Some Categorical Aspects of Fuzzy Topology

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Research in recent years has revealed that the construct of fuzzy topological spaces behaves quite differently from that of topological spaces with respect to certain categorical properties. In this paper we discuss some of these aspects. Since the topological construct \( L\text{-FTS} \) contains nontrivial both initially and finally closed full subconstructs, and each such construct gives rise to a natural autonomous theory of fuzzy topology, it can be said to some extent that fuzzy topology should consist of a system of closely related topology theories, including the classical topology theory as a special case, with each applying to one such subconstruct. Therefore in the first part of this paper the theory of sobriety is established for each finally and initially closed full subconstruct of \( L\text{-FTS} \) to illustrate this idea. The second topic of this paper is the relationship between the construct of stratified \( L\)-fuzzy topological spaces and several other familiar constructs in fuzzy topology, for example, the constructs of Sostak fuzzy topological spaces and \( L\)-fuzzifying topological spaces.

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PRELIMINARIES

After a development period of more than 30 years, fuzzy topology has become rather diverse in its topics as well as its methods. The recent book [17] is a comprehensive treatment of the general topology and lattice theoretic aspects of fuzzy topology. In this paper we deal with some categorical properties of the construct of fuzzy topological spaces.

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Throughout this paper $L$ always denotes a completely distributive lattice if not otherwise stated. Our references for category and categorical topology notions and results are [1, 7, 25]. We refer the reader to the Compendium [5] for lattice ideas, and our reference to basic results about frames is [11]. However, we recall some basic concepts about completely distributive lattices and topological constructs here.

Let $L$ be a complete lattice, $\alpha, \beta \in L$. We say that $\alpha$ is way below (wedge below) $\beta$, in symbols, $\alpha \ll \beta(\alpha \dashv \beta)$ or $\beta \gg (\beta \dashv \alpha)$, if for every directed (arbitrary) subset $D \subseteq L$, $\bigvee D \geq \beta$ implies $\alpha \leq \delta$ for some $\delta \in D$. A complete lattice is said to be continuous (completely distributive) if every element in $L$ is the supremum of all the elements way below (wedge below) it. For equational definitions of continuity and complete distributivity we refer the reader to [5].

By the definition of continuity (complete distributivity) it is easy to see that a complete lattice $L$ is continuous (completely distributive) iff the operator $\text{sup} : \text{Id}(L) \to L$ (sup: $\text{Low}(L) \to L$) taking every ideal (lower set) to its supremum has a left adjoint, where $\text{Id}(L) = \text{Low}(L)$ is the complete lattice of all the ideals (lower sets) in $L$ with respect to the inclusion ordering. Hence the way below (wedge below) relation $\ll (\dashv)$ in a continuous (completely distributive) lattice has the interpolation property, that is, $\alpha \ll \beta(\alpha \dashv \beta)$ implies there is some $\delta \in L$, $\alpha \ll \delta \ll \beta(\alpha \dashv \beta \delta \dashv \beta)$.

It is known [5] that a distributive continuous lattice $L$ is completely distributive iff $L$ has enough coprimes, i.e., the set $C(L)$ of coprimes in $L$ is sup-generating in $L$. It is easy to see that $\alpha \dashv \beta \ll \beta$ implies $\alpha \dashv \beta \delta$ for some $\delta \in D$, and if $\alpha$ is a coprime then $\alpha \ll \beta$ iff $\alpha \dashv \beta$.

Elements in $L^X$ are called $L$-fuzzy sets in $X$, and for every $\alpha \in L$ we also write $\alpha$ to denote the constant $L$-fuzzy set of $X$ with value $\alpha$. For each $\alpha \in L$, $U \subseteq X$, $\alpha \wedge U$ denotes the $L$-fuzzy set taking the value $\alpha$ at $x \in U$ and 0 at $x \notin U$. Such an $L$-fuzzy set is called a one-step function (or sometimes a leveled characteristic function [19]). And for all $\lambda \in L^X$ and $\alpha \in L$, let $\lambda_\alpha = \{x \in X \mid (\lambda(x) \gg \alpha)\}$ and $\lambda_{\geq \alpha} = \{x \in X \mid (\lambda(x) \geq \alpha)\}$, called respectively the strong $\alpha$-cut of $\lambda$ and the $\alpha$-cut of $\lambda$.

A functor $T : A \to B$ is said to be topological provided that every $T$-source $(X \xrightarrow{f_i} T(A_i))_{i \in J}$ has a unique $T$-initial lift $(A \xrightarrow{g_i} A_i)_{i \in J}$. A concrete category over Set is called a construct, and it is called a topological construct if the forgetful functor is topological. A topological construct $A$ is called well-fibered provided that it is fiber-small and provided that on any set of cardinality at most 1 there is exactly one $A$-structure on it.

Suppose $(A, U)$ is a fiber-small topological construct and $\xi, \eta$ are two $A$-structures on a set $X$; we say that $\xi$ is coarser than $\eta$ (in symbols $\xi \leq \eta$)
if \( \text{id}_X : (X, \eta) \rightarrow (X, \xi) \) is continuous. It is easy to see the binary relation \( \leq \) on \( \mathcal{A}(X) \), the \( U \)-fibres of \( X \), is a partial order, and that under this partial order \( \mathcal{A}(X) \) becomes a complete lattice.

An \( L \)-fuzzy topology on a set \( X \) is a subset \( \Delta \subseteq L^X \) which is closed under finite infs and arbitrary sups and contains all the constants, i.e., a stratified Chang–Goguen fuzzy topology. \( L \text{-FTS} \) denotes the construct of \( L \)-fuzzy topological spaces and continuous maps; clearly it is a well-fibered topological construct.

1. LOWEN FUNCTORS AND SUBUNIVERSES OF \( L \text{-FTS} \)

Let \( (X, \mathcal{T}) \) be a crisp topological space and \( \omega_L(\mathcal{T}) \) denote the collection of all the Scott continuous functions (the lower semicontinuous functions) from \( X \) to \( L \); then \( \omega_L(\mathcal{T}) \) is an \( L \)-fuzzy topology on \( X \). In this way we get a functor (indeed, a full embedding) \( \omega_L : \text{Top} \rightarrow L \text{-FTS} \).

\( \omega_L \) has a concrete right adjoint \( \iota_L : L \text{-FTS} \rightarrow \text{Top} \). \( \iota_L \) takes every \( L \)-fuzzy topological space \( (X, \Delta) \) to \( (X, \iota_L(\Delta)) \) where \( \iota_L(\Delta) \) is the coarsest topology on \( X \) making all \( \lambda \in \Delta \) Scott continuous. \( \omega_L \) also has a concrete left adjoint \( \rho_L : L \text{-FTS} \rightarrow \text{Top} \); \( \rho_L \) maps an \( L \)-fuzzy topological space \( (X, \Delta) \) to \( (X, \rho_L(\Delta)) \), where \( \rho_L(\Delta) \) is the finest topology on \( X \) such that \( \omega_L(\rho_L(\Delta)) \subseteq \Delta \).

Therefore \( \omega_L \) embeds \( \text{Top} \) in \( L \text{-FTS} \) as a both concretely reflective and coreflective full subconstruct; this means that \( \omega_L(\text{Top}) \) is closed with respect to final and initial structures in \( L \text{-FTS} \), hence we can identify \( \text{Top} \) with \( \omega_L(\text{Top}) \).

Remark. The functors \( \omega_L, \iota_L \) were first introduced by Lowen [20] in 1976 for \( L = [0, 1] \). In 1987 Liu and Luo [16] generalized them to the general case, i.e., the case in which \( L \) is a completely distributive lattice. The importance of these functors in fuzzy topology was first realized by Lowen [20], and they are called the Lowen functors in the literature.

Recall that \( \text{Top} \) has no simultaneously bireflective and bicoreflective nontrivial subconstruct [12]; this phenomenon sharply distinguishes fuzzy topology and classical topology on the categorical level. In the case \( L = [0, 1] \), Lowen and Wyys [23, 24] proved that \( L \text{-FTS} \) contains many such subconstructs, and a nice characterization and classification theorem of these subconstructs was also presented in their papers. For a general completely distributive lattice \( L \), \( L \text{-FTS} \) also has nontrivial both initially and finally closed full subconstructs other than \( \omega_L(\text{Top}) \); for example, the construct of Lowen spaces in [18] is one of such subconstruct.

A both finally closed and initially closed full subconstruct of \( L \text{-FTS} \) will be called a subuniverse of \( L \text{-FTS} \). In this section we present the characterization of subuniverses of \( L \text{-FTS} \) given in [23].
DEFINITION 1.1 [23]. Given \( \Sigma \subseteq L^L \), an \( L \)-fuzzy topology \( \Delta \) on \( X \) is said to be \( \Sigma \)-stable if for all \( \sigma \in \Sigma \) and \( \lambda \in \Delta \), \( \sigma \lambda \in \Delta \). In that case we shall also say that the space \((X, \Delta)\) is \( \Sigma \)-stable.

\( \text{Stab}(\Sigma) \) stands for the full subconstruct of \( L \)-FTS, the objects of which are all \( \Sigma \)-stable spaces.

A subconstruct \( \mathcal{A} \) is said to be fully stable if there exists a \( \Sigma \) such that \( \text{id}_L \in \Sigma \) and \( \mathcal{A} = \text{Stab}(\Sigma) \).

PROPOSITION 1.2 [8]. Let \( \mathcal{A} \) be a both concretely reflective and concretely coreflective full subconstruct of \( L \)-FTS; then \( \mathcal{A} \) is fully stable.

Proof. Let \( \Delta_0 \) be the fuzzy topology on \( L \) generated by \( \text{id}_L \) and the constants, and let \((L, \Sigma)\) be the \( \mathcal{A} \)-coreflection of \((L, \Delta_0)\); clearly \( \text{id}_L \in \Sigma \) and \( \mathcal{A} = \text{Stab}(\Sigma) \).

It is easy to see that the \( \Sigma \) in the above proposition contains \( \text{id}_L \) and is a composition closed \( L \)-fuzzy topology on \( L \); this means, for all \( \mu, \sigma \in \Sigma \), that we have \( \sigma \mu \in \Sigma \). Such an \( L \)-fuzzy topology on \( L \) is called a total \( L \)-fuzzy topology [23].

EXAMPLES. (1) Let \( \Sigma_L = [L \to L] \) denote the collection of all the Scott continuous functions from \( L \) to itself, i.e., the collection of all the functions which preserve directed sups. Then easily \( \Sigma_L \) is a total \( L \)-fuzzy topology on \( L \). Moreover, \( \Sigma_L = \omega_L(S) \) where \( S \) denotes the Scott topology on \( L \). It is easy to check that \( \omega_L(\text{Top}) = \text{Stab}(\Sigma_L) \) [19, 23].

(2) Suppose \( L \) is a linearly ordered lattice. Let \( \Sigma_N \) be the \( L \)-fuzzy topology on \( L \) generated by \( \{ \sigma \in \Sigma_L | \sigma \leq \text{id}_L \} \cup \{ \text{constants} \} \), then \( \Sigma_N \) is a total \( L \)-fuzzy topology on \( L \) and \( \text{Stab}(\Sigma_N) \) is the construct of Lowen spaces [18]. Particularly when \( L = [0, 1] \), \( \text{Stab}(\Sigma_N) \) is the construct FNS of fuzzy neighbourhood spaces [23].

(3) Let \( \Sigma_0 \) be the \( L \)-fuzzy topology on \( L \) generated as a subbase by \( \{ \text{id}_L \} \cup \{ \text{constants} \} \), then \( \Sigma_0 \) is a total \( L \)-fuzzy topology, the coarsest total fuzzy topology on \( L \), and \( \text{Stab}(\Sigma_0) = L \)-FTS.

LEMMA 1.3. Let \( \Sigma_1, \Sigma_2 \) be two total \( L \)-fuzzy topologies on \( L \) with \( \text{Stab}(\Sigma_1) = \text{Stab}(\Sigma_2) \), then \( \Sigma_1 = \Sigma_2 \).

Proof. The space \((L, \Sigma_1)\) is clearly \( \Sigma_1 \)-stable, hence it is \( \Sigma_2 \)-stable; thus \( \Sigma_2 \subseteq \Sigma_1 \) since \( \text{id}_L \in \Sigma_1 \). Analogously, we have \( \Sigma_1 \subseteq \Sigma_2 \).

PROPOSITION 1.4. Let \( \mathcal{A} \) be a subuniverse of \( L \)-FTS and \( \Sigma \) a total \( L \)-fuzzy topology on \( L \) such that \( \mathcal{A} = \text{Stab}(\Sigma) \); then \( \Sigma \subseteq \Sigma_L = [L \to L] \).

Proof. Since every one-point space is trivially \( \Sigma \)-stable, the subconstruct \( \mathcal{B} = \mathcal{A} \cap \omega_L(\text{Top}) \) is not empty. Clearly \( \mathcal{B} \) is both finally closed and finally closed in \( L \)-FTS. Thus it is a both finally and initially closed full subconstruct of \( \omega_L(\text{Top}) \), therefore \( \mathcal{B} = \omega_L(\text{Top}) \).
Now suppose that \( \sigma \in \Sigma \) is not Scott continuous; then \( (L, [L \to L]) \) is not \( \Sigma \)-stable. But this contradicts that \( (L, [L \to L]) \in \omega_L(\text{Top}) \).

**Lemma 1.5.** Let \( \Sigma \) be a total \( L \)-fuzzy topology on \( L \) coarser than \( [L \to L] \) and let \( \Sigma \) have a base consisting of elements preserving nonempty finite infs and nonempty sups. Then \( \text{Stab}(\Sigma) \) is a subuniverse of \( L\text{-FTS} \).

**Proof.** The concrete coreflectivity of \( \text{Stab}(\Sigma) \) in \( L\text{-FTS} \) can be proved exactly as in [23] for \( L = [0, 1] \).

As for the concrete reflectivity, let \( \Sigma^* \) be a base of \( \Sigma \) consisting of elements preserving nonempty infs and nonempty finite infs; for each \( L \)-fuzzy topological space \( (X, \Delta) \) let \( \Delta^* = \{ \lambda \in \Delta \mid \sigma \lambda \in \Delta \text{ for all } \sigma \in \Sigma \} = \{ \lambda \in \Delta \mid \sigma \lambda \in \Delta \text{ for all } \sigma \in \Sigma^* \} \). Then it is easy to see that \( \Delta^* \) is an \( L \)-fuzzy topology on \( X \) and \( (X, \Delta^*) \) is the \( \text{Stab}(\Sigma) \)-reflection of \( (X, \Delta) \).

If \( L \) is a linearly ordered complete lattice then every Scott continuous function \( L \to L \) preserves nonempty finite infs and nonempty sups; therefore we get the following.

**Theorem 1.6 [23].** If \( L \) is a linearly ordered complete lattice, then the subuniverses of \( L\text{-FTS} \) correspond bijectively to the total \( L \)-fuzzy topologies on \( L \) coarser than \( \Sigma_L = [L \to L] \).

**Remark.** The importance of the existence of nontrivial both initially and finally full subconstructs of \( L\text{-FTS} \) lies in that, as observed by Lowen and Wuyts [24], each such subconstruct of \( \text{FTS} \) gives rise to a perfectly viable and natural autonomous theory of fuzzy topology.” Putting this differently, fuzzy topology should consist of a system of closely related theories of topology, each such theory applying to a subuniverse of \( L\text{-FTS} \). Particularly, the theory of classical topology applies to \( \omega_L(\text{Top}) \) which is identified with \( \text{Top} \). In [37] the second author presented an example to illustrate this idea by introducing soberity for each subuniverse of \( L\text{-FTS} \) in the case in which \( L \) is a linearly ordered complete lattice, and this example will be generalized to the general case in the next section. Thus we can say that for every fixed completely distributive lattice \( L \), \( L \)-fuzzy topology is a generalization of topology, contrary to the related statements in [13, 14].

### 2. \( \Sigma \)-FRAMES

The results in this section are a generalization of those in [37], where \( L \) is assumed to be linearly ordered.

Let \( \Sigma \subseteq [L \to L] \) be a total \( L \)-fuzzy topology on \( L \); a \( \Sigma \)-frame is defined to be a triple \( (A, i_A, (\sigma^A)_{\sigma \in \Sigma}) \), where \( A \) is a frame, \( i_A: L \to A \) is a frame map, and \( (\sigma^A)_{\sigma \in \Sigma} \) is a collection of maps (not necessarily frame maps) \( A \to A \) with the conditions:
(1) If $\sigma$ is the constant map $[\alpha]$ from $L$ to $L$, then $\sigma^A(a) = i_A(\alpha)$ for each $a \in A$, i.e., $\sigma^A$ is the constant map with value $i_A(\alpha)$.

(2) $(\sigma_1 \wedge \sigma_2)^A = \sigma_1^A \wedge \sigma_2^A; (\vee_{i \in I} \sigma_i)^A = \vee_{i \in I} \sigma_i^A$.

A morphism (also called a $\Sigma$-frame map) between $\Sigma$-frames $(A, i_A, (\sigma^A)_{\sigma \in \Sigma})$ and $(B, i_B, (\sigma^B)_{\sigma \in \Sigma})$ is a frame map $f: A \rightarrow B$ such that

(i) $f i_A = i_B$;

(ii) $\sigma^B f = f \sigma^A$ for all $\sigma \in \Sigma$.

The category of $\Sigma$-frames and $\Sigma$-frame maps is denoted $\Sigma\text{ Frm}$.

Given a $\Sigma$-stable space $X$, for each $\sigma \in \Sigma$, $\lambda \in \Omega_L(X)$, the lattice of open sets in $X$, let $\sigma^X(\lambda) = \sigma \lambda$. It is easy to see that in this way $\Omega_L(X)$ becomes a $\Sigma$-frame, and we write $\Omega_\Sigma(X)$ for $(\Omega_L(X), i_X, (\sigma^X)_{\sigma \in \Sigma})$. Thus, we get a functor $\Omega_\Sigma: \text{Stab}(\Sigma) \rightarrow \Sigma\text{ Frm}^\text{op}$.

Conversely, given a $\Sigma$-frame $(A, i_A, (\sigma^A)_{\sigma \in \Sigma})$, define a $\Sigma$-point of $A$ to be a frame map $p: A \rightarrow L$ such that

(i) $p i_A = \text{id}_L$ and

(ii) for each $\sigma \in \Sigma$, $a \in A$, $p(\sigma^A(a)) = \sigma(p(a))$.

Or equivalently, $p$ is a $\Sigma$-frame map from $(A, i_A, (\sigma^A)_{\sigma \in \Sigma})$ to $(L, \text{id}_L, (\sigma^L)_{\sigma \in \Sigma})$, where $\sigma^L = \sigma$ for all $\sigma \in \Sigma$.

Let $\Sigma\text{ pt}A$ denote the set of all the $\Sigma$-points of $A$. For each $a \in A$, define $\Phi_\Sigma(a) \in L^{\Sigma\text{ pt}A}$ by $\Phi_\Sigma(a)(p) = p(a)$. It is routine to verify the following.

**Theorem 2.1.** (1) $\{\Phi_\Sigma(a) | a \in A\}$ is an $L$-fuzzy topology on $\Sigma\text{ pt}A$, and $(\Sigma\text{ pt}A, \Phi_\Sigma(A))$ is a $\Sigma$-stable space.

(2) $\Sigma\text{ pt}$ is a right adjoint of $\Omega_\Sigma$. The unit $\eta_X: X \rightarrow \Sigma\text{ pt}\Omega_\Sigma(X)$ of this adjunction is given by $\eta_X(x)(\lambda) = \lambda(x)$, and the counit $\epsilon_A: A \rightarrow \Omega_\Sigma(\Sigma\text{ pt}A)$ is given by $\epsilon_A(a) = \Phi_\Sigma(a)$.

A $\Sigma$-stable space $X$ is called $\Sigma$-sober if $\eta_X$ is bijective and hence a homeomorphism. A $\Sigma$-frame $A$ is called $\Sigma$-spatial if $\epsilon_A$ is injective and hence a frame isomorphism.

**Theorem 2.2.** (1) For each $\Sigma$-stable space $X$, $\Omega_\Sigma(X)$ is $\Sigma$-spatial.

(2) For each $\Sigma$-frame $A$, $\Sigma\text{ pt}A$ is $\Sigma$-sober.

(3) The category of $\Sigma$-sober spaces is dual to that of $\Sigma$-spatial $\Sigma$-frames.
Examples. (1) Recall that $\Sigma_0$ is the coarsest total $L$-fuzzy topology on $L$, i.e., $\Sigma_0$ is generated by $\{id_L\} \cup \{\text{constants}\}$; we say that the category $\Sigma_0 \text{ Frm}$ is isomorphic to $L \downarrow \text{ Frm}$, the category of $L$-fuzzy frames in [38, 37].

Proof. Given $\sigma \in \Sigma_0$, it is easy to see that there exist $\alpha, \beta \in L$ such that $\sigma = (\alpha \vee id_L) \wedge \beta$. Let $(A, i_A)$ be an $L$-fuzzy frame; define $\sigma^A : A \rightarrow A$ by

$$\sigma^A = ([\alpha]^A \vee id_A) \wedge [\beta]^A$$

where $[\alpha], [\beta]$ are the constant maps $L \rightarrow L$ with values $\alpha, \beta$ respectively. Then clearly $(A, i_A, (\sigma^A)_{\sigma \in \Sigma_0})$ is a $\Sigma_0$-frame, and a frame map $f : A \rightarrow B$ is a morphism in $L \downarrow \text{ Frm}$ iff it is a $\Sigma_0$-frame map. Thus in this way we get an isomorphism from the category $L \downarrow \text{ Frm}$ to that of $\Sigma_0$-frames. \[\square\]

(2) Recall that $\Sigma_L = [L \rightarrow L]$. Given a crisp topological space $X$, let $\Omega_L(X)$ denote the open set lattice of $\omega_L(X)$. For each $\sigma \in \Sigma_L$, define $\sigma^X : \Omega_L(X) \rightarrow \Omega_L(X)$ by $\sigma^X(\lambda) = \sigma\lambda$, then $\Omega_{\Sigma_L}(X) = (\Omega_L(X), i_X, (\sigma^X)_{\sigma \in \Sigma_L})$ becomes a $\Sigma_L$-frame. In this way we get a contravariant functor $\Omega_{\Sigma_L}$ from $\text{ Top}$ to $\Sigma_L \text{ Frm}$.

(3) [30] Let $\Sigma$ be a total $L$-fuzzy topology on $L$, then $(L, \Sigma)$ is $\Sigma$-sober.

Example (1) shows that $\Sigma_0$-soberity coincides with $L$-fuzzy sobriety in [38], and $\Sigma_0$-spatiality coincides with $L$-fuzzy spatiality in [38].

Theorem 2.3. Let $X$ be a crisp space, then $X$ is sober iff $\omega_L(X)$ is $\Sigma_L$-sober.

Proof. Let $\Omega(X)$ denote the open set lattice of $X$ and suppose that $p : \Omega(X) \rightarrow 2$ is a point. Define $p_L : \Omega_{\Sigma_L}(X) \rightarrow L$ by

$$p_L(\lambda) = \bigvee_{\lambda \in L} \alpha \wedge p(\lambda_a).$$

Clearly $p_L$ is a $\Sigma_L$-point of $\Omega_{\Sigma_L}(X)$, and $p_L$ is an extension of $p$.

Conversely, suppose that $p : \Omega_{\Sigma_L}(X) \rightarrow L$ is a $\Sigma_L$-point. We claim at first that for each $U \in \Omega(X) \subset \Omega_{\Sigma_L}(X)$, $p(U) \neq 0$ implies that $p(U) = 1$. Indeed, for each $\alpha \ll 1$ in $L$, the function $\alpha^* : L \rightarrow L$ defined by

$$\alpha^*(\beta) = \begin{cases} 
1, & \alpha \ll \beta, \\
0, & \text{otherwise},
\end{cases}$$

is in $\Sigma_L$, and for each $\lambda \in L^X$, $\alpha^* \lambda = \lambda_a = \{ x \in X | \lambda(x) \gg \alpha \}$. Since $p$ is a $\Sigma_L$-point of $\Omega_{\Sigma_L}(X)$, we get for each $\alpha \ll 1$

$$p(U) = p(U_a) = \alpha^*(p(U)).$$
Thus necessarily \( p(U) = 1 \). Therefore the restriction of \( p \) on \( \Omega(X) \) is a frame map to 2, and \( p \) is a determined by its restriction on \( \Omega(X) \) by the formula

\[
p(\lambda) = \bigvee_{\alpha \in \Lambda} \alpha \wedge p(\lambda_\alpha).
\]

Now it is easy to see that \( X \) is sober iff \( \omega_L(X) \) is \( \Sigma_L \)-sober.

**Example.** Let \( L = [0, 1] \) and \( S \) denote the Scott topology on \( L \), i.e., \( S = \{\{0, 1\}\} \cup \{(t, 1) | t \in [0, 1]\} \). Let \( X = (L, S) \); clearly \( X \) is sober. But as will be shown, \( \omega_L(X) \) is not \( \Sigma_L \)-sober. Write \( \Omega(X) \) for the open-set lattice of \( X \) and \( \Omega_L(X) \) for the open-set lattice of \( \omega_L(X) \). Define \( \xi: \Omega(X) \rightarrow [0, 1] \) by \( \xi(0, 1) = 1 \) and \( \xi((t, 1)) = (1 - t)^2 \) for all \( t \in [0, 1] \) clearly \( \xi \) is a frame map. Now define \( p_L: \Omega_L(X) \rightarrow L \) by

\[
p_L(\lambda) = \bigvee_{\alpha \in \Lambda} \alpha \wedge \xi(\lambda_\alpha),
\]

then \( p_L \in L \cdot \text{Pt}\Omega_L(X) \) and \( p_L \neq \eta_x(x) \) for all \( x \in X \) since

\[
p_L\left(\frac{1}{2} \wedge \left(\frac{1}{2}, 1\right)\right) = \frac{1}{4}
\]

where \( \frac{1}{2} \wedge \left(\frac{1}{2}, 1\right) \) is a Scott continuous function \([0, 1] \rightarrow [0, 1]\) defined by

\[
\frac{1}{2} \wedge \left(\frac{1}{2}, 1\right)(x) = \begin{cases} \frac{1}{2}, & x \in \left(\frac{1}{2}, 1\right), \\ 0, & \text{otherwise.} \end{cases}
\]

Roughly speaking, a topological space is called sober if it can be recovered from the lattice structures of its open-set lattice which is a frame. Clearly, the open-set lattice of a stratified \( L \)-fuzzy topological space \( X \) is an object in the comma category \( L \downarrow \text{Frm} \); and if \( X \) is moreover \( \Sigma \)-stable then the open-set lattice of \( X \) is a \( \Sigma \)-frame. Thus when a \( \Sigma \)-stable space \( X \) is regarded merely as an \( L \)-fuzzy topological space, the extra structure, i.e., the \( \Sigma \)-frame structure, of its open-set lattice is completely ignored. The above example and Theorem 2.3 justify to some extent the necessity of introducing different postulations of sobriety for different subuniverses of \( L \)-\text{FTS}. 

Being aware of the importance of the functor \( \omega_L: \text{Top} \rightarrow L \)-\text{FTS} in fuzzy topology, it is natural to ask whether there exists a natural functor from \text{Frm} to \( L \downarrow \text{Frm} \) which can be regarded as a counterpart of \( \omega_L \). In the following we show that if the way-below relation on \( L \) is multiplicative, i.e., \( \alpha \ll \beta, \alpha \ll \delta \) implies \( \alpha \ll \alpha \wedge \delta \), then such a functor does exist. From now on in this section \( L \) is always assumed to be a completely distributive lattice with a multiplicative way-below relation and \( C(L) \) denotes the set of coprimes in \( L \).
For each frame $A$, let
\[ \omega_L(A) = \{ \gamma : C(L) \to A | \forall \alpha \in C(L), \gamma(\alpha) = \bigvee_{\beta \in C(L), \beta \gg \alpha} \gamma(\beta) \}. \]

It is easily verified that $\omega_L(A)$ is closed with respect to the pointwise infimum of finite elements and the pointwise supremum of arbitrary collections of elements in $AC(L)$; thus $\omega_L(A)$ is a subframe of $AC(L)$.

Define $i_A : L \to \omega_L(A)$ by
\[ i_A(\alpha)(\beta) = \begin{cases} 1, & \beta \ll \alpha, \\ 0, & \text{otherwise.} \end{cases} \]
for all $\alpha \in L$, $\beta \in C(L)$, then $(\omega_L(A), i_A)$ becomes an object in $L \downarrow \text{Frm}$.

**Theorem 2.4.** The correspondence $A \mapsto (\omega_L(A), i_A)$ forms a left adjoint of the forgetful functor $U : L \downarrow \text{Frm} \to \text{Frm}$.

**Proof.** Given a frame $A$, define $\eta_A : A \to \omega_L(A)$ as follows: for all $a \in A$, $\alpha \in C(L)$,
\[ \eta_A(a)(\alpha) = \begin{cases} a, & \alpha \ll 1, \\ 0, & \text{otherwise.} \end{cases} \]
Clearly $\eta_A(a) \in \omega_L(A)$ for each $a \in A$ and $\eta_A$ is a frame map $A \to \omega_L(A)$.

In order to prove our conclusion it suffices to show that for every $L$-fuzzy frame $(B, i_B)$ and every frame map $f : A \to B$ there exists a unique $L$-fuzzy frame map $f^* : (\omega_L(A), i_A) \to (B, i_B)$ such that $f = U(f^*) \eta_A$.

**Existence of $f^*$.** For each $\gamma \in \omega_L(A)$, let
\[ f^*(\gamma) = \bigvee_{\alpha \in C(L)} \eta_B(\alpha) \land f(\gamma(\alpha)), \]
then $f^*$ is the desired $L$-fuzzy frame map, indeed,

(i) $f^*$ is a frame map.

(a) That $f^*$ preserves the bottom and the top elements is trivial by definition.

(b) $f^*$ preserves nonempty sups.
\[
\begin{align*}
  f^*(\bigvee_{i \in T} \gamma_i) &= \bigvee_{a \in C(L)} i_B(\alpha) \land f((\bigvee_{i \in T} \gamma_i)(\alpha)) \\
  &= \bigvee_{a \in C(L)} i_B(\alpha) \land (\bigvee_{i \in T} f(\gamma_i(\alpha))) \\
  &= \bigvee_{i \in T} \bigvee_{a \in C(L)} i_B(\alpha) \land f(\gamma_i(\alpha)) \\
  &= \bigvee_{i \in T} f^*(\gamma_i).
\end{align*}
\]
(c) $f^*$ preserves finite infs.

At first $f^*(\gamma_1 \land \gamma_2) \leq f^*(\gamma_1) \land f^*(\gamma_2)$ is obvious; conversely,

$$f^*(\gamma_1) \land f^*(\gamma_2) = \bigvee_{\alpha \in C(L)} \left[ i_B(\alpha) \land f(\gamma_1(\alpha)) \right] \land \bigvee_{\beta \in C(L)} \left[ i_B(\beta) \land f(\gamma_2(\beta)) \right]$$

$$= \bigvee_{\alpha, \beta \in C(L)} \left[ i_B(\alpha \land \beta) \land (f(\gamma_1(\delta_1)) \land f(\gamma_2(\delta_2))) \right]$$

$$\leq \bigvee_{\delta \in C(L), \delta \gg \alpha, \delta \gg \beta} \left[ i_B(\alpha \land \beta) \land f(\gamma_1(\delta) \land \gamma_2(\delta)) \right]$$

$$= f^*(\gamma_1 \land \gamma_2),$$

therefore $f^*(\gamma_1 \land \gamma_2) = f^*(\gamma_1) \land f^*(\gamma_2)$.

(ii) $f^* i_A = i_B$, indeed, for each $\alpha \in L$,

$$f^* i_A(\alpha) = \bigvee_{\beta \in C(L)} \left[ i_B(\beta) \land f(i_A(\alpha)(\beta)) \right] = i_B(\alpha).$$

The last equality holds since $\bigvee \{ \beta \in C(L)| i_A(\alpha)(\beta) = 1 \} = \alpha$.

(iii) $U(f^*) \eta_A = f$. Indeed, for each $a \in A$,

$$U(f^*) \eta_A(a) = \bigvee_{\alpha \in C(L)} \left[ i_B(\alpha) \land f(\eta_A(a)(\alpha)) \right] = f(a)$$

since $\bigvee \{ \alpha \in C(L)| \eta_A(a)(\alpha) = a \} = 1$.

*Uniqueness of $f^*$. We claim at first that for each $\gamma \in \omega_L(A)$,

$$\gamma = \bigvee_{\alpha \in C(L)} i_A(\alpha) \land \eta_A(\gamma(\alpha)),$$

Indeed, for all $\beta \in C(L),

$$i_A(\alpha)(\beta) \land \eta_A(\gamma(\alpha))(\beta) = \begin{cases} \gamma(\alpha), & \beta \ll \alpha, \\ 0, & \text{otherwise}, \end{cases}$$

thus $i_A(\alpha) \land \eta_A(\gamma(\alpha)) \leq \gamma$. On the other hand,

$$\bigvee_{\alpha \in C(L), \alpha \gg \beta} (i_A(\alpha) \land \eta_A(\gamma(\alpha)))(\beta) = \bigvee_{\alpha \in C(L), \alpha \gg \beta} \gamma(\alpha) = \gamma(\beta),$$

hence the claim holds. Therefore for every $L$-fuzzy frame map $g: (\omega_L(A), i_A) \rightarrow (B, i_B)$ which satisfies the conditions we have

$$g(\gamma) = \bigvee_{\alpha \in C(L)} gi_A(\alpha) \land U(g) \eta_A(\gamma(\alpha)) = \bigvee_{\alpha \in C(L)} i_B(\alpha) \land f(\gamma(\alpha)).$$
(ω_L(A), i_A) is called the free L-fuzzy frame generated by A.

**Corollary 2.5.** Let f : A → B be a frame map, then ω_L(f) : (ω_L(A), i_A) → (ω_L(B), i_B) is defined by ω_L(f)(γ) = fγ.

**Remark.** If the wedge below the relation ⩾ on L is multiplicative, then it can be checked that ω_L(A) is lattice isomorphic to {γ : L → A|γ(α) = ∨_{β ≪ α} γ(β) for each α ∈ L}.

For each σ ∈ Σ_L, define σ^A : ω_L(A) → ω_L(A) as follows:

Case 1. σ(0) = 0, then for all α ∈ C(L), γ ∈ ω_L(A),

σ^A(γ)(α) = ∨{γ(β)|β ∈ C(L), α ≪ σ(β)}.

Case 2. σ(0) = β ≠ 0, then

σ^A(γ)(α) = \begin{cases} 1 ∈ A, & \text{if } α ≪ β, \\ \bigvee \{γ(δ)|δ ∈ C(L), σ(δ) ≪ α\}, & \text{otherwise}. \end{cases}

Then we have the following.

**Proposition 2.6.** (ω_L(A), i_A, (σ^A)_{σ∈Σ_L}) is a Σ_L-frame, and for every frame map f : A → B, ω_L(f) is a Σ_L-frame map. Thus ω_L is a functor from Frm to Σ_LFrm, and it is a full embedding.

**Proof.** Similar to Proposition 4.4 in [37], omitted here.

**Theorem 2.7.** A frame A is spatial iff ω_L(A) is Σ_L-spatial and ω_L(ptA) is homeomorphic to Σ_Lpt(ω_L(A)).

**Proof.** (1) Suppose that A is spatial. We prove that ω_L(A) is Σ_L-spatial, i.e., the map Φ_{Σ_L} : ω_L → Φ_{Σ_L}(ω_L) is injective.

Let γ_1, γ_2 ∈ ω_L(A) be two distinct elements. There exists some α ∈ C(L) such that γ_1(α) ≠ γ_2(α). By the spatiality of A there is a frame map p : A → 2 such that p(γ_1(α)) ≠ p(γ_2(α)), say p(γ_1(α)) = 1, p(γ_2(α)) = 0, for example. Since γ_1(α) = ∨_{β ≪ α, β ∈ C(L)} γ_2(β) there is some δ ∈ C(L) with δ ≫ α and p(γ_1(δ)) = 1. Define p_L : ω_L(A) → L by the formula

p_L(γ) = ∨_{β ∈ C(L)} β ∧ p(γ(β)),

then p_L is a Σ_L-point of ω_L(A).

Case 1. α = δ, hence α ≪ α and

p_L(γ_1) = ∨_{β ∈ C(L)} β ∧ p(γ_1(β)) ≥ α,
while
\[ p_L(\gamma_2) = \bigvee_{\beta \in C(L)} \beta \wedge p(\gamma_2(\beta)) \not\geq \alpha. \]

**Case 2.** \( \alpha \neq \delta \), hence
\[ p_L(\gamma_1) = \bigvee_{\beta \in C(L)} \beta \wedge p(\gamma_1(\beta)) \geq \delta \]

while
\[ p_L(\gamma_2) = \bigvee_{\beta \in C(L)} \beta \wedge p(\gamma_2(\beta)) \not\geq \delta. \]

Thus by a combination of the above two cases \( \omega_L(A) \) is \( \Sigma_L \)-spatial.

(2) Suppose that \( \omega_L(A) \) is \( \Sigma_L \)-spatial; we prove that \( A \) is spatial. This follows from the observation that every \( \Sigma_L \) frame map from \( \omega_L(A) \) to \( L \) is uniquely determined by its restriction on \( A \), and this restriction is a point of \( A \), i.e., a frame map from \( A \) to 2.

(3) Define \( f : \omega_L(pt(A)) \rightarrow \Sigma_L.pt(\omega_L(A)) \) through the formula
\[ f(p)(\gamma) = \bigvee_{\alpha \in C(L)} \alpha \wedge p(\gamma(\alpha)). \]

It is easy to see that \( f \) is a bijection by Proposition 2.6, thus what remains is to prove that both \( f \) and \( f^{-1} \) are continuous. We prove the continuity of \( f \) for an example. Indeed, for each \( \gamma \in \omega_L(A) \), \( p \in pt(A) \),
\[ f^{-1}(\Phi_{\Sigma_L}(\gamma))(p) = \Phi_{\Sigma_L}(\gamma)(f(\gamma)) \]
\[ = f(p)(\gamma) \]
\[ = \bigvee_{\alpha \in C(L)} \alpha \wedge p(\gamma(\alpha)) \]
\[ = \bigvee_{\alpha \in C(L)} \alpha \wedge \Phi_{\Sigma_L}^{-1}(\gamma(\alpha))(p), \]

thus \( f \) is continuous and our conclusion follows. \( \qed \)

Therefore, if the way-below relation on \( L \) is multiplicative we can identify the sobriety of crisp spaces with \( \Sigma_L \)-sobriety and the spatiality of frames with \( \Sigma_L \)-spatiality; thus the classical theory of sobriety of spaces and spatiality of frames is a special case of the theory of \( \Sigma \)-sobriety and \( \Sigma \)-spatiality. Particularly, the following diagram commutes:

\begin{center}
\begin{tikzcd}
\text{Frm} \arrow[r, shift right=1em, \text{pt}] & \text{Top} \\
\omega_L \downarrow \quad & \quad \downarrow \omega_L \\
\Sigma_L \text{ Frm} \arrow[r, shift right=1em, \Sigma_L pt] & \omega_L(\text{Top}).
\end{tikzcd}
\end{center}
Conversely, given a crisp topological space $X$, $\Omega(X)$ denotes its open-set lattice and $\Omega_L(X)$ denotes the open-set lattice of $\omega_L(X)$. For each $\lambda \in \Omega_L(X)$, $\alpha \in C(L)$, let $\lambda^*(\alpha) = \{x \in X | \lambda(x) \gg \alpha\}$; clearly $\lambda^* \in \omega_L(\Omega(X))$.

For each $\gamma \in \omega_L(X)$, let $\gamma_\alpha = \bigvee_{\alpha \in C(L)} \alpha \wedge \gamma(\alpha)$. Recall that $\alpha \wedge \gamma(\alpha) \in L^X$ is defined by $\alpha \wedge \gamma(\alpha)(x) = \alpha$ if $x \in \gamma(\alpha)$ and by $\alpha \wedge \gamma(\alpha)(x) = 0$ if $x \notin \gamma(\alpha)$; then $\gamma_\alpha \in \Omega_L(X)$. It can be easily checked that $(\lambda^*)_\alpha = \lambda$, $(\gamma_\alpha)_\gamma = \gamma$, thus $\omega_L(\Omega(X))$ is lattice isomorphic to $\Omega_L(X)$. Moreover, we have

**Theorem 2.8.** For each $\sigma \in \Sigma_L$, $\sigma^{0(X)}(\gamma) = (\gamma \sigma)^*$; thus as $\Sigma_L$-frames $\omega_L(\Omega(X))$ and $\Omega_L(X)$ are isomorphic to each other. Particularly, the following diagram commutes:

$$
\begin{array}{ccc}
\text{Top} & \xrightarrow{\Omega} & \text{Frm} \\
\omega_L & \downarrow & \omega_L \\
L - \text{FTS} & \xrightarrow{\Omega_L} & L \downarrow \text{Frm}.
\end{array}
$$

For more about the sobriety of $L$-fuzzy topological spaces we refer the reader to [26, 27, 30, 37, 38].

### 3. The Relationship Between $L$-FTS and Other Constructs in Fuzzy Topology

Since the 1970s there have appeared in the literature many categories of a fuzzy topological nature, for example, the construct of (stratified) Šostak fuzzy topological spaces [28, 29], the construct of fuzzifying topological spaces [33, 28], the constructs of fuzzy neighborhood spaces and fuzzy uniform spaces in the sense of Lowen [21, 22], and the construct of super uniform spaces [6].

**Definition 3.1** [28, 33]. An $L$-fuzzifying topology on a set $X$ is a function $\tau: 2^X \rightarrow L$ with the following conditions:

1. $\tau(\varnothing) = \tau(X) = 1$.
2. $\tau(U \cap V) \geq \tau(U) \wedge \tau(V)$.
3. $\tau(\bigcup_{i \in T} U_i) \geq \bigwedge_{i \in T} \tau(U_i)$

A function $f:(X, \tau) \rightarrow (Y, \eta)$ between $L$-fuzzifying topological spaces is called continuous if $\tau(f^{-1}(U)) \geq \eta(U)$ for all $U \subseteq Y$. The construct of $L$-fuzzifying topological spaces is denoted $L$-FYS.

Briefly speaking, an $L$-fuzzifying topology on a set $X$ assigns to every subset of $X$ a degree of being open other than being definitely open or
not. A closely related construct of \textbf{\textit{L-FYS}} is the construct of topological \textit{L-fuzzifying} neighborhood spaces.

**Definition 3.2** [34]. An \textit{L-fuzzifying} neighborhood structure on a set \(X\) is a family of functions \(P = \{p_x: 2^X \rightarrow L | x \in X\}\) such that

\[(LN1) \quad p_x(X) = 1.\]
\[(LN2) \quad p_x(U \cap V) = p_x(U) \land p_x(V).\]
\[(LN3) \quad p_x(U) \neq 0 \text{ implies } x \in U.\]

\((X, P)\) is called an \textit{L-fuzzifying} neighborhood space, and it is called topological if it satisfies moreover

\[(LN4) \quad p_x(U) = \bigvee_{x \in V \subseteq U} p_y(V).\]

It is proved in [34] that \((LN4)\) is equivalent to

\[(LN4') \quad p_x(U) = \bigvee_{x \in V \subseteq U} (p_x(V) \land p_y(U)).\]

A function \(f: (X, P) \rightarrow (Y, Q)\) between \textit{L-fuzzifying} neighborhood spaces is called continuous if for all \(x \in X, U \subseteq Y, p_x(f^{-1}(U)) \geq q_{f(x)}(U).\)

It is proved in [34] that the construct of \textit{L-fuzzifying} topological spaces is concretely isomorphic to that of topological \textit{L-fuzzifying} neighborhood spaces.

**Definition 3.3** [28, 29]. By a Šostak \textit{L-fuzzy} topology on a set \(X\) we mean a function \(\tau: L^X \rightarrow L\) with the following conditions:

\[(1) \quad \tau(0) = \tau(1_X) = 1.\]
\[(2) \quad \tau(\lambda \land \gamma) \geq \tau(\lambda) \land \tau(\gamma).\]
\[(3) \quad \tau(\bigvee_{t \in T} \lambda_t) \geq \bigwedge_{t \in T} \tau(\lambda_t).\]

Continuous functions between Šostak \textit{L-fuzzy} topological spaces are defined similarly to those between \textit{L-fuzzifying} topological spaces. The construct of Šostak \textit{L-fuzzy} topological spaces is denoted \textbf{\textit{SL-FTS}}.

A Šostak \textit{L-fuzzy} topological space \((X, \tau)\) is called stratified if \(\tau(\alpha) = 1\) for every constant fuzzy set \(\alpha \in L^X\). The construct of stratified Šostak \textit{L-fuzzy} topological spaces is denoted \textbf{\textit{SSL-FTS}}.

In [36] the second author analyzed the relationship between the above mentioned constructs by means of a construction called the cotower extension of topological constructs. We recall this construction here.

Let \(C\) be a fiber-small topological construct, and \(L\) a completely distributive lattice. A \textit{cotower} (indexed by \(L\)) in \(C\) is a pair \((X, \Gamma)\), where \(X\) is a set, and \(\Gamma: L \rightarrow C(X)\) is a function from \(L\) to the complete lattice of all
the $C$-structures on $X$ such that for all $\alpha \in L$,

$$\{(X, \Gamma(\alpha)) \xrightarrow{id_X} (X, \Gamma(\beta))\}_{\beta \sim \alpha}$$

is an initial source: A map $f: (X, \Gamma) \to (Y, \Xi)$ between cotowers in $C$ is called continuous if $f: (X, \Gamma(\alpha)) \to (Y, \Xi(\alpha))$ is continuous for every $\alpha \in L$. The construct of all the cotowers indexed by $L$ is denoted $C^c(L)$, called the co-tower extension of $C$.

Cotower extensions are analogous to the tower extensions of topological constructs introduced in [35]. It is routine to verify the following.

**Proposition 3.4.** $C^c(L)$ is topological and the initial structures in $C^c(L)$ are computed levelwise.

Let $F: \mathbf{A} \to \mathbf{B}$ be a functor between two fiber-small topological constructs which preserves initial sources; then by the above proposition $F$ induces a functor $A^c(L) \to B^c(L)$, and this functor still preserves initial sources.

Given a $C$-object $(X, \xi)$, let $\omega_L(\xi): L \to C(X)$ be defined as

$$\omega_L(\xi)(\alpha) = \begin{cases} \xi, & \alpha < 1, \\ \text{the indiscrete structure on } X, & \text{otherwise}. \end{cases}$$

It is easy to see that $\omega_L$ induces a concrete full embedding of $C$ in $C^c(L)$, and we have more.

**Theorem 3.5 [36].** The functor $\omega_L: C \to C^c(L)$ has a concrete left adjoint and a concrete right adjoint. Hence $C$ is a both concretely reflective and concretely coreflective subconstruct of $C^c(L)$.

The major results in [36] about the relationship between several constructs in fuzzy topology are listed in the following examples.

**Examples.** (1) Let $L = [0, 1]$; the topological constructs $\textbf{Top}^c(L)$ have appeared in the literature under different names: the construct $\textbf{FNS}$ of fuzzy neighborhood spaces in Lowen [22, 31]; the construct of probabilistic topological spaces in Brock and Kent [3]; the construct of fuzzifying topological spaces in [33, 39].

(2) If $L = [0, 1]$, then $\textbf{PrTop}^c(L)$ is isomorphic to the construct of fuzzy neighborhood convergence spaces in [2], and it is isomorphic to the construct of probabilistic pretopological spaces in [3, 9].
(3) If $L = [0, 1]$, then the cotower extension of the construct of uniform spaces is isomorphic to the construct of the Lowen fuzzy uniform spaces [4, 21]. Since the canonical functor from the construct of uniform spaces to that of topological spaces preserves initial sources, it induces a functor from the construct of fuzzy uniform spaces to that of fuzzy neighborhood spaces, and this functor coincides with the functor in [21, 22].

(4) [15] The construct of super uniform spaces introduced in [6] is concretely isomorphic to a full subconstruct of the cotower extension of the construct of fuzzy uniform spaces in the sense of Lowen. Hence there exists a natural functor from the construct of super uniform spaces to that of stratified Šostak fuzzy topological spaces since the latter is concretely isomorphic to the co-tower extension of stratified fuzzy topological spaces (see (5) below) and the embedding of $\mathbf{FNS}$ in $\mathbf{FTS}$ preserves initial sources.

(5) The construct of (stratified) Šostak fuzzy topological spaces is concretely isomorphic to the cotower extension of the construct of (stratified) Chang–Goguen spaces [36].

Thus a lot of constructs in fuzzy topology can be expressed as the cotower extension of some simpler constructs. Recently Herrlich and Zhang proved the following.

THEOREM 3.6 [10].

1. If $L$ is an atomic complete Boolean algebra, $L$-$\mathbf{FTS}$ is concretely isomorphic to $\mathbf{Top}^c(L)$.

2. If $L$ is a finite linearly ordered lattice with at least three elements, $L$-$\mathbf{FTS}$ cannot be expressed as the co-tower extension of any topological constructs.

A slight improvement of the proof of the above theorem in [10] yields the following.

PROPOSITION 3.7. If $L$ is a finite distributive lattice, then $L$-$\mathbf{FTS}$ is concretely isomorphic to $\mathbf{Top}^c(L)$ iff $L$ is Boolean.

In the following we discuss the relationship between the construct of $L$-fuzzifying topological spaces and that of $L$-fuzzy topological spaces.
Lemma 3.8 [18]. Let \((\Gamma(\alpha))_{\alpha \in L}\) be a tower of topologies on a set \(X\), then the operator \(\circ : L^X \rightarrow L^X\) defined by
\[
\lambda^\circ = \bigvee_{\alpha \in L} \alpha \land \text{int}_a \lambda_{[\alpha]} = \bigvee_{\alpha \in L} \alpha \land \text{int}_a \lambda_{[\alpha]},
\]
where \(\text{int}_a\) is the interior operator on \(X\) with respect to \(\Gamma(\alpha)\), \(\lambda_a = \{x \in X| \lambda(x) \geq \alpha\}\), and \(\lambda_{[\alpha]} = \{x \in X| \lambda(x) \geq \alpha\}\), satisfies the following conditions:

1. For every \(\alpha \in L\), \(\alpha^\circ = \alpha\);
2. \(\lambda \leq \mu\) implies \(\lambda^\circ \leq \mu^\circ\);
3. \(\lambda^\circ \leq \lambda\);
4. \((\lambda^\circ)^\circ = \lambda^\circ\).

Hence it induces a stratified fuzzy topology on \(X\), denoted \(\delta(\Gamma)\). And an \(L\)-fuzzy set \(\lambda\) is open in \(\delta(\Gamma)\) iff \(\lambda_a \in \Gamma(\alpha)\) for all \(\alpha \in L\).

An operator \(\circ : L^X \rightarrow L^X\) satisfying the above conditions is called a stratified \(L\)-fuzzy interior operator.

By the above lemma it is easy to see that \(\delta\) is functorial from \(\text{Top}^e(L)\) to \(L\text{-FTS}\), and it is a full embedding. Hence the construct of \(L\)-fuzzifying topological spaces can be embedded in \(L\text{-FTS}\) as a full subconstruct and moreover, as will be pointed out below, this embedding is very nice: \(\text{Top}^e(L)\) is concretely reflective and coreflective in \(L\text{-FTS}\), hence it is initially and finally closed in \(L\text{-FTS}\).

Lemma 3.9 [34]. Let \((X, P)\) be a topological \(L\)-fuzzifying neighborhood space, then the operator \(\circ : L^X \rightarrow L^X\) defined by
\[
\lambda^\circ(x) = \bigvee_{\alpha \in L} (\alpha \land p_x(\lambda_{[\alpha]})) = \bigvee_{\alpha \in L} (\alpha \land p_x(\lambda_{[\alpha]}))
\]
is a stratified \(L\)-fuzzy interior operator on \(X\), thus it induces an \(L\)-fuzzy topology on \(X\), denoted \(\gamma(P)\).

Theorem 3.10 [18]. Let \((X, \Delta)\) be a stratified \(L\)-fuzzy topological space. Then the following are equivalent.

1. For all \(\lambda \in \Delta, \alpha \in L, \alpha \land \lambda_a \in \Delta\).
2. \(\Delta\) has a basis consisting of one-step functions, that is to say, the collection of the elements of the form \(\alpha \land U, \alpha \in L, U \subseteq X\), is a basis for \(\Delta\).
(3) $\Delta$ has a subbasis consisting of one-step functions.

(4) There exists a cotower of topologies $(\Gamma(\alpha))_{\alpha \in L}$ on $X$ such that $\Delta = \delta(\Gamma)$.

(5) There exists a topological $L$-fuzzifying neighborhood structure $P$ on $X$ such that $\Delta = \gamma(P)$.

(6) The $L$-fuzzy interior operator corresponding to $\Delta$ satisfies

$$\lambda^\circ = \bigvee_{\alpha \in L} (\lambda \wedge (\lambda_\alpha)^\circ) = \bigvee_{\alpha \in L} (\alpha \wedge (\lambda_\alpha)^\circ).$$

A stratified $L$-fuzzy topological space $(X, \Delta)$ satisfying the above equivalent conditions is called a Lowen space in [18] since when $L = [0, 1]$ these spaces coincide with the fuzzy neighborhood spaces introduced by Lowen [22].

By (3) in the above theorem it is easy to verify the following.

**Theorem 3.11** [34, 18]. $\mathbf{Top}^c(L)$ is concretely reflective and co-reflective in $L$-$\mathbf{FTS}$.

For more about Lowen spaces we refer the reader to [18]. In the case $L = [0, 1]$, the concrete reflectivity and co-reflectivity of $\mathbf{FNS}$ in $\mathbf{FTS}$ were proved in [32].

Thus we have the diagram

$$\mathbf{Top}^{r,c} \rightarrow \mathbf{Top}^c(L) = L$-FYS $\rightarrow L$-$\mathbf{FTS}$ $\rightarrow \mathbf{SSL}$-$\mathbf{FTS} = L$-$\mathbf{FTS}^c(L),$$

where $r, c$ mean respectively concretely reflective and concretely coreflective.

The results in this section show that the construct $L$-$\mathbf{FTS}$ plays a more basic and more important role in fuzzy topology.

**REFERENCES**


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