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On subgroups of minimal topological groups

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Abstract

A topological group is *minimal* if it does not admit a strictly coarser Hausdorff group topology. The *Roelcke uniformity* (or *lower uniformity*) on a topological group is the greatest lower bound of the left and right uniformities. A group is *Roelcke-precompact* if it is precompact with respect to the Roelcke uniformity. Many naturally arising non-Abelian topological groups are Roelcke-precompact and hence have a natural compactification. We use such compactifications to prove that some groups of isometries are minimal. In particular, if $\mathbb{U}_1$ is the Urysohn universal metric space of diameter 1, the group $\mathrm{Iso}(\mathbb{U}_1)$ of all self-isometries of $\mathbb{U}_1$ is Roelcke-precompact, topologically simple and minimal. We also show that every topological group is a subgroup of a minimal topologically simple Roelcke-precompact group of the form $\mathrm{Iso}(M)$, where M is an appropriate non-separable version of the Urysohn space.

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1. Introduction

This paper was motivated by the following questions:

Question 1.1 *(V. Pestov, A. Arhangelskii, 1980's).* What are subgroups of minimal topological groups?

Question 1.2 *(W. Roelcke, 1990).* What are subgroups of lower precompact topological groups?

We now explain and discuss the notions of a minimal group and of a lower precompact group.

Compact spaces X can be characterized among all Tikhonov spaces by each of the following two properties: (1) X is minimal, in the sense that X admits no strictly coarser Tikhonov (or Hausdorff) topology; (2) X is absolutely closed, which means that X is closed in any Tikhonov space Y containing X as a subspace. One can consider the notions of minimality and absolute closedness also for other classes of spaces. For example, for the class of Hausdorff spaces one gets the notions of H-minimal and H-closed spaces which are no longer equivalent to each other or to

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compactness but are closely related: a space is H-minimal iff it is H-closed and semiregular, and a space is compact iff it is H-minimal and satisfies the Urysohn separation axiom. See the survey [35] for a discussion of these notions.

Let us now consider the case of topological groups. All topological groups are assumed to be Hausdorff, unless otherwise explicitly stated. A topological group is *minimal* if it does not admit a strictly coarser Hausdorff group topology.[1] A topological group is *absolutely closed* if it is closed in every topological group containing it as a topological subgroup. A topological group G is absolutely closed if and only if it is *Rajkov-complete*, or *upper complete*, that is complete with respect to the upper uniformity which is defined as the least upper bound $\mathcal{L} \vee \mathcal{R}$ of the left and the right uniformities on G. Recall that the sets $\{(x, y)\colon x^{-1}y \in U\}$, where U runs over a base at unity of G, constitute a base of entourages for the left uniformity $\mathcal{L}$ on G. In the case of the right uniformity $\mathcal{R}$, the condition $x^{-1}y \in U$ is replaced by $yx^{-1} \in U$. We shall call Rajkov-complete groups simply *complete*. The *Rajkov completion* $\widehat{G}$ of a topological group G is the completion of G with respect to the upper uniformity $\mathcal{L} \vee \mathcal{R}$. For every topological group G the space $\widehat{G}$ has a natural structure of a topological group. The group $\widehat{G}$ can be defined as a unique (up to an isomorphism) complete group containing G as a dense subgroup. A group is *Weil-complete* if it is complete with respect to the left uniformity $\mathcal{L}$ (or, equivalently, with respect to the right uniformity $\mathcal{R}$). Every Weil-complete group is complete, but not vice versa.

Unlike the category of Hausdorff spaces, where "minimal" implies "absolutely closed", minimal groups need not be absolutely closed (that is, complete). If G is a minimal group, then its Rajkov completion $\widehat{G}$ also is minimal. On the other hand, if G is a dense subgroup of a minimal group H, then G is minimal if and only if for every closed normal subgroup $N \neq \{1\}$ of H we have $G \cap N \neq \{1\}$ ([3,36,41]; see historical remarks in [7, Section 2.1]). Thus the study of minimal groups can be reduced to the study of complete minimal groups: a group G is minimal if and only if its Rajkov completion $\widehat{G}$ is minimal, and for every closed normal subgroup $N \neq \{1\}$ of $\widehat{G}$ we have $G \cap N \neq \{1\}$. Compact groups are complete minimal, and in the Abelian case the converse is also true, according to a deep theorem of Prodanov and Stoyanov [37,9]: every complete minimal Abelian group is compact. In the non-Abelian case, the class of complete minimal groups properly contains the class of compact groups. There exist non-compact minimal Lie groups [10,39], and actually a discrete infinite group can be minimal [16,25]. It is natural to ask how big the difference is between the class of compact groups and the class of complete minimal groups. For example, one can ask if the class of complete minimal groups is closed under infinite products (this question, to the best of my knowledge, is still open; the answer is positive for groups with a trivial center [21]), or if the relations between cardinal invariants of compact groups remain valid for complete minimal groups, etc.

If G is a topological group, we denote by $\mathcal{N}(G)$ the filter of neighbourhoods of the neutral element. Besides the left, right, and upper uniformities (denoted by $\mathcal{L}$, $\mathcal{R}$, and $\mathcal{L} \vee \mathcal{R}$, respectively), every topological group has yet another compatible uniformity $\mathcal{L} \wedge \mathcal{R}$, the greatest lower bound of $\mathcal{L}$ and $\mathcal{R}$. (Note that in general the greatest lower bound of two compatible uniformities on a topological space need not be compatible with the topology.) If $U \in \mathcal{N}(G)$, the cover $\{UxU\colon x \in G\}$ is $\mathcal{L} \wedge \mathcal{R}$-uniform, and every $\mathcal{L} \wedge \mathcal{R}$-uniform cover of G has a refinement of this form. The uniformity $\mathcal{L} \wedge \mathcal{R}$ is called the *lower uniformity* in [38]; we shall call it the *Roelcke uniformity*, in honour of Walter Roelcke who was the first to introduce and investigate this notion.

A uniform space X is *precompact* if its completion is compact or, equivalently, if for every entourage U the space X can be covered by finitely many U-small sets. A topological group G is *precompact* if one of the following equivalent conditions holds:[2] (1) $(G, \mathcal{L})$ is precompact; (2) $(G, \mathcal{R})$ is precompact; (3) $(G, \mathcal{L} \vee \mathcal{R})$ is precompact; (4) for every $U \in \mathcal{N}(G)$ there exists a finite set $F \subset G$ such that $FU = UF = G$. Every Tikhonov space is a subspace of a compact space, but not every topological group is a subgroup of a compact group: the subgroups of compact groups are precisely precompact groups. Let us say that a topological group G is *Roelcke-precompact* if it is precompact with respect to the Roelcke uniformity $\mathcal{L} \wedge \mathcal{R}$. Thus G is Roelcke-precompact iff for every $U \in \mathcal{N}(G)$ there exists a finite set $F \subset G$ such that $UFU = G$. The *Roelcke completion* of a topological group G is the completion of G with respect to the Roelcke uniformity $\mathcal{L} \wedge \mathcal{R}$. If G is Roelcke-precompact, the Roelcke completion $R(G)$ of G will be also called the *Roelcke compactification*.

Precompact groups are Roelcke-precompact, but not vice versa [38]. For example, the unitary group of a Hilbert space or the group $\mathrm{Sym}(E)$ of all permutations of a discrete set E, both considered with the pointwise convergence

[1] The survey [7] on minimal groups contains a lot of information and more than a hundred references.

[2] Answering a question of Walter Roelcke, I proved that these conditions are also equivalent to: (5) for every $U \in \mathcal{N}(G)$ there exists a finite set $F \subset G$ such that $FUF = G$. This was later rediscovered by S. Solecki and other authors. A short proof can be found in [4, Proposition 4.3].

topology, are Roelcke-precompact but not precompact. While the left, right and upper uniformities of a subgroup of a topological group are induced by the corresponding uniformities of the group, this is not so for the Roelcke uniformity, and a subgroup of a Roelcke-precompact group need not be Roelcke-precompact. This justifies Question 1.2.

The aim of the present paper is to provide a complete answer to Questions 1.1 and 1.2. Let us say that a group G is *topologically simple* if G has no closed normal subgroups besides G and $\{1\}$.

Main Theorem 1.3. *Every topological group G is isomorphic to a subgroup of a complete minimal group which is Roelcke-precompact, topologically simple and has the same weight as G.*

"Isomorphic" means here "isomorphic as a topological group". The *weight* of a topological space X is the cardinal $w(X) = \min\{|\mathcal{B}|: \mathcal{B} \text{ is a base for } X\}$. A group G is *totally minimal* [8] if all Hausdorff quotient groups of G are minimal. Since minimal topologically simple groups are totally minimal, we could write "totally minimal" instead of "minimal" in our Main Theorem.

Let $Q = [0, 1]^\omega$ be the Hilbert cube, and let $\mathrm{Homeo}(Q)$ be the topological group of all self-homeomorphisms of Q. The group $H = \mathrm{Homeo}(Q)$ is universal [44], [27, Theorem 2.2.6], in the sense that every topological group G with a countable base is isomorphic to a topological subgroup of H. Therefore, for groups with a countable base a natural way to prove Theorem 1.3 would be to prove that the group $\mathrm{Homeo}(Q)$, which is known to be simple, is Roelcke-precompact and minimal. I do not know if $\mathrm{Homeo}(Q)$ indeed has these properties:

Problem 1.4. Is the group $\mathrm{Homeo}(Q)$ Roelcke-precompact or minimal?

There is another universal topological group with a countable base, namely the group $\mathrm{Iso}(\mathbb{U})$ of all self-isometries of the Urysohn universal metric space $\mathbb{U}$ [47], [27, Theorem 2.3.1], [50, Theorem 6.1]. The group $\mathrm{Iso}(\mathbb{U})$ is not Roelcke-precompact [50, p. 344]; I do not know whether it is minimal or not.

Problem 1.5. Is the group $\mathrm{Iso}(\mathbb{U})$ minimal?

We consider the "bounded version" $\mathbb{U}_1$ of the space $\mathbb{U}$ and show that the group $\mathrm{Iso}(\mathbb{U}_1)$ is Roelcke-precompact, topologically simple and minimal. This proves Theorem 1.3 for groups with a countable base. For groups of uncountable weight the argument is similar, but we must consider non-separable analogues of the space $\mathbb{U}_1$.

Recall some definitions. A bijection between metric spaces is an *isometry* if it is distance-preserving. For a metric space M we denote by $\mathrm{Iso}(M)$ the topological group of all isometries of M onto itself, equipped with the topology of pointwise convergence (which coincides in this case with the compact-open topology). Let d be the metric on M. The sets of the form $U_{F,\epsilon} = \{g \in \mathrm{Iso}(M): d(g(x), x) < \epsilon \text{ for all } x \in F\}$, where F is a finite subset of M and $\epsilon > 0$, constitute a base at the unity of $\mathrm{Iso}(M)$.

A metric space M is *ω-homogeneous* if every isometry $f : A \to B$ between finite subsets A, B of M can be extended to an isometry of M onto itself. The Urysohn universal metric space $\mathbb{U}$ is the unique (up to an isometry) complete separable metric space which has either of the following two properties: (1) $\mathbb{U}$ is ω-homogeneous and contains an isometric copy of every separable metric space; (2) $\mathbb{U}$ is *finitely injective*: if L is a finite metric space, $K \subset L$ and $f : K \to \mathbb{U}$ is an isometric embedding, then f can be extended to an isometric embedding of L into $\mathbb{U}$. For the equivalence of the conditions (1) and (2), see Proposition 1.6 below (we consider there the bounded version $\mathbb{U}_1$ of $\mathbb{U}$, but the proof for $\mathbb{U}$ is the same). Actually $\mathbb{U}$ is *compactly injective* as well: in the definition of finite injectivity, one can replace finite metric spaces $K \subset L$ by arbitrary compact metric spaces [17], see also [32, Lemma 5.1.19 and Proposition 5.1.20].

We now introduce the bounded version of the space $\mathbb{U}$. The *diameter* of a metric space (M, d) is the number $\sup\{d(x, y): x, y \in M\}$. Let us say that a metric space M is *Urysohn* if its diameter is equal to 1 and it is *finitely injective with respect to spaces of diameter* $\leqslant 1$, that is, the following holds: if L is a finite metric space of diameter $\leqslant 1$, $K \subset L$ and $f : K \to M$ is an isometric embedding, then f can be extended to an isometric embedding of L into M. It suffices if this property holds for $L = K \cup \{p\}$. Thus a metric space M of diameter 1 is Urysohn iff for any finite sequence $x_1, \dots, x_n$ of points of M and any sequence $a_1, \dots, a_n$ of positive numbers $\leqslant 1$ such that $|a_i - a_j| \leqslant d(x_i, x_j) \leqslant a_i + a_j$ $(i, j = 1, \dots, n)$ there exists $y \in M$ such that $d(y, x_i) = a_i$ $(i = 1, \dots, n)$. Using the notion of a Katětov function that will be introduced later in Section 3, we can reformulate this condition as follows:

for every finite $X \subset M$ and every Katětov function $f : X \to [0,1]$ there exists $y \in M$ such that $d(x,y) = f(x)$ for every $x \in X$.

Remark. A notation like Urysohn$_{\leqslant 1}$ might have been more appropriate for what we have called Urysohn (note that the unbounded space $\mathbb{U}$ is not Urysohn according to our definition!). However, we shall use the shorter term, in hope that no confusion will arise. Let us again bring to the reader's attention that *all Urysohn spaces have diameter* 1.

Proposition 1.6. *Let M be a metric space of diameter* 1:

(1) *if M is Urysohn, then M contains an isometric copy of every countable metric space of diameter $\leqslant 1$. If M moreover is complete, then it contains an isometric copy of every separable metric space of diameter $\leqslant 1$*;
(2) *if M is ω-homogeneous and contains an isometric copy of every finite metric space of diameter $\leqslant 1$, then M is Urysohn*;
(3) *if M_1 and M_2 are complete separable Urysohn spaces, then every isometry between finite subsets $A \subset M_1$ and $B \subset M_2$ extends to an isometry between M_1 and M_2*;
(4) *a complete separable metric space of diameter* 1 *is Urysohn if and only if it is ω-homogeneous and contains an isometric copy of every finite metric space of diameter $\leqslant 1$*;
(5) *there exists a unique* (*up to an isometry*) *complete separable Urysohn space $\mathbb{U}_1$. The space $\mathbb{U}_1$ is the unique complete separable metric space of diameter $\leqslant 1$ which is ω-homogeneous and contains an isometric copy of every separable metric space of diameter $\leqslant 1$*;
(6) *there exists a non-complete separable ω-homogeneous Urysohn space which contains an isometric copy of every separable metric space of diameter $\leqslant 1$.*

This is essentially due to Urysohn [52]. The last item was added by Katětov [20], who answered a question of Urysohn that had remained open for more than 60 years.

Proof. (1) is obvious (use induction). To prove (2), suppose that $K \subset L$ are finite metric spaces, $\mathrm{diam}(L) \leqslant 1$, and let $f : K \to M$ be an isometric embedding. Pick an isometric embedding $g : L \to M$, and use ω-homogeneity of M to find an isometry h of M such that h extends the isometry $f(g_{|K})^{-1} : g(K) \to f(K)$. Then $hg : L \to M$ is an isometric embedding that extends f. For (3), enumerate dense countable subsets in M_1 and M_2 and use the "back-and-forth" (or "shuttle") method to extend the given isometry between A and B to an isometry between dense subsets of M_1 and M_2. Then use completeness to obtain an isometry between M_1 and M_2. Applying (3) in the case when $M_1 = M_2$, we see that every complete separable Urysohn space is ω-homogeneous. Thus (4) and uniqueness in (5) follow from (1)–(3). The existence of $\mathbb{U}_1$ is a special case of Theorem 3.2 that we shall prove later; the idea of the proof is due to Katětov. The existence of a non-complete Urysohn space easily follows from Katětov's methods presented in this paper; we refer the reader to [20] for details. □

For the history of invention of the universal Urysohn space, see [1,43,18]. According to Alexandrov [1], Urysohn was thinking about the universal space in the very last days of his life, and, after finishing another project on 14 August 1924, was going to work on two further papers: on metrization of normal spaces with a countable base and on the universal space. He wrote just the first page of the first of these papers. It was dated 17 August 1924, the day of his death.

For more on the Urysohn space, see [12,23,24,27,29,31,32,50,51], and papers in this volume. We mention the striking result of Vershik: for a generic point d of the Polish space of metrics on a countable set X the completion of (X,d) is isometric to the Urysohn space $\mathbb{U}$ [53–55]. Similarly, for a generic shift-invariant metric d on $\mathbf{Z}$ (= the group of integers) the completion of the metric group $(\mathbf{Z}, d)$ is isometric to $\mathbb{U}$ [5].

The proof of Theorem 1.3 consists of two parts. We first prove that every topological group can be embedded in the group $\mathrm{Iso}(M)$ of isometries of a complete ω-homogeneous Urysohn space M, and then prove that such groups of isometries are minimal, Roelcke-precompact and topologically simple.

Theorem 1.7. *For every topological group G there exists a complete ω-homogeneous Urysohn metric space M of the same weight as G such that G is isomorphic to a subgroup of* $\mathrm{Iso}(M)$.

Theorem 1.8. *If M is a complete ω-homogeneous Urysohn metric space, then the group* $\mathrm{Iso}(M)$ *is complete, Roelcke-precompact, minimal and topologically simple. The weight of* $\mathrm{Iso}(M)$ *is equal to the weight of M.*

Theorem 1.3 follows from Theorems 1.7 and 1.8.

The proof of Theorem 1.7 depends on Katětov's construction that leads to a canonical embedding of any metric space M into a finitely injective space. In the non-separable case this construction must be complemented by a construction of a natural embedding of a metric space into an ω-homogeneous space. We use Graev metrics on free groups for this.

The proof of Theorem 1.8 is based on the study of the Roelcke compactifications of groups of isometries. The Roelcke compactifications of some topological groups of importance admit an explicit description and are equipped with additional structures. For example, for the unitary group $U_s(H)$, where H is a Hilbert space and the subscript s indicates the strong operator topology (= the topology inherited from the Tikhonov product H^H), the Roelcke compactification can be identified with the unit ball Θ in the algebra of bounded linear operators on H [46]. The ball Θ is equipped with the weak operator topology. This is the topology inherited from H^H, where each factor H carries the weak topology. Another case when the Roelcke compactification can be explicitly described is the following. Let K be a zero-dimensional compact space such that all non-empty clopen subspaces of K are homeomorphic to K. For example, K might be the Cantor set. Let $G = \mathrm{Homeo}(K)$, equipped with the compact-open topology. Then $R(G)$ is the set of all closed relations R on K (= closed subsets of K^2) such that the domain and the range of R is equal to K [49]. Yet another example of a topological group G for which $R(G)$ is known is the group $G = \mathrm{Homeo}_+[0,1]$ of all orientation-preserving self-homeomorphisms of the closed interval $I = [0,1]$. In that case $R(G)$ can be identified with the closure of the set of graphs of elements of G in the space of closed subsets of the square I^2, see the picture in [27, Example 2.5.4].

The proof of Theorem 1.8 leans on the study of the Roelcke compactification $R(G)$ for $G = \mathrm{Iso}(M)$, where M is a complete ω-homogeneous Urysohn metric space. In this case $R(G)$ can be identified with the space of metric spaces covered by two isometric copies of M, see Sections 6 and 7 below. Equivalently, $\Theta = R(G)$ can be identified with a certain subset of $I^{M\times M}$ that we now are going to specify.

A *semigroup* is a set with an associative binary operation. Let S be a semigroup with the multiplication $(x,y) \mapsto xy$. An element $x \in S$ is an *idempotent* if $x^2 = x$. We say that a self-map $x \mapsto x^*$ of S is an *involution* if $x^{**} = x$ and $(xy)^* = y^*x^*$ for all $x, y \in S$. An element $x \in S$ is *symmetrical* if $x^* = x$, and a subset $A \subset S$ is *symmetrical* if $A^* = A$. An *ordered semigroup* is a semigroup with a partial order $\leqslant$ such that the conditions $x \leqslant x'$ and $y \leqslant y'$ imply $xy \leqslant x'y'$.

Denote by I the closed unit interval $[0,1]$. Let $\uplus$ be the associative operation on I defined by $x \uplus y = \min(x+y, 1)$. Let X be a set, and let $S = I^{X\times X}$ be the set of all functions $f: X^2 \to I$. We make S into an ordered semigroup with an involution. Define an operation $(f,g) \mapsto f \bullet g$ on S by

$$f \bullet g(x,y) = \inf\{f(x,z) \uplus g(z,y):\ z \in X\} \quad (x,y \in X).$$

This operation is associative, since for $f, g, h \in S$ and $x, y \in X$ both $(f \bullet g) \bullet h(x,y)$ and $f \bullet (g \bullet h)(x,y)$ are equal to

$$\inf\{f(x,z) \uplus g(z,u) \uplus h(u,y):\ z, u \in X\}.$$

Define an involution $f \mapsto f^*$ on S by $f^*(x,y) = f(y,x)$.

Let (M,d) be a complete ω-homogeneous Urysohn metric space, and let $G = \mathrm{Iso}(M)$. The Roelcke compactification Θ of G can be identified with a closed subsemigroup of $I^{M\times M}$ and has a natural structure of an ordered semigroup with an involution. Namely, Θ can be viewed as the set of all functions $f \in I^{M\times M}$ which are bi-Katětov in the sense of Definition 6.1. Such functions can be described in terms of the structure of an ordered semigroup with an involution on $I^{M\times M}$: a function $f \in I^{M\times M}$ is bi-Katětov if and only if

$$f \bullet d = d \bullet f = f, \quad f^* \bullet f \geqslant d, \quad f \bullet f^* \geqslant d.$$

The metric d is the unity of Θ, and the constant 1 is a zero element of Θ, in the sense that $f \bullet 1 = 1 \bullet f = 1$ for every $f \in \Theta$ (in fact, for every $f \in I^{M\times M}$).

Note that Θ is a compact space and a semigroup, but it might be misleading to call it a "compact semigroup", since the semigroup operation on Θ is not (even separately) continuous. However, both the topology and the algebraic structure on Θ will play an important role in our proofs.

The Roelcke compactification Θ of $G = \mathrm{Iso}(M)$ is used to prove Theorem 1.8 in the following way. Let $f: G \to G'$ be a surjective morphism of topological groups. To prove that G is minimal and topologically simple, we must prove that either f is a homeomorphism or $|G'| = 1$. Extend f to a map $F: \Theta \to \Theta'$, where Θ' is the Roelcke compactification of G'. Let $S = F^{-1}(e')$, where e' is the unity of G'. Then S is a closed subsemigroup of Θ which is invariant under inner automorphisms. To every closed subsemigroup of Θ an idempotent can be assigned in a canonical way. Let p be the idempotent corresponding to S. Since S is invariant under inner automorphisms, so is p. We show that certain idempotents in Θ are in a one-to-one correspondence with closed subsets of M (Proposition 6.4). Since there are no non-trivial G-invariant closed subsets of M, it follows that p is trivial: it is either the unity of Θ or the constant 1. Accordingly, either f is a homeomorphism or $G' = \{e'\}$.

The same method was used in [46] and [49] to give alternative proofs of Stoyanov's theorem that the unitary group of a Hilbert space is totally minimal and of Gamarnik's theorem that the group of homeomorphisms of the Cantor set is minimal, see Remarks 2 and 3 in Section 9 below.

Under the conditions of Theorem 1.8, the group $\mathrm{Iso}(M)$ has the *fixed point on compacta* (*f.p.c.*) property. This deep result is due to V. Pestov [29–32].[3] A topological group G has the f.p.c. property, or is *extremely amenable*, if every compact G-space has a G-fixed point. As pointed out by Pestov, his theorem, combined with Theorem 1.7 of the present paper, implies that every topological group is a subgroup of an extremely amenable group.

We prove Theorem 1.7 in Section 5 and Theorem 1.8 in Section 8.

Another version of Question 1.1 is the following (see [2, Problem VI.6], [26, Problem 519]): is every topological group a quotient of a minimal topological group? I have earlier announced that the answer is positive. Moreover, I claimed that every topological group is a group retract of a minimal topological group. In other words, for every topological group G there exist a minimal topological group $G' \supseteq G$ and a morphism $r: G' \to G$ such that $r^2 = r$ (it follows that G is a quotient of G'). My announcement appears as Theorem 3.3F.2 in [6]. However, my announcement was premature, and my "proof" contained a gap. A complete proof has been recently found by M. Megrelishvili [22].

Megrelishvili's construction shows that every Weil-complete group is a group retract of a Weil-complete minimal group. This result, combined with the fact that every topological group is a quotient of a Weil-complete group [45],[4] implies that *every topological group is a quotient of a Weil-complete minimal group.* Indeed, given any topological group G, represent G as a quotient of a Weil-complete group G', and then, using Megrelishvili's theorem, represent G' as a group retract (and hence as a quotient) of a Weil-complete minimal group.

2. Invariant pseudometrics on groups

A pseudometric d on a group G is *left-invariant* if $d(xy, xz) = d(y, z)$ for all $x, y, z \in G$. Right-invariant pseudometrics are defined similarly. A pseudometric is *two-sided invariant* if it is left-invariant and right-invariant. Let e be the unity of G. A non-negative real function p on G is a *seminorm* if it satisfies the following conditions: (1) $p(e) = 0$; (2) $p(xy) \leqslant p(x) + p(y)$ for all $x, y \in G$; (3) $p(x^{-1}) = p(x)$ for all $x \in G$. If p is a seminorm on G, define a left-invariant pseudometric d by $d(x, y) = p(x^{-1}y)$. We thus get a one-to-one correspondence between seminorms and left-invariant pseudometrics. Given a left-invariant pseudometric d, the corresponding seminorm p is defined by $p(x) = d(x, e)$. A seminorm p is *invariant* if it is invariant under inner automorphisms, that is $p(yxy^{-1}) = p(x)$ for every $x, y \in G$. Invariant seminorms correspond to two-sided invariant pseudometrics.

Now let G be a topological group. Then the topology of G is determined by the collection of all continuous left-invariant pseudometrics [15, Theorem 8.2]. Equivalently, for every neighbourhood U of unity there exists a continuous seminorm p on G such that the set $\{x \in G: p(x) < 1\}$ is contained in U.

Theorem 2.1. *For every topological group G there exists a metric space M such that $w(G) = w(M)$ and G is isomorphic* (*as a topological group*) *to a subgroup of* $\mathrm{Iso}(M)$.

This theorem has been rediscovered many times by various authors, see [48] and historical remarks in [27,28].

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[3] The setting considered in these papers and books is not exactly the same as in Theorem 1.8 (detailed proofs are given either for the separable case or for unbounded metrics), but, as noted in [29], the same argument works for bounded metrics *verbatim*.

[4] It was proved in [45] that the free topological group of any stratifiable space is Weil-complete. Since every topological space is the image of a stratifiable space under a quotient (even open) map [19], it follows that every topological group is a quotient of a Weil-complete group.

1st proof. There exists a family $D = \{d_\alpha\colon \alpha \in A\}$ of continuous left-invariant pseudometrics on G which determines the topology of G and has the cardinality $|A| = w(G)$. Replacing, if necessary, each $d \in D$ by $\inf(d, 1)$, we may assume that all pseudometrics in D are bounded by 1. For every $\alpha \in A$ let M_α be the metric space associated with the pseudometric space (G, d_α), and let $M = \bigoplus_{\alpha\in A} M_\alpha$ be the disjoint sum of the spaces M_α. There is an obvious metric on M which extends the metric of each M_α: if two points of M are in distinct pieces M_α and M_β, define the distance between them to be 1. The left action of G on itself yields for every $\alpha \in A$ a natural continuous homomorphism $G \to \mathrm{Iso}(M_\alpha)$. The homomorphism $G \to \prod_{\alpha\in A} \mathrm{Iso}(M_\alpha)$ thus obtained is a homeomorphic embedding. It remains to note that the group $\prod_{\alpha\in A} \mathrm{Iso}(M_\alpha)$ can be identified with a topological subgroup of $\mathrm{Iso}(M)$. □

2nd proof. Let B be the Banach space of all bounded real functions on G which are uniformly continuous with respect to the right uniformity. The natural left action of G on B, defined by the formula $gf(h) = f(g^{-1}h)$ $(g, h \in G,\ f \in B)$, yields an isomorphic embedding of G into $\mathrm{Iso}(B)$. The weight of B may exceed the weight of G, but it is easy to find a G-invariant subspace B' of B such that B' determines the topology of G and $w(B') = w(G)$. Then the natural homomorphism $G \to \mathrm{Iso}(B')$ still is a homeomorphic embedding. □

Let us discuss invariant seminorms on free groups. For a set X we denote by $S(X)$ the set of all words of the form $x_1^{\epsilon_1} \ldots x_n^{\epsilon_n}$, where $n \geqslant 0$, $x_i \in X$ and $\epsilon_i = \pm 1$, $1 \leqslant i \leqslant n$. In other words, $S(X)$ is the free monoid[5] on the set $X \cup X^{-1}$, where X^{-1} is a disjoint copy of X. A word $w \in S(X)$ is *irreducible* if it does not contain subwords of the form $x^\epsilon x^{-\epsilon}$. We consider the *free group* $F(X)$ on a set X as the set of all irreducible words in $S(X)$. Every word $w \in S(X)$ represents a uniquely defined element $w' \in F(X)$ which can be obtained from w by consecutive deletion of subwords of the form $x^\epsilon x^{-\epsilon}$. In this situation we say that the words w and w' are *equivalent*. For $u, v \in S(X)$ we denote by $u|v$ the product of u and v in the semigroup $S(X)$, that is the word obtained by writing v after u (without cancelations). If u and v are irreducible, we denote by uv their product in the group $F(X)$, that is the irreducible word equivalent to $u|v$.

Let (X, d) be a metric space. A real function f on X is *non-expanding*, or 1*-Lipschitz*, if $|f(x) - f(y)| \leqslant d(x, y)$ for every $x, y \in X$. Let k be a non-negative non-expanding function on X. We shall describe a two-sided invariant pseudometric $Gr(d, k)$ on the free group $F(X)$ which is called the *Graev pseudometric* [13]. The corresponding invariant seminorm p is characterized by the following property: p is the greatest invariant seminorm on $F(X)$ such that $p(x) = k(x)$ and $p(x^{-1}y) \leqslant d(x, y)$ for every $x, y \in X$. We shall need later the following explicit construction of the seminorm p.

It will be convenient to define the function p on the entire set $S(X)$. Given a word $w = x_1^{\epsilon_1} \ldots x_n^{\epsilon_n} \in S(X)$, we define a *w-pairing* to be a collection E of pairwise disjoint two-element subsets of the set $J = \{1, \ldots, n\}$ such that: (1) if $\{a, b\} \in E$ and $\{i, j\} \in E$, where $a < b$ and $i < j$, then the intervals $[a, b]$ and $[i, j]$ are either disjoint or one of them is contained in the other (this means that the cases $a < i < b < j$ and $i < a < j < b$ are excluded); (2) if $\{i, j\} \in E$, then $\epsilon_i = -\epsilon_j$. To put it less formally, some pairs of letters of the word w are connected by arcs, each letter is connected with at most one other letter, each arc connects a pair of letters of the form x and y^{-1} $(x, y \in X)$, and the arcs do not intersect each other. Given a w-pairing E, define the *Graev sum* $s_E = s_E(w)$ by

$$s_E = \sum \bigl\{d(x_i, x_j)\colon \{i, j\} \in E,\ i < j\bigr\} + \sum \bigl\{k(x_i)\colon i \in J \setminus \cup E\bigr\},$$

and let $p(w)$ be the minimum of the numbers s_E, taken over the finite set of all w-pairings E.

We claim that $p(w) = p(w')$ if the words $w, w' \in S(X)$ are equivalent. It suffices to consider the case when $w = u|v$ and $w' = u|x^\epsilon x^{-\epsilon}|v$. We show that for every w'-pairing E' there exists a w-pairing E such that $s_E \leqslant s_{E'}$, and vice versa. In one direction this is obvious: given a w-pairing E, which we consider as a system of arcs connecting the letters of the word w, add one more arc which connects the letters x^ϵ and $x^{-\epsilon}$ of the word w'. We get a w'-pairing E' for which $s_E = s_{E'}$. Conversely, let a w'-pairing E' be given. We must construct a w-pairing E for which $s_E \leqslant s_{E'}$. As above, we consider E' as a system of arcs. The word w is obtained from w' by deleting the subword $x^\epsilon x^{-\epsilon}$. To get E, we replace the arcs which go from the letters x^ϵ and $x^{-\epsilon}$ and leave the other arcs unchanged. Consider the following cases.

[5] A *monoid* is a semigroup with a neutral element. We require that monoid morphisms should preserve the neutral element.

Case 1. There is an arc in E' connecting the letters x^ϵ and $x^{-\epsilon}$. Then just delete this arc to get E. We have $s_E = s_{E'}$.

Case 2. The letters x^ϵ and $x^{-\epsilon}$ are connected in E', but not with each other. Let x^ϵ be connected with $y^{-\epsilon}$ and $x^{-\epsilon}$ be connected with z^ϵ. Replace these two connections by one connection between $y^{-\epsilon}$ and z^ϵ. The sums s_E and $s_{E'}$ differ by the terms $d(y,z)$ and $d(y,x)+d(x,z)$, hence the triangle inequality implies that $s_E \leqslant s_{E'}$.

Case 3. One of the letters x^ϵ and $x^{-\epsilon}$, say x^ϵ, is connected in E' and the other is unpaired. Let x^ϵ be connected with $y^{-\epsilon}$. Delete this connection and leave the letter $y^{-\epsilon}$ unpaired in E. The sums s_E and $s_{E'}$ differ by the terms $k(y)$ and $d(x,y)+k(x)$. Since the function k is non-expanding, we have $k(y) \leqslant d(x,y)+k(x)$ and hence $s_E \leqslant s_{E'}$.

Case 4. Both x^ϵ and $x^{-\epsilon}$ are unpaired in E'. Then all arcs are left without change. The sum s_E is obtained from $s_{E'}$ by omitting the non-negative term $2k(x)$, hence $s_E \leqslant s_{E'}$.

We have thus proved the claim that $p(w) = p(w')$ for equivalent words $w, w' \in S(X)$. It follows that the restriction of p to $F(X)$ is indeed a seminorm: if $u, v \in F(X)$, then $p(uv) = p(u|v) \leqslant p(u) + p(v)$. It is easy to see that $p(u) = p(u^{-1})$ for every $u \in F(X)$. We show that p is invariant under inner automorphisms. If $u \in S(X)$, $x \in X$, $\epsilon = \pm 1$ and $w = x^\epsilon|u|x^{-\epsilon}$, then $p(w) \leqslant p(u)$, since every u-pairing can be extended in an obvious way to a w-pairing with the same Graev sum. It follows that for every $u, v \in F(X)$ we have $p(uvu^{-1}) = p(u|v|u^{-1}) \leqslant p(v)$, and by symmetry of the relation of being conjugate in $F(X)$ also the opposite inequality holds. Thus $p(uvu^{-1}) = p(v)$, which means that the seminorm p is invariant.

Let Y be a pseudometric space, and let $\mathrm{Iso}(Y)$ be the group of all distance-preserving permutations of Y, equipped with the topology of pointwise convergence. Then $\mathrm{Iso}(Y)$ is a topological group, not necessarily Hausdorff. For later use we note here the following:

Lemma 2.2. *Let (X,d) be a metric space, and let k be a non-expanding function on X. Let $D = Gr(d,k)$ be the Graev pseudometric on the free group $G = F(X)$. Let $H_1 \subset \mathrm{Iso}(X)$ be the topological group of all isometries of X which preserve the function k, and let $H_2 \subset \mathrm{Iso}(G)$ be the topological group (not necessarily Hausdorff) of all automorphisms of G which preserve the pseudometric D. Then the natural homomorphism $\varphi \mapsto \varphi^*$ from H_1 to H_2 is continuous.*

Proof. It suffices to show that for every $w \in G$ the map $\varphi \mapsto \varphi^*(w)$ from H_1 to (G,D) is continuous at the unity. If $w = x_1^{\epsilon_1} \dots x_n^{\epsilon_n}$, then $\varphi^*(w) = \varphi(x_1)^{\epsilon_1} \dots \varphi(x_n)^{\epsilon_n}$, and we have $D(\varphi^*(w), w) \leqslant \sum_{i=1}^n d(\varphi(x_i), x_i)$. Let $\epsilon > 0$ be given. If $\varphi \in H_1$ is close enough to the unity, then $d(\varphi(x_i), x_i) < \epsilon/n$, $1 \leqslant i \leqslant n$, and therefore $D(\varphi^*(w), w) < \epsilon$. □

3. Katětov's construction of Urysohn extensions

Definition 3.1. Let M be a subspace of a metric space L. We say that M is *g-embedded* in L if there exists a continuous homomorphism $e : \mathrm{Iso}(M) \to \mathrm{Iso}(L)$ such that for every $\varphi \in \mathrm{Iso}(M)$ the isometry $e(\varphi) : L \to L$ is an extension of φ.

Let M be a g-embedded subspace of a metric space L. A homomorphism $e : \mathrm{Iso}(M) \to \mathrm{Iso}(L)$ satisfying the condition of Definition 3.1 is a homeomorphic embedding, since the inverse map $e(\varphi) \mapsto \varphi = e(\varphi)|M$ is continuous. It follows that $\mathrm{Iso}(M)$ is isomorphic to a topological subgroup of $\mathrm{Iso}(L)$.

In this section we prove the following theorem:

Theorem 3.2. *Let M be a metric space of diameter $\leqslant 1$. There exists a complete Urysohn metric space L containing M as a subspace such that $w(L) = w(M)$ and M is g-embedded in L.*

It follows that for every topological group G there exists a complete Urysohn metric space M of the same weight as G such that G is isomorphic to a subgroup of $\mathrm{Iso}(M)$. This is weaker than Theorem 1.7, since in the non-separable case the metric space M need not be ω-homogeneous. In the next section we shall prove that every metric space M can be g-embedded into an ω-homogeneous metric space L. Using this fact, we show that the Urysohn space L in Theorem 3.2 can be additionally assumed ω-homogeneous (Theorem 5.1). This yields Theorem 1.7, see Section 5.

The arguments of [47,50] show that Theorem 3.2 essentially follows from Katětov's construction of Urysohn extensions [20]. For the reader's convenience we give a detailed proof.

Let (X,d) be a metric space of diameter $\leqslant 1$. We say that a function $f: X \to [0,1]$ is *Katětov* if $|f(x)-f(y)| \leqslant d(x,y) \leqslant f(x)+f(y)$ for all $x,y \in X$. A function f is Katětov if and only if there exists a metric space $X' = X \cup \{p\}$ of diameter $\leqslant 1$ containing X as a subspace such that for every $x \in X$ $f(x)$ is equal to the distance between x and p. Let $E(X)$ be the set of all Katětov functions on X, equipped with the sup-metric d_X^E defined by $d_X^E(f,g) = \sup\{|f(x)-g(x)|: x \in X\}$. If Y is a non-empty subset of X and $f \in E(Y)$, define $g = \kappa_Y(f) \in E(X)$ by

$$g(x) = \inf\bigl(\bigl\{d(x,y)+f(y):\ y \in Y\bigr\} \cup \{1\}\bigr) = \inf\bigl\{d(x,y) \uplus f(y):\ y \in Y\bigr\}$$

for every $x \in X$. It is easy to check that g is indeed a Katětov function on X and that g extends f; one can define g as the largest 1-Lipschitz function $X \to [0,1]$ that extends f. The map $\kappa_Y: E(Y) \to E(X)$ is an isometric embedding. Let

$$E(X,\omega) = \bigcup\bigl\{\kappa_Y\bigl(E(Y)\bigr):\ Y \subset X,\ Y \text{ is finite and non-empty}\bigr\} \subset E(X).$$

For every $x \in X$ let $h_x \in E(X)$ be the function on X defined by $h_x(y) = d(x,y)$. Note that $h_x = \kappa_{\{x\}}(0)$ and hence $h_x \in E(X,\omega)$. The map $x \mapsto h_x$ is an isometric embedding of X into $E(X,\omega)$. Thus we can identify X with a subspace of $E(X,\omega)$.

Lemma 3.3. *If $x \in X$ and $f \in E(X)$, then $d_X^E(f,h_x) = f(x)$.*

Proof. Since f is a Katětov function, for every $y \in Y$ we have $f(y) - d(x,y) \leqslant f(x)$ and $d(x,y) - f(y) \leqslant f(x)$. Hence $d_X^E(f,h_x) = \sup\{|f(y)-d(x,y)|:\ y \in X\} \leqslant f(x)$, and at $y = x$ the equality is attained. □

Lemma 3.4. *Let $Z = Y \cup \{p\}$ be a finite metric space of diameter $\leqslant 1$. Every isometric embedding $j: Y \to X$ extends to an isometric embedding of Z into $E(X,\omega)$.*

Proof. We may assume that Y is a subspace of X and that $j(y) = y$ for every $y \in Y$. Let $f \in E(Y)$ be the Katětov function defined by $f(y) = \nu(y,p)$ for every $y \in Y$, where ν is the metric on Z. Let $g = \kappa_Y(f) \in E(X,\omega)$. We claim that the extension of j over Z which maps p to g is an isometric embedding. It suffices to check that $d_X^E(h_y,g) = \nu(y,p)$ for every $y \in Y$. Fix $y \in Y$. Let $h_y^* \in E(Y)$ be the restriction of h_y to Y. According to Lemma 3.3 we have $d_Y^E(h_y^*,f) = f(y)$. Since $h_y = \kappa_Y(h_y^*)$, $g = \kappa_Y(f)$ and the map $\kappa_Y: E(Y) \to E(X)$ is distance-preserving, it follows that $d_X^E(h_y,g) = d_Y^E(h_y^*,f) = f(y) = \nu(y,p)$, as claimed. □

Lemma 3.5. *Any metric space X of diameter $\leqslant 1$ is g-embedded in $E(X,\omega)$.*

Proof. It is clear that every isometry $\varphi: Y \to Z$ between any two metric spaces can be extended to an isometry $\varphi^*: E(Y,\omega) \to E(Z,\omega)$. Such an extension is unique, since every point in $E(Y,\omega)$ (or, more generally, in $E(Y)$) is uniquely determined by its distances from the points of Y (Lemma 3.3), and similarly for Z. In particular, every isometry $\varphi \in \mathrm{Iso}(X)$ uniquely extends to an isometry $\varphi^* \in \mathrm{Iso}(E(X,\omega))$. The map $\varphi \mapsto \varphi^*$ is a homomorphism of groups. We show that this homomorphism is continuous. Fix $f \in E(X,\omega)$ and $\epsilon > 0$. Pick a finite subset Y of X and $g \in E(Y)$ so that $f = \kappa_Y(g)$. Let U be the set of all $\varphi \in \mathrm{Iso}(X)$ such that $d(\varphi(y),y) < \epsilon$ for every $y \in Y$. Then U is a neighbourhood of unity in $\mathrm{Iso}(X)$. It suffices to show that $d_X^E(\varphi^*(f),f) < \epsilon$ for every $\varphi \in U$. Fix $\varphi \in U$. Let $g_\varphi = g \circ \varphi^{-1} \in E(\varphi(Y))$. Then $\varphi^*(f) = \kappa_{\varphi(Y)}(g_\varphi)$. Thus for every $x \in X$ we have

$$\varphi^*(f)(x) = \inf\bigl\{d(x,z) \uplus g_\varphi(z):\ z \in \varphi(Y)\bigr\} = \inf\bigl\{d\bigl(x,\varphi(y)\bigr) \uplus g(y):\ y \in Y\bigr\}.$$

Since

$$f(x) = \inf\bigl\{d(x,y) \uplus g(y):\ y \in Y\bigr\},$$

it follows that

$$\bigl|\varphi^*(f)(x) - f(x)\bigr| \leqslant \sup\bigl\{\bigl|d\bigl(x,\varphi(y)\bigr) - d(x,y)\bigr|:\ y \in Y\bigr\} \leqslant \max\bigl\{d\bigl(y,\varphi(y)\bigr):\ y \in Y\bigr\} < \epsilon,$$

whence $d_X^E(\varphi^*(f),f) < \epsilon$. □

Let α be an ordinal, and let $\mathcal{M} = \{M_\beta\colon \beta < \alpha\}$ be a family of metric spaces such that M_β is a subspace of M_γ for all $\beta < \gamma < \alpha$. We say that the family $\mathcal{M}$ is *continuous* if $M_\beta = \bigcup_{\gamma<\beta} M_\gamma$ for every limit ordinal $\beta < \alpha$, $\beta > 0$.

Proposition 3.6. *Let $\{M_\beta\colon \beta \leqslant \alpha\}$ be an increasing continuous chain of metric spaces. If M_β is g-embedded in $M_{\beta+1}$ for every $\beta < \alpha$, then M_0 is g-embedded in M_α.*

Proof. For every $\beta < \alpha$ pick a continuous homomorphism $e_\beta : \mathrm{Iso}(M_\beta) \to \mathrm{Iso}(M_{\beta+1})$ such that $e_\beta(\varphi)$ extends φ for every $\varphi \in \mathrm{Iso}(M_\beta)$. By transfinite recursion on $\beta \leqslant \alpha$ define a homomorphism $\Lambda_\beta : \mathrm{Iso}(M_0) \to \mathrm{Iso}(M_\beta)$ such that $\Lambda_\beta(\varphi)$ extends $\Lambda_\gamma(\varphi)$ for every $\varphi \in \mathrm{Iso}(M_0)$ and $\gamma < \beta \leqslant \alpha$. Let Λ_0 be the identity map of $\mathrm{Iso}(M_0)$. If $\beta = \gamma + 1$, put $\Lambda_\beta = e_\gamma \Lambda_\gamma$. If β is a limit ordinal, let $\Lambda_\beta(\varphi)$ be the isometry of M_β such that for every $\gamma < \beta$ its restriction to M_γ is equal to $\Lambda_\gamma(\varphi)$. We prove by induction on β that each homomorphism Λ_β is continuous. This is obvious for non-limit ordinals. Assume that β is limit. To prove that $\Lambda_\beta : \mathrm{Iso}(M_0) \to \mathrm{Iso}(M_\beta)$ is continuous, it suffices to show that for every $x \in M_\beta$ the map $\varphi \mapsto \Lambda_\beta(\varphi)(x)$ from $\mathrm{Iso}(M_0)$ to M_β is continuous. Fix $x \in M_\beta$. Pick $\gamma < \beta$ so that $x \in M_\gamma$. Then $\Lambda_\beta(\varphi)(x) = \Lambda_\gamma(\varphi)(x)$ for every $\varphi \in \mathrm{Iso}(M_0)$. The map Λ_γ is continuous by the assumption of induction, hence the map $\varphi \mapsto \Lambda_\beta(\varphi)(x) = \Lambda_\gamma(\varphi)(x)$ also is continuous. Thus $\Lambda_\alpha : \mathrm{Iso}(M_0) \to \mathrm{Iso}(M_\alpha)$ is a continuous homomorphism such that $\Lambda_\alpha(\varphi)$ extends φ for every $\varphi \in \mathrm{Iso}(M_0)$. This means that M_0 is g-embedded in M_α. □

Put $X_0 = X$, $X_{n+1} = E(X_n, \omega)$. We consider each X_n as a subspace of X_{n+1}, so we get an increasing sequence $X_0 \subset X_1 \subset \cdots$ of metric spaces. Let $X_\omega = \bigcup\{X_n\colon n \in \omega\}$.

Proposition 3.7. *The space X_ω is Urysohn, and X is g-embedded in X_ω.*

Proof. Let $Z = Y \cup \{p\}$ be a finite metric space of diameter $\leqslant 1$, and let $j : Y \to X_\omega$ be an isometric embedding. Pick $n \in \omega$ so that $j(Y) \subset X_n$. In virtue of Lemma 3.4, there exists an isometric embedding of Z into $X_{n+1} \subset X_\omega$ which extends j. This means that X_ω is Urysohn. The second assertion of the proposition follows from Lemma 3.5 and Proposition 3.6. □

Proposition 3.8. *(See [20].) The weight of X_ω is equal to the weight of X.*

Proof. It suffices to show that for every metric space X the weight of $E(X, \omega)$ is equal to the weight of X. Let $w(X) = \tau$, and let A be a dense subset of X of cardinality τ. Let $\gamma = \{\kappa_Y(E(Y))\colon Y \subset A,\ Y \text{ is finite}\}$. Then γ is a family of separable subspaces of $E(X, \omega)$, $|\gamma| = \tau$ and $\bigcup \gamma$ is dense in $E(X, \omega)$ (see the proof of Lemma 1.8 in [20]). Hence $E(X, \omega)$ has a dense subspace of cardinality τ. □

Proposition 3.9. *Every metric space is g-embedded in its completion.*

Proof. Let M be a metric space, $\overline{M}$ be its completion. Every isometry $\varphi \in \mathrm{Iso}(M)$ uniquely extends to an isometry $\varphi^* \in \mathrm{Iso}(\overline{M})$. We show that the homomorphism $\varphi \mapsto \varphi^*$ is continuous. Let d be the metric on $\overline{M}$. Fix $x \in \overline{M}$ and $\epsilon > 0$. Pick $y \in M$ so that $d(x, y) < \epsilon$. Let $U = \{\varphi \in \mathrm{Iso}(M)\colon d(\varphi(y), y) < \epsilon\}$. Then U is a neighbourhood of unity in $\mathrm{Iso}(M)$. If $\varphi \in U$, then $d(\varphi^*(x), x) \leqslant d(\varphi^*(x), \varphi^*(y)) + d(\varphi^*(y), y) + d(y, x) < 3\epsilon$. This implies the continuity of the homomorphism $\varphi \mapsto \varphi^*$. □

Proposition 3.10. *(See [52], [31, Lemma 3.4.10], [32, Lemma 5.1.17], [14, Section 3.11$\frac{2}{3}{}_+$].) The completion of any Urysohn metric space is Urysohn.*

Proof. Let (M, d) be a complete metric space containing a dense Urysohn subspace A. We must prove that M is Urysohn.

Let Y be a finite subset of M, and let $f \in E(Y)$ be a Katětov function. It suffices to prove that there exists $z \in M$ such that $d(y, z) = f(y)$ for every $y \in Y$. Pick a sequence $\{a_n\colon n \in \omega\} \subset A$ such that:

if $A_n = \{a_k\colon k \leqslant n\}$ and $r_n = d(a_{n+1}, A_n)$, $n = 0, 1, \ldots,$ then the series $\sum r_n$ converges;
every $y \in Y$ is a cluster point of the sequence $\{a_n\colon n \in \omega\}$.

To construct such a sequence, enumerate Y as $Y = \{y_1, \dots, y_s\}$, and for every k and j ($k \in \omega$, $1 \leqslant j \leqslant s$) pick a point $x_k^j \in A$ such that $d(x_k^j, y_j) < 2^{-k}$. Then $d(x_{k+1}^j, x_k^j) < 2^{1-k}$ for every k and j, and the sequence

$$x_0^1, x_0^2, \dots, x_0^s, x_1^1, \dots, x_1^s, x_2^1, \dots$$

has the required properties.

Let $g = \kappa_Y(f) \in E(M)$. We construct by induction a sequence $\{z_n\colon n \in \omega\}$ of points of A such that:

(1) if $k \leqslant n$, then $d(z_n, a_k) = g(a_k)$;
(2) $d(z_{n+1}, z_n) \leqslant 2r_n$ for every $n \in \omega$.

Pick $z_0 \in A$ so that $d(z_0, a_0) = g(a_0)$. This is possible since A is Urysohn. Suppose that the points $z_0, \dots, z_n$ have been constructed so that the conditions 1 and 2 are satisfied. Consider two Katětov functions f_n and g_n on the set $A_{n+1} = \{a_k\colon k \leqslant n+1\}$: let $f_n(x) = d(z_n, x)$ for every $x \in A_{n+1}$, and let $g_n = g|_{A_{n+1}}$. By (1), the functions f_n and g_n coincide on A_n, hence the distance between them in the space $E(A_{n+1})$ is equal to

$$\begin{aligned}\bigl|f_n(a_{n+1}) - g_n(a_{n+1})\bigr| &= \sup\bigl\{\bigl|f_n(a_{n+1}) - f_n(x) - g_n(a_{n+1}) + g_n(x)\bigr|\colon x \in A_n\bigr\} \\ &\leqslant \sup\bigl\{\bigl|f_n(a_{n+1}) - f_n(x)\bigr|\colon x \in A_n\bigr\} + \sup\bigl\{\bigl|g_n(a_{n+1}) - g_n(x)\bigr|\colon x \in A_n\bigr\} \\ &\leqslant 2d(a_{n+1}, A_n) = 2r_n.\end{aligned}$$

Let X_n be the metric space $A_{n+1} \cup \{f_n\}$, considered as a subspace of $E(A_{n+1})$. In virtue of Lemma 3.3, the map of X_n to A which leaves each point of A_{n+1} fixed and sends f_n to z_n is an isometric embedding. Since A is Urysohn, this map can be extended to an isometric embedding of $X_n \cup \{g_n\}$ to A. Let z_{n+1} be the image of g_n. Then $d(z_{n+1}, z_n) = d^E_{A_{n+1}}(g_n, f_n) \leqslant 2r_n$. In virtue of Lemma 3.3, for every $k \leqslant n+1$ we have $d(z_{n+1}, a_k) = g_n(a_k) = g(a_k)$. Thus the conditions 1 and 2 are satisfied, and the construction is complete.

Since the series $\sum r_n$ converges, it follows from (2) that the sequence $\{z_n\colon n \in \omega\}$ is Cauchy and hence has a limit in the complete space M. Let $z = \lim z_n$. By (1), we have $d(z, a_k) = g(a_k)$ for every $k \in \omega$. Since Y is contained in the closure of the set $\{a_n\colon n \in \omega\}$, it follows that $d(z, y) = g(y) = f(y)$ for every $y \in Y$. □

Proof of Theorem 3.2. Let M be a metric space of diameter $\leqslant 1$, and let M_ω be the Urysohn extension of M constructed above. Consider the completion L of M_ω. Proposition 3.10 implies that L is Urysohn. Proposition 3.8 shows that $w(L) = w(M)$. Finally, M is g-embedded in M_ω (Proposition 3.7) and M_ω is g-embedded in L (Proposition 3.9), so M is g-embedded in L. Thus L has the properties required by Theorem 3.2. □

4. Graev metrics and ω-homogeneous extensions

In this section we prove the following:

Theorem 4.1. *Every metric space can be g-embedded into an ω-homogeneous metric space of the same weight and the same diameter.*

The proof is based on the construction of Graev metrics described in Section 2. We apply this construction to metric spaces of relations. A *relation* on a set X is a subset of X^2. If R and S are relations on X, then the composition $R \circ S$ (or simply RS) is defined by $R \circ S = \{(x, y)\colon \exists z((x, z) \in S \text{ and } (z, y) \in R)\}$. The inverse relation R^{-1} is defined by $R^{-1} = \{(x, y)\colon (y, x) \in R\}$. The set of all relations on a set X is a semigroup with an involution: the multiplication is given by the composition, and the involution is given by the map $R \mapsto R^{-1}$. The unity of this semigroup is the diagonal Δ of X^2.

We use the notation of Section 2. In particular, if k is a non-expanding function $\geqslant 0$ on a metric space (X, d), then $Gr(d, k)$ is the Graev pseudometric on the free group $F(X)$. We consider the group $F(X)$ as a subset of the free monoid $S(X)$ on the set $X \cup X^{-1}$.

Proof of Theorem 4.1. Let (M, d) be a metric space. We first construct a g-embedding of M into a metric space M^* such that $w(M^*) = w(M)$ and every isometry between finite subsets of M extends to an isometry of M^*.

For every isometry $f: A \to B$ between finite non-empty subsets of M consider the graph $R = \{(a, f(a)):\ a \in A\}$ of f, and let Γ be the set of all such graphs. Thus a non-empty finite subset $R \subset M^2$ is an element of Γ iff for any two pairs $(x_1, y_1), (x_2, y_2) \in R$ we have $d(x_1, x_2) = d(y_1, y_2)$. Equip M^2 with the metric d_2 defined by $d_2((x_1, y_1), (x_2, y_2)) = d(x_1, x_2) + d(y_1, y_2)$, and let d_H be the corresponding Hausdorff metric on the set of finite subsets of M^2. If R and S are two non-empty finite subsets of M^2 and $a \geqslant 0$, then $d_H(R, S) \leqslant a$ iff for every $p \in R$ there exists $q \in S$ such that $d_2(p, q) \leqslant a$, and for every $p \in S$ there exists $q \in R$ such that $d_2(p, q) \leqslant a$.

Let k be the non-expanding function on (Γ, d_H) defined by $k(R) = \max\{d(x, y):\ (x, y) \in R\}$. Let G be the free group on Γ, equipped with the Graev pseudometric $D = Gr(d_H, k)$. To avoid confusion of multiplication in G with composition of relations, we assign to each $R \in \Gamma$ a symbol t_R, and consider elements of G as irreducible words of the form $x_1^{\epsilon_1} \dots x_n^{\epsilon_n}$, where $x_i = t_{R_i}$. Similarly, we consider elements of the semigroup $S(\Gamma)$ as words of the same form. Let $\Delta = \{(x, x):\ x \in M\}$ be the diagonal of M^2. The set $\Gamma' = \Gamma \cup \{\emptyset\} \cup \{\Delta\}$ is a symmetrical subsemigroup of the semigroup of all relations on M. Let Φ: $G \to \Gamma'$ be the map defined by the following rule: if $w = t_{R_1}^{\epsilon_1} \dots t_{R_n}^{\epsilon_n} \in G$ is a non-empty irreducible word, then $\Phi(w) = R_1^{\epsilon_1} \circ \dots \circ R_n^{\epsilon_n}$. If $a, b \in M$, then $(a, b) \in \Phi(w)$ iff there exists a chain $c_0 = b, c_1, \dots, c_n = a$ of points of M such that for every $i = 1, \dots, n$ we have either $\epsilon_i = 1$ and $(c_i, c_{i-1}) \in R_i$ or $\epsilon_i = -1$ and $(c_{i-1}, c_i) \in R_i$. For the empty word $e_G \in G$ we put $\Phi(e_G) = \Delta$.

Note that the definition of $\Phi(w)$ makes sense also without the assumption that the word w be irreducible, so we can assume that Φ is defined on the set $S(\Gamma)$ of all words of the form $t_{R_1}^{\epsilon_1} \dots t_{R_n}^{\epsilon_n}$. Recall that $w_1|w_2$ denotes the word obtained by writing w_2 after w_1 (without cancelations). We have $\Phi(w_1|w_2) = \Phi(w_1) \circ \Phi(w_2)$.

Lemma 4.2. *If $w \in S(\Gamma)$ and u is the irreducible word equivalent to w, then $\Phi(u) \supset \Phi(w)$.*

Proof. It suffices to prove that $\Phi(w') \supset \Phi(w)$ if w' is obtained from w by canceling one pair of letters. Let $w = u|t_R^{\epsilon} t_R^{-\epsilon}|v$ and $w' = u|v$. Since R^ϵ is a functional relation, we have $R^\epsilon \circ R^{-\epsilon} \subset \Delta$ and hence $\Phi(w') = \Phi(u) \circ \Phi(v) = \Phi(u) \circ \Delta \circ \Phi(v) \supset \Phi(u) \circ R^\epsilon \circ R^{-\epsilon} \circ \Phi(v) = \Phi(w)$. □

For every $w \in G$ we have $\Phi(w^{-1}) = \Phi(w)^{-1}$. We claim that $\Phi(w_1 w_2) \supset \Phi(w_1) \circ \Phi(w_2)$ for every $w_1, w_2 \in G$. Indeed, the product $w_1 w_2 \in G$ is the irreducible word equivalent to $w_1|w_2$, therefore $\Phi(w_1 w_2) \supset \Phi(w_1|w_2) = \Phi(w_1) \circ \Phi(w_2)$ by Lemma 4.2.

For every $a, b \in M$ let $H_{a,b} \subset G$ be the set of all $w \in G$ such that $(a, b) \in \Phi(w)$. We claim that $H_{a,b}^{-1} = H_{b,a}$ and $H_{b,c} H_{a,b} \subset H_{a,c}$ for every $a, b, c \in M$. This follows from the properties of Φ established in the preceding paragraph. Indeed, pick $w_1 \in H_{b,c}$ and $w_2 \in H_{a,b}$. Then $(a, b) \in \Phi(w_2)$ and $(b, c) \in \Phi(w_1)$, hence $(a, c) \in \Phi(w_1) \circ \Phi(w_2) \subset \Phi(w_1 w_2)$ and $w_1 w_2 \in H_{a,c}$. This proves the inclusion $H_{b,c} H_{a,b} \subset H_{a,c}$. The equality $H_{a,b}^{-1} = H_{b,a}$ is proved similarly.

Note that $t_R \in H_{a,b}$ if and only if $(a, b) \in R$, since $\Phi(t_R) = R$. Note also that $e_G \in H_{a,b}$ if and only if $a = b$, since $\Phi(e_G) = \Delta$.

Consider the following equivalence relation $\sim$ on $G \times M$: a pair (g, a) is equivalent to a pair (h, b) iff $h^{-1} g \in H_{a,b}$. Since $e_G \in H_{a,a}$, $H_{a,b}^{-1} = H_{b,a}$ and $H_{b,c} H_{a,b} \subset H_{a,c}$ for all $a, b, c \in M$, the relation $\sim$ is reflexive, symmetric and transitive and thus is indeed an equivalence relation. Let L be the quotient set $G \times M/{\sim}$. The group G acts on $G \times M$ by the rule $g \cdot (h, a) = (gh, a)$. The relation $\sim$ is invariant under this action, so there is a uniquely defined left action of G on L which makes the canonical map $G \times M \to L$ into a morphism of G-sets. Let $i: M \to L$ be the map which sends each point $a \in M$ to the class of the pair $(1, a)$. If $a \neq b$, then the pairs $(1, a)$ and $(1, b)$ are not equivalent, since $e_G \notin H_{a,b}$. The map i is therefore injective, and we can consider M as a subspace of L, identifying M with $i(M)$. Every $x \in L$ can be written in the form $x = g \cdot a$ (or simply $x = ga$), where $g \in G$ and $a \in M$.

Let $a, b \in M$. The set of all $g \in G$ such that $ga = b$ is equal to $H_{a,b}$. If $R \in \Gamma$ is a relation containing the pair (a, b), then $t_R \in H_{a,b}$ and hence $t_R a = b$. It follows that the action of G on L is transitive. Moreover, for every isometry $f: A \to B$ between finite subsets of M there exists $g \in G$ such that the self-map $x \to gx$ of L extends f. Indeed, if $R \in \Gamma$ is the graph of f, then $t_R \in H_{a, f(a)}$ and hence $t_R a = f(a)$ for every $a \in A$. Thus $g = t_R$ has the required property.

We now define a G-invariant pseudometric ν on L which extends the metric d on M. Let p be the Graev seminorm on G corresponding to the pseudometric $D = Gr(d_H, k)$. We have $p(w) = D(w, e_G)$ for every $w \in G$. For every $x, y \in L$ let

$$\nu(x, y) = \inf\bigl\{p(g):\ g \in G,\ gx = y\bigr\}.$$

Then ν is a pseudometric on L. Since the seminorm p is invariant under inner automorphisms, the pseudometric ν is G-invariant. Indeed, for $x, y \in L$ and $h \in G$ we have $\nu(hx, hy) = \inf\{p(g)\colon ghx = hy\} = \inf\{p(h^{-1}gh)\colon h^{-1}ghx = y\} = \inf\{p(g')\colon g'x = y\} = \nu(x, y)$. We claim that ν extends the metric d on M: $d(a, b) = \nu(a, b)$ for every $a, b \in M$. Since for $w \in G$ the condition $wa = b$ is equivalent to $w \in H_{a,b}$, we have $\nu(a, b) = \inf\{p(w)\colon w \in H_{a,b}\}$. If $R = \{(a, b)\}$, then $t_R \in H_{a,b}$ and $p(t_R) = k(R) = d(a, b)$. It follows that $\nu(a, b) \leqslant d(a, b)$. It remains to prove the opposite inequality, which is equivalent to the following assertion:

Lemma 4.3. *If $a, b \in M$ and $w \in H_{a,b}$, then $p(w) \geqslant d(a, b)$.*

Proof. Let $w = t_{R_1}^{\epsilon_1} \dots t_{R_n}^{\epsilon_n}$. We argue by induction on n, the length of w. If $n = 0$, then $w = e_G$, and we noted that $e_G \in H_{a,b}$ implies $a = b$. If $n = 1$, then $w = t_R^{\epsilon}$ and $p(w) = k(R)$. Since $w \in H_{a,b}$, the relation R contains either (a, b) or (b, a) and hence $p(w) = k(R) \geqslant d(a, b)$. Assume that $n > 1$. It suffices to show that there exists $u \in H_{a,b}$ of length $< n$ such that $p(u) \leqslant p(w)$.

We use the construction of the Graev seminorm p described in Section 2. Let E be a w-pairing for which $p(w)$ is attained. In other words, E is a disjoint system of two-element subsets of the set $J = \{1, \dots, n\}$ such that for the Graev sum

$$s_E = \sum\bigl\{d_H(R_i, R_j)\colon \{i, j\} \in E,\ i < j\bigr\} + \sum\bigl\{k(R_i)\colon i \in J \setminus \cup E\bigr\}$$

we have $p(w) = s_E$. Considering the pair $(i, j) \in E$ with the least possible value of $|i - j|$ (“the shortest arc”), we see that at least one of the following three cases must occur:

(1) there exists an i such that $\{i, i + 1\} \in E$;
(2) there exists an i such that $\{i, i + 2\} \in E$ and $i + 1 \in J \setminus \bigcup E$;
(3) there exists an i such that $i, i + 1 \in J \setminus \bigcup E$.

In cases (1) or (3) we replace the subword $t_{R_i}^{\epsilon_i} t_{R_{i+1}}^{\epsilon_{i+1}}$ of w by the letter t_S, where $S = R_i^{\epsilon_i} \circ R_{i+1}^{\epsilon_{i+1}}$. In case (2) we replace the subword $t_{R_i}^{\epsilon_i} t_{R_{i+1}}^{\epsilon_{i+1}} t_{R_{i+2}}^{\epsilon_{i+2}}$ of w by the letter t_S, where $S = R_i^{\epsilon_i} \circ R_{i+1}^{\epsilon_{i+1}} \circ R_{i+2}^{\epsilon_{i+2}}$. In all cases we get a word w' of length $< n$. To justify the usage of the symbol t_S, we must show that $S \in \Gamma$, which reduces to the fact that $S \neq \emptyset$. Had S been empty, the same would have been true for $\Phi(w) = R_1^{\epsilon_1} \circ \dots \circ R_n^{\epsilon_n}$. On the other hand, since $w \in H_{a,b}$, we have $(a, b) \in \Phi(w) \neq \emptyset$.

Let $u \in G$ be the irreducible word equivalent to w'. The length of u is less than n. We show that $u \in H_{a,b}$ and $p(u) \leqslant p(w)$.

By Lemma 4.2 we have $\Phi(w') \subset \Phi(u)$. Plainly $\Phi(w) = R_1^{\epsilon_1} \circ \dots \circ R_n^{\epsilon_n} = \Phi(w')$. Since $w \in H_{a,b}$, we have $(a, b) \in \Phi(w)$. Thus $(a, b) \in \Phi(w) = \Phi(w') \subset \Phi(u)$ and $u \in H_{a,b}$, as required.

We prove that $p(u) \leqslant p(w)$. As in Section 2, we define $p(w')$ even if the word w' is reducible, and we have $p(u) = p(w') = \inf s_{E'}$, where E' runs over the set of all w'-pairings. The w-pairing E in an obvious way yields a w'-pairing E', which coincides with E outside the changed part of w and leaves the new letter t_S unpaired. The Graev sums s_E and $s_{E'}$ differ only by the term $k(S)$ in the sum $s_{E'}$ and the terms $d_H(R_i, R_{i+1})$ (case 1) or $d_H(R_i, R_{i+2}) + k(R_{i+1})$ (case 2) or $k(R_i) + k(R_{i+1})$ (case 3) in the sum s_E. According to Lemma 4.4 below, we have $s_{E'} \leqslant s_E$. Thus $p(u) = p(w') \leqslant s_{E'} \leqslant s_E = p(w)$. □

Lemma 4.4. *Let $\epsilon, \delta \in \{-1, 1\}$,*

(1) *if $R_1, R_2 \in \Gamma$ and $S = R_1^{\epsilon} R_2^{-\epsilon}$ is non-empty, then $k(S) \leqslant d_H(R_1, R_2)$;*
(2) *if $R_1, R_2, R_3 \in \Gamma$ and $S = R_1^{\epsilon} R_2^{\delta} R_3^{-\epsilon}$ is non-empty, then $k(S) \leqslant d_H(R_1, R_3) + k(R_2)$;*
(3) *if $R_1, R_2 \in \Gamma$ and $S = R_1^{\epsilon} R_2^{\delta}$ is non-empty, then $k(S) \leqslant k(R_1) + k(R_2)$.*

Proof. Since $k(R) = k(R^{-1})$ and $d_H(R, T) = d_H(R^{-1}, T^{-1})$ for every $R, T \in \Gamma$, we may assume that $\epsilon = \delta = 1$. Pick $(a, b) \in S$ so that $k(S) = d(a, b)$. Case (1) follows from (2) (take for R_2 in (2) a sufficiently large finite part of Δ), so let us consider case (2). There exist $x, y \in M$ such that $(a, x) \in R_3^{-1}$, $(x, y) \in R_2$ and $(y, b) \in R_1$. Since $(x, a) \in R_3$, there exists a pair $(u, v) \in R_1$ such that $d(a, v) + d(u, x) \leqslant d_H(R_1, R_3)$. The relation R_1, being an element of Γ, is the graph of a partial isometry, so from $(y,b) \in R_1$ and $(u,v) \in R_1$ it follows that $d(v,b) = d(u,y)$. Note that $d(x,y) \leqslant k(R_2)$. Thus we have $k(S) = d(a,b) \leqslant d(a,v) + d(v,b) = d(a,v) + d(u,y) \leqslant d(a,v) + d(u,x) + d(x,y) \leqslant d_H(R_1,R_3) + k(R_2)$, as required. Case (3) is easy: there exists a point $c \in M$ such that $(a,c) \in R_2$ and $(c,b) \in R_1$, hence $k(S) = d(a,b) \leqslant d(a,c) + d(c,b) \leqslant k(R_2) + k(R_1)$. □

We have thus proved that the pseudometric ν on L extends the metric d on M. Let (M^*, d^*) be the metric space associated with the pseudometric space (L,ν). The metric space (M,d) can be naturally identified with a subspace of (M^*, d^*). We show that M is g-embedded in M^*.

In virtue of the functorial nature of the construction of M^*, every isometry φ of M naturally extends to an isometry φ^* of M^*. The map $\varphi \mapsto \varphi^*$ from $\mathrm{Iso}(M)$ to $\mathrm{Iso}(M^*)$ is a homomorphism of groups. We claim that this homomorphism is continuous. This follows from the fact that at each step of our construction new spaces are obtained from the old ones via functors "with finite support": every element of Γ is a *finite* relation on M, and every word $w \in G$ involves only finitely many elements of Γ. Given an isometry $\varphi \in \mathrm{Iso}(M)$, the isometry $\varphi^* \in \mathrm{Iso}(M^*)$ can be obtained step by step in the following way. First we consider the isometry φ_1 of the metric space (Γ, d_H) corresponding to φ; the isometry φ_1 preserves the function k on Γ and gives rise to the automorphism φ_2 of the group $G = F(\Gamma)$ which preserves the Graev pseudometric D; then we get the isometry φ_3 of L which maps the class of each pair (g,x) $(g \in G,\ x \in M)$ to the class of the pair $(\varphi_2(g), \varphi(x))$; finally we get the isometry $\varphi_4 = \varphi^*$ of M^*. We show step by step that φ_i depends continuously on φ. For $i = 1$ this is straightforward: use the fact that Γ consists of *finite* subsets of M^2. For $i = 2$ apply Lemma 2.2 with $X = \Gamma$. Let us consider the case $i = 3$. Pick a point $x = ga \in L$ $(g \in G,\ a \in M)$. It suffices to check that $\nu(\varphi_3(x), x)$ is small if φ is close to the identity. We have $\nu(\varphi_3(x), x) = \nu(\varphi_2(g)\varphi(a), ga) \leqslant \nu(\varphi_2(g)\varphi(a), g\varphi(a)) + \nu(g\varphi(a), ga) = \nu(g^{-1}\varphi_2(g)\varphi(a), \varphi(a)) + \nu(\varphi(a), a)$. By the definition of ν, the first term of the last sum does not exceed $p(g^{-1}\varphi_2(g)) = D(\varphi_2(g), g)$ and hence is arbitrarily small if φ is close enough to the identity. The same is true for second term, and we are done. Finally, φ_4 is the image of φ_3 under the natural morphism $\mathrm{Iso}(L) \to \mathrm{Iso}(M^*)$, and the case $i = 4$ follows.

We have thus proved that M is g-embedded in M^*. We saw that each isometry between finite subsets of M extends to an isometry of L and hence also to an isometry of M^*. It is easy to see that $w(M^*) = w(M)$. If the diameter C of M is finite, replace the metric d^* of M^* by $\inf(d^*, C)$. This operation can make the group $\mathrm{Iso}(M^*)$ only larger, and the diameter of M^* becomes equal to that of M.

To finish the proof of Theorem 4.1, iterate the construction of M^*. We get an increasing chain $M_0 = M \subset M_1 = M^* \subset M_2 = M_1^* \subset \cdots$ of metric spaces such that each M_n is g-embedded in M_{n+1}, every isometry between finite subsets of M_n extends to an isometry of M_{n+1}, $w(M_n) = w(M)$ and $\operatorname{diam} M_n = \operatorname{diam} M$, $n = 0, 1, \ldots$. Consider the space $M_\omega = \bigcup_{n\in\omega} M_n$. We have $w(M_\omega) = w(M)$ and $\operatorname{diam} M_\omega = \operatorname{diam} M$. In virtue of Proposition 3.6, each M_n is g-embedded in M_ω. Since every finite subset of M_ω is contained in some M_n, it is clear that M_ω is ω-homogeneous. □

Remarks.

1. If $a, b \in M$ are distinct and $S = \{(b,b)\}$, the pairs $(1,a)$ and (t_S, a) represent distinct points of L that have the same image in M^*. Early versions of this paper contained the false statement that ν itself is a metric and $M^* = L$. I am indebted to the referee for catching this error.
2. The referee raised the question whether the methods of this section could be used to prove the following result by S. Solecki [40] and A.M. Vershik [56] that extends an earlier result by Hrushovski: for every finite metric space A there exists another finite metric space A^* containing A such that all partial isometries[6] of A extend to isometries of A^*. I do not know the answer. A partial answer is provided by Pestov's paper [34] where the Hrushovski–Solecki–Vershik theorem is proved with the aid of pseudometrics on groups, and the notion of a residually finite group is used to construct isometric embeddings of finite metric spaces into finite metric groups. A similar technique was used in [33].

5. Proof of Theorem 1.7

In this section we prove Theorem 1.7.

[6] A *partial isometry* of A is an isometry between two subsets of A.

Theorem 5.1. *Let M be a metric space of diameter $\leqslant 1$. There exists a complete ω-homogeneous Urysohn metric space L containing M as a subspace such that $w(L) = w(M)$ and M is g-embedded in L.*

Proof. Consider two cases.

Case 1. M is separable. According to Theorem 3.2, there exists a complete separable Urysohn space L such that M is a g-embedded subspace of L. According to Proposition 1.6, $L = \mathbb{U}_1$ is ω-homogeneous.

Case 2. M is not separable. Let $\tau = w(M)$. Applying in turn Theorems 3.2 and 4.1, construct an increasing continuous chain $\{M_\alpha\colon \alpha \leqslant \omega_1\}$ of metric spaces of weight τ and diameter $\leqslant 1$ such that $M_0 = M$, each M_α is g-embedded in $M_{\alpha+1}$ ($\alpha < \omega_1$), and $M_{\alpha+1}$ is complete Urysohn for α even and ω-homogeneous for α odd. Let $L = M_{\omega_1} = \bigcup_{\alpha<\omega_1} M_\alpha$. Proposition 3.6 implies that each M_α is g-embedded in L. The space L is Urysohn, being the union of the increasing chain $\{M_{2\alpha+1}\colon \alpha < \omega_1\}$ of Urysohn spaces. For similar reasons the space L is ω-homogeneous. Finally, since every countable subset of L is contained in some M_α, $\alpha < \omega_1$, and all spaces $M_{2\alpha+1}$ are complete, every Cauchy sequence in L converges, which means that L is complete. Thus L has the properties required by Theorem 5.1. □

Proof of Theorem 1.7. Let G be a topological group. According to Theorem 2.1, there exists a metric space (M, d) such that $w(M) = w(G)$ and G is isomorphic to a subgroup of $\mathrm{Iso}(M)$. We may assume that M has diameter $\leqslant 1$: otherwise replace the metric d by $\inf(d, 1)$. Theorem 5.1 implies that there exists a complete ω-homogeneous Urysohn metric space L such that $w(L) = w(M)$ and $\mathrm{Iso}(M)$ is isomorphic to a subgroup of $\mathrm{Iso}(L)$. Then $w(L) = w(G)$ and G is isomorphic to a subgroup of $\mathrm{Iso}(L)$, as required. □

6. Semigroups of bi-Katětov functions

Let (M, d) be a complete metric space of diameter $\leqslant 1$.

Definition 6.1. A function $f : M \times M \to I = [0, 1]$ is *bi-Katětov* if for every $x \in M$ the functions $f(x, \cdot)$ and $f(\cdot, x)$ on M are Katětov (see Section 3).

Thus a function $f : M^2 \to I$ is bi-Katětov if and only if for every $x, y, z \in M$ we have

$$\begin{aligned} \left|f(x, y) - f(x, z)\right| &\leqslant d(y, z) \leqslant f(x, y) + f(x, z), \\ \left|f(y, x) - f(z, x)\right| &\leqslant d(y, z) \leqslant f(y, x) + f(z, x). \end{aligned}$$

Let Θ be the compact space of all bi-Katětov functions on M^2, equipped with the topology of pointwise convergence. In the next section we shall prove that the Roelcke completion of the group $\mathrm{Iso}(M)$ can be identified with Θ, provided that the complete metric space M is Urysohn and ω-homogeneous. In the present section we study the structure of an ordered semigroup with an involution on Θ.

Recall that we defined in Section 1 an associative operation $\bullet$ on the set $S = I^{M\times M}$. If $f, g \in S$ and $x, y \in M$, then

$$f \bullet g(x, y) = \inf\bigl\{f(x, z) \uplus g(z, y)\colon z \in M\bigr\}.$$

The involution $f \mapsto f^*$ on S is defined by $f^*(x, y) = f(y, x)$. Every idempotent in S satisfies the triangle inequality. If $f \in S$ is zero on the diagonal of M^2, then f is an idempotent in S if and only if f satisfies the triangle inequality. A function $f \in S$ is a pseudometric on X if and only if f is zero on the diagonal and f is a symmetrical idempotent. In particular, we have $d = d^* = d \bullet d$.

The semigroup S has a natural partial order: for $p, q \in S$ the inequality $p \leqslant q$ means that $p(x, y) \leqslant q(x, y)$ for all $x, y \in M$. This partial order is compatible with the semigroup structure: if $p_1 \leqslant p_2$ and $q_1 \leqslant q_2$, then $p_1 \bullet q_1 \leqslant p_2 \bullet q_2$.

It is clear that the set Θ of all bi-Katětov functions is closed under the involution. It is easy to verify that Θ also is closed under the operation $\bullet$. This fact also can be deduced from the following proposition:

Proposition 6.2. *A function $f : M^2 \to I$ is bi-Katětov if and only if*

$$f \bullet d = d \bullet f = f, \quad f^* \bullet f \geqslant d, \quad f \bullet f^* \geqslant d,$$

where d is the metric on M.

Proof. The condition $f \bullet d = f$ (respectively, $d \bullet f = f$) holds if and only if the function $f(x, \cdot)$ (respectively, $f(\cdot, x)$) is non-expanding for every $x \in X$. The condition $f^* \bullet f \geqslant d$ (respectively, $f \bullet f^* \geqslant d$) holds if and only if $d(y,z) \leqslant f(x,y) + f(x,z)$ (respectively, $d(y,z) \leqslant f(y,x) + f(z,x)$) for all $x, y, z \in X$. □

Let S be any ordered semigroup with an involution, and let $d \in S$ be a symmetrical idempotent. The set S_d of all $x \in S$ such that

$$xd = dx = x, \quad x^*x \geqslant d, \quad xx^* \geqslant d$$

is closed under the multiplication and under the involution and can be considered as a semigroup with the unity d. Indeed, we have $d \in S_d$ since $d = d^* = d^2$, and it is clear that d is the unity of S_d. If $x, y \in S_d$, then $xyd = xy = dxy$ and $(xy)^*xy = y^*x^*xy \geqslant y^*dy = y^*y \geqslant d$; similarly, $xy(xy)^* \geqslant d$ and hence $xy \in S_d$. Thus S_d is a semigroup. If $x \in S_d$, then $x^*d = x^*d^* = (dx)^* = x^*$ and similarly $dx^* = x^*$. It follows that S_d is symmetrical.

The arguments of the preceding paragraph and Proposition 6.2 show that Θ is a semigroup with the unity d. In general, the operation $(f, g) \mapsto f \bullet g$ need not be continuous (not even continuous on the left or on the right).

Proposition 6.3. *Let S be a closed subsemigroup of Θ, and let T be the set of all $f \in S$ such that $f \geqslant d$. If $T \neq \emptyset$, then T has a greatest element p, and p is an idempotent.*

Proof. We claim that every non-empty closed subset of Θ has a maximal element. Indeed, if C is a non-empty linearly ordered subset of Θ, then C has a least upper bound b in Θ, and b belongs to the closure of C. Thus our claim follows from Zorn's lemma.

The set T is a closed subsemigroup of Θ. Let p be a maximal element of T. For every $q \in T$ we have $p \bullet q \geqslant p \bullet d = p$, whence $p \bullet q = p$. It follows that p is idempotent and that $p = p \bullet q \geqslant d \bullet q = q$. Thus p is the greatest element of T. □

We now describe all idempotents in Θ which are $\geqslant d$. For every closed non-empty subset F of M let $b_F \in \Theta$ be the bi-Katětov function defined by $b_F(x,y) = \inf\{d(x,z) \uplus d(z,y)\colon z \in F\}$. If $F = \emptyset$, let $b_F = 1$, that is the function on M^2 which is identically equal to 1. (Note that 1 is *not* the unity of Θ; on the contrary, $f \bullet 1 = 1 \bullet f = 1$ for every $f \in I^{M \times M}$, so 1 might be called a zero element of Θ.)

Proposition 6.4. *If F is a closed subset of M, then b_F is an idempotent $\geqslant d$ in Θ, and every idempotent $\geqslant d$ in Θ is equal to b_F for some closed $F \subset M$.*

Proof. Let F be a closed subset of M. It is clear that $b_F \geqslant d$. If $F \neq \emptyset$, then $b_F \bullet b_F(x,y) = \inf\{d(x,z_1) \uplus d(z_1,u) \uplus d(u,z_2) \uplus d(z_2,y)\colon u \in M,\ z_1, z_2 \in F\} = \inf\{d(x,z) \uplus d(z,y)\colon z \in F\} = b_F(x,y)$ for every $x, y \in M$. Thus b_F is an idempotent. The same is obviously true if $F = \emptyset$.

Conversely, let p be an idempotent in Θ such that $p \geqslant d$. Let $F = \{x \in M\colon p(x,x) = 0\}$. The function $p : M^2 \to I$, being non-expanding in each argument, is continuous, hence F is closed in M. We claim that $p = b_F$.

We first show that $p \leqslant b_F$. This is evident if $F = \emptyset$, so assume that $F \neq \emptyset$. For every $x, y, z \in M$ we have $p(x,y) \leqslant d(x,z) + p(z,y) \leqslant d(x,z) + d(z,y) + p(z,z)$, since the functions $p(\cdot, y)$ and $p(z, \cdot)$ are non-expanding. It follows that $p(x,y) \leqslant \inf(\{d(x,z) + d(z,y) + p(z,z)\colon z \in F\} \cup \{1\}) = b_F(x,y)$.

We prove that $b_F \leqslant p$. Fix $x, y \in M$. We must show that $b_F(x,y) \leqslant p(x,y)$. This is evident if $p(x,y) = 1$, so assume that $p(x,y) < 1$. Fix $\epsilon > 0$ so that $p(x,y) + \epsilon < 1$. Since $p \bullet p = p$, for every $u, v \in M$ we have $p(u,v) = \inf(\{p(u,z) + p(z,v)\colon z \in M\} \cup \{1\})$. Hence we can construct by induction a sequence of points $z_1, z_2, \dots$ in M such that

$$p(x,z_1) + p(z_1,y) < p(x,y) + \epsilon/2,$$

$$p(z_1,z_2) + p(z_2,y) < p(z_1,y) + \epsilon/4,$$

$$p(z_2,z_3) + p(z_3,y) < p(z_2,y) + \epsilon/8,$$

$$\dots$$

$$p(z_n,z_{n+1}) + p(z_{n+1},y) < p(z_n,y) + \epsilon/2^{n+1},$$

$$\dots$$

Adding the first n inequalities, we get

$$p(x,z_1)+\sum_{i=1}^{n-1}p(z_i,z_{i+1})+p(z_n,y)<p(x,y)+\epsilon. \tag{I}$$

It follows that the series $\sum_{i=1}^{\infty}p(z_i,z_{i+1})$ converges. Since $d\leqslant p$, the series $\sum_{i=1}^{\infty}d(z_i,z_{i+1})$ also converges. This implies that the sequence $z_1,z_2,\dots$ is Cauchy and hence has a limit in M. Let $z=\lim z_i$. Since the series $\sum_{i=1}^{\infty}p(z_i,z_{i+1})$ converges, we have $\lim p(z_i,z_{i+1})=0$ and hence $p(z,z)=0$. Thus $z\in F$.

Since p is an idempotent, it satisfies the triangle inequality: $p(u,v)\leqslant p(u,w)+p(w,v)$ for all $u,v,w\in M$. The inequality (I) therefore implies that $p(x,z_n)+p(z_n,y)<p(x,y)+\epsilon$ for every n. Passing to the limit, we get $p(x,z)+p(z,y)\leqslant p(x,y)+\epsilon$. Thus $b_F(x,y)\leqslant d(x,z)+d(z,y)\leqslant p(x,z)+p(z,y)\leqslant p(x,y)+\epsilon$. Since ϵ was arbitrary, it follows that $b_F(x,y)\leqslant p(x,y)$. □

Remark. We shall see later in this section that the elements of Θ (= bi-Katětov functions on M^2) admit a geometric interpretation: they correspond to metric spaces covered by two isometric copies of M. If F is a closed subset of M, the function b_F considered above corresponds to the amalgam of two copies of M with the copies of F amalgamated. This description, together with the geometric description of the operation $\bullet$ on Θ provided in the last paragraph of this section, makes it obvious that each b_F is an idempotent.

Let $G=\mathrm{Iso}(M)$. For every isometry $\varphi\in G$ let $i(\varphi)\in\Theta$ be the bi-Katětov function defined by $i(\varphi)(x,y)=d(x,\varphi(y))$. It is easy to check that the map $i:G\to\Theta$ is a homeomorphic embedding. We claim that the embedding $i:G\to\Theta$ is a morphism of monoids with an involution. This means that $i(e_G)=d$, $i(\varphi^{-1})=i(\varphi)^*$ and $i(\varphi\psi)=i(\varphi)\bullet i(\psi)$ for all $\varphi,\psi\in G$. The first equality is obvious. For the second, note that $i(\varphi^{-1})(x,y)=d(x,\varphi^{-1}(y))=d(y,\varphi(x))=i(\varphi)(y,x)=i(\varphi)^*(x,y)$. For the third, note that $i(\varphi\psi)(x,y)=d(x,\varphi\psi(y))=\inf\{d(x,\varphi(z))+d(\varphi(z),\varphi\psi(y))\colon z\in M\}=\inf\{d(x,\varphi(z))+d(z,\psi(y))\colon z\in M\}=\inf\{i(\varphi)(x,z)+i(\psi)(z,y)\colon z\in M\}=i(\varphi)\bullet i(\psi)(x,y)$.

Thus we can identify G with a subgroup of Θ. There are natural left and right actions of G on Θ, defined by $(g,p)\mapsto g\bullet p$ and $(g,p)\mapsto p\bullet g$ ($g\in G$, $p\in\Theta$), respectively.

Proposition 6.5. *The maps $(g,p)\mapsto g\bullet p$ and $(g,p)\mapsto p\bullet g$ from $G\times\Theta$ to Θ are continuous. If $p\in\Theta$ and $x,y\in M$, then $g\bullet p(x,y)=p(g^{-1}(x),y)$ and $p\bullet g(x,y)=p(x,g(y))$.*

Proof. We have $g\bullet p(x,y)=\inf\{d(x,g(z))\uplus p(z,y)\colon z\in M\}$. Taking $z=g^{-1}(x)$, we see that the right side is $\leqslant p(g^{-1}(x),y)$. On the other hand, for every $z\in M$ we have $d(x,g(z))+p(z,y)=d(g^{-1}(x),z)+p(z,y)\geqslant p(g^{-1}(x),y)$, whence the opposite inequality. The continuity of the left action easily follows from the explicit formula that we have just proved. The argument for the right action is similar. □

Let us show that all invertible elements of Θ are in $i(G)$.

It will be useful to establish a one-to-one correspondence between elements of Θ and other objects which we call M-triples. Let $s=(h_1,h_2,L)$ be a triple such that L is a metric space of diameter $\leqslant 1$, $h_i:M\to L$ is an isometric embedding ($i=1,2$) and $L=h_1(M)\cup h_2(M)$. We say that s is an *M-triple*. Two M-triples (h_1,h_2,L) and (h_1',h_2',L') are *isomorphic* if there exists an isometry $g:L\to L'$ such that $h_i'=gh_i$, $i=1,2$.

Given an M-triple $s=(h_1,h_2,L)$, let $f_s\in\Theta$ be the bi-Katětov function defined by $f_s(x,y)=\rho_L(h_1(x),h_2(y))$, where ρ_L is the metric on L. It is easy to verify that we get in this way a one-to-one correspondence between Θ and the set of classes of isomorphic M-triples. The subset $i(G)$ of Θ corresponds to the set of classes of triples $s=(h_1,h_2,L)$ such that $h_1(M)=h_2(M)=L$. Indeed, if $\varphi\in G$, then for the M-triple $s=(\mathrm{id}_M,\varphi,M)$ we have $f_s=i(\varphi)$. Conversely, every M-triple $s=(h_1,h_2,L)$ such that $h_1(M)=h_2(M)=L$ is isomorphic to the triple $(\mathrm{id}_M,\varphi,M)$, where $\varphi=h_1^{-1}h_2$ is an isometry of M. Thus s corresponds to $\varphi\in G$.

Proposition 6.6. *The set of invertible elements of Θ coincides with $i(G)$.*

Proof. Let $f\in\Theta$ be invertible. Let $s=(h_1,h_2,L)$ be an M-triple corresponding to f. This means that (L,ρ) is a metric space, h_1 and h_2 are distance-preserving maps from M to L, $L=h_1(M)\cup h_2(M)$ and $f(x,y)=$

$\rho(h_1(x), h_2(y))$ for all $x, y \in M$. We saw that elements of G correspond to triples s satisfying the condition $h_1(M) = h_2(M) = L$. Thus we must verify this condition.

Let g be the inverse of f. Then $f \bullet g = g \bullet f = d$. For every $x \in M$ we have $\inf\{f(x, y) + g(y, x)\colon y \in M\} = f \bullet g(x, x) = d(x, x) = 0$ and hence $\rho(h_1(x), h_2(M)) = \inf\{f(x, y)\colon y \in M\} = 0$. This means that $h_1(x)$ belongs to the closure of $h_2(M)$ in L. Since M is complete and h_2 is an isometric embedding, $h_2(M)$ is closed in L. It follows that $h_1(x) \in h_2(M)$. Since $x \in M$ was arbitrary, we have $h_1(M) \subset h_2(M)$. Similarly, $h_2(M) \subset h_1(M)$ and therefore $h_1(M) = h_2(M) = L$. □

The operation $\bullet$ has the following description in terms of M-triples. Let $p, q \in \Theta$. There exists a quadruple $s = (h_1, h_2, h_3, L)$ such that (L, ρ) is a metric space of diameter $\leqslant 1$, $L = L_1 \cup L_2 \cup L_3$, $h_i : M \to L_i$ is an isometry ($i = 1, 2, 3$), $(h_1, h_2, L_1 \cup L_2)$ is an M-triple corresponding to p and $(h_2, h_3, L_2 \cup L_3)$ is an M-triple corresponding to q. The bi-Katětov function f corresponding to the M-triple $(h_1, h_3, L_1 \cup L_3)$ depends on s, and the largest function f over all quadruples s such as above is equal to $p \bullet q$. Indeed, we have $f(x, y) = \rho(h_1(x), h_3(y)) \leqslant \inf\{\rho(h_1(x), h_2(z)) \uplus \rho(h_2(z), h_3(y))\colon z \in M\} = \inf\{p(x, z) \uplus q(z, y)\colon z \in M\} = p \bullet q(x, y)$. To see that the function $p \bullet q$ can be attained, consider two disjoint copies M' and M'' of M. For $x \in M$ denote by x' the copy of x in M', and use similar notation for M''. Let ρ be the pseudometric on $X = M \cup M' \cup M''$ defined by $\rho(x, y) = \rho(x', y') = \rho(x'', y'') = d(x, y)$, $\rho(x, y') = p(x, y)$, $\rho(x', y'') = q(x, y)$ and $\rho(x, y'') = p \bullet q(x, y)$. The triangle inequality for ρ is easily verified. (The space X is the amalgam (in the class of spaces of diameter $\leqslant 1$) of the spaces $M \cup M'$ and $M' \cup M''$ with the subspace M' amalgamated, see [51] for a definition.) Let L be the metric space associated with the pseudometric space (X, ρ). Let L_1, L_2, L_3 be the images of M, M', M'' in L, respectively. Let $h_i : M \to L_i$ be the obvious isometry, $i = 1, 2, 3$. The quadruple $s = (h_1, h_2, h_3, L)$ has the properties considered above, and the bi-Katětov function corresponding to the M-triple $(h_1, h_3, L_1 \cup L_3)$ is equal to $p \bullet q$.

7. The Roelcke compactification of groups of isometries

Let (M, d) be a complete ω-homogeneous Urysohn metric space, and let $G = \mathrm{Iso}(M)$. In the next section we shall prove that G is minimal and topologically simple. The idea of the proof is to explicitly describe the Roelcke compactification of G. It turns out that the Roelcke completion of G can be identified with the compact space Θ of all bi-Katětov functions on M^2.

In the preceding section we defined the embedding $i : G \to \Theta$ by $i(\varphi)(x, y) = d(x, \varphi(y))$. The space Θ, being compact, has a unique compatible uniformity. Let $\mathcal{U}$ be the coarsest uniformity on G which makes the map $i : G \to \Theta$ uniformly continuous. We say that $\mathcal{U}$ is the uniformity induced by i. The uniform space $(G, \mathcal{U})$ is isomorphic to $i(G)$, considered as a uniform subspace of Θ. We are going to prove that $\mathcal{U}$ is the Roelcke uniformity on G (Theorem 7.3).

Let us explain the idea of the proof. Let $\varphi, \varphi' \in G$. We want to prove that φ and φ' are “sufficiently close” in Θ if and only if $\varphi' \in U\varphi U$, where U is a “small” neighbourhood of the unity. Thus we are led to the following question: under what conditions does the equation $\varphi' = \psi_1\varphi\psi_2$ have a solution with “small” ψ_1 and ψ_2? Here “small” means that points of a given finite subset $A \subset M$ are moved by less than ϵ. Observe that similar questions for the equations $\varphi' = \varphi\psi$ or $\varphi' = \psi\varphi$ have an obvious answer: $\varphi' \in \varphi U$ iff φ and φ' move points of A “almost in the same way”, that is, $d(\varphi(x), \varphi'(x)) < \epsilon$ for every $x \in A$; similarly, $\varphi' \in U\varphi$ iff the inverse maps φ^{-1} and φ'^{-1} move points of A “almost in the same way”. The equation $\varphi' = \psi_1\varphi\psi_2$ with two unknowns ψ_1 and ψ_2 looks more complicated. However, the answer to the above question is easy also in this case: the condition $\varphi' \in U\varphi U$ means that the finite metric spaces $A \cup \varphi(A)$ and $A \cup \varphi'(A)$ are close to each other in the Gromov–Hausdorff metric.

We shall need the notion of the Gromov–Hausdorff metric only for finite metric spaces with a given enumeration (it differs from the usual notion dealing with non-enumerated spaces). Let $X = \{x_1, \dots, x_n\}$ and $Y = \{y_1, \dots, y_n\}$ be two such spaces. The *Gromov–Hausdorff distance for enumerated spaces* between X and Y, denoted by $d_{GH}^{en}(X, Y)$, is the infimum of the numbers $\max\{D(x_i, y_i)\colon i = 1, \dots, n\}$, taken over all pseudometrics D on $X \cup Y$ (we assume that X and Y are disjoint) such that D induces the given metrics on X and Y. If X and Y have diameter $\leqslant 1$, we may assume that the same is true for $(X \cup Y, D)$, otherwise replace D by $D \wedge 1$. Since the Urysohn space (M, d) contains an isometric copy of every finite metric space of diameter $\leqslant 1$ (Proposition 1.6), it follows that $d_{GH}^{en}(X, Y)$ is the infimum of the numbers $\max\{d(a_i, b_i)\colon i = 1, \dots, n\}$, where $a_i, b_i \in M$ ($1 \leqslant i \leqslant n$) are such that the correspondences $x_i \mapsto a_i$ and $y_i \mapsto b_i$ are isometric embeddings of X and Y into M, respectively.

Proposition 7.1. *Let (X, d_X) and (Y, d_Y) be two enumerated finite metric spaces, $X = \{x_1, \dots, x_n\}$, $Y = \{y_1, \dots, y_n\}$. Let*

$$\epsilon = \max\{|d_X(x_i, x_j) - d_Y(y_i, y_j)|\colon\ i, j = 1, \dots, n\}.$$

Then $d_{GH}^{en}(X, Y) = \epsilon/2$.

Proof. The inequality $\geqslant$ is obvious: if D is a pseudometric on $X \cup Y$ extending d_X and d_Y and $\epsilon = |d_X(x_i, x_j) - d_Y(y_i, y_j)|$, then at least one of the numbers $D(x_i, y_i)$ and $D(x_j, y_j)$ must be $\geqslant \epsilon/2$. To prove the reverse inequality, we construct a pseudometric D on $Z = X \cup Y$ extending d_X and d_Y such that

$$D(x_i, y_i) = \epsilon/2, \quad i = 1, \dots, n.$$

The function D is defined by these requirements on X^2, Y^2, and the set $\{(x_i, y_i)\colon\ i = 1, \dots, n\}$. To see that D can be extended to a pseudometric on Z, it suffices to verify that for any sequence $z_1, \dots, z_s$ of points of Z such that all the expressions $D(z_i, z_{i+1})$ $(1 \leqslant i < s)$ and $D(z_1, z_s)$ are defined the inequality

$$D(z_1, z_s) \leqslant \sum_{i=1}^{s-1} D(z_i, z_{i+1}) \tag{A}$$

holds. Then the required extension is given by the formula

$$D(z, z') = \inf \sum_{i=1}^{s-1} D(z_i, z_{i+1}),$$

where the infimum is taken over all chains $z_1 = z, z_2, \dots, z_s = z'$ such that all the terms $D(z_i, z_{i+1})$ are defined. An easy argument using induction shows that (A) follows from its special case: for any "quadrangle" in Z of the form x_i, y_i, y_j, x_j each of the four numbers $d_X(x_i, x_j)$, $D(x_i, y_i)$, $d_Y(y_i, y_j)$, and $D(x_j, y_j)$ does not exceed the sum of the three others. This case is obvious: for example, since $d_X(x_i, x_j) - d_Y(y_i, y_j) \leqslant \epsilon$, we have

$$d_X(x_i, x_j) \leqslant d_Y(y_i, y_j) + \epsilon = D(x_i, y_i) + d_Y(y_i, y_j) + D(x_j, y_j). \qquad \square$$

Corollary 7.2. *Let (X, d) be an Urysohn metric space. Let $a_1, \dots, a_n, b_1, \dots, b_n \in X$, and suppose that*

$$|d(a_i, a_j) - d(b_i, b_j)| \leqslant 2\epsilon$$

for all $i, j = 1, \dots, n$. Then there exist points $c_1, \dots, c_n \in X$ such that $d(c_i, c_j) = d(b_i, b_j)$ and $d(a_i, c_i) \leqslant \epsilon$ for all $i, j = 1, \dots, n$.

We now are in a position to prove the main result of this section. Recall that (M, d) is a complete ω-homogeneous Urysohn metric space, $G = \mathrm{Iso}(M)$, and Θ is the space of bi-Katětov functions on M^2 considered in the previous section.

Theorem 7.3. *The range of the embedding $i : G \to \Theta$ is dense in Θ. The uniformity $\mathcal{U}$ on G induced by the embedding i coincides with the Roelcke uniformity $\mathcal{L} \wedge \mathcal{R}$. Therefore, G is Roelcke-precompact, and the Roelcke compactification of G can be identified with Θ.*

Proof. If A is a finite subset of M and $\epsilon > 0$, let $U_{A,\epsilon} = \{\psi \in G\colon\ d(\psi(x), x) < \epsilon$ for every $x \in A\} \in \mathcal{N}(G)$. Let $W_{A,\epsilon}$ be the set of all pairs $(f, g) \in \Theta^2$ such that $|f(x, y) - g(x, y)| < \epsilon$ for all $x, y \in A$. The sets of the form $W_{A,\epsilon}$ constitute a base of entourages of the uniformity on Θ. If $(f, g) \in W = W_{A,\epsilon}$, we say that f and g are *W-close*. Our proof proceeds in three parts.

(a) We prove that $i(G)$ is dense in Θ. Let $f \in \Theta$, and let Of be a neighbourhood of f in Θ. We must prove that $i(\varphi) \in Of$ for some $\varphi \in G$.

We may assume that Of is the set of all $g \in \Theta$ such that g is $W_{A,\epsilon}$-close to f:

$$Of = \{g \in \Theta\colon\ |g(x, y) - f(x, y)| < \epsilon \text{ for all } x, y \in A\},$$

where A is a finite subset of M and $\epsilon > 0$. Let $A = \{a_1, \ldots, a_n\}$. We claim that there exist points $b_1, \ldots, b_n \in M$ such that $d(b_i, b_j) = d(a_i, a_j)$ and $d(a_i, b_j) = f(a_i, a_j)$, $1 \leqslant i, j \leqslant n$. Indeed, since f is bi-Katětov, the formulas above define a pseudometric on the set $F = \{a_1, \ldots, a_n, b_1, \ldots, b_n\}$, where $b_1, \ldots, b_n$ are new points. Since M is Urysohn, the embedding of A into M extends to a distance-preserving map from F to M.

Since M is ω-homogeneous, there exists an isometry φ of M such that $\varphi(a_i) = b_i$, $1 \leqslant i \leqslant n$. Let $g = i(\varphi)$. For every $i, j \in [1, n]$ we have $g(a_i, a_j) = d(a_i, \varphi(a_j)) = d(a_i, b_j) = f(a_i, a_j)$. Thus $g \in Of$. This proves that $i(G)$ is dense in Θ.

(b) We prove that the uniformity $\mathcal{U}$ is coarser than $\mathcal{L} \wedge \mathcal{R}$.

Whenever a topological group H acts continuously on a compact space X (on the left), for every $x \in X$ the orbit map $h \mapsto hx$ from H to X is right-uniformly continuous. We saw that G acts continuously on Θ (Proposition 6.5). The embedding $i : G \to \Theta$ can be viewed as the orbit map corresponding to d, the neutral element of Θ. It follows that i is $\mathcal{R}$-uniformly continuous. Similarly, i is $\mathcal{L}$-uniformly continuous (use the right action of G on Θ, or, alternatively, use the involution on Θ to deduce $\mathcal{L}$-uniform continuity of i from its $\mathcal{R}$-uniform continuity). Therefore, the uniformity $\mathcal{U}$ is coarser than both $\mathcal{L}$ and $\mathcal{R}$ and hence coarser than $\mathcal{L} \wedge \mathcal{R}$.

(c) We prove that $\mathcal{U}$ is finer than $\mathcal{L} \wedge \mathcal{R}$. It suffices to show that for every $U \in \mathcal{N}(G)$ there exists an entourage W of the uniformity on Θ (in other words, a neighbourhood of the diagonal of Θ^2) with the following property: if $\varphi, \varphi' \in G$ are such that $i(\varphi)$ and $i(\varphi')$ are W-close, then $\varphi' \in U\varphi U$. Assume that $U = U_{A,\epsilon}$. We claim that $W = W_{A,2\epsilon}$ has the required property.

Let $\varphi, \varphi' \in G$ be such that $i(\varphi)$ and $i(\varphi')$ are $W_{A,2\epsilon}$-close. This means that

$$\delta = \max\left\{\left|d\left(x, \varphi(y)\right) - d\left(x, \varphi'(y)\right)\right|:\ x, y \in A\right\} < 2\epsilon.$$

Let $A = \{a_1, \ldots, a_n\}$, $b_i = \varphi(a_i)$ and $c_i = \varphi'(a_i)$, $i = 1, \ldots, n$. We have $d(b_i, b_j) = d(a_i, a_j) = d(c_i, c_j)$ and $|d(a_i, b_j) - d(a_i, c_j)| \leqslant \delta$ for all i and j. In virtue of Corollary 7.2, there exist points $a'_1, \ldots, a'_n, b'_1, \ldots, b'_n \in M$ such that the correspondence $a_i \mapsto a'_i$, $b_i \mapsto b'_i$ is distance-preserving and $d(a'_i, a_i) \leqslant \delta/2 < \epsilon$, $d(b'_i, c_i) \leqslant \delta/2 < \epsilon$.

Since M is ω-homogeneous, there exists an isometry ψ_1 of M such that $\psi_1(a_i) = a'_i$ and $\psi_1(b_i) = b'_i$, $i = 1, \ldots, n$. We have $\psi_1 \in U$, since each a_i is moved by less than ϵ. Put $\psi_2 = \varphi^{-1}\psi_1^{-1}\varphi'$. For every $i = 1, \ldots, n$ we have $d(\psi_2(a_i), a_i) = d(\varphi'(a_i), \psi_1\varphi(a_i)) = d(c_i, b'_i) < \epsilon$, hence $\psi_2 \in U = U_{A,\epsilon}$. Thus $\varphi' = \psi_1\varphi\psi_2 \in U\varphi U$, as required. □

Recall that a non-empty collection $\mathcal{F}$ of non-empty subsets of a set X is a *filter base* on X if for every $A, B \in \mathcal{F}$ there is $C \in \mathcal{F}$ such that $C \subset A \cap B$. If X is a topological space, $\mathcal{F}$ is a filter base on X and $x \in X$, then x is a *cluster point* of $\mathcal{F}$ if every neighbourhood of x meets every member of $\mathcal{F}$, and $\mathcal{F}$ *converges* to x if every neighbourhood of x contains a member of $\mathcal{F}$. If $\mathcal{F}$ and $\mathcal{G}$ are two filter bases on G, let $\mathcal{F}\mathcal{G} = \{AB:\ A \in \mathcal{F},\ B \in \mathcal{G}\}$.

For every $p \in \Theta$ let $\mathcal{F}_p = \{G \cap V:\ V$ is a neighbourhood of p in $\Theta\}$. In other words, $\mathcal{F}_p$ is the trace on G of the filter of neighbourhoods of p in Θ. If $p, q \in \Theta$, it is not true in general that $\mathcal{F}_p\mathcal{F}_q$ converges to $p \bullet q$. However, we have the following result, which will be used in the proof of Theorem 1.8:

Proposition 7.4. *If $p, q \in \Theta$, then $p \bullet q$ is a cluster point of the filter base $\mathcal{F}_p\mathcal{F}_q$.*

Proof. Let U_1, U_2, U_3 be neighbourhoods of p, q and $p \bullet q$, respectively. We must show that U_3 meets the set $(U_1 \cap i(G))(U_2 \cap i(G))$.

We may assume that for some finite set $A = \{a_1, \ldots, a_n\} \subset M$ and $\epsilon > 0$ we have

$$U_1 = \left\{f \in \Theta:\ \left|f(x, y) - p(x, y)\right| < \epsilon \text{ for all } x, y \in A\right\},$$
$$U_2 = \left\{f \in \Theta:\ \left|f(x, y) - q(x, y)\right| < \epsilon \text{ for all } x, y \in A\right\},$$
$$U_3 = \left\{f \in \Theta:\ \left|f(x, y) - p \bullet q(x, y)\right| < \epsilon \text{ for all } x, y \in A\right\}.$$

We saw in the last paragraph of the preceding section that there exist a metric space (L, ρ) and isometric embeddings $h_i : M \to L$ $(i = 1, 2, 3)$ such that $p(x, y) = \rho(h_1(x), h_2(y))$, $q(x, y) = \rho(h_2(x), h_3(y))$ and $p \bullet q(x, y) = \rho(h_1(x), h_3(y))$ for all $x, y \in M$. Let $X = h_1(A) \cup h_2(A) \cup h_3(A)$. Since M is Urysohn, there exists an isometric embedding of X into M which extends the isometry $h_1^{-1} : h_1(A) \to A$. It follows that there exist points $b_1, \ldots, b_n, c_1 \ldots, c_n \in M$ such that $d(b_i, b_j) = d(c_i, c_j) = d(a_i, a_j)$, $d(a_i, b_j) = p(a_i, a_j)$, $d(b_i, c_j) = q(a_i, a_j)$ and

$d(a_i, c_j) = p \bullet q(a_i, a_j)$ for all i, j. Since M is ω-homogeneous, there exists an isometry $\varphi \in G$ such that $\varphi(a_i) = b_i$, $1 \leqslant i \leqslant n$. Let $x_i = \varphi^{-1}(c_i)$. Using again the ω-homogeneity of M, we find an isometry $\psi \in G$ such that $\psi(a_i) = x_i$, $1 \leqslant i \leqslant n$. Note that $\varphi\psi(a_i) = c_i$ and $d(a_i, x_j) = d(\varphi(a_i), \varphi(x_j)) = d(b_i, c_j) = q(a_i, a_j)$ for all i, j. We claim that $i(\varphi) \in U_1$, $i(\psi) \in U_2$ and $i(\varphi\psi) \in U_3$. Indeed, we have $i(\varphi)(x, y) = d(x, \varphi(y)) = p(x, y)$ for all $x, y \in A$ and hence $i(\varphi) \in U_1$. The other two cases are considered similarly. Thus $i(\varphi\psi) \in ((U_1 \cap i(G))(U_2 \cap i(G))) \cap U_3 \neq \emptyset$. □

If H is a group and $g \in H$, we denote by l_g (respectively, r_g) the left shift of H defined by $l_g(h) = gh$ (respectively, the right shift defined by $r_g(h) = hg$).

Proposition 7.5. *Let H be a topological group, and let K be the Roelcke completion of H. Let $g \in H$. Each of the following self-maps of H extends to a self-homeomorphism of K*:

(1) *the left shift l_g*;
(2) *the right shift r_g*;
(3) *the inversion $g \mapsto g^{-1}$*;
(4) *the inner automorphism $h \mapsto ghg^{-1}$.*

Proof. Let $\mathcal{L}$ and $\mathcal{R}$ be the left and the right uniformity on H, respectively. In each of the cases (1)–(4) the map $f : H \to H$ under consideration is an automorphism of the uniform space $(H, \mathcal{L} \wedge \mathcal{R})$. This is obvious for the cases (3) and (4). For the cases (1) and (2), observe that the uniformities $\mathcal{L}$ and $\mathcal{R}$ are invariant under left and right shifts, hence the same is true for their greatest lower bound $\mathcal{L} \wedge \mathcal{R}$. It follows that in all cases f extends to an automorphism of the completion K of the uniform space $(H, \mathcal{L} \wedge \mathcal{R})$. □

For the group G and its Roelcke completion Θ the validity of Proposition 7.5 can be seen directly. Recall that the embedding $i : G \to \Theta$ is a morphism of monoids with an involution (see the two paragraphs before Proposition 6.5). The involution $f \mapsto f^*$ on Θ is continuous and hence coincides with the extension of the inversion on G given by Proposition 7.5. For every $g \in G$ let L_g, R_g and Inn_g be the self-maps of Θ defined by $L_g(p) = g \bullet p$, $R_g(p) = p \bullet g$ and $\mathrm{Inn}_g(p) = g \bullet p \bullet g^{-1}$. These maps are extensions over Θ of the left shift l_g of G, the right shift r_g, and the inner automorphism $l_g \circ r_{g^{-1}}$, respectively. In virtue of Proposition 6.5, the maps L_g and R_g are continuous, and the same is true for $\mathrm{Inn}_g = L_g \circ R_{g^{-1}}$.

An *inner automorphism* of Θ is a map of the form Inn_g, $g \in G$. Proposition 6.5 shows that $\mathrm{Inn}_g(p)(x, y) = p(g^{-1}(x), g^{-1}(y))$ for all $p \in \Theta$ and $x, y \in M$. It follows that for every closed $F \subset M$ we have $\mathrm{Inn}_g(b_F) = b_{g(F)}$, where b_F is the idempotent corresponding to F (see Proposition 6.4).

Proposition 7.6. *There are precisely two idempotents in Θ which are $\geqslant d$ and are invariant under all inner automorphisms*: *the unity d and the constant* 1.

Proof. According to Proposition 6.4, every idempotent $\geqslant d$ is of the form b_F for some closed $F \subset M$. If b_F is invariant under inner automorphisms, then $b_{g(F)} = \mathrm{Inn}_g(b_F) = b_F$ and hence $g(F) = F$ for every $g \in G$. Since the action of G on M is transitive, no proper non-empty subset of M is G-invariant. Thus either $F = M$ or $F = \emptyset$. Accordingly, either $b_F = d$ or $b_F = 1$. □

8. Proof of Theorem 1.8

We preserve the notation of the preceding section: M is a complete ω-homogeneous Urysohn metric space, $G = \mathrm{Iso}(M)$, Θ is the set of all bi-Katětov functions on M^2. We saw that G is Roelcke-precompact and that Θ can be identified with the Roelcke compactification of G (Theorem 7.3). In this section we prove that G is minimal and topologically simple.

Proposition 8.1. *For every topological group H the following conditions are equivalent*:

(1) *H is minimal and topologically simple*;

(2) *if* $f: H \to H'$ *is a continuous onto homomorphism of topological groups, then either* f *is a homeomorphism or* $|H'| = 1$.

Proposition 8.2. *The group* G *has no compact normal subgroups other than* $\{e\}$.

We shall prove later that actually G has no non-trivial closed normal subgroups.

Proof. Let $H \neq \{e\}$ be a normal subgroup of G. We show that H is not compact.

Fix $a \in M$ and $f \in H$ such that $f(a) \neq a$. Let $r = d(f(a), a)$, and let $S = \{x \in M\colon\ d(x, a) = r\}$ be the sphere of radius r centered at a. We claim that the orbit Ha contains S. Fix $x \in S$. Since M is ω-homogeneous, there exists an isometry $g \in G$ which leaves the point a fixed and maps $f(a)$ to x. Let $h = gfg^{-1}$. Since H is normal, we have $h \in H$ and hence $x = h(a) \in Ha$. Thus $S \subset Ha$, as claimed.

Since M is Urysohn, we can construct by induction an infinite sequence $x_1, x_2, \ldots$ of points in S such that all the pairwise distances between distinct members of this sequence are equal to r. Since $S \subset Ha$, it follows that Ha is not compact. Hence H is not compact. □

Let (L, ρ) be a metric space. A self-map $f : L \to L$ is *non-expanding* if $\rho(f(x), f(y)) \leqslant \rho(x, y)$ for all $x, y \in L$.

Lemma 8.3. *Let* (L, ρ) *be a metric space, and let* F *be the semigroup of all non-expanding self-maps of* L*, equipped with the topology of pointwise convergence. Then the map* $(f, g) \mapsto f \circ g$ *from* F^2 *to* F *is continuous. Thus* F *is a topological semigroup.*

This lemma and Proposition 8.4 below are well known. We include a proof for the reader's convenience.

Proof. It suffices to show that for every $x \in L$ the map $(f, g) \mapsto f(g(x))$ from F^2 to L is continuous. Fix $f_0, g_0 \in F$, $x \in L$ and $\epsilon > 0$. Let $y = g_0(x)$, $Of_0 = \{f \in F\colon\ \rho(f(y), f_0(y)) < \epsilon\}$ and $Og_0 = \{g \in F\colon\ \rho(g(x), y) < \epsilon\}$. If $f \in Of_0$ and $g \in Og_0$, then $\rho(f(g(x)), f_0(g_0(x))) \leqslant \rho(f(g(x)), f(y)) + \rho(f(y), f_0(y)) < \rho(g(x), y) + \epsilon < 2\epsilon$. □

Proposition 8.4. *If* L *is a complete metric space, then the group* $\mathrm{Iso}(L)$ *is complete.*

Recall that we call a topological group complete if it is complete with respect to the upper uniformity.

Proof. Let $X = L^L$ be the set of all self-maps of L, equipped with the product uniformity. The group $H = \mathrm{Iso}(L)$ can be considered as a subset of X. The uniformity $\mathcal{U}$ on H induced by the product uniformity on X coincides with the left uniformity $\mathcal{L}$. Indeed, a basic entourage for $\mathcal{U}$ has the form $W_{A,\epsilon} = \{(f, g) \in H^2\colon\ \rho(f(x), g(x)) < \epsilon$ for all $x \in A\}$, where ρ is the metric on L, A is a finite subset of L and $\epsilon > 0$. Let $U_{A,\epsilon} = \{f \in H\colon\ \rho(f(x), x) < \epsilon$ for all $x \in A\}$. Then $U_{A,\epsilon}$ is a basic neighbourhood of unity in H, and $W_{A,\epsilon} = \{(f, g) \in H^2\colon\ g^{-1}f \in U_{A,\epsilon}\}$ is a basic entourage for $\mathcal{L}$. Thus $\mathcal{U} = \mathcal{L}$. It follows that the map $g \to g^{-1}$ from H to X induces the right uniformity on H, and the map $j : H \to X^2$ defined by $j(g) = (g, g^{-1})$ induces the upper uniformity $\mathcal{L} \vee \mathcal{R}$. Since X^2 is complete, to prove that H is complete it suffices to show that $j(H)$ is closed in X^2. Let F be the set of all non-expanding self-maps of L. Then F is closed in X. The map $(f, g) \mapsto f \circ g$ from F^2 to F is continuous (Lemma 8.3). Since $j(G) = \{(f, g) \in F^2\colon\ fg = gf = \mathrm{id}_L\}$, it follows that $j(G)$ is closed in F^2 and hence in X^2. □

We say that a metric space L is *homogeneous* if every point of L can be mapped to every other point by an isometry of L onto itself.

Lemma 8.5. *If* L *is a homogeneous metric space, then* $w(\mathrm{Iso}(L)) = w(L)$.

Proof. For every metric space X we have $w(\mathrm{Iso}(X)) \leqslant w(X)$. If X is homogeneous, then for every $a \in X$ the map $f \to f(a)$ from $\mathrm{Iso}(X)$ to X is onto, whence $w(X) \leqslant w(\mathrm{Iso}(X))$. □

We are now ready to prove Theorem 1.8:

If M is a complete ω-homogeneous Urysohn metric space, then the group $G = \mathrm{Iso}(M)$ is complete, Roelcke-precompact, minimal and topologically simple. The weight of G is equal to the weight of M.

Proof. We saw that G is Roelcke-precompact (Theorem 7.3). Proposition 8.4 shows that G is complete, and Lemma 8.5 shows that $w(G) = w(M)$. Let $f : G \to G'$ be a continuous onto homomorphism. According to Proposition 8.1, to prove that G is minimal and topologically simple, it suffices to prove that either f is a homeomorphism or $|G'| = 1$.

Since G is Roelcke-precompact, so is G'. Let Θ' be the Roelcke compactification of G'. The homomorphism f extends to a continuous map $F : \Theta \to \Theta'$. Let e' be the unity of G', and let $S = F^{-1}(e') \subset \Theta$.

Claim 1. S is a subsemigroup of Θ.

Let $p, q \in S$. In virtue of Proposition 7.4, there exist filter bases $\mathcal{F}_p$ and $\mathcal{F}_q$ on G such that $\mathcal{F}_p$ converges to p (in Θ), $\mathcal{F}_q$ converges to q and $p \bullet q$ is a cluster point of the filter base $\mathcal{F}_p\mathcal{F}_q$. The filter bases $\mathcal{F}'_p = F(\mathcal{F}_p)$ and $\mathcal{F}'_q = F(\mathcal{F}_q)$ on G' converge to $F(p) = F(q) = e'$, hence the same is true for the filter base $\mathcal{F}'_p\mathcal{F}'_q = F(\mathcal{F}_p\mathcal{F}_q)$. Since $p \bullet q$ is a cluster point of $\mathcal{F}_p\mathcal{F}_q$, $F(p \bullet q)$ is a cluster point of the convergent filter base $F(\mathcal{F}_p\mathcal{F}_q)$. A convergent filter on a Hausdorff space has only one cluster point, namely the limit. Thus $F(p \bullet q) = e'$ and hence $p \bullet q \in S$.

Claim 2. The semigroup S is closed under involution.

In virtue of Proposition 7.5, the inversion on G' extends to an involution $x \mapsto x^*$ of Θ'. Since $F(p^*) = F(p)^*$ for every $p \in G$, the same holds for every $p \in \Theta$. Let $p \in S$. Then $F(p^*) = F(p)^* = e'$ and hence $p^* \in S$.

Claim 3. If $g \in G$ and $g' = f(g)$, then $F^{-1}(g') = g \bullet S = S \bullet g$.

We saw that the left shift $h \mapsto gh$ of G extends to a continuous self-map $L = L_g$ of Θ defined by $l(p) = g \bullet p$ (Proposition 6.5). According to Proposition 7.5, the self-map $x \mapsto g'x$ of G' extends to a self-homeomorphism L' of Θ'. The maps $F \circ L$ and $L' \circ F$ from Θ to Θ' coincide on G and hence everywhere. Replacing g by g^{-1}, we see that $F \circ L^{-1} = (L')^{-1} \circ F$. Thus $F^{-1}(g') = F^{-1}L'(e') = LF^{-1}(e') = g \bullet S$. Using right shifts, we similarly conclude that $F^{-1}(g') = S \bullet g$.

Claim 4. S is invariant under inner automorphisms of Θ.

We have just seen that $g \bullet S = S \bullet g$ for every $g \in G$, hence $g \bullet S \bullet g^{-1} = S$.

Let $T = \{f \in S\colon\ f \geqslant d\}$. Note that $i(e_G) = d \in T \neq \emptyset$. According to Proposition 6.3, there is a greatest element p in T, and p is idempotent. Since inner automorphisms of Θ preserve the order on Θ and the unity d, Claim 4 implies that p is invariant under inner automorphisms. In virtue of Proposition 7.6, either $p = d$ or $p = 1$. We shall show that either f is a homeomorphism or $|G'| = 1$, according to which of the cases $p = d$ or $p = 1$ holds.

Consider first the case $p = d$.

Claim 5. If $p = d$, then all elements of S are invertible in Θ.

Let $f \in S$. Then $f^* \bullet f \in S$ and $f \bullet f^* \in S$, since S is a symmetrical semigroup. According to Proposition 6.2, we have $f^* \bullet f \geqslant d$ and $f \bullet f^* \geqslant d$. Since $p = d$, there are no elements $> d$ in S. Thus the inequalities $f^* \bullet f \geqslant d$ and $f \bullet f^* \geqslant d$ are actually equalities. It follows that f^* is the inverse of f.

Claim 6. If $p = d$, then $S = \{e\}$.

Claim 5 and Proposition 6.6 imply that S is a subgroup of G. This subgroup is normal (Claim 4) and compact, since S is closed in Θ. Proposition 8.2 implies that $S = \{e\}$.

Claim 7. If $p = d$, then $f : G \to G'$ is a homeomorphism.

Claims 6 and 3 imply that $G = F^{-1}(G')$ and that the map $f : G \to G'$ is bijective. Since F is a map between compact spaces, it is perfect, and hence so is the map $f : G = F^{-1}(G') \to G'$. Thus f, being a perfect bijection, is a homeomorphism.

Now consider the case $p = 1$.

Claim 8. If $1 \in S$, then $G' = \{e'\}$.

Let $g \in G$ and $g' = f(g)$. We have $g \bullet 1 = 1 \in S$. On the other hand, Claim 3 implies that $g \bullet 1 \in g \bullet S = F^{-1}(g')$. Thus $g' = F(g \bullet 1) = F(1) = e'$. □

9. Remarks

1. Let M be a complete ω-homogeneous Urysohn metric space, and let $G = \mathrm{Iso}(M)$. In Section 7 we identified the Roelcke completion of G with the set Θ of all bi-Katětov functions on M^2. The set Θ was equipped with structures of three kinds: topology, order, semigroup structure. The proof of Theorem 1.8 was based on the interplay between these three structures. We now establish a natural one-to-one correspondence between Θ and a set of closed relations on a compact space. This correspondence will be an isomorphism for all three structures on Θ.

Let K be a compact space. A *closed relation* on K is a closed subset of K^2. Let $E(K)$ be the compact space of all closed relations on K, equipped with the Vietoris topology. The set $E(K)$ has a natural partial order. If $R, S \in E(K)$, then the composition $R \circ S$ is a closed relation, since $R \circ S$ is the image of the closed subset $\{(x, z, y)\colon (x, z) \in S,\ (z, y) \in R\}$ of K^3 under the projection $K^3 \to K^2$ which is a closed map. Thus $E(K)$ is a semigroup with involution. In general the map $(R, S) \mapsto R \circ S$ from $E(K)^2$ to $E(K)$ is not separately continuous (neither left nor right continuous).

We denote by $\mathrm{Homeo}(K)$ the group of all self-homeomorphisms of K, equipped with the compact-open topology. For every $h \in \mathrm{Homeo}(K)$ let $\Gamma(h) = \{(x, h(x))\colon x \in K\}$ be the graph of h. The map $h \mapsto \Gamma(h)$ from $\mathrm{Homeo}(K)$ to $E(K)$ is a homeomorphic embedding and a morphism of monoids with an involution. The uniformity induced on $\mathrm{Homeo}(K)$ by this embedding is coarser than the Roelcke uniformity.

Now let K be the compact space of all non-expanding functions $f : M \to I = [0, 1]$, considered as a subspace of the product I^M. There is a natural left action of G on K, defined by $gf(x) = f(g^{-1}(x))$ $(g \in G,\ f \in K,\ x \in M)$. This action gives rise to a morphism $G \to \mathrm{Homeo}(K)$ of topological groups which is easily seen to be a homeomorphic embedding. Let $j : G \to E(K)$ be the composition of this embedding with the map $h \mapsto \Gamma(h)$ from $\mathrm{Homeo}(K)$ to $E(K)$. If $g \in G$, then $j(g)$ is the relation $\{(f, gf)\colon f \in K\}$. Let Φ be the closure of $j(G)$ in $E(K)$. Let Θ and $i : G \to \Theta$ be the same as in Sections 6 and 7.

Theorem 9.1. *The uniformity on G induced by the embedding $j : G \to E(K)$ coincides with the Roelcke uniformity, hence Φ can be identified with the Roelcke compactification of G. The set Φ is a subsemigroup of $E(K)$. There exists a unique homeomorphism $H : \Phi \to \Theta$ such that $i = Hj$. The map H is an isomorphism of ordered semigroups.*

We omit the detailed proof and confine ourselves by a description of the isomorphism H. If $R \in \Phi$, let $H(R)$ be the bi-Katětov function on M^2 defined by $H(R)(x, y) = \sup\{|q(x) - p(y)|\colon (p, q) \in R\}$, $x, y \in M$. If $f \in \Theta$, the relation $H^{-1}(f)$ is defined by $H^{-1}(f) = \{(p, q) \in K^2\colon |q(x) - p(y)| \leqslant f(x, y) \text{ for all } x, y \in M^2\}$.

Let us see what some of the results about Θ obtained in Section 6 mean in terms of relations on K. Functions $p \in \Theta$ which are $\geqslant d$ correspond (via the isomorphism H) to relations $R \in \Phi$ which contain the diagonal of K^2 or, in other words, are reflexive. Thus Proposition 6.2 implies that for every $R \in \Phi$ the relations RR^{-1} and $R^{-1}R$ are reflexive. This is equivalent to the fact that for every $R \in \Phi$ the domain and the range of R is equal to K.

According to Proposition 6.4, each idempotent $\geqslant d$ in Θ has the form b_F for some closed $F \subset M$. Note that each b_F is symmetrical. Symmetrical idempotents $\geqslant d$ in Θ correspond to relations in Φ which are reflexive, symmetrical and transitive or, in other words, are equivalence relations. For every closed $F \subset M$ let $R_F = H(b_F)$ be the equivalence relation corresponding to the idempotent b_F. Two non-expanding functions $f, g \in K$ are R_F-equivalent if and only if $f|F = g|F$. Proposition 6.4 implies that an equivalence relation R on K belongs to Φ if and only if $R = R_F$ for some closed $F \subset M$.

2. Let H be a Hilbert space, and let $G = U(H)$ be the group of all unitary operators on H, equipped with the pointwise convergence topology. L. Stoyanov proved that G is totally minimal [42,9]. The methods of the present paper yield an alternative proof of this theorem. Let $\mathcal{B}(H)$ be the algebra of all bounded linear operators on H. The *weak operator topology* on $\mathcal{B}(H)$ is the coarsest topology such that for every $x, y \in H$ the function $A \mapsto (Ax, y)$ on $\mathcal{B}(H)$ is continuous. Let $T = \{A\colon \|A\| \leqslant 1\}$ be the unit ball in $\mathcal{B}(H)$, equipped with the weak operator topology. The Roelcke compactification of the group G can be identified with T [46]. The set T is a semigroup, and the idempotents in T are the orthogonal projectors. The proof of the fact that G is totally minimal proceeds similarly to the proof of Theorem 1.8. Let us indicate the main steps. Let $f : G \to G'$ be a surjective morphism of topological groups. To prove that G is totally minimal, it suffices to prove that f is a quotient map. Extend f to a map $F : T \to T'$, where T' is the Roelcke compactification of G'. Let e' be the unity of G', and let $S = F^{-1}(e')$ be the kernel of F. Then S is a closed subsemigroup of T. It turns out that every closed subsemigroup of T contains a least idempotent. Let p be the least idempotent in S. Since S is invariant under inner automorphisms of Θ, so is p. It follows that either $p = 1$ or $p = 0$. If $p = 1$, then $S \subset G$, $G = F^{-1}(G')$ and the map f is perfect. If $p = 0$, then $G' = \{e'\}$. See [46] for more details.

3. Our method of proving minimality, based on the consideration of the Roelcke compactifications, can be applied to some groups of homeomorphisms. A zero-dimensional compact space X is *h-homogeneous* if all non-empty clopen subsets of X are homeomorphic to each other. Let K be a zero-dimensional h-homogeneous compact space, and let $G = \mathrm{Homeo}(K)$. Then G is minimal and topologically simple [49]. Let us sketch a proof of this fact which closely follows the proof of Theorem 1.8. In the special case when $K = 2^\omega$ is the Cantor set, the minimality of $\mathrm{Homeo}(K)$ was proved by Gamarnik [11].

Let T be the compact space of all closed relations R on K such that the domain and the range of R are equal to K. The map $h \mapsto \Gamma(h)$ from G to T induces the Roelcke uniformity on G, and the range $\Gamma(G)$ of this map is dense in T. Thus the Roelcke compactification of G can be identified with T. We noted that the set $E(K)$ of all closed relations on K is an ordered semigroup with an involution. The set T is a closed symmetrical subsemigroup of $E(K)$. Let Δ be the diagonal in K^2. A relation $R \in E(K)$ is an equivalence relation if and only if R is a symmetrical idempotent and $R \supset \Delta$. Let S be a closed subsemigroup of $E(K)$, and let S_1 be the set of all $R \in S$ such that $R \supset \Delta$. The proof of Proposition 6.3 shows that the set S_1, if it is non-empty, has a largest element P, and P is an idempotent. If S is symmetrical, then so is P, hence P is an equivalence relation.

Now let $f : G \to G'$ be a surjective morphism of topological groups. We show that either f is a homeomorphism or $|G'| = 1$. According to Proposition 8.1, this means that G is minimal and topologically simple. Extend f to a map $F : T \to T'$, where T' is the Roelcke compactification of G'. Let e' be the unity of G', and let $S = F^{-1}(e')$. Then S is a closed symmetrical subsemigroup of T. Let P be the largest element in the set $S_1 = \{R \in S\colon \Delta \subset R\}$. Then P is an equivalence relation on K. Since S is G-invariant, so is P. But there are only two G-invariant closed equivalence relations on K, namely Δ and K^2. If $P = \Delta$, then $S \subset G$, $G = F^{-1}(G')$ and f is perfect. Since G has no non-trivial compact normal subgroups, we conclude that f is a homeomorphism. If $P = K^2$, then $S = T$ and $G' = \{e'\}$.

It is not clear if a similar argument can be used when K is a Hilbert cube and $G = \mathrm{Homeo}(K)$, see Problem 1.4.

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