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DISCRETE MATHEMATICS

Research Problems

The purpose of the research problems section is the presentation of unsolved problems in discrete mathematics. Older problems are acceptable if they are not as widely known as they deserve. Problems should be submitted using the format as they appear in the journal and sent to

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Readers wishing to make comments dealing with technical matters about a problem that has appeared should write to the correspondent for that particular problem. Comments of a general nature about previous problems should be sent to Professor Alspach.

Problem 230. Posed by Frank Ruskey.

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Following Knuth [1], define an *extended binary tree* to be a rooted ordered tree in which every node has either two children (internal nodes) or has no children (leaf nodes). Let $\boldsymbol{B}_n$ denote the set of all extended binary trees with n internal nodes (and hence $n+1$ leaves). The cardinality of $\boldsymbol{B}_n$ is the nth Catalan number $\binom{2n}{n}/(n+1)$. For a tree T in $\boldsymbol{B}_n$, let $\beta(T)$ denote the sum over all $\binom{n+1}{2}$ pairs of leaves in T of the lengths of the paths joining the leaves. Now let β_n denote the sum of $\beta(T)$ over all trees in $\boldsymbol{B}_n$. Using recurrence relations and generating functions, it was shown in [2] that

$$\beta_n = n2^{2n-1}.$$

Such a simple expression suggests that there might be a direct combinatorial proof using bijections. The problem is to find such a proof.

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Reference

[1] D.E. Knuth, Fundamental Algorithms (Addison-Wesley, Reading, MA, 1973).
[2] F. Ruskey, On the average length of paths in random binary trees, Congr. Numer.. 34 (1982) 373–380.

Problem 231. Posed by V. Petrovic.

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Let A be a set of $2k$, $k \geqslant 2$, distinct positive integers. It is desired to partition A into two subsets A_0 and A_1 each with cardinality k so that the sum of any $k-1$ elements of A_i is not an element of A_{i-1}, $i=0$ or 1. It is not possible to find such a partition when A is $\{1,3,4,5,6,7\}$ or any of $\{1,2,3,4,5,x\}$, $x \geqslant 7$. Can it be done in all other cases?

Problem 232. Posed by Arunabha Sen.

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A graph $G=(V,E)$ is called *perfect* if G and each of its induced subgraphs has chromatic number equal to the maximum cardinality of its cliques. An *independent* set in G is a set of vertices no two of which are adjacent. Assume that the vertices of G are labelled $1,2,\ldots,n$, where $|V|=n$. An *incidence vector* of an independent set is a vector $\boldsymbol{v}=(v_1,v_2,\ldots,v_n)$ with $v_i=1$ when vertex i belongs to the independent set and $v_i=0$ othewise.

Let $I_1,I_2,\ldots,I_p$ denote the independent sets of G (including the empty set). Define a $2n \times np$ matrix A as follows. If $j=rp+s$, $1 \leqslant s \leqslant p$, $0 \leqslant r \leqslant n-1$, then the jth column of A has a 1 in row $r+1$, 0's in all other rows amongst rows $\{1,2,\ldots,n\}$, and the incidence vector of I_s in rows $\{n+1, n+2,\ldots,2n\}$.

Let constants $C_1,C_2,\ldots,C_n$ satisfying $C_1 \leqslant C_2 \leqslant \cdots \leqslant C_n$ be given. Define c_i, $1 \leqslant i \leqslant np$, by $c_i=(t_i-1)\ C_{\lceil i/p \rceil}$, where t_i is the number of 1's in column i of A. The problem is to minimize $\boldsymbol{cx}$ subject to $A\boldsymbol{x}=\boldsymbol{1}$ and $\boldsymbol{x} \geqslant \boldsymbol{0}$, where $\boldsymbol{1}$ denotes the vector of all 1's and $\boldsymbol{c}=(c_1,c_2,\ldots,c_{np})$.

Conjecture. The above optimization problem has an integral optimal solution.

Problem 233. Posed by Liang Sun.

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The maximum degree and the edge chromatic number of a graph G are denoted by $\Delta(G)$ and $\chi'(G)$, respectively. Let $n>1$ be an integer and $N=\{0,1,\ldots,n-1\}$. Let $S\subset N$ such that $0\notin S$ and $i\in S$ implies $n-i\in S$. The *circulant* graph $G(n,S)$ of order n with *symbol* S has vertices $v_0, v_1, \ldots, v_{n-1}$ and an edge joining v_i and v_j if and only if $j-i\in S$, where the latter calculation is carried out modulo n.

Conjecture. If $G=G(n,S)$ is a connected circulant graph of odd order, then

$$\chi'(G=v_k)=\Delta(G-v_k),$$

where $G-v_k$ denotes the graph obtained from G by removing the vertex v_k and all edges incident with it.

Sun and Xu [2] have proved the conjecture is true when n is an odd prime power. Alspach [1] has asked whether or not every connected Cayley graph with even degree on an abelian group has a Hamilton decomposition. If the answer to Alspach's problem is yes, then the above conjecture is true.

References

[1] B. Alspach, Research Problem 59, Discrete Math. 50 (1984) 115.
[2] L. Sun and J. Xu, The edge chromatic number of two classes of graphs, J. Beijing Inst. Tech. (1992).