Q, and $\mathcal{Z}_0 = \{C\colon C$ is a component of $Z\}$ endowed with the quotient topology is a totally disconnected compact metric space;*

(2) *$Z'_0 = \bigcup\{C\colon C \in \mathcal{Z}_0$ and C is not an isolated point of $\mathcal{Z}_0\} \subseteq Z$ is an invariant closed set, $\mathcal{Z}'_0 = \{C\colon C \in \mathcal{Z}_0$ and $C \subset Z'_0\}$ endowed with the quotient topology is a Cantor set, and Z'_0 is a quotient Cantor set of continua;*

(3) *$Z''_0 = \bigcup\{C\colon C \in Z'_0$ such that, for some sequence $\{E_i\}_{i=0}^{\infty}$, each member of which is a component of $\bigcap_{i\geqslant 1} \widetilde{F}_{-1,-i}(Q)$ that intersects both s and s', and for*

some sequence $\{n_i\}_{i=1}^{\infty}$ *of positive integers,* $C = \lim_{i\to\infty} \widetilde{F}_{n_i,0}^{-1}(E_i)$ *(in the quotient topology)*$\} \subseteq Z_0'$ *is an invariant closed set,* $\mathcal{Z}_0'' = \{C\colon C$ *is a component of* $Z_0''\}$ *endowed with the quotient topology is a Cantor set, and* Z_0'' *is a quotient Cantor set of continua*; *and*

(4) $Z_0''' = \overline{\bigcup\{C\colon C \text{ is a point of continuity of } Z_0''\}} \subseteq Z_0''$ *is an invariant closed set, and* $\mathcal{Z}_0''' = \{C\colon C$ *is a component of* $Z_0''\}$ *endowed with the quotient topology is a Cantor set,* Z_0''' *is a quotient Cantor set of continua, and if* $x \in C \in \mathcal{Z}_0'''$*, then* $x \in \overline{Z_0'''\backslash C}$.

Proof. Suppose that $M(\geqslant 2)$ is the fold number of F. Let $N = N_F$, and let

$$\tau = \inf\bigl\{|x-y|\colon x \in Q,\ \ y \in L^+(\overline{x})\bigr\} > 0, \quad \text{and}$$
$$0 < \tau' < \inf\bigl\{|F(x)-F(y)|\colon x \in Q,\ \ y \in L^+(\overline{x})\bigr\}, \quad \tau' < \tau/2.$$

Let $B = \overline{F(Q)\backslash Q}$. Choose $\varepsilon' > 0$ so that if $\widehat{F}_1, \ldots, \widehat{F}_N$ is a collection of N homeomorphisms on $\mathbb{R}^2$ such that

(i) $\widehat{F}_i(L(\overline{x}))$ is in $L^+(\overline{x})$ for each $1 \leqslant i \leqslant N$;

(ii) for each $q \in \mathbb{R}^2$, $|F(q) - \widehat{F}_i(q)| < \varepsilon'$ and if $i_1, \ldots, i_N$ is a permutation of the finite sequence $1, \ldots, N$, then the composition $\widehat{F}_{i_1} \circ \cdots \circ \widehat{F}_{i_N}$ has the property that if $q \in D_{\varepsilon'}(F(Q) - Q)$, then $\widehat{F}_{i_1} \circ \cdots \circ \widehat{F}_{i_N}(q)$ is in $L^+(\overline{x})$ (in other words, each $\widehat{F}_i$ is chosen so close to F that the resulting composition of N homeomorphisms satisfies an appropriately modified version of (Fi4));

(iii) for each i, F_i is a horseshoe map on Q having the same fold number as F (i.e., if K is an arc in Q that intersects both opposite sides s and s' of Q, then $Q \cap F_i(K)$ contains at least M components each of which intersects both s and s', $F_i(s \cup s') \cap Q = \emptyset$, and $F_i(Q)$ does not intersect either of the other two sides of Q); and

(iv) $\varepsilon' < \tau'$.

There is $\varepsilon' > \varepsilon > 0$ such that if $\widehat{F}$ is a plane homeomorphism and $|F(q) - \widehat{F}(q)| < \varepsilon$ for each $q \in Q$, then $\widehat{F}(F^{-1}(D_\varepsilon(Q))) \subset D_{\varepsilon'}(Q)$ and $\widehat{F}^{-1}(\overline{Q}) \subset D_\varepsilon(F^{-1}(Q))$. Then suppose that for each integer j, F_j is a homeomorphism on $\mathbb{R}^2$, and for each $q \in \mathbb{R}^2$, $|F(q) - F_j(q)| < \varepsilon$. Consider

$$Z = \bigl\{q_0 \in Q \mid \text{the trajectory } \{q_j\}_{j=-\infty}^{\infty} \subset Q\bigr\}.$$

Note that $Z = \cdots \cap \widetilde{F}_{-1,-2}(Q) \cap \widetilde{F}_{-1,-1}(Q) \cap Q \cap \widetilde{F}_{0,0}^{-1}(Q) \cap \widetilde{F}_{1,0}^{-1}(Q) \cap \cdots$.

Then, as in the unperturbed case, $Q \cap F_n(Q) = Q \cap \widetilde{F}_{n,n}(Q)$ contains at least M components each of which intersects both s and s' as does $Q \cap F(Q)$; $Q \cap F_n(Q \cap F_{n-1}(Q)) = Q \cap F_n(Q) \cap \widetilde{F}_{n,n-1}(Q)$ contains at least M^2 components each of which intersects both s and s' as does $Q \cap F(Q) \cap F^2(Q)$, etc. Thus, for each $m \leqslant n$, $Q \cap (\bigcap_{j=m}^{n} \widetilde{F}_{n,j}(Q))$ contains at least M^{n-m+1} components each of which intersects both s and s' as does $\bigcap_{j=0}^{n-m+1} F^j(Q)$. Also, if for each $n \geqslant m$, $E_{n,m} = \{C\colon C$ is a component of $Q \cap (\bigcap_{j=m}^{n} \widetilde{F}_{n,j}(Q))$ that intersects both s and $s'\}$, then $E_{n,m}$ has at least M^n distinct members.

Further,

$$Q \cap \left(\bigcap_{j=m-1}^{n} \widetilde{F}_{n,j}(Q) \right) \subset Int_Q \left(Q \cap \left(\bigcap_{j=m}^{n} \widetilde{F}_{n,j}(Q) \right) \right).$$

As before, $Q \cap (\bigcap_{j=1}^{\infty} \widetilde{F}_{-1,-j}(Q)) = \widetilde{B}_0$ is a nowhere dense, closed set each component of which intersects either s or s', and it follows that

$$Z_0 = Q \cap \left(\bigcap_{j=1}^{\infty} \widetilde{F}_{-1,-j}(Q) \right) \cap \left(\bigcap_{j=0}^{\infty} \widetilde{F}_{j,0}^{-1}(Q) \right)$$

is a nowhere dense, closed set contained in the interior of Q. Applying the properties of the quotient map, $\mathcal{Z}_0 = \{C\colon C$ is a component of $Z_0\}$ endowed with the quotient topology is a totally disconnected, compact metric space. Furthermore, $Z_0' = \bigcup\{C\colon C \in \mathcal{Z}_0$ and C is not an isolated point of $\mathcal{Z}_0\} \subseteq Z_0$ is a closed set, and $\mathcal{Z}_0' = \{C\colon C \in \mathcal{Z}_0$ and $C \subset Z_0'\}$ endowed with the quotient topology is a Cantor set so that Z_0' is a quotient Cantor set of continua.

Consider then $Z_0'' = \bigcup\{C\colon C \in \mathcal{Z}_0'$ such that, for some sequence $\{E_i\}_{i=0}^{\infty}$, each member of which is a component of $\widetilde{B}_0$ that intersects both s and s', and for some sequence $\{n_i\}_{i=1}^{\infty}$ of positive integers, $C = \lim_{i\to\infty} \widetilde{F}_{n_i,0}^{-1}(E_i)$ (in the quotient topology)$\}$. By construction, Z'' is closed. Note that if for each n, $\mathcal{G}_n = \{G_{n,1}, \ldots, G_{n,\beta_n}\} = \{G\colon G$ is a component of $\widetilde{F}_{-1,-n}(Q) \cap \widetilde{F}_{n,0}^{-1}(Q)$ with nonempty interior$\}$, then $\mathcal{G}_n$ is a mutually disjoint cover of $\widetilde{F}_{n,0}^{-1}(\widetilde{B}_0)$ by closed neighborhoods. (If a component C of $\widetilde{F}_{-1,-n}(Q) \cap \widetilde{F}_{n,0}^{-1}(Q)$ has empty interior, then $\widetilde{F}_{n+1,0}(C) \cap Q = \emptyset$. Thus, we can assume without loss of generality that each component of $\widetilde{F}_{-1,-n}(Q) \cap \widetilde{F}_{n,0}^{-1}(Q)$ has nonempty interior.) Suppose further that $\mathcal{G}_n' = \{G_{n,1}, \ldots, G_{n,\alpha_n}\} \subseteq \mathcal{G}_n = \{G_{n,1}, \ldots, G_{n,\beta_n}\}$, where $\mathcal{G}_n'$ contains all those members of $\mathcal{G}_n$ which contain some $\widetilde{F}_{n,0}^{-1}(C)$, where C is a component of $\widetilde{B}_0$ that intersects both s and s' (and the listing of members of $\mathcal{G}_n$ reflects this property). Then $\mathcal{G}' = \bigcup_{n\geqslant 0} \mathcal{G}_n'$ is a countable base for the set Z_0'', where each $G_{n,i}$ in $\mathcal{G}_n'$ contains at least M^n members of $\mathcal{G}_{n+1}'$, but $\mathcal{G}_{n+1}'$ may contain neighborhoods not contained in $\bigcup \mathcal{G}_n'$. For each $C \in \mathcal{Z}_0''$, there are a positive integer N_C and a unique nested sequence $\{G_{n,i_n}\}_{n\geqslant N_C}$ of members of $\mathcal{G}'$ such that $C = \bigcap_{n\geqslant N_C} G_{n,i_n} \in \mathcal{G}_n'$. From this, it follows that the set $Z_0'' \subseteq Z_0'$, that $\mathcal{Z}_0''$ is an upper semicontinuous decomposition of Z_0'', and that Z_0'' is a quotient Cantor set of continua. The last part then follows as before. □

The next two results tell us that for the fluid flow diffeomorphism itself, the situation is simpler than in the two previous theorems, i.e., for each quadrilateral Q_i on which F is a hyperbolic horseshoe, the corresponding invariant set Z_i in Q_i *is* a Cantor set, and thus the various decompositions considered in the two previous results are trivial and all the same, namely Z_i itself. The hyperbolicity makes the difference.

Corollary 3.6. *Suppose that F is the fluid flow diffeomorphism, and $i \in \mathbb{Z}$. Then $Z_i = \{q \in Q_i \mid$ for all $n \in \mathbb{Z}$, $F^n(q) \in Q_i\}$ is a Cantor set contained in the interior of Q_i.*

Proof. That Z_i is a nowhere dense, invariant, closed set with the upper semicontinuous decomposition $\mathcal{Z}_i = \{C\colon C$ is a component of $Z_i\}$ which is a totally disconnected, compact metric space in the quotient topology, which contains a Cantor set, and which is contained in the interior of Q_i, follows from a previous theorem. (That for almost every point q in Q_i there is some positive integer n_q such that $F^{n_q}(q) \notin Q_i$ follows automatically from the assumption that F is a hyperbolic horseshoe on Q_i.) However, because F is diffeomorphism and F is a hyperbolic horseshoe on Q with Q an isolating neighborhood for Z_i, Z_i is a Cantor set. □

Theorem 3.7. *Suppose that F is the fluid flow diffeomorphism. There is $\varepsilon > 0$ such that if for each integer j, F_j is a diffeomorphism on $\overline{\mathbb{R}^2}$ such that for each $q \in \mathbb{R}^2$, $|F(q) - F_j(q)| < \varepsilon$ and $|DF(q) - DF_j(q)| < \varepsilon$, then $Z_i = \{q \in Q_i \mid \widetilde{F}_{n,0}(q) \in Q_i$ for all $n \in \widetilde{\mathbb{N}}$ and $(\widetilde{F}_{-1,-n})^{-1}(q) \in Q_i$ for all $n \in \mathbb{N}\}$ is a Cantor set in the interior of Q_i.*

Proof. Suppose that M is the fold number of F, and s and s' denote the opposite sides of Q_i that $F(Q_i)$ intersects. Choose $\varepsilon > 0$ so that if F' is a diffeomorphism on $\overline{\mathbb{R}^2}$, and for each $q \in \mathbb{R}^2$, $|F(q) - F'(q)| < \varepsilon$, and $|DF(q) - DF'(q)| < \varepsilon$, then F' satisfies properties (Fi1)–(Fi4). (This choice of ε needs to be modified later in the proof, but these are the constraints we need now.) Then suppose that for each integer j, F_j is a diffeomorphism on $\mathbb{R}^2$ such that for each $q \in \mathbb{R}^2$, $|F(q) - F_j(q)| < \varepsilon$, and $|DF(q) - DF_j(q)| < \varepsilon$.

We also assume that ε "works" for Proposition 3.3, i.e., for each pair of integers i, j, if for each $q \in \mathbb{R}^2$, $|F(q) - F_j(q)| < \varepsilon$ and $|DF(q) - DF_j(q)| < \varepsilon$, then for almost every $q \in Q_i$, $\lim_{n\to\infty} \widetilde{F}_{n,0}(q) = \overline{\infty}$ and, in particular, the x-coordinate of $\widetilde{F}_{n,0}(q)$ tends to $+\infty$ as $n \to \infty$.

Note that for each integer n, $Q_i \cap F_n(Q_i)$ contains at least M components intersecting both s and s'. Then, as in the unperturbed case, for each $m \leqslant n$, $Q_i \cap (\bigcap_{j=m}^{n} \widetilde{F}_{n,j}(Q_i))$ contains at least M^{n-m+1} components intersecting both s and s' as does $\bigcap_{j=0}^{n-m+1} F^j(Q_i)$, each extending from side s to opposite side s'. Further,

$$Q_i \cap \left(\bigcap_{j=m-1}^{n} \widetilde{F}_{n,j}(Q_i) \right) \subset Int_{Q_i}\left(Q_i \cap \left(\bigcap_{j=m}^{n} \widetilde{F}_{n,j}(Q_i) \right) \right).$$

As before, $Q_i \cap (\bigcap_{j=1}^{\infty} \widetilde{F}_{-1,-j}(Q_i)) = \widetilde{B}_0$ consists of a closed set with uncountably many components and each component intersects either the side s or the side s', and does not intersect either of the other two sides. Now for each $n \in \mathbb{N}$,

$$\widetilde{B}_0 \cap \left(\bigcap_{j=0}^{0} (\widetilde{F}_{j,0})^{-1}(Q_i) \right) \supset \widetilde{B}_0 \cap \left(\bigcap_{j=0}^{1} (\widetilde{F}_{j,0})^{-1}(Q_i) \right) \supset \cdots$$
$$\supset \widetilde{B}_0 \cap \left(\bigcap_{j=0}^{\infty} (\widetilde{F}_{j,0})^{-1}(Q_i) \right) = Z_i.$$

In fact, each component of $\widetilde{B}_0 = Q_i \cap (\bigcap_{j=1}^{\infty} \widetilde{F}_{-1,-j}(Q_i))$ contains at least M components of $\widetilde{B}_0 \cap (\bigcap_{j=0}^{0} (\widetilde{F}_{j,0})^{-1}(Q_i))$, each of which contains at least M components of $\widetilde{B}_0 \cap (\bigcap_{j=0}^{1} (\widetilde{F}_{j,0})^{-1}(Q_i))$, etc., so that each component of $\widetilde{B}_0$ contains at least M^n components of $\widetilde{B}_0 \cap (\bigcap_{j=0}^{n-1} (\widetilde{F}_{j,0})^{-1}(Q_i))$, with nesting of these components occurring at each step.

Thus, $Z_i = \widetilde{B}_0 \cap (\bigcap_{j=0}^{\infty} (\widetilde{F}_{j,0})^{-1}(Q_i))$ is a closed set with uncountably many components, none of which has interior in Q_i (because the trajectory of almost every point in Q_i converges to $\overline{\infty}$), and none of which intersects ∂Q_i. Note that Z_i is also the set $\{q \in Q_i \mid \widetilde{F}_{n,0}(q) \in Q_i$ for all $n \in \widetilde{\mathbb{N}}$ and $(\widetilde{F}_{-1,-n})^{-1}(q)$ for all $n \in \mathbb{N}\}$.

Recall that, applying the previous corollary, $C_{Q_i} = \{q \in Q_i \mid F^n(q) \in Q_i$ for all $n \in \mathbb{Z}\}$ is a Cantor set contained in the interior of Q_i, and F is hyperbolic on C_{Q_i}. Then F is expansive on C_{Q_i}, and there is a constant $c > 0$ such that if $p \neq q \in C_{Q_i}$, there is some integer $n_{p,q} = n$ such that $|F^n(p) - F^n(q)| \geqslant c$. Also, we can extend the splitting $\mathbb{E}_x^s \times \mathbb{E}_x^u$ from on C_{Q_i} to a neighborhood V of C_{Q_i} in Q_i. There is some c' such that if $x \in C_{Q_i}$, then $D_{c'}(x) \subset V$, and if we let $\mathbb{B}_x(c') = \mathbb{E}_x^s(c') \times \mathbb{E}_x^u(c') \subset T_x(\mathbb{R}^2)$ denote all vectors in $\mathbb{E}_x^s \times \mathbb{E}_x^u$ of norm less than c', and $w_x(c') = \{x + \boldsymbol{v} \in \mathbb{R}^2 \mid \boldsymbol{v} \in \mathbb{B}_x(c')\} \subset \mathbb{R}^2$, then $w_x(c') \subset V$, and $w_x(c') = D_{c'}(x)$.

Next, cover C_{Q_i} with a finite collection $\{u_1, u_2, \ldots, u_m\}$ of mutually disjoint open sets in $\mathbb{R}^2$ with the following properties:

(i) for $1 \leqslant k \leqslant m$, $u_k \cap C_{Q_i} = \overline{u_k} \cap C_{Q_i}$;
(ii) $\sup\{diam(u_k)\} < \alpha < \max\{c/64, c'/64\}$; and
(iii) $\inf\{|x - y| \mid x \in u_k,\ y \in u_{k'},\ k \neq k'\} > \alpha' > 0$.

There is a Lebesgue number α'' such that if $q \in C_{Q_i}$, then $D_{\alpha''}(q) \subset u_k$ for some k.

Choose $\varepsilon' > 0$ such that $\varepsilon' < \min\{c/64, \alpha/16, \alpha'/16, \alpha''/16\}$. Since Q_i is an isolating neighborhood for C_{Q_i}, we apply the shadowing theorem and find $\delta > 0$ and $\eta > 0$ such that if $\{x_j\}_{j=-\infty}^{\infty}$ is a δ-chain for F with $\inf\{|x_j - y| \mid y \in C_{Q_i},\ j \in \mathbb{Z}\} < \eta$, then there is a unique point y in C_{Q_i} such that y ε'-shadows $\{x_j\}_{j=-\infty}^{\infty}$. Finally, modify the choice of ε once again:

(i) Choose $\varepsilon < \min\{\delta, \eta, \varepsilon'\}$.
(ii) There are a positive integer K and $0 < \mu < 1$ such that for x in C_{Q_i},

$$|DF^K(x)(\boldsymbol{v}^s)| \leqslant C\mu^K|\boldsymbol{v}^s| \quad \text{for } \boldsymbol{v}^s \in \mathbb{E}_x^s, \text{ and}$$
$$|DF^{-K}(x)(\boldsymbol{v}^u)| \leqslant C\mu^K|\boldsymbol{v}^u| \quad \text{for } \boldsymbol{v}^u \in \mathbb{E}_x^u.$$

There is $\varepsilon > 0$ such that if for each z in C_{Q_i} and for each integer n, $|F(z) - F_n(z)| < \varepsilon$ and $|DF(z) - DF_n(z)| < \varepsilon$, then if $\{i_1, \ldots, i_K\}$ denotes a collection of K integers and $G = F_{i_1} \circ \cdots \circ F_{i_K}$, then there is some $a > 1$ such that if x is in C_{Q_i}, and

(a) $\boldsymbol{v}_x^s \in \mathbb{E}_x^s$ and $\boldsymbol{v}_x^u \in \mathbb{E}_x^u$ with $|\boldsymbol{v}_x^u| \geqslant |\boldsymbol{v}_x^s|$, then $|\boldsymbol{v}^u_{F^K(x)}| \geqslant a\,|\boldsymbol{v}^s_{F^K(x)}|$ and $|\boldsymbol{v}^u_{G(x)}| \geqslant a\,|\boldsymbol{v}^s_{G(x)}|$
(b) $\boldsymbol{v}_x^s \in \mathbb{E}_x^s$ and $\boldsymbol{v}_x^u \in \mathbb{E}_x^u$ with $|\boldsymbol{v}_x^u| \leqslant |\boldsymbol{v}_x^s|$, then $|\boldsymbol{v}^u_{F^{-K}(x)}| \leqslant (1/a)|\boldsymbol{v}^s_{F^{-K}(x)}|$, and $|\boldsymbol{v}^u_{G^{-1}(x)}| \leqslant (1/a)|\boldsymbol{v}^s_{G^{-1}(x)}|$.

(Of course, this ε still satisfies the first part of the proof. Thus, we are merely adjusting the size of ε downward again, if necessary.) As before, suppose that for each integer j, F_j is a diffeomorphism on $\overline{\mathbb{R}^2}$ such that for each $q \in \mathbb{R}^2$, $|F(q) - F_i(q)| < \varepsilon$, and $|DF(q) - DF_i(q)| < \varepsilon$.

Then the resulting set

$$Z_i = Q_i \cap \left(\bigcap_{j=1}^{\infty} \widetilde{F}_{-1,-j}(Q_i)\right) \cap \left(\bigcap_{j=0}^{\infty} (\widetilde{F}_{j,0})^{-1}(Q_i)\right)$$

is a closed set with uncountably many components contained in the interior of Q_i. Note that if $\{q_j\}_{j=-\infty}^{\infty}$ is the trajectory of a point q_0 in Z_i, then $\{q_j\}_{j=-\infty}^{\infty} \subset Z_i$, and $\{q_j\}_{j=-\infty}^{\infty}$ is a δ-chain for F (since $|F(q_j) - F_j(q_j)| = |F(q_j) - q_{j+1}| < \varepsilon < \delta$).

Suppose that Z_i contains a nondegenerate continuum E. If $q = q_0 \in E$, then the trajectory $\{q_j\}_{j=-\infty}^{\infty}$ is contained in Q_i. Let $E_0 = E$, and for each positive integer n,

$$\begin{aligned} E_n &= \widetilde{F}_{n,0}(E) \subset \widetilde{F}_{n,0}(Z_i) \\ &= \left(\bigcap_{j=-n}^{\infty} \widetilde{F}_{n,-j}(Q_i)\right) \cap Q_i \cap \left(\bigcap_{j=n+1}^{\infty} (\widetilde{F}_{j,n+1})^{-1}(Q_i)\right) \subseteq Q_i, \quad \text{and} \\ E_{-n} &= (\widetilde{F}_{-1,-n})^{-1}(E_0) \subset (\widetilde{F}_{-1,-n})^{-1}(Z_i) \\ &= \left(\bigcap_{j=-n+1}^{\infty} \widetilde{F}_{-n-1,-j}(Q_i)\right) \cap Q_i \cap \left(\bigcap_{j=n}^{\infty} (\widetilde{F}_{j,-n})^{-1}(Q_i)\right) \subseteq Q_i. \end{aligned}$$

Choose $p_0 \neq q_0$ in E. Then there are unique points y_p and y_q in C_{Q_i} such that y_p and y_q ε'-shadow the trajectories $\{p_j\}_{j=-\infty}^{\infty}$ and $\{q_j\}_{j=-\infty}^{\infty}$, respectively. If $y_p \neq y_q$, then because F is expansive on C_{Q_i}, there is some integer l such that $|F^l(y_p) - F^l(y_q)| \geqslant c$. Then consider E_l: since $|F^l(y_p) - p_l| < \varepsilon' < c/64$, $|F^l(y_q) - q_l| < \varepsilon' < c/64$, and p_l and q_l are in E_l, $diam(E_l) > 31c/32$. Note that for each x_l in E_l, the corresponding trajectory $\{x_j\}_{j=-\infty}^{\infty}$ ("centered" at x_l rather than x_0), is contained in Z_i. But this means we have a problem: for each x_l in E_l, there is a unique point y_x in C_{Q_i} which ε'-shadows the trajectory $\{x_j\}_{j=-\infty}^{\infty}$, and $\varepsilon' < \alpha''$, so that $x_l \in D_{\alpha''}(y_x) \subset \bigcup_{j=1}^{m} u_j$. Since $diam(u_j) < c/64$, it follows that for some w_l and z_l in E_l, the corresponding y_w and y_z are in *different* members u_j and $u_{j'}$ of the cover of C_{Q_i}, respectively, and for each member x_l of E_l, the associated y_x is in *some* u_k. But this is a contradiction to E_l being a nondegenerate continuum. Then $y_p = y_q = y$, and y ε'-shadows each trajectory $\{p_j\}_{j=-\infty}^{\infty}$, where $p_0 \in E_0$. Then $diam(E_n) < \varepsilon'$ for each integer n.

Hence, for each $x \in C_{Q_i}$,

$$\begin{aligned} \mathbb{B}_x(\varepsilon') &= \mathbb{E}_x^s(\varepsilon') \times \mathbb{E}_x^u(\varepsilon') \subset T_x(\mathbb{R}^2), \quad \text{and} \\ w_x(\varepsilon') &= \{x + \boldsymbol{v} \in \mathbb{R}^2 \mid \boldsymbol{v} \in \mathbb{B}_x(\varepsilon')\} \subset Q_i. \end{aligned}$$

In particular, for each integer n, $E_n \subset w_y(\varepsilon') = D_{\varepsilon'}(y)$.

Suppose that for each $n \geqslant 1$, $\mathcal{G}_n = \{G_{n,1}, \ldots, G_{n,\beta_n}\} = \{G\colon G$ is a component of $\widetilde{F}_{-1,-n}(Q) \cap \widetilde{F}_{n,0}^{-1}(Q)$ with nonempty interior$\}$, and $\mathcal{G}'_n = \{G_{n,1}, \ldots, G_{n,\alpha_n}\} \subseteq \mathcal{G}_n = \{G_{n,1}, \ldots, G_{n,\beta_n}\}$, where $\mathcal{G}'_n$ contains all those members of $\mathcal{G}_n$ which contain some $\widetilde{F}_{n,0}^{-1}(C)$, where C is a component of $\widetilde{B}_0 = \bigcap_{n\geqslant 1} \widetilde{F}_{-1,-n}^{-1}(Q_i)$ that intersects both s and s' (and the listing of members of $\mathcal{G}_n$ reflects this property). As before, without loss of generality, we can assume that each $\mathcal{G}'_n$ is a mutually disjoint cover of $\widetilde{F}_{-1,-n}(Q) \cap \widetilde{F}_{n,0}^{-1}(Q)$, and if $\mathcal{G}''_n = \{G \cap Z_i\colon G \in \mathcal{G}'_n\}$, then $\bigcup_{n\geqslant 1} \mathcal{G}''_n$ is a clopen base for Z_i. Note that if for some n, a continuum K is contained in some $G \in \mathcal{G}'_n$ and K intersects both $\widetilde{F}_{n,0}^{-1}(s)$ and $\widetilde{F}_{n,0}^{-1}(s')$, then $\widetilde{F}_{n,0}(K)$ intersects both s and s'. Likewise, if t and t' denote the other two sides of Q_i (i.e., the sides that are neither s nor s') and K intersects both $\widetilde{F}_{-1,-n}(t)$ and $\widetilde{F}_{-1,-n}(t')$, then $\widetilde{F}_{-1,-n}^{-1}(K)$ intersects both t and t'.

But then we again reach a contradiction: The splitting is preserved under the application of F, and each DF_n has been chosen so close to DF that the stretching and contracting directions are quite close to those of F. Thus, E_0 is either not contained in $W^s(y)$ or it is not contained in $W^u(y)$, and in at least one of the positive or negative directions, the continuum E_0 is eventually "stretched across" some u_k, which means that, after a sufficient number of applications of the appropriate F_n's (either in the forward direction or the negative), the image of the continuum E_0 must intersect Z_i and $\mathbb{R}^2\backslash Q_i$. Thus, E_0 cannot be nondegenerate. Using a similar argument, it follows that no point of Z_i is isolated, and it follows that Z_i is a Cantor set. □

Theorem 3.8. *Suppose that F is a homeomorphism of $\overline{\mathbb{R}^2}$, Q is a quadrilateral in $\mathbb{R}^2$, F is a horseshoe map on Q, for almost every point q in Q there is some positive integer n_q such that $F^{n_q}(q) \notin Q$, and F has the lockout property on Q. In the Hausdorff metric, the sequence $Q, F(Q), F^2(Q), \ldots$ of continua in $\overline{\mathbb{R}^2}$ has a unique limit point $\widetilde{B}$, $\widetilde{B}$ is an invariant, nowhere dense continuum, and $\widetilde{B}$ is the closure of the set $\{x \in \mathbb{R}^2 \mid F^{-n}(x) \in Q$ for all sufficiently large $n\}$. Furthermore, $\widetilde{B}$ contains an invariant, indecomposable continuum Λ, Λ is the largest indecomposable continuum contained in $\widetilde{B}$, and Λ contains the quotient Cantor set Z''' defined in Theorem* 3.4.

Proof. Suppose Q has sides s_1, s_2, s_3, s_4, with s_2 and s_4 denoting the opposite sides which do not intersect $F(Q)$, and s_1 and s_3 denoting the opposite sides that do intersect $F(Q)$. By Theorem 3.4, $\bigcap_{i\geqslant 0} F^i(Q)$ is a closed, nowhere dense set in Q, each component of $\bigcap_{i\geqslant 0} F^i(Q)$ intersects either s_1 or s_3, and $\bigcap_{i\geqslant 0} F^i(Q)$ has uncountably many components. Let $\widetilde{B}_0 = \bigcap_{i\geqslant 0} F^i(Q)$. For each $n \geqslant 0$, $\widetilde{B}_n = \bigcap_{i\geqslant n} F^i(Q) = F^n(\widetilde{B}_0)$ is a closed, nowhere dense set in $F^n(Q)$, and each component of $\widetilde{B}_n$ intersects either the side $F^n(s_1)$ of $F^n(B)$ or the side $F^n(s_3)$. For each n, $\widetilde{B}_n \subset \widetilde{B}_{n+1}$. Consider $\overline{\bigcup_{n\geqslant 0} \widetilde{B}_n} = \widetilde{B}$. Since $\widetilde{B}_n \subset \widetilde{B}_{n+1}$ for each n, $\widetilde{B}$ is the Hausdorff limit of the sequence $\widetilde{B}_0, \widetilde{B}_1, \widetilde{B}_2, \ldots$. Since $F(\widetilde{B}_n) = \widetilde{B}_{n+1}$, $F(\widetilde{B}) = \widetilde{B}$.

Let $\widetilde{Q}$ denote an accumulation point (relative to the Hausdorff metric) of the sequence $Q, F(Q), F^2(Q), \ldots$ of continua in $\overline{\mathbb{R}^2}$. Then there is an increasing sequence $n_1, n_2, \ldots$ of positive integers such that $F^{n_1}(Q), F^{n_2}(Q), \ldots$ converges to $\widetilde{Q}$. Now $\widetilde{B} \subseteq \widetilde{Q}$, but we need to show that $\widetilde{Q} \subseteq \widetilde{B}$. For $0 \leqslant m \leqslant n$, let $E_{m,n} = \bigcap_{j=m}^{n} F^j(Q)$. There is a subsequence $n_{\sigma_1}, n_{\sigma_2}, \ldots$ of $n_1, n_2, \ldots$ such that $E_{n_{\sigma_1}, n_{\sigma_2}}, E_{n_{\sigma_1}, n_{\sigma_2}}, \ldots$ converges to $\widetilde{B}$ (relative to the Hausdorff metric). If $\varepsilon_1, \varepsilon_2, \ldots$ is a sequence of positive numbers that converges to 0, then $\overline{D_{\varepsilon_1}(E_{n_{\sigma_1}, n_{\sigma_2}})}, \overline{D_{\varepsilon_2}(E_{n_{\sigma_1}, n_{\sigma_2}})}, \ldots$ converges to $\widetilde{B}$ (relative to the Hausdorff metric).

Suppose that n is a positive integer, $n \geqslant 2$. Then if $x \in F^n(Q) \cap Q$, but x is not in $\bigcap_{j=0}^{n-1} F^j(Q)$, then there is some least $j > 0$ such that x is not in $F^j(Q)$. Thus, x is in $(\bigcap_{k=0}^{j-1} F^k(Q))\backslash F^j(Q)$, and $F^{-j}(x)$ is not in Q. But this means that $F^j(F^{-j}(x)) = x$ is not in Q, because of the lockout property. This is a contradiction. It follows that $F^n(Q) \cap Q \subset \bigcap_{j=0}^{n-1} F^j(Q)$, and $F^n(Q) \cap Q = \bigcap_{j=0}^{n} F^j(Q)$.

Now the sequence $F^{n_{\sigma_1}}(Q), F^{n_{\sigma_2}}(Q), \dots$ converges to $\widetilde{Q}$, and if ε_i denotes the least upper bound of the Hausdorff distances from $F^{n_{\sigma_i}}(Q)$ to $F^{n_{\sigma_j}}(Q)$ for $j > i$, then the sequence $\varepsilon_1, \varepsilon_2, \dots$ converges to 0. Further, for each i,

$$\begin{aligned} F^{n_{\sigma_{i+1}}}(Q) &= \bigl(F^{n_{\sigma_{i+1}}}(Q) \cap F^{n_{\sigma_i}}(Q)\bigr) \cup \bigl(F^{n_{\sigma_{i+1}}}(Q) \backslash F^{n_{\sigma_i}}(Q)\bigr) \\ &= E_{n_{\sigma_i}, n_{\sigma_{i+1}}} \cup \bigl(F^{n_{\sigma_{i+1}}}(Q) \backslash F^{n_{\sigma_i}}(Q)\bigr). \end{aligned}$$

Then

$$E_{n_{\sigma_i}, n_{\sigma_{i+1}}} \subset E_{n_{\sigma_i}, n_{\sigma_{i+1}}} \cup \bigl(F^{n_{\sigma_{i+1}}}(Q) \backslash F^{n_{\sigma_i}}(Q)\bigr) = F^{n_{\sigma_{i+1}}}(Q) \subset \overline{D_{\varepsilon_i}(E_{n_{\sigma_i}, n_{\sigma_{i+1}}})}.$$

Thus, $\widetilde{Q} = \widetilde{B}$.

Let $\widetilde{\mathcal{B}}'_0 = \{C\colon C$ is a component of $\widetilde{B}_0$ that intersects both s and $s'\}$, and $\widetilde{B}'_0 = \bigcup \widetilde{\mathcal{B}}'_0$. Then $\widetilde{\mathcal{B}}'_0$, when endowed with the quotient topology, is a totally disconnected, compact metric space. For each $n > 0$, let $\mathcal{H}_n = \{H_{n,1}, H_{n,2}, \dots, H_{n,\gamma_n}\}$ be a listing of the components of $\bigcap_{i=0}^{n} F^i(Q)$ that intersect both s_1 and s_3. Thus, $C \in \widetilde{\mathcal{B}}'_0$ if and only if there is some unique sequence $i_1, i_2, \dots$ of positive integers such that $C = \bigcap_{n \geqslant 1} H_{n,i_n}$. Since $F(H_{n,i_n}) \cap Q$ contains at least M components that intersect both s_1 and s_3, and each $H_{n,j}$ contains at least one of these M components, $\widetilde{\mathcal{B}}'_0$ is a Cantor set, and $\widetilde{B}'_0$ is a quotient Cantor set.

Let $\widetilde{L}_0$ denote the closure of $\bigcup\{C\colon C$ is a point of continuity of $\widetilde{\mathcal{B}}'_0\}$. Then $\widetilde{L}_0$ is a quotient Cantor set of continua with respect to the upper semicontinuous decomposition $\widetilde{\mathcal{L}}_0 = \{C\colon C$ is a component of $\widetilde{L}_0\}$. It is easy to check that the quotient Cantor set Z''' of Theorem 3.4 is contained in $\widetilde{L}_0$. For each $n \geqslant 0$, let $\widetilde{L}_n = F^n(\widetilde{L}_0)$, and let $\widetilde{\mathcal{L}}_n = \{F^n(C)\colon F^n(C)$ is a component of $\widetilde{L}_n\}$. Thus, each $\widetilde{L}_n$ is a quotient Cantor set.

Consider $\overline{\bigcup_{n \geqslant 0} \widetilde{L}_n} = \Lambda$. Since $\widetilde{L}_n \subset \widetilde{L}_{n+1}$ for each n, Λ is the Hausdorff limit of the sequence $\widetilde{L}_0, \widetilde{L}_1, \widetilde{L}_2, \dots$. Since $F(\widetilde{L}_n) = \widetilde{L}_{n+1}$, $F(\widetilde{L}) = \widetilde{L}$. Let $1/2 > \varepsilon_1 > 0$ be a positive number such that $D_{\varepsilon_1}(\widetilde{L}_0) \cup Q \subset \bigcup \mathcal{H}_1$. Let $\mathcal{M}_1 = \{H_{1,j} \cap D_{\varepsilon_1}(\widetilde{L}_0)\colon 1 \leqslant j \leqslant \gamma_1\}$. Inductively, having chosen ε_{i-1}, choose ε_i to be a positive number less than $\varepsilon_{i-1}/2$ and less than $1/2^i$ such that $D_{\varepsilon_i}(\widetilde{L}_0) \cup Q \subset \bigcup \mathcal{H}_i$. Let $\mathcal{M}_i = \{H_{1,j} \cap D_{\varepsilon_i}(\widetilde{L}_0)\colon 1 \leqslant j \leqslant \gamma_1\}$. Thus, each $\widetilde{\mathcal{M}}_i = \{L \cap \widetilde{L}_0\colon L \in \mathcal{M}_i\}$ is a clopen cover of $\widetilde{L}_0$ relative to the topology inherited by the subspace $\widetilde{L}_0$, and $\widetilde{\mathcal{M}} = \bigcup_{i \geqslant 1} \widetilde{\mathcal{M}}_i$ is a basis for $\widetilde{L}_0$. For each i, let $\mathcal{L}_{0,i} = \widetilde{\mathcal{M}}_i$, and for each $n \geqslant 0$, let $\mathcal{L}_{n,i} = \{F^n(L)\colon L \in \widetilde{\mathcal{M}}_i\}$.

For each n, $\widetilde{L}_n$ is a collection of components. For each point x in $\widetilde{L}_n$, x is contained in some component $R_{x,n}$ of $\widetilde{L}_n$. Then $R_{x,n} \subseteq R_{x,n+1}$ for each x, each n, and $F(R_{x,n}) = R_{F(x),n+1}$. Note that $R_{x,n} \subset \widetilde{L}_n \subset F^n(Q)$, and $R_{x,n}$ "runs through" $F^n(Q)$, in the sense that it intersects the opposite sides $F^n(s_1)$ and $F^n(s_3)$ of the nth image of the quadrilateral Q. Clearly, R_x is first category in Λ and connected. For each $n > m \geqslant 1$, each $R_{x,m}$ is some $E_{x,m,n} \in \mathcal{L}_{m,n}$, and $E_{x,m,n}$ is a component of $\bigcap_{m \leqslant l \leqslant n} F^l(Q)$. Also, for each m, n, $F^n(R_{x,m})$ "runs through" $\bigcap_{m+n \geqslant i \geqslant m} F^i(Q)$ in the sense that if $L \in \mathcal{L}_{m,n}$, then $L \cap F^n(R_{x,m})$ contains at least one component that intersects both the opposite sides $F^m(s_1)$ and $F^m(s_3)$. Since $\mathcal{L}_m$ is a basis for $\widetilde{L}_m$, R_x is dense in Λ. Further, if $R_x \cap R_y \neq \emptyset$ for some x, y in $\bigcup_{m \geqslant 0} \widetilde{L}_m$, then $R_x = R_y$. Thus, $\mathcal{R} = \{R_x\colon x \in \bigcup_{m \geqslant 0} \widetilde{L}_m\}$ partitions the set $\bigcup_{m \geqslant 0} \widetilde{L}_m$. It follows that $\bigcup_{m \geqslant 0} \widetilde{L}_m$ is connected, that Λ is a continuum, and that $\mathcal{R}$ is uncountable.

Suppose that Λ is decomposable. Then there exists some proper subcontinuum H of Λ such that H has interior relative to the subspace Λ. Choose x from $\widetilde{L}_0$. Then R_x intersects both $Int_\Lambda(H)$ and $Int_\Lambda(\Lambda\backslash H)$, and there is some n such that

(i) $R_{x,n} \cap (F^n(s_1) \cup F^n(s_3))$ does not intersect some component C of $R_{x,n} \cap H$, and
(ii) $C \cap Int_\Lambda(H) \neq \emptyset$.

Choose $m > n$ so that

(i) $E_{x,n,m} \cap (F^n(s_1) \cup F^n(s_3))$ does not intersect some component C' of $E_{x,n,m} \cap H$, and
(ii) $C' \cap Int_\Lambda(H) \neq \emptyset$.

But H is a proper subcontinuum of Λ, and by the choice of $E_{x,n,m}$, the component C' of $E_{x,n,m} \cap H$, which intersects $Int_\Lambda(H)$, but does not contain H, is clopen in H (since $E_{x,n,m} \cap (F^n(s_1) \cup F^n(s_3)) \cap C' = \emptyset$). Then H is not connected. Thus, we have a contradiction, and it must be the case that Λ is indecomposable. □

Remark. Note that the continua in the set $\widetilde{B}_0$ need not be arcs, since F may not be a diffeomorphism. The components of $\widetilde{B}_0$ could even be hereditarily indecomposable.

Theorem 3.9. *Suppose that F is a plane homeomorphism, Q is a quadrilateral in the plane such that F is a horseshoe on Q, for almost every point q in Q there is some positive integer n_q such that $F^{n_q}(q) \notin Q$, and that F satisfies the strong lockout property on Q, i.e., there are a positive integer integer $N_F, \tau > 0$, and a point $\overline{x}$ in $\mathbb{R}$ such that $F(L(\overline{x})) \subset L^+(\overline{x})$, $Q \subset L^-(\overline{x})$, $d(F(L(\overline{x})), L(\overline{x})) > \tau$, and if $q \in Q$ and $F(q)$ is not in Q, then $F^{N_F}(q)$ is in $L^+(\overline{x})$. There is $\varepsilon > 0$ such that if for each negative integer j, F_j is a homeomorphism on $\mathbb{R}^2$ such that $|F(x) - F_j(x)| < \varepsilon$ for each x in $\mathbb{R}^2$, then the sequence $Q, \widetilde{F}_{-1,-1}(Q), \widetilde{F}_{-1,-2}(Q), \ldots$ of continua in $\overline{\mathbb{R}^2}$ converges in the Hausdorff metric to a unique limit point $\widetilde{B}$. Considered as a subset of $\overline{\mathbb{R}^2}$, $\widetilde{B}$ is a nowhere dense continuum, and it is the closure of the set $\{x_0 \in \mathbb{R}^2 \mid \widetilde{F}^{-1}_{-1,-n}(x_0) \in Q$ for all sufficiently large $n\}$. Furthermore, $\widetilde{B}$ contains an indecomposable continuum Λ, Λ is the largest indecomposable continuum contained in $\widetilde{B}$, and Λ contains the quotient Cantor set Z''' discussed in Theorem* 3.5.

Proof. This proof is just an appropriately modified version of the last proof. Thus we omit the proof. □

Theorem 3.10. *Suppose that F is the fluid flow diffeomorphism, and $i \in \mathbb{Z}$. There is $\varepsilon > 0$ such that if for each integer $j < 0$, F_j is an area preserving diffeomorphism on $\overline{\mathbb{R}^2}$ such that for each $q \in \mathbb{R}^2$,*

$$\left|F(q) - F_j(q)\right| < \varepsilon \quad \text{and} \quad \left|DF(q) - DF_j(q)\right| < \varepsilon,$$

then the Hausdorff limit of the sequence $Q_i, F_{-1}(Q_i), \widetilde{F}_{-1,-2}(Q_i), \ldots$ of continua in $\overline{\mathbb{R}^2}$ is an indecomposable continuum $\widetilde{\Lambda}_i$. Furthermore, $\widetilde{\Lambda}_i\backslash\{\overline{\infty}\}$ is the set $\{x \in \mathbb{R}^2 \mid (\widetilde{F}_{-1,-n})^{-1}(x) \in Q_i$ for all sufficiently large $n\}$, and $\widetilde{\Lambda}_i$ is contained in the boundary $\partial\widetilde{S}^+(x_0)$ of $\widetilde{S}^+(x_0)$.

Proof. Suppose Q_i has sides s_1, s_2, s_3, s_4, with s_2 and s_4 denoting the opposite sides which do not intersect $F(Q_i)$, and s_1 and s_3 denoting the opposite sides that do intersect $F(Q_i)$. As in the last proof, choose $\varepsilon' > 0$ so that if F' is a diffeomorphism on $\overline{\mathbb{R}^2}$, and for each $q \in \mathbb{R}^2$, $|F(q) - F'(q)| < \varepsilon'$ and $|DF(q) - DF_j(q)| < \varepsilon'$, then F'satisfies properties (Fi1)–(Fi4); for almost every $q \in Q_i$, $\lim_{n\to\infty} \widetilde{F}_{n,0}(q) = \overline{\infty}$; and, in particular, the x-coordinate of $\widetilde{F}_{n,0}(q)$ tends to $+\infty$ as $n \to \infty$ (Proposition 3.3). There is $\varepsilon > 0$ such that Theorem 3.7 applies, and $\varepsilon < \varepsilon'$. Then suppose that for each negative integer j, F_j is a diffeomorphism on $\overline{\mathbb{R}^2}$, and for each $q \in \mathbb{R}^2$, $|F(q) - F_j(q)| < \varepsilon$ and $|DF(q) - DF_j(q)| < \varepsilon$. Then $C_{Q_i} = \{q \in Q_i \mid \widetilde{F}_{n,0}(q) \in Q_i$ for all $n \in \widetilde{\mathbb{N}}$ and $(\widetilde{F}_{-1,-n})^{-1}(q)$ for all $n \in \mathbb{N}\}$ is a Cantor set in the interior of Q_i.

As in the last proof, $Q_i \cap (\bigcap_{j<0} \widetilde{F}_{-1,j}(Q_i))$ is a a closed set with uncountably many nowhere dense components. Because of Theorem 3.7, and arguments similar (only simpler, since we only have to consider the forward direction) to those used in the proof of Theorem 3.7, it follows that $\widetilde{B}_0 = Q_i \cap (\bigcap_{j<0} \widetilde{F}_{-1,j}(Q_i))$ is a Cantor set of continua in Q_i, i.e., $\widetilde{\mathcal{B}}_0 = \{C\colon C$ is a component of $\widetilde{B}_0\}$ is a decomposition of $\widetilde{B}_0$, which is a Cantor set with respect to the topology induced by the Hausdorff metric. For each $n > 0$, $\widetilde{B}_n = \bigcap_{j\leqslant -n} \widetilde{F}_{-1,j}(Q_i)$ is a Cantor set of continua in $\widetilde{F}_{-1,-n}(Q_i)$. For each n, $\widetilde{B}_n \subset \widetilde{B}_{n+1}$. Consider $\overline{\bigcup_{n\geqslant 0} \widetilde{B}_n} = \widetilde{B}$. Since $\widetilde{B}_n \subset \widetilde{B}_{n+1}$ for each n, $\widetilde{\Lambda}_i$ is the Hausdorff limit of the sequence $\widetilde{B}_0, \widetilde{B}_1, \widetilde{B}_2, \ldots,$ and $\widetilde{\Lambda}_i$ is also the Hausdorff limit of the sequence $Q_i, \widetilde{F}_{-1,-1}(Q_i), \widetilde{F}_{-1,-2}(Q_i)\ldots.$

We can partition each $\widetilde{B}_n$ into its components: denote this collection of components as $\mathcal{B}_n$. For each point x in $\widetilde{B}_n$, x is contained in some component $R_{x,n}$ of $\widetilde{B}_n$. Then $R_{x,n} \subseteq R_{x,n+1}$ for each x, each n, and $\widetilde{F}_{-1,-n-1} \circ (\widetilde{F}_{-1,n})^{-1}(R_{x,n}) = R_{z,n+1}$, where $z = \widetilde{F}_{-1,-n-1} \circ (\widetilde{F}_{-1,n})^{-1}(x)$. Note that $R_{x,n} \subset \widetilde{B_n} \subset \widetilde{F}_{-1,-n}(Q_i)$. Clearly, R_x is first category in $\widetilde{B}$ and connected. For each $n > 1$, each $R_{x,0}$ is in the interior (relative to the subspace Q_i) of some component of $Q_i \cap (\bigcap_{1\leqslant l\leqslant n} \widetilde{F}_{-1,-l}(Q_i))$, and moreover, the collection $\{Q_i \cap (\bigcap_{1\leqslant l\leqslant n} \widetilde{F}_{-1,-l}(Q_i))\}_{n=1}^{\infty}$ forms a neighborhood base in Q_i for the component $R_{x,0}$. Thus, for each $m \geqslant 0$ and each $n > m$, each $R_{x,m}$ is in the interior (relative to the subspace $\widetilde{F}_{-1,-m}(Q_i)$) of some component $E_{x,m,n}$ of $Q_i \cap (\bigcap_{1\leqslant l\leqslant n} \widetilde{F}_{-1,-l}(Q_i))$ and, moreover, the collection $\{Q_i \cap (\bigcap_{m\leqslant l\leqslant n} \widetilde{F}_{-1,-l}(Q_i))\}_{n=1}^{\infty}$ forms a neighborhood base for the component $R_{x,m}$ of $\widetilde{B_m}$. In fact, if $\xi > 0$, there is some N_0 such that if $n \geqslant N_0$, $x \in \widetilde{B}_0$, then $D_\xi(R_{x,0}) \supseteq Q_i \cap (\bigcap_{1\leqslant l\leqslant n} \widetilde{F}_{-1,-l}(Q_i))$. Thus, for each m, there is some $\xi'_m > 0$ such that for each x in $\widetilde{B}_m$, $(\widetilde{F}_{-1,-m})(D_{\xi'}(R_{z,0})) \subseteq D_\xi(R_{x,m})$, where $z = \widetilde{F}_{-1,-m-1} \circ (\widetilde{F}_{-1,m})^{-1}(x)$. It follows that if x is in some $\widetilde{B}_m$, $\lim_{n\to\infty} \nu(R_{x,m}, E_{x,m,n}) = 0$. Thus, R_x is dense in $\widetilde{B}$.

Further, if $R_x \cap R_y \neq \emptyset$ for some x, y in $\widetilde{B}$, then $R_x = R_y$. Thus, $\mathcal{R} = \{R_x\colon x \in \bigcup_{n\geqslant 0} \widetilde{B}_n\}$ partitions the set $\bigcup_{n\geqslant 0} \widetilde{B}_n$. It follows that $\bigcup_{n\geqslant 0} \widetilde{B}_n$ is connected, that $\widetilde{B}$ is a continuum, and that $\mathcal{R}$ is uncountable. Then, finishing with essentially the same argument as in the proof of Theorem 3.8, it follows that $\widetilde{\Lambda}_i$ is an indecomposable continuum. □

The last proof is just an appropriately modified version of the corresponding proof in the last section. That is also the case with the remaining results in this section, and thus we omit the proofs.

Theorem 3.11 (Intermingling Theorem). *Suppose that F is a diffeomorphism of $\mathbb{R}^2$, $\widetilde{p}_1$ and $\widetilde{p}_2$ are hyperbolic saddle fixed points for F, and F satisfies the following conditions*:

(1) *for $i = 1, 2$, $W^u(\widetilde{p}_i)$ intersects transversally $W^s(\widetilde{p}_i)$ in a point other than $\widetilde{p}_i$,*
(2) *for $i = 1, 2$, $\overline{W^u(\widetilde{p}_i)}$ is nowhere dense in $\mathbb{R}^2$,*
(3) *$\widetilde{p}_1 \notin \overline{W^u(\widetilde{p}_2)}$, but $W^u(\widetilde{p}_1)$ intersects $W^s(\widetilde{p}_2)$ transversally at some point, and*
(4) *for $i = 1, 2$, and for some quadrilateral Q_i containing p_i in its interior, F satisfies the strong lockout property on Q_i.*

Then

(a) *each of $\Lambda_1 = \overline{W^u(\widetilde{p}_1)}$ and $\Lambda_2 = \overline{W^u(\widetilde{p}_2)}$ is an indecomposable continuum and $\Lambda_1 \supset \Lambda_2$, but $\Lambda_1 \neq \Lambda_2$;*
(b) *there is $\varepsilon > 0$ such that if for each integer $j < 0$, F_j is a diffeomorphism on $\overline{\mathbb{R}^2}$ such that for each $q \in \mathbb{R}^2$, $|F(q) - F_j(q)| < \varepsilon$, and $|DF(q) - DF_j(q)| < \varepsilon$, then for some open set o_i containing $\widetilde{p}_i$, the sequence $\overline{o_i}, F_{-1,-1}(\overline{o_i}), F_{-1,-2}(\overline{o_i}), \ldots$ converges in the Hausdorff metric on $\overline{\mathbb{R}^2}$ to a unique limit point $\widetilde{\Lambda}_i$, which is an indecomposable continuum; and*
(c) *for $i = 1, 2$, $\widetilde{\Lambda}_i = \overline{\{q\colon (\widetilde{F}_{-1,-n})^{-1}(q) \in o_i \text{ for all sufficiently large } n\}}$, and $\widetilde{\Lambda}_2 \subset \widetilde{\Lambda}_1$, but $\widetilde{\Lambda}_2 \neq \widetilde{\Lambda}_1$.*

Proof. That $\Lambda_1 = \overline{W^u(\widetilde{p}_1)}$ and $\Lambda_2 = \overline{W^u(\widetilde{p}_2)}$ are indecomposable continua follows from Barge's Theorem, if we note that each branch of each $W^u(\widetilde{p}_i)$ is dense in Λ_i because of the Lambda Lemma. It also follows from the Lambda Lemma that $\Lambda_2 \subset \Lambda_1$. Since $\widetilde{p}_1 \notin \overline{W^u(\widetilde{p}_2)}$, $\Lambda_2 \neq \Lambda_1$.

Applying the Horseshoe Theorem, for $i = 1, 2$, we can find an open set o_i contained in Q_i and containing $\widetilde{p}_i$ such that $\overline{o_i}$ is homeomorphic to $[0,1] \times [0,1]$, and such that for some positive integer N, F^N is a hyperbolic horseshoe on $\overline{o_i}$. Without loss of generality, we can think of $\overline{o_i}$ as being a quadrilateral in the plane. Since F^N is a hyperbolic horseshoe on $\overline{o_i}$, for almost every point x in o_i, there is some positive integer N_x such that $F^{N_x}(x)$ is not in $\overline{o_i}$. Thus, Theorem 3.9 applies to F^N on $\overline{o_i}$, and we can choose ε' so small that Theorem 3.9 is satisfied for F^N on $\overline{o_i}$ relative to ε'. Then choose $\varepsilon < \varepsilon'$ so that for any N diffeomorphisms $\widehat{F}_1, \widehat{F}_2, \ldots, \widehat{F}_N$ satisfying $|F(q) - \widehat{F}_j(q)| < \varepsilon$, and $|DF(q) - D\widehat{F}_j(q)| < \varepsilon$ for each q in $\mathbb{R}^2$, then $|F^N(q) - \widehat{\widehat{F}}_{1,N}(q)| < \varepsilon'$, and $|DF^N(q) - D\widehat{\widehat{F}}_{1,N}(q)| < \varepsilon'$ for each q in $\mathbb{R}^2$. The result follows. □

Corollary 3.12 (Intermingling theorem for the fluid flow map). *Suppose that F is the fluid flow diffeomorphism. Then*

(a) *$\Lambda_i = \overline{W^{u+}(p_i)} = \overline{W^{u-}(p_i)}$ and $\Lambda_{i+1} = \overline{W^{u+}(p_{i+1})} = \overline{W^{u-}(p_{i+1})}$ are indecomposable continua which both contain the point $\overline{\infty}$, but neither intersect any Cyl_j°;*
(b) *Λ_i is contained in the boundary $\partial S^+(x_i)$ of $S^+(x_i)$ and $\Lambda_i \setminus \{\overline{\infty}\} = \{q\colon F^{-n}(q)$ is in Q_i for all sufficiently large $n\}$;*

(c) *there is $\varepsilon > 0$ such that if for each integer $j < 0$, F_j is an area preserving diffeomorphism on $\overline{\mathbb{R}^2}$ such that for each $q \in \mathbb{R}^2$, $|F(q) - F_j(q)| < \varepsilon$ and $|DF(q) - DF_j(q)| < \varepsilon$, then in the Hausdorff metric, the sequence Q_i, $F_{-1}(Q_i)$, $\widetilde{F}_{-1,-2}(Q_i)$, $\ldots$ of continua in $\overline{\mathbb{R}^2}$ has a unique limit point $\widetilde{\Lambda}_i$ which contains $\overline{\infty}$ and is an indecomposable continuum, but does not intersect any Cyl_j°;*

(d) *$\widetilde{\Lambda}_i$ is contained in the boundary $\partial\widetilde{S}^+(x_i)$ of $\widetilde{S}^+(x_i)$ and*

$$\widetilde{\Lambda}_i \backslash \{\overline{\infty}\} = \{q\colon (\widetilde{F}_{-1,-n})^{-1}(q) \text{ is in } Q_i \text{ for all sufficiently large } n\},$$

and

(e) *for $j > i$, $\Lambda_i \supset \Lambda_j$, but $\Lambda_i \neq \Lambda_j$.*

Proposition 3.13. *Suppose that F is a diffeomorphism of $\mathbb{R}^2$, $\widetilde{p}_1$ and $\widetilde{p}_2$ are hyperbolic saddle fixed points for F, there is a connected segment U_1 of the unstable manifold $W^u(\widetilde{p}_1)$ of $\widetilde{p}_1$ and a connected segment S_2 of the stable manifold of $\widetilde{p}_2$ which have the same end points, one component J of $\mathbb{R}^2 \backslash (U_1 \cup S_2)$ contains a fixed point z_0, and another component J' $(\neq J)$ of $\mathbb{R}^2 \backslash (U_1 \cup S_2)$ contains another fixed point z_1. Then each arc from z_0 to z_1 must intersect $\overline{W^u(\widetilde{p}_1)}$.*

Proof. Suppose there is an arc γ from z_0 to z_1 which does not intersect $\overline{W^u(\widetilde{p}_1)}$. Then γ must intersect S_2. Then for each $n > 0$, $F^n(U_1)$ is a connected segment of the unstable manifold $W^u(\widetilde{p}_1)$ of $\widetilde{p}_1$ and $F^n(S_2)$ is a connected segment of the stable manifold $W^s(\widetilde{p}_2)$ of $\widetilde{p}_2$ which have the same end points, and $F^n(J)$ is the component of $\mathbb{R}^2 \backslash (U_1 \cup S_2)$ that contains the fixed point z_0, and $F^n(J')$ is the component of $\mathbb{R}^2 \backslash (F^n(U_1) \cup F^n(S_2))$ contains the fixed point z_1. Then γ must intersect $F^n(S_2)$ for each n, so there is a point $x_n \in F^n(S_2) \cap \gamma$. Since as n increases, the length of $F^n(S_2)$ converges to 0 and the segments $F^n(S_2)$ converge to the point $\widetilde{p}_2$, the sequence $x_1, x_2, \ldots$ converges to $\widetilde{p}_2$. But this means that $\widetilde{p}_2$ is in $\overline{W^u(\widetilde{p}_1)} \cap \gamma$. Thus, we have a contradiction. □

Finally, putting all the fluid flow results together, we have our main fluid flow result.

Theorem 3.14 (Main fluid flow theorem, with and without noise). *Suppose that F is the fluid flow diffeomorphism. There is $\varepsilon > 0$ such that if for each integer $j < 0$, F_j is an area preserving diffeomorphism on $\mathbb{R}^2$ such that for each $q \in \mathbb{R}^2$, $|F(q) - F_j(q)| < \varepsilon$ and $|DF(q) - DF_j(q)| < \varepsilon$, then for each $i \in \mathbb{Z}$,*

(1) *$C_{Q_i} = \{x\colon F^n(x) \in Q_i$ for all $n \in \mathbb{Z}\}$ is an invariant Cantor set in Q_i;*

(2) *$Z_i = \{x_0\colon$ the trajectory $\{x_j\}_{j=-\infty}^{\infty}$ is contained in $Q_i\}$ is a Cantor set in Q_i;*

(3) *$\Lambda_i = \overline{W^u(p_i)}$ is an invariant indecomposable continuum, in the Hausdorff metric the sequence Q_i, $F(Q_i)$, $F^2(Q_i)$, $\ldots$ of continua in $\overline{\mathbb{R}^2}$ has the unique limit point Λ_i, and Λ_i contains $\overline{\infty}$, but does not intersect any Cyl_j°;*

(4) *in the Hausdorff metric, the sequence Q_i, $F_{-1}(Q_i)$, $\widetilde{F}_{-1,-2}(Q_i)$, $\ldots$ of continua in $\overline{\mathbb{R}^2}$ has a unique limit point $\widetilde{\Lambda}_i$ which is an indecomposable continuum and which contains $\overline{\infty}$, but does not intersect any Cyl_j°; and*

(5) *each arc from Cyl_i to Cyl_{i+1} must intersect Λ_i and $\widetilde{\Lambda}_i$.*

Remark 3.1. Thus, in the noisy case, each continuum $\widetilde{\Lambda}_i$ gets "caught around" all subsequent cylinders, as does each continuum Λ_i. It follows that the complement of each $\widetilde{\Lambda}_i$ consists of infinitely many connected components.

It is frequently the case that when p is a hyperbolic saddle fixed point for an area preserving diffeomorphism on a 2-manifold that each branch of the unstable manifold $W^u(p)$ is dense in some open set o containing p. For example, this is the case for the hyperbolic toral automorphisms. In this situation, Barge's Theorem does not hold because condition (B1) is not satisfied. However, in such a case, an indecomposable continuum is still involved, as the next theorem demonstrates. This result has been independently obtained by Marcy Barge [3], who used other techniques to prove it. It may describe the situation when we consider our fluid flow diffeomorphism F, when we restrict F to $\mathbb{R}^2$, to be the lift of a diffeomorphism $\widetilde{F}$ on $S^1 \times \mathbb{R}$, with projection map $\varphi : \mathbb{R}^2 \to S^1 \times \mathbb{R}$ defined so that $\varphi F = \widetilde{F}\varphi$. Then $\widetilde{\Lambda} = \overline{\varphi(\Lambda_0)}$ is a continuum in $S^1 \times \mathbb{R}$, $\varphi(p_0) = \widetilde{p}_0$ is a saddle fixed point for $\widetilde{F}$, and $W^u(\widetilde{p}_0)$ just keeps "wrapping around" itself. Thus, it may well be dense in an open set that contains $\widetilde{p}_0$.

Theorem 3.15. *Suppose that X is a compact 2-manifold (with or without boundary), $F : X \to X$ is a diffeomorphism, and p is a hyperbolic saddle point for F which is not in the boundary ∂X of X (if X has a boundary). Further, suppose that one branch $W^{u+}(p)$ of the unstable manifold of p intersects the stable manifold $W^s(p)$ in a point other than p, but that $W^{u+}(p)$ and $W^s(p)$ do not contain an arc in their intersection. (Homoclinic tangencies are allowed.) Then there is a homeomorphism $h : X \to X$, a continuous surjection $f : X \to X$, and an indecomposable continuum Λ in X such that*

(1) *$fh = Ff$ (that is, h factors over F),*
(2) *$h(\Lambda) = \Lambda$,*
(3) *$f(\Lambda) = \overline{W^{u+}(p)}$, and*
(4) *for $x \in X$, the preimage $f^{-1}(x)$ is either a single point or an arc.*

Proof. Since F is a diffeomorphism, $W^u(p)$ is a smooth curve, and it is a one-to-one image of the reals. Thus, at each point x of $W^u(p)$ there is a unique tangent line L_x and a unique line N_x perpendicular to L_x. There is a one-to-one, continuous function $\alpha : \mathbb{R} \to W^u(p)$ such that $\alpha([0, \infty)) = W^{u+}(p)$. Without loss of generality, assume that $F(W^{u+}(p)) = W^{u+}(p)$ (otherwise replace F by F^2). There is a continuous function $\beta : \mathbb{R} \to (0, 1]$ such that (1) $\beta(0) = 1$, (2) $\beta(x) = \beta(-x)$, (3) $\lim_{x\to\pm\infty} \beta(x) = 0$. (We might take the function $\beta(x) = 1/(1 + x^2)$, for example.) Next "slice" the space X along $W^u(p)$ and at each point $x = \alpha(t_x)$ of $W^u(p)$, insert a closed line segment $\widetilde{N_x}$ into the space X of length $\widetilde{\beta}(t_x) \leqslant \beta(t_x)$ which is centered at x and lies along the line N_x, with the resulting function $\widetilde{\beta} : [0, \infty) \to (0, 1]$ continuous and decreasing. (We assume that these inserted intervals $\widetilde{N_x}$ are all disjoint.) Let $\mathcal{N} = \{\widetilde{N_x} \colon x \in W^u(p)\}$ and let $\mathcal{N}^+ = \{\widetilde{N_x} \colon x \in W^{u+}(p)\}$. There is a continuous one-to-one surjection $\phi : \mathbb{R} \times [-1, 1] \to \bigcup \mathcal{N}$ such that for each (t, ε), $\phi(t, \varepsilon) = \alpha(t)$, and $\phi(\{t\} \times [-1, 1]) = \widetilde{N}_t$. The result of all this is a new space $\widetilde{X}$, which is homeomorphic to X in a natural way, but which has the property that

X is a quotient of $\widetilde{X}$. Precisely, let $\mathcal{M} = \mathcal{N} \cup \{\{x\}\colon x \notin \bigcup \mathcal{N}\}$. Then X is the quotient space $\widetilde{X}/\mathcal{M}$ and the map $\tilde{f}\colon \widetilde{X} \to X$ defined by $\tilde{f}(z) = x$ for each x in $\bigcup \mathcal{N}$, $z \in \widetilde{N_x}$, and $\tilde{f}(x) =$ for $x \notin (\bigcup \mathcal{N})$ is continuous and monotone. Thus, the map $\tilde{f}$ is the quotient map of $\widetilde{X}$ to X. By construction, there is a homeomorphism $g\colon \widetilde{X} \to X$ (and, of course, $g \neq \tilde{f}$). Note that $\phi(\mathbb{R} \times \{1\}) \cap \tilde{f}^{-1}(W^s(p)\backslash\{p\}) \neq \emptyset$.

There is a homeomorphism $h\colon \widetilde{X} \to \widetilde{X}$ induced by F: Define $h(x) = \tilde{f}^{-1} F \tilde{f}(x)$ for $x \notin \bigcup \mathcal{N}$, and for $z = \phi(t, \varepsilon)$, $z \in \widetilde{N_x} \in \mathcal{N}$, define $h(z)$ to be the unique point in $\widetilde{N_{F(x)}} \cap (\phi(\mathbb{R} \times \{\varepsilon\})$ (i.e., $h(\widetilde{N_x}) = \widetilde{N_{F(x)}}$ for $\widetilde{N_x} \in \mathcal{N}$, and $h(\phi(\mathbb{R} \times \{\varepsilon\}) = \phi(\mathbb{R} \times \{\varepsilon\})$ for $\varepsilon \in [-1, 1]$). Thus, h is one-to-one and onto. A little checking reveals that h is also continuous, so that h is a homeomorphism. Further, $\tilde{f} h = F \tilde{f}$.

Finally, we need to prove that $\overline{\phi([0, \infty) \times \{1\})} = \Lambda$ is an indecomposable continuum. Since $\phi([0, \infty) \times \{1\})$ is connected, and X is compact, Λ is a continuum. Now $W^{u+}(p)$ is dense in $\overline{W^{u+}(p)}$, and, because of the hyperbolicity at p, for each t_y in $[0, \infty)$, $\alpha([t_y, \infty))$ is dense in $\overline{W^{u+}(p)}$. Further, for $0 \leqslant s \leqslant t$, $\alpha([s, t])$ is nowhere dense in $\overline{W^{u+}(p)}$. Then $\phi([t_y, \infty) \times \{1\}) \subseteq \tilde{f}^{-1}(W^{u+}(p))$ and is dense in $\overline{\phi([t_y, \infty) \times \{1\})} = \Lambda$, and for $\varepsilon \in [-1, 1]$, $t_y \in [0, \infty)$, $\overline{\phi([t_y, \infty) \times \{\varepsilon\})} \supseteq \overline{\phi([0, \infty) \times \{1\})} = \Lambda$. Let $C = \phi([0, \infty) \times \{1\})$, and for $\varepsilon \in [-1, 1]$, let $C_\varepsilon = \phi([0, \infty) \times \{\varepsilon\})$. Note that $\tilde{f}|C_\varepsilon\colon C_\varepsilon \to W^{u+}(p)$ is one-to-one and onto.

Suppose that Λ is decomposable. Then there exists some proper subcontinuum H of Λ such that H has interior relative to the subspace Λ. There exist 4 numbers $t_1 < t_2 < t_3 < t_4$ such that $\alpha(t_i) = y_i$, $C \cap \tilde{f}^{-1}(y_i) = z_i$, and z_1, z_3 are not in H, while z_2, z_4 are in H.

Suppose that O is an open set in X that contains p, is homeomorphic to an open disk, and is sufficiently small that $W^{u+}(p)$ separates O, as well as any open subset of O that contains p. Then $\tilde{f}^{-1}(O)$ is an open set that contains $\widetilde{N_p}$. There is a positive integer N such that $h^{-N}(z_i)$ is in $\widetilde{f^{-1}(O)}$ for each $1 \leqslant i \leqslant 4$. For $i = 1$, and $i = 3$, there are arcs M_1 and M_3 such that M_i and $\widetilde{N_{h^{-N}(z_i)}}$ intersect in an arc, M_i contains $h^{-N}(z_i)$ in its interior, both endpoints of M_i are in $\bigcup_{|\varepsilon|<1} C_\varepsilon$, and $(M_1 \cup M_3) \cap h^{-N}(H) = \emptyset$. There is some segment $[t_z, t_{z'}]$ in $\mathbb{R}$ such that $\phi([t_z, t_{z'}] \times \{1/2\})$ intersects each M_i and is contained in $\tilde{f}^{-1}(O)$, and there is some t_0 such that $\phi([0, t_0] \times \{1/2\})$ intersects each M_i and is contained in $\tilde{f}^{-1}(O)$. Then $S = M_1 \cup M_3 \cup \phi([t_z, t_{z'}] \times \{1/2\}) \cup \phi([0, t_0] \times \{1/2\})$ separates $\tilde{f}^{-1}(O)$, and one component of $\tilde{f}^{-1}(O)\backslash S$ contains the point $h^{-N}(z_2)$ of $h^{-N}(H)$ but not the point $h^{-N}(z_4)$ of $h^{-N}(H)$. But S does not intersect $h^{-N}(H)$, while it does separate $h^{-N}(H)$ (since $h^{-N}(z_2)$ is "inside" S and $h^{-N}(z_4)$ is "outside" S). Then S separates $h^{-N}(H)$, which is a contradiction to $h^{-N}(H)$ being a subcontinuum of Λ. □

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