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# Induction of Hamiltonian Poisson actions

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*Abstract*: We propose a Poisson–Lie analog of the symplectic induction procedure, using an appropriate Poisson generalization of the reduction of symplectic manifolds with symmetry. Having as basic tools the equivariant momentum maps of Poisson actions, the double group of a Poisson–Lie group and the reduction of Poisson manifolds with symmetry, we show how one can induce a Poisson action admitting an equivariant momentum map. We prove that, under certain conditions, the dressing orbits of a Poisson–Lie group can be obtained by Poisson induction from the dressing orbits of a Poisson–Lie subgroup.

*Keywords*: Poisson–Lie groups, induction of Poisson actions, dressing orbits.

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## 1. Introduction

Poisson manifolds occur as phase spaces in Hamiltonian mechanics and have important applications to the theory of completely integrable systems. This is, in particular, the case of bihamiltonian manifolds, that is manifolds equipped with two Poisson structures $\pi_1$ and $\pi_2$ such that $[\pi_1, \pi_2] = 0$, see [9, 10, 15, 16, 19]. The algebras of observables in quantum mechanics are also relevant to Poisson geometry, as explained in [12].

A Lie group equipped with a Poisson structure such that the corresponding group operation be a Poisson map, is called Poisson–Lie group. This particularly interesting and rich structure has first been studied in [5] and [20] (see also [14] and the monograph [22]). Poisson–Lie groups arise naturally in problems of quantum field theory and integrable systems. For example, a solution of the quantum Yang–Baxter equation defines a "quantum group" in the sense of [6] which, by definition, is a Hopf algebra. Formally, the "classical limit" of a quantum group is a Poisson–Lie group.

On the other hand, there exist integrable systems, as for example the KdV equations, for which Poisson–Lie groups provide a deeper insight. For such systems, the dressing transformation groups play the rôle of "hidden symmetry" groups. According to [20], the dressing transformation group does not in general preserve the Poisson structure on the phase space. Furthermore, it carries a natural Poisson structure defined by the Riemann–Hilbert problem entering the definition of the dressing transformations, and it turns out that this Poisson structure makes the dressing transformation group into a Poisson–Lie group.

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In the same context, the Hamiltonian actions of Poisson–Lie groups have clarified several aspects of the soliton equations. Indeed, the dressing transformations of the soliton equations which admit a Lax representation, are generated by the monodromy matrix [2], which in this case is a momentum mapping in the sense of [13].

Our aim in this paper is to generalize and study the procedure of symplectic induction [11, 7, 8] in the context of Poisson–Lie groups and Poisson manifolds. As we shall explain, this generalization is possible in the following sense: given a Poisson–Lie group $(G, \pi_G)$, a Poisson–Lie subgroup $(H, \pi_H) \hookrightarrow (G, \pi_G)$, a Poisson manifold $(P, \pi_P)$ and a Hamiltonian action $H \times P \to P$ with equivariant momentum mapping $P \to H^*$, one can construct a new Poisson manifold $(P_{\text{ind}}, \pi_{\text{ind}})$ on which the Poisson–Lie group $(G, \pi_G)$ acts in a Hamiltonian way. This statement is our basic result and it is given by Theorem 4.3. As in the symplectic case, an appropriate reduction procedure (for Poisson manifolds now) is needed. This is easily obtained putting together known facts about Poisson reduction [13, 22], see Theorem 2.1. We also need appropriate Poisson generalizations of the natural Hamiltonian actions of a Lie group $G$ and a Lie subgroup $H \subset G$ on the cotangent bundle $T^*G$ from which the induced manifold is constructed [8]. Propositions 3.1 and 3.3 describe these actions in the Poisson setting.

We finally prove that the Poisson induction procedure can be used in order to find Poisson generalizations of the modified cotangent bundles [8] and of the symplectic induction of coadjoint orbits [3, 4].

**Conventions.** If $(P, \pi_P)$ is a Poisson manifold, then $\pi_P^\sharp: T^*P \to TP$ is the map defined by $\alpha(\pi_P^\sharp(\beta)) = \pi_P(\alpha, \beta)\ \forall \alpha, \beta \in T^*P$. Let now $\sigma: G \times P \to P$ (resp. $\sigma: P \times G \to P$) be a left (resp. right) Poisson action of the Poisson–Lie group $(G, \pi_G)$ on $(P, \pi_P)$, and let us denote by $\sigma(X)$ the infinitesimal generator of the action and by $G^*$ the dual group of $G$. Then, we say that $\sigma$ is Hamiltonian if there exists a differentiable map $J: P \to G^*$, called momentum mapping, satisfying the following equation, for each $X \in \mathfrak{g}$:

$$\sigma(X) = \pi_P^\sharp(J^* X^l) \quad (\text{resp. } \sigma(X) = -\pi_P^\sharp(J^* X^r)).$$

In the previous equation $X^l$ (resp. $X^r$) is the left (resp. right) invariant 1-form on $G^*$ whose value at the identity is equal to $X \in \mathfrak{g} \cong (\mathfrak{g}^*)^*$. The momentum mapping is said to be equivariant, if it is a morphism of Poisson manifolds with respect to the Poisson structure $\pi_P$ on $P$ and the canonical Poisson structure on the dual group of the Poisson Lie group $(G, \pi_G)$. Left and right infinitesimal dressing actions $\lambda: \mathfrak{g}^* \to \mathfrak{X}(G)$ and $\rho: \mathfrak{g}^* \to \mathfrak{X}(G)$ of $\mathfrak{g}$ on $G^*$ are defined by

$$\lambda(\xi) = \pi_G^\sharp(\xi^l) \quad \text{and} \quad \rho(\xi) = -\pi_G^\sharp(\xi^r) \qquad \forall \xi \in \mathfrak{g}^*.$$

Similarly, one defines infinitesimal left and right dressing actions of $\mathfrak{g}$ on $G^*$. In the case where the vector fields $\lambda(\xi)$ (or, equivalently, $\rho(\xi)$) are complete for all $\xi \in \mathfrak{g}^*$, we have left and right actions of $(G^*, \pi_{G^*})$ on $(G, \pi_G)$ denoted also by $\lambda$ and $\rho$ respectively, and we say that $(G, \pi_G)$ is a complete Poisson–Lie group.

## 2. Reduction of Poisson manifolds

The reduction of symplectic manifolds with symmetry has been systematically studied in [18]. The importance of this procedure for Hamiltonian dynamics is already very clear as it describes in a unified way several properties of Hamiltonian systems. The Poisson generalization of reduction with symmetry has been carried out in [13] for the special case of a Poisson action of a Poisson–Lie group on a symplectic manifold, admiting a momentum map. On the other hand, reduction of Poisson manifolds with symmetry under the Hamiltonian action of an ordinary Lie group can be found in [22]. Here we will study a somewhat more general situation where a Poisson–Lie group acts in a Hamiltonian way on a Poisson manifold. Before we state the reduction theorem for Poisson manifolds with symmetry, we recall the notion of sub-characteristic distribution. If $(P, \pi_P)$ is a Poisson manifold and $N$ a submanifold of $P$, then we define the sub-characteristic distribution of $N$ as

$$\mathcal{C}N = \pi_P^\sharp((TN)^\circ) \cap TN \tag{2.1}$$

where $(TN)^\circ$ is the annihilator of the tangent bundle $TN$:

$$(T_xN)^\circ = \left\{\alpha \in T_x^*P \;\middle|\; \alpha(v) = 0, \;\; \forall v \in T_xN\right\}.$$

We will deal only with Poisson actions of Poisson–Lie groups admiting equivariant momentum mappings. Although this seems to be a strong condition on the Poisson action, it has been proved ([3]) that, at least for Poisson actions on symplectic manifolds, one can, under reasonable conditions, be reduced to the equivariant case.

**Theorem 2.1.** *Let $(P, \pi_P)$ be a Poisson manifold and $\sigma: G \times P \to P$ a Poisson action of the connected Poisson–Lie group $(G, \pi_G)$ on $(P, \pi_P)$ admiting an equivariant momentum mapping $J: P \to G^*$. Let $u \in G^*$ be an element such that*: (1) *$u$ is a regular value for all the restrictions of $J$ to the symplectic leaves of $P$*; (2) *the submanifold $J^{-1}(u)$ has a clean intersection with the symplectic leaves of $P$. Then, if $G_u$ is the isotropy subgroup of $u$ with respect to the left dressing action of $G$ on $G^*$, the sub-characteristic distribution of $J^{-1}(u)$ defines a regular foliation* (*that is of constant dimension*) *whose leaves are the orbits of $G_u$. Furthermore, if this foliation is defined by a submersion $s: J^{-1}(u) \to P_u$, then the manifold $P_u$ possesses a well-defined Poisson structure whose symplectic distribution is the projection of $\mathcal{S}(P) \cap TJ^{-1}(u)$, where $\mathcal{S}(P)$ is the symplectic distribution of $(P, \pi_P)$.*

**Proof.** We observe that the existence of a momentum mapping for the action $\sigma$, implies that the orbit $G \cdot x$, for each $x \in P$, is contained in the symplectic leaf $S(x)$ through $x$ and for each $x \in J^{-1}(u)$, the orbit $G_u \cdot x$ is contained in $S(x)_u = S(x) \cap J^{-1}(u)$. Furthermore, we have $\pi_P^\sharp(x)\big((T_xJ^{-1}(u))^\circ\big) = T_x(G \cdot x)$ and the submanifold $J^{-1}(u)$ has a clean intersection with the orbits of $G$ in $P$: $T(G \cdot x) \cap TJ^{-1}(u) = T(G_u \cdot x)$. After these remarks, the details of the proof are as in [13] and [22]. $\square$

The reduction described in Theorem 2.1 is called leafwise reduction because the reduced Poisson structure is obtained by reducing each symplectic leaf of $P$ by the standard procedure of symplectic geometry.

## 3. Hamiltonian actions on the double Lie group

Let $G$ be a Lie group and $i: H \hookrightarrow G$ a closed Lie subgroup. We have a right action of $H$ on $G$ given by right multiplication, $(g, h) \mapsto gh \;\; \forall g \in G, h \in H$, and a left action of $G$ on itself given by left multiplication, $(g, g') \mapsto gg' \ \forall g, g' \in G$. The cotangent lifts of these two actions are the basis of the symplectic induction [8] and in the left trivialization $T^*G \cong G \times \mathfrak{g}^*$ they are given by the relations

$$((g,\mu),h) \mapsto (gh, \mathrm{Coad}(h^{-1})\,\mu) \qquad \forall (g,\mu) \in T^*G,\ h \in H, \tag{3.1}$$

$$(g,(g',\mu)) \mapsto (gg',\mu) \qquad \forall g \in G,\ (g',\mu) \in T^*G. \tag{3.2}$$

These actions are Hamiltonian and their equivariant momentum mappings are respectively given by

$$T^*G \ni (g,\mu) \mapsto -i^*\mu \in \mathfrak{h}^* \qquad \forall (g,\mu) \in T^*G, \tag{3.3}$$

$$T^*G \ni (g,\mu) \mapsto \mathrm{Coad}(g)\,\mu \in \mathfrak{g}^* \qquad \forall (g,\mu) \in T^*G \tag{3.4}$$

where $\mathfrak{h}$ is the Lie algebra of $H$ and $i^*\colon \mathfrak{g}^* \to \mathfrak{h}^*$ is the canonical projection. We will generalize in this section the previous Hamiltonian actions in the context of Poisson–Lie groups. This generalization will provide the basis for Poisson induction, as we will see in the sequel.

Let $(G, \pi_G)$ be a connected, simply connected and complete Poisson–Lie group and $i\colon (H, \pi_H) \hookrightarrow (G, \pi_G)$ a closed Poisson–Lie subgroup. Then, if $D(G)$ is the double group of $G$, we find, by [13, Proposition II.36], that the right action $r\colon D(G) \times H \to D(G)$ given by right multiplication

$$r(d,h) = dh \quad \forall d \in D(G),\ h \in H \tag{3.5}$$

is a Poisson action for the symplectic structure $\pi_+$ on $D(G)$ and the Poisson structure $\pi_H$ on $H$. We recall here that in the case we are studying the double group $D(G)$ is gobally isomorphic to the product $G \times G^*$ with the group law given by the relation

$$(g,u)\cdot(h,v) = \big(g\rho_{u^{-1}}(h), \lambda_{h^{-1}}(u)v\big) \qquad \forall (g,u),\ (h,v) \in D. \tag{3.6}$$

Furthermore, there exist two Poisson structures, $\pi_+$ (symplectic) and $\pi_-$ (Poisson–Lie) on $D(G)$ given by

$$\pi_\pm(d) = \tfrac{1}{2}(R_d\pi_0 \pm L_d\pi_0),$$

where $\pi_0 \in \Lambda^2\mathfrak{d}$ is the bivector defined by $\pi_0(\xi_1 + X_1, \xi_2 + X_2) = \xi_1(X_2) - \xi_2(X_1)\ \forall \xi_i + X_i \in \mathfrak{d}^*, i = 1, 2$, see [13] for more details. In the defining equation of $\pi_\pm$, $L_d$ and $R_d$ are the extensions to multivector fields, of left and right multiplication in $D$.

In fact, the right Poisson action given by (3.5) is Hamiltonian:

**Proposition 3.1.** *The right Poisson action given by* (3.5) *is Hamiltonian with equivariant momentum mapping* $J_r\colon D(G) \to H^*$ *which can be taken equal to*

$$J_r = s \circ i^* \circ p_2$$

*where* $s\colon H^* \to H^*$ *is the inversion on the dual group* $H^*$, $i^*\colon G^* \to H^*$ *is the projection of dual groups induced by the inclusion* $i\colon H \hookrightarrow G$, *and* $p_2\colon D(G) \to G^*$ *is the projection onto the second factor.*

**Proof.** The infinitesimal generator of the right action (3.5) is given by the relation $r(Y)(d) = T_eL_d(i_*Y, 0)\ \forall Y \in \mathfrak{h}, d \in D(G)$. Setting now $(J_r^*Y^r)(d) = (\eta_1 + Y_1) \circ T_gL_{g^{-1}} \circ T_dR_{u^{-1}}$, where

$d = gu$, $\eta_1 + Y_1 \in \mathfrak{d}^* \cong \mathfrak{g}^* \oplus \mathfrak{g}$, and $\pi_+^\sharp(J_r^* Y^r)(d) = (J_r^* Y^r)_d^\sharp$, one finds

$$(J_r^* Y^r)_d^\sharp = (T_g R_u \circ T_e L_g)\big[\big(Y_1 - T_g L_{g^{-1}} \lambda(\eta_1)(g)\big) \oplus \big(T_u R_{u^{-1}} \rho(Y_1)(u) - \eta_1\big)\big]. \tag{3.7}$$

The elements $\eta_1$ and $Y_1$ of the previous expression are calculated using the definition of the momentum map $J_r$. One finds $\eta_1 = 0$ and $Y_1 = -i_* \mathrm{Coad}(w^{-1})Y$, $w = J_r(d) = (i^* u)^{-1}$. We proceed by recalling the following useful properties of Poisson–Lie groups [3]:

**Lemma 3.2.** (1) *If* $w = (i^* u)^{-1}$, *then* $i_* \mathrm{Coad}(w^{-1})Y = \mathrm{Coad}(u) i_* Y \;\; \forall u \in G^*, Y \in \mathfrak{h}$.

(2) *The left and right dressing vector fields on the Poisson–Lie group* $(G, \pi_G)$ *are related as follows*:

$$\rho(\mathrm{Coad}(g)\xi)(g) = -\lambda(\xi)(g)$$

*for each* $g \in G$, $\xi \in \mathfrak{g}^*$.

(3) *If the map* $\phi \colon G \times G^* \to D(G)$ *given by* $\phi(g,u) = gu$ *is a global diffeomorphism and* $X \in \mathfrak{g}$, $u \in G^*$, *then*

$$\mathrm{Ad}_{D(G)}(u)(X \oplus 0) = T_e \rho_{u^{-1}}(X) \oplus \big(-T_u R_{u^{-1}} \lambda(X)(u)\big)$$

*where* $\rho_u$ *is the right dressing transformation of* $G^*$ *on* $G$ *and* $\lambda(X)$ *the infinitesimal generator of the left dressing transformation of* $G$ *on* $G^*$.

Replacing now in (3.7) the values of $\eta_1$ and $Y_1$, using Lemma 3.2 and the fact that the tangent at the identity of the dressing transformations equals to the coadjoint representation, we find

$$\begin{aligned} -(J_r^* Y^r)_d^\sharp &= (T_g R_u \circ T_e L_g)\big(\mathrm{Ad}_{D(G)}(u)(i_* Y \oplus 0)\big) \\ &= T_e L_d (i_* Y \oplus 0), \end{aligned}$$

which proves that $J_r$ is indeed an equivariant (because it is a Poisson morphism) momentum map for the right action $r$. □

Using analogous techniques, one can prove the following:

**Proposition 3.3.** *The left action* $l \colon (G, \pi_G) \times (D(G), \pi_+) \to (D(G), \pi_+)$ *given by*

$$l_k(d) = \lambda_u(k \lambda_{u^{-1}}(g)) \cdot u = \lambda_{\rho_{g^{-1}}(u)}(k) g \cdot u \qquad \forall k \in G,\; d = gu \in D(G) \tag{3.8}$$

*is Hamiltonian with equivariant momentum map* $J_l \colon D(G) \to G^*$ *such that*

$$J_l(d) = \rho_{g^{-1}}(u) \qquad \forall d = gu \in D(G). \tag{3.9}$$

In the trivial case where the Poisson structure $\pi_G$ is zero, one has $G^* = \mathfrak{g}^*$ and the dressing transformations of $G^*$ on $G$ are trivial. Furthermore, the dressing transformations of $G$ on $G^*$ reduce to the coadjoint action of $G$ on $\mathfrak{g}^*$ and the group law on $D(G) = G \times \mathfrak{g}^*$ is simply the semi-direct product structure on $T^*G = G \ltimes \mathfrak{g}^*$. Then, the Hamiltonian actions and their equivariant momentum mappings described in Propositions 3.1 and 3.3 reduce to the actions and momentum mappings given by the relations (3.1), (3.3) and (3.2), (3.4) respectively, because $\lambda_u = \mathrm{id} \;\; \forall u \in G^*$.

## 4. Induction of Hamiltonian Poisson actions

We recall first [3] for the reader's convenience some properties of the Hamiltonian actions of Poisson–Lie groups, very useful in what follows.

**Proposition 4.1.** (1) *Let $\sigma\colon P\times G\to P$ be a right Poisson action of the connected, simply connected and complete Poisson–Lie group $(G,\pi_G)$ on the Poisson manifold $(P,\pi_P)$, admiting an equivariant momentum mapping $J\colon P\to G^*$. Then, the map $\tilde\sigma\colon G\times P\to P$ defined as*

$$\tilde\sigma(g,p)=\sigma\big(p,[\lambda_{J(p)}(g)]^{-1}\big)\qquad\forall g\in G,\ p\in P \tag{4.1}$$

*is a left Poisson action. Furthermore, $J$ is an equivariant momentum map for $\tilde\sigma$.*

(2) *Let $\sigma_i\colon G\times P_i\to P_i$, $i=1,2$ be left Poisson actions admiting equivariant momentum mappings $J_i\colon P_i\to G^*$, where $G$ is as previously. Then the map $\sigma\colon G\times P\to P$, $P=P_1\times P_2$ defined by*

$$\sigma(g,p)=\big(\sigma_1(\lambda_{J_2(p_2)}(g),p_1),\sigma_2(g,p_2)\big),\qquad p=(p_1,p_2)\in P \tag{4.2}$$

*is a left Poisson action with respect to the Poisson structure $\pi_P=\pi_1\oplus\pi_2$ on $P$. Furthermore, $J=\tilde m\circ(J_1\times J_2)\colon P\to G^*$ is an equivariant momentum mapping for $\sigma$, where $\tilde m\colon G^*\times G^*\to G^*$ is the group multiplication in $G^*$.*

Consider a Poisson–Lie group $(G,\pi_G)$ and let $i\colon(H,\pi_H)\hookrightarrow(G,\pi_G)$ be a closed Poisson–Lie subgroup. In order to simplify the discussion, we assume that $(G,\pi_G)$ is complete, connected and simply connected, so the double group $D(G)$ of $G$ will be isomorphic to $G\times G^*$ with the group law given by (3.6).

Let $\sigma\colon(H,\pi_H)\times(P,\pi_P)\to(P,\pi_P)$ be a left Hamiltonian action of $(H,\pi_H)$ on the Poisson manifold $(P,\pi_P)$ with equivariant momentum mapping $J\colon P\to H^*$. By Proposition 4.1, we have a left Poisson action $\tilde r\colon(H,\pi_H)\times(D(G),\pi_+)\to(D(G),\pi_+)$ canonically associated to the right Poisson action of Proposition 3.1, and if

$$(\check P,\pi_{\check P})=(P,\pi_P)\times(D(G),\pi_+), \tag{4.3}$$

we also have a left Poisson action $\check\sigma\colon(H,\pi_H)\times(\check P,\pi_{\check P})\to(\check P,\pi_{\check P})$ admiting an equivariant momentum mapping $\check J\colon\check P\to H^*$ given by

$$\check J(p,d)=J(p)\,J_r(d)\qquad\forall(p,d)\in\check P. \tag{4.4}$$

Explicitly, the action $\check\sigma$ is given by

$$\check\sigma_h(p,d)=\big(\sigma(\lambda_{J_r(d)}(h),p),\tilde r_h(d)\big)\qquad\forall(p,d)\in\check P,\ h\in H \tag{4.5}$$

where $\tilde r$ is the left action with

$$\tilde r_h(d)=d\lambda_{J_r(d)}(h)^{-1}\qquad\forall d\in D(G),\ h\in H. \tag{4.6}$$

We now observe that the momentum mapping $\check J\colon\check P\to H^*$ is a submersion, so each element of the dual group $H^*$ is a regular value for $\check J$. In particular, if $e^*$ is the unit of $H^*$, then $\check J^{-1}(e^*)$ is a submanifold of $\check P$. Using the fact that $\pi_+$ is symplectic, we find that the symplectic leaves of

$(\check{P}, \pi_{\check{P}})$ are of the form $S \times D(G)$, where $S$ is a symplectic leaf of $P$. This means that $e^*$ is a regular value for all the restrictions of $\check{J}$ to the symplectic leaves of $\check{P}$.

Next, we consider the intersections of the submanifold $\check{J}^{-1}(e^*)$ with the symplectic leaves of $\check{P}$. Using the expression (4.4) of $\check{J}$ and Proposition 3.1 we find

$$\check{J}^{-1}(e^*) = \{(p, gu) \in \check{P} = P \times D(G) \mid J(p) = i^*(u)\}. \tag{4.7}$$

On the other hand, the symplectic leaf $S(m)$ through $m = (p, d) \in \check{P}$ is equal to $S(m) = S(p) \times D(G)$, and

$$\check{J}^{-1}(e^*) \cap S(m) = \{(p, gu) \in S(p) \times D(G) \mid J(p) = i^*(u)\}. \tag{4.8}$$

We see now that $T_n \check{J}^{-1}(e^*) \cap T_n S(m) = T_n\big(\check{J}^{-1}(e^*) \cap S(m)\big)$ for each point $n \in \check{J}^{-1}(e^*) \cap S(m)$ which confirms that $\check{J}^{-1}(e^*)$ has a clean inersection with the symplectic leaves of $\check{P}$. Furthermore, the isotropy subgroup of $e^*$ with respect to the left dressing transformations of $H$ on $H^*$ is the group $H$ itself, and if we assume that the action of $H$ on $P$ is proper, then all the conditions of Theorem 2.1 are fulfilled. The quotient manifold

$$P_{\text{ind}} = \frac{\check{J}^{-1}(e^*)}{H} \tag{4.9}$$

which by construction is a Poisson manifold, is called induced Poisson manifold. We will denote its Poisson structure as $\pi_{\text{ind}}$.

In order to construct a Poisson action of $(G, \pi_G)$ on $(P_{\text{ind}}, \pi_{\text{ind}})$, we first study some properties of the Poisson actions and their momentum mappings of Propositions 3.1 and 3.3.

**Proposition 4.2.** *Let* $\check{l}: G \times \check{P} \to \check{P}$ *be the action defined by*

$$\check{l}_k(p, d) = (p, l_k(d)) \qquad \forall k \in G,\ (p, d) \in \check{P} \tag{4.10}$$

*where the action* $l: G \times D(G) \to D(G)$ *is given by* (3.8). *Then,* $\check{l}$ *is a Poisson action with equivariant momentum map* $\check{L}: \check{P} \to G^*$ *given by*

$$\check{L}(p, d) = J_l(d). \tag{4.11}$$

*Furthermore, the following identities are valid*:

(1) $J_r \circ l_k = J_r \ \ \forall k \in G$,
(2) $J_l \circ r_h = J_l \ \ \forall h \in H$,
(3) $r_h \circ l_k = l_k \circ r_h \ \ \forall k \in G, h \in H$,
(4) $\check{l}_k \circ \check{\sigma}_h = \check{\sigma}_h \circ \check{l}_k \ \ \forall k \in G, h \in H$.

**Proof.** The fact that $\check{l}$ is a Poisson action is evident. In order to prove that $\check{L}$ defined in (4.11) is a momentum map for $\check{l}$, it is sufficient to apply Proposition 4.1(2) choosing $\sigma_1$ as the trivial action of $G$ on $P$ and $\sigma_2 = l$. In that case, the constant map $P \to G^*$ which to each point associates the identity of $G^*$, is an equivariant momentum map for $\sigma_1$.

Let now $k \in G$, $h \in H$ and $d = gu = u_1 g_1 \in D(G)$. Then

$$(J_r \circ l_k)(d) = J_r\big(\lambda_{\rho_{g^{-1}(u)}}(k) g \cdot u\big) = s(i^*(u)) = J_r(d).$$

We check now relation (2):

$$(J_l \circ r_h)(d) = J_l(dh) = J_l(u_1 g_1 h) = u_1 = J_l(d),$$

because $J_l$ coincides with the projection $p_2^t\colon D(G) \to G^*$ defined by $p_2^t(u_1 g_1) = u_1$ (see [13]). We omit the proof of (3) which is based on similar techniques. We finally check the validity of (4) making use of the commutativity between $r_h$ and $l_k$. □

We now observe that the momentum map $\check{J}$ given by (4.4) is invariant under the action $\check{l}$: $(\check{J} \circ \check{l}_k)(p, d) = J(p)J_r(l_k(d)) = \check{J}(p, d)$ $\forall k \in G$, $(p, d) \in \check{P}$, thanks to relation (1) of Proposition 4.2. Thus, we obtain an action $\check{l}\colon G \times \check{J}^{-1}(e^*) \to \check{J}^{-1}(e^*)$ which commutes with the action $\check{\sigma}$ of $H$ on the submanifold $\check{J}^{-1}(e^*)$ (we recall that $\check{J}$ is equivariant, so we have an action $\check{\sigma}\colon H \times \check{J}^{-1}(e^*) \to \check{J}^{-1}(e^*)$). Consequently, we have a left action $l_{\text{ind}}\colon G \times P_{\text{ind}} \to P_{\text{ind}}$ of $G$ on the induced manifold $P_{\text{ind}}$. We will show that this action is Poisson. To this end, it is more convenient to reformulate the Poisson property of an action in terms of differentiable functions. Thus, using the Lie bracket on the 1-forms on a Poisson manifold $(P, \pi_P)$ and the infinitesimal expression of the Poisson property of an action, one finds that the action $\sigma\colon (G, \pi_G) \times (P, \pi_P) \to (P, \pi_P)$ is Poisson if and only if

$$\sigma(X)\{F, H\} = \{\sigma(X)F, H\} + \{F, \sigma(X)H\} + (\sigma \wedge \sigma)\,\delta(X)(dF \otimes dH) \tag{4.12}$$

for each $X \in \mathfrak{g}$, $F, H \in C^\infty(P)$, where $\delta\colon \mathfrak{g} \to \Lambda^2\mathfrak{g}$ is the linearization of $\pi_G$ at the identity of $G$. In our case, if $s\colon \check{J}^{-1}(e^*) \to P_{\text{ind}}$ is the projection, and $i_e\colon \check{J}^{-1}(e^*) \hookrightarrow \check{P}$ the canonical inclusion, then the following equation is valid

$$s^*\{F, H\} = i_e^*\{\tilde{F}, \tilde{H}\} \tag{4.13}$$

for each $F, H \in C^\infty(P_{\text{ind}})$, where $\tilde{F}$, $\tilde{H}$ are arbitrary local extensions of $s^*F$, $s^*H$ respectively, such that $d\tilde{F}, d\tilde{H}$ vanish on the subcharacteristic distribution $\mathcal{C}\check{J}^{-1}(e^*)$ (see [17, 22]). Taking into account the fact that the infinitesimal generator $l_{\text{ind}}(X)$ is obtained by projection of $\check{l}(X)$ $\forall X \in \mathfrak{g}$, one can write

$$\begin{aligned} s^*(l_{\text{ind}}(X)\{F, H\}) &= \check{l}(X)(s^*\{F, H\}) \\ &= \check{l}(X)(i_e^*\{\tilde{F}, \tilde{H}\}) \\ &= i_e^*\big(\{\check{l}(X)\tilde{F}, \tilde{H}\} + \{\tilde{F}, \check{l}(X)\tilde{H}\} + (\check{l} \wedge \check{l})\,\delta(X)(d\tilde{F} \otimes d\tilde{H})\big) \\ &= s^*\big(\{l_{\text{ind}}(X)F, H\} + \{F, l_{\text{ind}}(X)H\} \\ &\quad + (l_{\text{ind}} \wedge l_{\text{ind}})\,\delta(X)(d\tilde{F} \otimes d\tilde{H})\big), \end{aligned}$$

which confirms our assertion. Note that we used the fact that the function $\check{l}(X)\tilde{F}$ is an extension of $\check{l}(X)\,s^*F = s^*(l_{\text{ind}}(X)F)$ whose differential vanishes on vector fields taking their values in $\mathcal{C}\check{J}^{-1}(e^*)$: $d(\check{l}(X)\tilde{F})(\check{\sigma}(Y)) = \check{\sigma}(Y)\,\check{l}(X)\tilde{F} = \check{l}(X)\,\check{\sigma}(Y)\tilde{F} = 0$, thanks to the commutativity between the actions $\check{l}$ et $\check{\sigma}$ (Proposition 4.2(4)) and to the fact that $d\tilde{F}$ vanishes on $\mathcal{C}\check{J}^{-1}(e^*)$.

Consider now the momentum mapping $\check{L}\colon \check{P} \to G^*$ given by (4.11). By the invariance of the momentum mapping $J_l$ under the action $r$, we easily find that $\check{L}$ is invariant under the action $\check{\sigma}$. Thus, $\check{L}$ projects to a well-defined differentiable map $J_{\text{ind}}\colon P_{\text{ind}} \to G^*$. Then, the defining equation $\check{l}(X) = \pi^\sharp_{\check{P}}(\check{L}^* X^l)$ of $\check{L}$, shows clearly that we also have $l_{\text{ind}}(X) = \pi^\sharp_{\text{ind}}(J^*_{\text{ind}} X^l)$ $\forall X \in \mathfrak{g}$, which means that $J_{\text{ind}}$ is an equivariant momentum map for $l_{\text{ind}}$. We have proved:

**Theorem 4.3.** *Let $(G, \pi_G)$ be a Poisson–Lie group, $(H, \pi_H)$ a closed Poisson–Lie subgroup of $(G, \pi_G)$ and $\sigma: (H, \pi_H) \times (P, \pi_P) \to (P, \pi_P)$ a proper left Poisson action on the Poisson manifold $(P, \pi_P)$, admiting the equivariant momentum map $J: P \to H^*$. If $(G, \pi_G)$ is complete, connected and simply connected, then there exists a Poisson manifold $(P_{\text{ind}}, \pi_{\text{ind}})$, obtained in* (4.9) *by reduction through the momentum mapping given by* (4.4)*, and a left Poisson action $l_{\text{ind}}: (G, \pi_G) \times (P_{\text{ind}}, \pi_{\text{ind}}) \to (P_{\text{ind}}, \pi_{\text{ind}})$, induced by the action* (4.10)*, admiting an equivariant momentum map given by* (4.11)*. The manifold $(P_{\text{ind}}, \pi_{\text{ind}})$ is called induced Poisson manifold.*

## Examples

**Poisson induction from a point.** We consider the case where the Poisson manifold $(P, \pi_P)$ is a point with the zero Poisson structure: $P = \{\text{point}\}$ and the Poisson action of $(H, \pi_H)$ is trivial with the momentum mapping $J: P \to H^*$ given by a fixed element $u_0 \in H^*$: $J(p) = u_0$. The equivariance condition for such a momentum mapping is equivalent to the invariance of $u_0$ under the left dressing transformations of $H$ on $H^*$. Choosing now a Lie group morphism $s^*: H^* \to G^*$ which commutes with left dressing transformations, we obtain a map $j = s^* \circ J: P \to G^*$ and let $j(p) = w_0$. This defines, according to [3], a diffeomorphism $I: \check{J}^{-1}(e^*) \to P \times G \times H^0$ given by $I(p, gu) = (p, guw_0^{-1})$, where $H^\circ \subset G^*$ is the fibre over the identity of the canonical projection $G^* \to H^*$. Under this identification, the induced Poisson manifold is diffeomorphic to the associated bundle $G \times_H H^\circ$, which carries a natural symplectic structure obtained either by Poisson reduction of the symplectic manifold $D(G)$ [3] or by the construction of the symplectic groupoid of the reduced Poisson space $G/H$ [23]. The Poisson induction procedure modifies this natural structure in the following manner. If $Q: D(G) \to D(G)$ is the diffeomorphism given by right multiplication with the element $w_0^{-1}$, then a direct calculation shows that $Q(\pi_+(d)) = \pi_+(Q(d)) + L_d \pi_-(w_0^{-1})$, which means that the Poisson induction from a point, leads to a modification of the canonical symplectic structure of the symplectic groupoid of $G/H$ (or, using the terminology of [3], Poisson cotangent bundle of $G/H$) identified with the associated bundle $G \times_H H^\circ$. Clearly, the modification term vanishes when $u_0 = e^*$, because $\pi_-$ is a Poisson–Lie structure on $D(G)$. This is the exact Poisson analog of the modified cotangent bundle of a homogeneous space $G/H$ [8].

**Poisson induced orbits.** We are placed now in the case where $P = H \cdot v$, the orbit of the element $v \in H^*$ under the right dressing transformations of $H$ on $H^*$. This action is Hamiltonian with momentum mapping given by the inclusion of $P$ in $H^*$. Let $w \in G^*$ be an element of the dual group of $G$ such that $i^* w = v$. We make the assumption that the fibre of $i^*: G^* \to H^*$ over $v$ is contained in the orbit of $w$ under the dressing action of the subgroup $H$

$$wH^\circ \subset H \cdot w. \tag{4.14}$$

The constraint submanifold $\check{J}^{-1}(e^*)$ consists in pairs $(p, gu) \in P \times D(G)$ for which $p = i^* u$ and the action of $H$ on $\check{J}^{-1}(e^*)$ is given by

$$\check{\sigma}_h(p, gu) = \big(\rho_{h^{-1}}(p), gh^{-1}\rho_{h^{-1}}(u)\big),$$

and therefore the equivalence class $[p, gu]$ must be written as $[p, gu] = \rho_{g^{-1}}(u)$. But if we write $p = \rho_{k^{-1}}(v)$, $k \in H$, then $u = \rho_{k^{-1}}(wu^\circ)$, $u^\circ \in H^\circ$, because $H^\circ$ is invariant under the dressing action of $H$. Taking into account the condition (4.14), we find

$$P_{\mathrm{ind}} = G \cdot w$$

that is the orbit of $w \in G^*$ under the right dressing transformations of $G$, is obtained by Poisson induction on the orbit of $v = i^* w \in H^*$ under the dressing transformations of $H$.

**Remark.** The previous example is a Poisson generalization of one of the main results of [4] concerning the geometry of the coadjoint orbits of a semi-direct product. Indeed, it is shown in [4] that each coadjoint orbit of a semi-direct product can be obtained by symplectic induction on a coadjoint orbit of a conveniently choosen subgroup. The symplectic construction, concerning semi-direct products, is obtained from the Poisson one discussed here, if one takes (in the notation of [4]) $G = K \times_\rho V$ and $H = K_p \times_\rho V$, both with the zero Poisson structure. Let us note that for a semi-direct product, the condition (4.14) is satisfied for all the elements $w$.

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